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Eigenvalue problem for fully nonlinear second-order elliptic PDE on balls

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Abstract

We study the eigenvalue problem for positively homogeneous, of degree one, elliptic ODE on finite intervals and PDE on balls. We establish the existence and completeness results for principal and higher eigenpairs, i.e., pairs of an eigenvalue and its corresponding eigenfunction.

1. Introduction

We consider the eigenvalue problem for fully nonlinear elliptic PDE

$$\begin{cases} F\left(D^2u, Du, u, x\right) + \mu u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $u : \bar{\Omega} \to \mathbb{R}$ and $\mu \in \mathbb{R}$ represent the unknown function (eigenfunction) and constant (eigenvalue), respectively, and $F : \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, where $\mathbb{S}^N$ denotes the space of real symmetric $N \times N$ matrices.

Recently there has been much interest in eigenvalue problems for fully nonlinear PDE since the work of P.-L. Lions [15]. See [3,13,4,18,1,17] for these developments. See also [2,8,12] for some earlier related works. In this regard, most of work has been devoted to the questions concerning principal eigenvalues, while recent work by Esteban, Felmer and Quaas [10] (see also [4]) has established the existence of other eigenvalues beyond the principal eigenvalues and of the corresponding eigenfunctions in the one-dimensional or the radially symmetric problem. In this paper we extend the scope of the work of Esteban, Felmer and Quaas [10] to the eigenvalue problem set in the L^q framework.

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We thus study (1.1) in the one-dimensional or radially symmetric domains. That is, in what follows, we are concerned with the case where Ω is an open interval (a, b), with $-\infty < a < b < \infty$, or an open ball $B_R = B_R(0)$ in $\mathbb{R}^N$ of radius $R \in (0, \infty)$ with center at the origin.

We now introduce our basic assumptions (F1)–(F3) on the function F. Given constants $\lambda \in (0, \infty)$ and $\Lambda \in [\lambda, \infty]$, $P^{\pm}$ denote the Pucci operators defined as the functions on $\mathbb{S}^N$ given, respectively, by $P^+(M) \equiv P^+(M; \lambda, \Lambda) = \sup\{\operatorname{tr} AM\colon\ A \in \mathbb{S}^N,\ \lambda I_N \leqslant A \leqslant \Lambda I_N\}$ and $P^-(M) = -P^+(-M)$, where I_N denotes the $N \times N$ identity matrix and the relation, $X \leqslant Y$, is the standard order relation between $X, Y \in \mathbb{S}^N$. Note that if $N = 1$ and $\Lambda = \infty$, then $P^+(m) = \lambda m$ for $m \leqslant 0$ and $P^+(m) = \infty$ for $m > 0$.

(F1) The function $F : \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a Carathéodory function, i.e., the function $x \mapsto F(M, p, u, x)$ is measurable for any $(M, p, u) \in \mathbb{S}^N \times \mathbb{R}^{N+1}$ and the function $(M, p, u) \mapsto F(M, p, u, x)$ is continuous for a.a. $x \in \Omega$.

(F2) There exist constants $\lambda \in (0, \infty)$, $\Lambda \in [\lambda, \infty]$, $q \in [1, \infty]$ and functions $\beta, \gamma \in L^q(\Omega)$ such that

$$F(M_1, p_1, u_1, x) - F(M_2, p_2, u_2, x) \leqslant P^+(M_1 - M_2) + \beta(x)|p_1 - p_2| + \gamma(x)|u_1 - u_2|$$

for all $(M_1, p_1, u_1), (M_2, p_2, u_2) \in \mathbb{S}^N \times \mathbb{R}^{N+1}$ and a.a. $x \in \Omega$.

(F3) $F(tM, tp, tu, x) = tF(M, p, u, x)$ for all $t \geqslant 0$, all $(M, p, u) \in \mathbb{S}^N \times \mathbb{R}^{N+1}$ and a.a. $x \in \Omega$.

Of course, if $\Lambda = \infty$ and $M_1 \nleqslant M_2$, then the inequality in condition (F2) is trivially satisfied.

We make an additional assumption in the multi-dimensional case.

(F4) The function F is radially symmetric in the sense that for any $(m, l, q, u) \in \mathbb{R}^4$ and a.a. $r \in (0, R)$, the function

$$\omega \mapsto F\big(m\omega \otimes \omega + l(I_N - \omega \otimes \omega), q\omega, u, r\omega\big)$$

is constant on the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Here and henceforth $x \otimes x$ denotes the matrix in $\mathbb{S}^N$ with the (i, j) entry given by $x_i x_j$ if $x \in \mathbb{R}^N$.

We study the eigenvalue problem (1.1) in the Sobolev space $W^{2,q}(\Omega)$. For any pair $(\mu, \varphi) \in \mathbb{R} \times W^{2,1}(\Omega)$ which satisfies the PDE in the almost everywhere sense and the boundary condition of (1.1) in the pointwise sense, we call μ and φ an *eigenvalue* and *eigenfunction* of (1.1), respectively, provided $\varphi(x) \not\equiv 0$. We call such a pair an *eigenpair* of (1.1).

We state our main results in this paper.

Theorem 1.1. *Let $N = 1$ and $\Omega = (a, b)$, and assume that* (F1), (F2)*, with $\Lambda = \infty$, and* (F3) *hold. Then*:

(i) *For any $n \in \mathbb{N}$, there exist eigenpairs $(\mu_n^+, \varphi_n^+), (\mu_n^-, \varphi_n^-) \in \mathbb{R} \times W^{2,q}(a, b)$ of* (1.1) *and finite sequences $(x_{n,j}^+)_{j=0}^n, (x_{n,j}^-)_{j=0}^n \subset [a, b]$ such that*

$$\begin{cases} a = x_{n,0}^+ < x_{n,1}^+ < \cdots < x_{n,n}^+ = b, \quad a = x_{n,0}^- < x_{n,1}^- < \cdots < x_{n,n}^- = b, \\ (-1)^{j-1}\varphi_n^+(x) > 0 \quad \text{in } \big(x_{n,j-1}^+, x_{n,j}^+\big) \text{ for } j = 1, \ldots, n, \\ (-1)^j \varphi_n^-(x) > 0 \quad \text{in } \big(x_{n,j-1}^-, x_{n,j}^-\big) \text{ for } j = 1, \ldots, n. \end{cases}$$

(ii) *The eigenpairs (μ_n^+, φ_n^+) and (μ_n^-, φ_n^-) are complete in the sense that for any eigenpair $(\mu, \varphi) \in \mathbb{R} \times W^{2,q}(a, b)$ of* (1.1)*, there exist $n \in \mathbb{N}$ and $\theta > 0$ such that either $(\mu, \varphi) = (\mu_n^+, \theta\varphi_n^+)$ or $(\mu, \varphi) = (\mu_n^-, \theta\varphi_n^-)$ holds.*

For $q \in [1, \infty]$, let $W_{\mathrm{r}}^{2,q}(B_R)$ denote the space of those functions $\varphi \in W^{2,q}(B_R)$ which are radially symmetric. We may identify any function f in $W_{\mathrm{r}}^{2,q}(B_R)$ with a function g on $[0, R]$ such that $f(x) = g(|x|)$ for a.a. $x \in B_R$ and we employ the standard abuse of notation: $f(x) = f(|x|)$ for $x \in B_R$. We set $\lambda_* = \lambda/\Lambda$ and $q_* = N/(\lambda_* N + 1 - \lambda_*)$ if $\Lambda < \infty$. Note that $0 < \lambda_* \leqslant 1$ and $q_* \in [1, N)$.

Theorem 1.2. *Let $N \geqslant 2$ and $\Omega = B_R$, and assume that* (F1), (F2) *with $\Lambda < \infty$,* (F3) *and* (F4) *hold. Assume that $q > \max\{N/2, q_*\}$ and that $\beta \in L^N(B_R)$ if $q < N$. Then*:

(i) *For each* $n \in \mathbb{N}$, *there exist eigenpairs* $(\mu_n^+, \varphi_n^+), (\mu_n^-, \varphi_n^-) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(B_R)$ *of* (1.1) *and finite sequences* $(r_{n,j}^{\pm})_{j=0}^n \subset [0, R]$ *such that*

$$\begin{cases} 0 = r_{0,n}^+ < r_{n,1}^+ < \cdots < r_{n,n}^+ = R, \quad 0 = r_{0,n}^- < r_{n,1}^- < \cdots < r_{n,n}^- = R, \\ (-1)^{j-1}\varphi_n^+(r) > 0 \quad \text{in } (r_{n,j-1}^+, r_{n,j}^+) \text{ for } j = 1, \dots, n, \\ (-1)^j \varphi_n^-(r) > 0 \quad \text{in } (r_{n,j-1}^-, r_{n,j}^-) \text{ for } j = 1, \dots, n, \\ \varphi_n^+(0) > 0 > \varphi_n^-(0). \end{cases}$$

(ii) *The eigenpairs* (μ_n^+, φ_n^+) *and* (μ_n^-, φ_n^+) *are complete in the sense that for any eigenpair* $(\mu, \varphi) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(B_R)$ *of* (1.1), *there exist* $n \in \mathbb{N}$ *and* $\theta > 0$ *such that either* $(\mu, \varphi) = (\mu_n^+, \theta\varphi_n^+)$ *or* $(\mu, \varphi) = (\mu_n^-, \theta\varphi_n^-)$ *is valid.*

A comparison of these results with those of [10] might be in order. The results above treat the same eigenvalue problems as in [10]. The main differences are twofold: one is our weaker regularity assumptions on F and the other is in the method of proof. In the above results the regularity of F is imposed through (F1) and (F2), where the functions β and γ are assumed to be in some L^q space. We use here fairly elementary arguments to prove the existence of the principal eigenvalues and the higher eigenvalues based, respectively, on the so-called inverse power method and on the monotonicity on the domains of the eigenvalues.

Another feature of this article is this. Regarding the regularity hypotheses (F1) and (F2) on F in case $N \geqslant 2$, our requirement on β in Theorem 1.2 is only that $\beta \in L^q(B_R) \cap L^N(B_R)$. From the viewpoint of the existence of a solution, this requirement seems relatively sharp in comparison with the known results [11,14,19,9,5,6]. See Theorem 7.5 in this connection. We refer also to [16] for some recent results concerning regularity of axially symmetric solutions of uniformly elliptic Hessian equations.

In general, condition (F4) on F is different from Eq. (1.1) being Hessian. Let $N \geqslant 2$. For simplicity of the argument, we assume that F depends only on $M \in \mathbb{S}^N$. According to [20], the uniformly elliptic equation (1.1) is called Hessian (cf. [7]) if the function $F : \mathbb{S}^N \to \mathbb{R}$ is invariant under conjugation of the orthogonal matrices, i.e., $F(Q^{-1}MQ) = F(M)$ for all $M \in \mathbb{S}^N$ and $Q \in O(N)$, where $O(N)$ denotes the group of orthogonal matrices of order N. This condition is stated as $F(M)$ being a symmetric function of the eigenvalues of M.

Note that the eigenvalues of the matrix $M = m\omega \otimes \omega + l(I_N - \omega \otimes \omega)$, with $\omega \in S^{N-1}$ and $m, l \in \mathbb{R}$, are m and l. If $F(M)$ is a symmetric function of the eigenvalues of M, then

$$F\big(m\omega \otimes \omega + l(I_N - \omega \otimes \omega)\big), \quad \text{with } \omega \in S^{N-1} \text{ and } m, l \in \mathbb{R},$$

is a function of m, l. That is, if (1.1) is Hessian, then F satisfies (F4).

If $N = 2$, then any symmetric matrix $M \in \mathbb{S}^N$ can be represented as $M = \lambda_1 \omega \otimes \omega + \lambda_2(I_2 - \omega \otimes \omega)$, where λ_1 and λ_2 are the eigenvalues of M and $\omega \in S^1$ is an eigenvector corresponding to λ_1, and therefore, we see that (1.1) is Hessian if and only if F satisfies (F4).

However, if $N \geqslant 3$, then there are functions F which satisfy (F4) but are not invariant under conjugation of the matrices $Q \in O(N)$. For such an example see Appendix A.

The rest of this article is organized as follows. Section 2 is devoted to the study of the solvability of the Dirichlet problem for fully nonlinear ODE on a finite interval as well as some estimates of solutions of fully nonlinear ODE. In Section 3 we establish the existence of principal eigenpairs of fully nonlinear (homogeneous) ODE, and in Section 4 we present basic properties of eigenpairs of fully nonlinear ODE. Section 5 is devoted to completing the proof of one of the main results, Theorem 1.1. In Section 6, we turn the multi-dimensional radially symmetric problem (1.1) into one-dimensional problem. Section 7 collects several estimates on radial functions including the $W^{2,q}$ estimates of radial solutions of fully nonlinear PDE. Section 8 is devoted to the proof of Theorem 1.2. In Appendix A, we give an example of F which satisfies (F1)–(F4) but is not invariant under conjugation of the orthogonal matrices when $N \geqslant 3$ (i.e., (1.1) is not a Hessian equation).

2. Solvability of the Dirichlet problem in one dimension

In this section we deal with the one-dimensional case and study the solvability of the Dirichlet problem

$$F\big(u'', u', u, x\big) = 0 \quad \text{in } (a, b), \tag{2.1}$$

$$u(a) = u(b) = 0, \tag{2.2}$$

where $u' = \mathrm{d}u/\mathrm{d}x$ and $u'' = \mathrm{d}^2u/\mathrm{d}x^2$.

We assume throughout this section that (F1) and (F2), with $q = 1$ and $\Lambda = \infty$, hold. We thus use $P^{\pm}(m)$ to denote $P^{\pm}(m; \lambda, \infty)$ in this section.

In what follows, we use the following notation. For any function $u \in W^{2,1}(a, b)$, $F[u](x) := F(u''(x), u'(x), u(x), x)$ and $P^{\pm}[u](x) = P^{\pm}(u''(x))$. In particular, we have $F[0](x) = F(0, 0, 0, x)$. A function $u \in W^{2,1}(a, b)$ is said to be a subsolution (resp., supersolution) of (2.1) if $F[u](x) \geqslant 0$ (resp., $F[u](x) \leqslant 0$) a.e. in (a, b).

The following lemma is an adaptation of [10, Lemma 2.1].

Lemma 2.1. *There is a function $g_F : \mathbb{R}^2 \times (a, b) \to \mathbb{R}$ such that for a.a. $x \in (a, b)$ and all $(m, p, u) \in \mathbb{R}^3$, we have $m = g_F(p, u, x)$ (resp., $m < g_F(p, u, x)$ or $m > g_F(p, u, x)$) if and only if $F(m, p, u, x) = 0$ (resp., $F(m, p, u, x) < 0$ or $F(m, p, u, x) > 0$). The function g_F satisfies*

$$\big|g_F(p_1, u_1, x) - g_F(p_2, u_2, x)\big| \leqslant \lambda^{-1}\big(\beta(x)|p_1 - p_2| + \gamma(x)|u_1 - u_2|\big)$$

for all $(p_1, u_1), (p_2, u_2) \in \mathbb{R}^2$ and a.a. $x \in (a, b)$. Moreover, we have

$$\big|g_F(0, 0, x)\big| \leqslant \lambda^{-1}\big|F(0, 0, 0, x)\big| \quad \textit{for a.a. } x \in (a, b).$$

Proof. Observe by (F1) and (F2) that for a.a. $x \in (a, b)$ and any $(p, u) \in \mathbb{R}^2$, the function $m \mapsto F(m, p, u, x)$ is continuous on $\mathbb{R}$ and, if $m_1, m_2 \in \mathbb{R}$ and $m_1 < m_2$, then we have

$$F(m_1, p, u, x) - F(m_2, p, u, x) \leqslant \lambda(m_1 - m_2),$$

which implies that the function $m \mapsto F(m, p, u, x)$ is (strictly) increasing on $\mathbb{R}$ and has the range $\mathbb{R}$. Hence, for a.a. $x \in (a, b)$ and any $(p, u) \in \mathbb{R}^2$, there exists a unique $g_F = g_F(p, u, x)$ such that $m = g_F(p, u, x)$ (resp., $m > g_F(p, u, x)$ or $m < g_F(p, u, x)$) if and only if $F(m, p, u, x) = 0$ (resp., $F(m, p, u, x) > 0$ or $F(m, p, u, x) < 0$).

Next we check the Lipschitz property of the function $g_F : \mathbb{R}^2 \times (a, b) \to \mathbb{R}$. Let $(p_1, u_1), (p_2, u_2) \in \mathbb{R}^2$ and set $g_i = g_F(p_i, u_i, x)$, with $i = 1, 2$. If $g_1 < g_2$, then, by (F2), we have

$$\begin{aligned} 0 &= F(g_1, p_1, u_1, x) - F(g_2, p_2, u_2, x) \\ &\leqslant \lambda(g_1 - g_2) + \beta(x)|p_1 - p_2| + \gamma(x)|u_1 - u_2| \quad \text{for a.a. } x \in (a, b), \end{aligned}$$

which ensures the required Lipschitz property of g_F. Moreover, for a.a. $x \in (a, b)$, we get similarly to the above,

$$F(0, 0, 0, x) \leqslant -\lambda g_F(0, 0, x) \quad \text{if } g_F(0, 0, x) > 0,$$

and

$$-F(0, 0, 0, x) \leqslant \lambda g_F(0, 0, x) \quad \text{otherwise},$$

and we have $|g_F(0, 0, x)| \leqslant \lambda^{-1}|F(0, 0, 0, x)|$ for a.a. $x \in (a, b)$. □

Let g_F be the function from Lemma 2.1. It is clear that (2.1) is equivalent to the ordinary differential equation (ODE for short) of the normal form

$$u''(x) = g_F\big(u'(x), u(x), x\big) \quad \text{in } (a, b). \tag{2.3}$$

Together with this observation and Lemma 2.1, the standard theory of ODE guarantees the existence of a solution to the Cauchy problem for (2.1) as stated in the following.

Theorem 2.2. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $c \in [a, b]$. Assume that the function $F[0] \in L^1(a, b)$. Then there exists a unique solution $u \in W^{2,1}(a, b)$ of (2.1) satisfying $u(c) = \alpha_1$ and $u'(c) = \alpha_2$.*

We remark that the mapping $(\alpha_1, \alpha_2) \mapsto u$ from $\mathbb{R}^2$ to $C([a,b])$ is continuous, where u is the solution of (2.1) given by the above theorem. We omit here giving the proof of the above theorem and this remark on the continuous dependence of the solution of (2.1).

In what follows, given a function f on $[a,b]$, we denote by f_+ and f_- the functions $x \mapsto \max\{f(x), 0\}$ and $x \mapsto \max\{-f(x), 0\}$, respectively.

Lemma 2.3. *Let $c \in [a,b]$, $f \in L^1(a,b)$ and $u \in W^{2,1}(a,b)$. Assume that*

$$\lambda u''(x) + \beta(x)\left|u'(x)\right| + f(x) \geqslant 0 \quad \text{a.a. } x \in (a,b).$$

Then we have

$$\begin{aligned}(u')_-(x) \leqslant{}& (u')_-(c)\exp\left(\int_c^x \lambda^{-1}\beta(r)\,\mathrm{d}r\right)\\ &+ \int_c^x \lambda^{-1} f_+(r)\exp\left(\int_r^x \lambda^{-1}\beta(t)\,\mathrm{d}t\right)\mathrm{d}r \quad \text{for all } x \in [c,b],\end{aligned} \tag{2.4}$$

$$\begin{aligned}(u')_+(x) \leqslant{}& (u')_+(c)\exp\left(\int_x^c \lambda^{-1}\beta(r)\,\mathrm{d}r\right)\\ &+ \int_x^c \lambda^{-1} f_+(r)\exp\left(\int_x^r \lambda^{-1}\beta(t)\,\mathrm{d}t\right)\mathrm{d}r \quad \text{for all } x \in [a,c],\end{aligned} \tag{2.5}$$

and, if $u(a) \leqslant 0$ and $u(b) \leqslant 0$,

$$\max_{[a,b]} u \leqslant (b-a)\exp\left(\left\|\lambda^{-1}\beta\right\|_{L^1(a,b)}\right)\left\|\lambda^{-1} f_+\right\|_{L^1(a,b)}. \tag{2.6}$$

To see the role of the above lemma in the context of (2.1), it is worth noting that, if $f(x) \geqslant 0$, the inequality $\lambda u''(x) + \beta(x)|u'(x)| + f(x) \geqslant 0$ is equivalent to the inequality $P^+[u](x) + \beta|u'(x)| + f(x) \geqslant 0$.

The assertion (2.6) can be regarded as a weak version of the Aleksandrov–Bakelman–Pucci maximum principle.

In the following arguments, we use the fact that if f is absolutely continuous on $[a,b]$, then f_+ and f_- are absolutely continuous on $[a,b]$ and, for a.a. $x \in (a,b)$,

$$\begin{cases} (f_+)'(x) = f'(x) \quad \text{and} \quad (f_-)'(x) = 0 & \text{if } f(x) > 0,\\ (f_+)'(x) = 0 \quad \text{and} \quad (f_-)'(x) = -f'(x) & \text{if } f(x) < 0,\\ (f_+)'(x) = (f_-)'(x) = 0 & \text{if } f(x) = 0.\end{cases}$$

Proof of Lemma 2.3. We write $\hat\beta$ and $\hat f$ for $\lambda^{-1}\beta$ and $\lambda^{-1}f$, respectively. Setting $v = (u')_-$ and $w = (u')_+$, we observe that $v' \leqslant \hat\beta v + \hat f_+$ and $w' \geqslant -\hat\beta w - \hat f_+$ a.e. in (a,b). Hence, (2.4) and (2.5) are consequences of Gronwall's inequality.

For the proof of (2.6), we may assume that $\max_{[a,b]} u > 0$. We may moreover assume by replacing the interval $[a,b]$ by a smaller interval that $u(x) > 0$ for all $x \in (a,b)$. We choose a point c in (a,b) so that $u(c) = \max_{[a,b]} u$, and apply (2.5), to obtain

$$\max_{[a,c]} (u')_+ \leqslant \exp\left(\|\hat\beta\|_{L^1(a,c)}\right)\|\hat f_+\|_{L^1(a,c)},$$

and moreover

$$u(c) \leqslant u(c) - u(a) = \int_a^c u'(r)\,\mathrm{d}r \leqslant \int_a^c (u')_+(r)\,\mathrm{d}r \leqslant (b-a)\exp\left(\|\hat\beta\|_{L^1(a,b)}\right)\|\hat f_+\|_{L^1(a,b)},$$

which completes the proof. □

Let $u, v \in W^{2,1}(a,b)$, and observe that for a.a. $x \in (a,b)$,

$$F[u](x) - F[v](x) \leqslant P^+[u-v](x) + \beta(x)\left|u'(x) - v'(x)\right| + \gamma(x)\left|u(x) - v(x)\right|. \tag{2.7}$$

Henceforth we fix any $\kappa \geqslant 0$, and define the function F_κ on $\mathbb{R}^3 \times (a,b)$ by

$$F_\kappa(m,p,u,x) = F(m,p,u,x) - \kappa u.$$

As above, for any $u, v \in W^{2,1}(a,b)$ and a.a. $x \in (a,b)$, we have

$$\begin{aligned} F_\kappa[u](x) - F_\kappa[v](x) \leqslant{}& P^+[u-v](x) + \beta(x)\left|u'(x) - v'(x)\right| \\ &+ \big(\gamma(x) - \kappa\big)_+\big(u(x) - v(x)\big) \quad \text{if } u(x) \geqslant v(x). \end{aligned} \tag{2.8}$$

We set

$$\sigma = \sigma_\kappa := (b-a)\exp\big(\lambda^{-1}\|\beta\|_{L^1(a,b)}\big)\big\|\lambda^{-1}(\gamma-\kappa)_+\big\|_{L^1(a,b)}, \tag{2.9}$$

and note that $\lim_{\kappa\to\infty}\sigma_\kappa = 0$.

The following comparison principle holds for (2.1).

Theorem 2.4. *Let $f, g \in L^1(a,b)$ and $u, v \in W^{2,1}(a,b)$. Assume that $\sigma_\kappa < 1$, $u(x) \leqslant v(x)$ for $x = a, b$, and*

$$F_\kappa[v](x) + g(x) \leqslant F_\kappa[u](x) + f(x) \quad \textit{for a.a. } x \in (a,b).$$

Then

$$\max_{[a,b]}(u-v) \leqslant \frac{b-a}{(1-\sigma_\kappa)}\exp\big(\big\|\lambda^{-1}\beta\big\|_{L^1(a,b)}\big)\big\|\lambda^{-1}(f-g)_+\big\|_{L^1(a,b)}.$$

Proof. Set $w = u - v$. As in the proof of Lemma 2.3, we may assume that $\max_{[a,b]} w > 0$ and $w(x) > 0$ in (a,b). By (2.8), we get for a.a. $x \in (a,b)$,

$$P^+[w](x) + \beta(x)\left|w'(x)\right| + \big(\gamma(x) - \kappa\big)_+ w(x) + (f-g)_+(x) \geqslant 0.$$

Applying Lemma 2.3 yields

$$\max_{[a,b]} w \leqslant (b-a)\exp\big(\|\hat\beta\|_{L^1(a,b)}\big)\big\|\lambda^{-1}\big((\gamma-\kappa)_+ w + (f-g)_+\big)\big\|_{L^1(a,b)},$$

where $\hat\beta = \lambda^{-1}\beta$. Hence, we get

$$\max_{[a,b]} w \leqslant \sigma_\kappa \max_{[a,b]} w + (b-a)\exp\big(\|\hat\beta\|_{L^1(a,b)}\big)\big\|\lambda^{-1}(f-g)_+\big\|_{L^1(a,b)},$$

from which we easily obtain the desired bound on $\max_{[a,b]} w$. □

A simple consequence of the above theorem is the following.

Corollary 2.5. *Let $u \in W^{2,1}(a,b)$ and $v \in W^{2,1}(a,b)$ be, respectively, a subsolution and a supersolution of (2.1), with F replaced by F_κ. Assume $\sigma_\kappa < 1$. If $u(x) \leqslant v(x)$ for $x = a, b$, then $u(x) \leqslant v(x)$ for all $x \in [a,b]$.*

Next, we state and prove a strong comparison principle for (2.1).

Theorem 2.6. *Let $u, v \in W^{2,1}(a,b)$ satisfy*

$$F[v](x) \leqslant F[u](x) \quad \textit{for a.a. } x \in (a,b)$$

and $u(x) \leqslant v(x)$ in $[a,b]$. Then either $u(x) \equiv v(x)$ or $u(x) < v(x)$ holds in (a,b). Furthermore if $u(x) < v(x)$ in (a,b), then

$$\max\big\{(v-u)(a), (v-u)'(a)\big\} > 0 \quad \textit{and} \quad \max\big\{(v-u)(b), -(v-u)'(b)\big\} > 0.$$

Proof. Set $w = v - u$ and observe that

$$P^-[w] - \beta|w'| - \gamma w \leqslant 0 \quad \text{a.e. in } (a,b).$$

It is enough to show that if either $\max\{w(a), w'(a)\} \leqslant 0$, or $\max\{w(b), -w'(b)\} \leqslant 0$, or $w(c) = 0$ for some $c \in (a,b)$, then $w(x) \equiv 0$ in $[a,b]$. Moreover, it is enough to show that if either $\max\{w(a), w'(a)\} \leqslant 0$ or $\max\{w(b), -w'(b)\} \leqslant 0$, then $w(x) \equiv 0$ in $[a,b]$. Indeed, observing that if $w(c) = 0$ for some $c \in (a,b)$, then $w(c) = w'(c) = 0$ and applying the above assertion in the intervals $[a,c]$ and $[c,b]$, we deduce that $w(x) \equiv 0$ in both of two intervals $[a,c]$ and $[c,b]$.

We consider the case where $w(a) \leqslant 0$ and $w'(a) \leqslant 0$. Since $w \geqslant 0$ in $[a,b]$, we have indeed $w(a) = w'(a) = 0$. Since $z := -w$ satisfies $P^+[z] + \beta|z'| + \gamma w \geqslant 0$ a.e. in $[a,b]$, we deduce from Lemma 2.3 that for all $r \in [a,b]$,

$$\big(w'\big)_+(r) \leqslant \exp\big(\|\hat{\beta}\|_{L^1(a,b)}\big) \int_a^r \hat{\gamma}(t) w(t)\,\mathrm{d}t,$$

where $\hat{\beta} = \lambda^{-1}\beta$ and $\hat{\gamma} = \lambda^{-1}\gamma$. Integrating this over $[a,x]$, we get for $x \in [a,b]$,

$$w(x) \leqslant \exp\big(\|\hat{\beta}\|_{L^1(a,b)}\big) \int_a^x \mathrm{d}r \int_a^r \hat{\gamma}(t) w(t)\,\mathrm{d}t \leqslant (b-a)\exp\big(\|\hat{\beta}\|_{L^1(a,b)}\big) \int_a^x \hat{\gamma}(t) w(t)\,\mathrm{d}t.$$

From this, using Gronwall's inequality, we see that $w(x) \equiv 0$ in $[a,b]$.

An argument parallel to the above ensures that if $\max\{w(b), -w'(b)\} = 0$, then $w(x) \equiv 0$ in $[a,b]$. □

Theorem 2.7. *Let $\kappa \in [0,\infty)$. Assume that $F[0] \in L^1(a,b)$ and $\sigma_\kappa < 1$, where σ_κ is the constant defined by* (2.9)*. Then there is a unique solution $u \in W^{2,1}(a,b)$ of the Dirichlet problem* (2.1) *and* (2.2)*, with F_κ in place of F. Moreover, if $\beta, \gamma, F[0] \in L^q(a,b)$ for some $q \in (1,\infty]$, then $u \in W^{2,q}(a,b)$.*

Proof. The uniqueness assertion is a direct consequence of Corollary 2.5. It is thus enough to show the existence of a solution in $W^{2,1}(a,b)$ of (2.1) and (2.2), with F_κ in place of F.

For any $d \in \mathbb{R}$, we denote by $u(x;a,d)$ the unique solution in $W^{2,1}(a,b)$ of the Cauchy problem for (2.1), with F_κ in place of F, satisfying the initial condition $(u(a;a,d), u'(a;a,d)) = (0,d)$, where $u'(x;a,d) := \partial u(x;a,d)/\partial x$. As we have remarked after Theorem 2.2, we know that the function $d \mapsto u(b;a,d)$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.

Let $d_1, d_2 \in \mathbb{R}$ be such that $d_1 > d_2$. Set $w(x) = u(x;a,d_1) - u(x;a,d_2)$ for $x \in [a,b]$. Since $w \in C^1([a,b])$ and $w'(a) = d_1 - d_2 > 0$, there is a point $c \in (a,b]$ such that $w'(x) > 0$ for all $(a,c]$.

Fix such a point $c \in (a,b]$. Noting that $w'(x) > 0$ and $w(x) > 0$ for all $x \in (a,c]$ and $P^+[w] + \beta|w'| + (\gamma - \kappa)_+ w \geqslant 0$ a.e. in (a,c), we find by Lemma 2.3 that for all $x \in [a,c]$,

$$d_1 - d_2 = \big(w'\big)_+(a) \leqslant \mathrm{e}^{\hat{B}}\left(w'(x) + w(x)\int_a^x \lambda^{-1}\big(\gamma(t) - \kappa\big)_+\,\mathrm{d}t\right), \tag{2.10}$$

where $\hat{B} := \|\lambda^{-1}\beta\|_{L^1(a,b)}$.

We show that $w'(x) > 0$ for all $x \in [a,b]$. Indeed, if this is not the case, there is a point $e \in (a,b]$ such that $w'(e) = 0$ and $w'(x) > 0$ for all $x \in [a,e)$. Using Lemma 2.3 again, we get for all $x \in [a,e]$,

$$w'(x) \leqslant \mathrm{e}^{\hat{B}} w(e) \int_x^e \lambda^{-1}\big(\gamma(t) - \kappa\big)_+\,\mathrm{d}t = \mathrm{e}^{\hat{B}} w(e)\big\|\lambda^{-1}(\gamma - \kappa)_+\big\|_{L^1(a,b)}. \tag{2.11}$$

Integrating (2.11) over (a,e), we get $w(e) \leqslant \sigma_\kappa w(e)$, which yields $w(e) \leqslant 0$. This is a contradiction, and we conclude that $w'(x) > 0$ for all $x \in [a,b]$, which shows that (2.10) holds with $c = b$. Integrating (2.10) over (a,b), we get

$$(b-a)(d_1 - d_2) \leqslant \mathrm{e}^{\hat{B}} w(b)\big(1 + (b-a)\big\|\lambda^{-1}(\gamma - \kappa)_+\big\|_{L^1(a,b)}\big).$$

That is,

$$u(b;a,d_1) - u(b;a,d_2) \geqslant \frac{(b-a)(d_1 - d_2)}{\mathrm{e}^{\hat{B}}(1 + (b-a)\|\lambda^{-1}(\gamma - \kappa)_+\|_{L^1(a,b)})}.$$

This strict monotonicity and the continuity of the function $d \mapsto u(b; a, d)$ guarantee that there is a unique $d_* \in \mathbb{R}$ such that $u(b; a, d_*) = 0$. The function $u(x; a, d_*)$ of x is a solution of (2.1) and (2.2), with F_κ in place of F.

Now, we assume that $\beta, \gamma, F[0] \in L^q(a, b)$ for some $q \in (1, \infty]$. Observe by (F2) that both $\varphi = u$ and $\varphi = -u$ satisfy

$$\lambda\varphi''(x) + \beta(x)\big|\varphi'(x)\big| + \big(\gamma(x) + \kappa\big)\big|\varphi(x)\big| + \big|F[0](x)\big| \geqslant 0 \quad \text{for a.a. } x \in (a, b).$$

Hence,

$$\big|u''(x)\big| \leqslant \lambda^{-1}\big(\beta(x)\big|u'(x)\big| + \big(\gamma(x) + \kappa\big)\big|u(x)\big| + \big|F[0](x)\big|\big) \quad \text{for a.a. } x \in (a, b).$$

Noting that $u \in C^1([a, b])$, we conclude that $u'' \in L^q(a, b)$ and, accordingly, $u \in W^{2,q}(a, b)$. □

Remark 2.8. The same assertion as Theorem 2.7 concerning the existence, uniqueness and regularity of solutions $u \in W^{2,1}(a, b)$ of the Dirichlet problem for (2.1) is valid under the general boundary condition $u(a) = d_1$ and $u(b) = d_2$, where $d_1, d_2 \in \mathbb{R}$ are any given constants. Indeed, one can prove this assertion in the same fashion as in the proof above.

3. Principal eigenvalues in one dimension

This section is devoted to the existence of principal eigenpairs of (1.1) in one dimension under hypotheses (F1)–(F3).

Throughout this section we assume that $N = 1$, $\Omega = (a, b)$, where $-\infty < a < b < \infty$, and (F1), (F2) with $\Lambda = \infty$ and (F3) hold. We remark that, by assumption (F3), we have $F[0] = 0$.

We fix a constant $\kappa \geqslant 0$ so that

$$\sigma = \sigma_\kappa := (b - a)\exp\big(\big\|\lambda^{-1}\beta\big\|_{L^1(a,b)}\big)\big\|\lambda^{-1}(\gamma - \kappa)_+\big\|_{L^1(a,b)} < 1, \tag{3.1}$$

and, as before, set $F_\kappa(m, p, u, x) := F(m, p, u, x) - \kappa u$. We consider the eigenvalue problem

$$\begin{cases} F_\kappa\big(u'', u', u, x\big) + \nu u = 0 & \text{in } (a, b), \\ u(a) = u(b) = 0. \end{cases} \tag{3.2}$$

We prove here the following proposition, which is obviously a special case (i.e., the case $n = 1$) of Theorem 1.1.

Theorem 3.1. *There exist eigenpairs $(\nu^+, \varphi^+), (\nu^-, \varphi^-) \in \mathbb{R} \times W^{2,q}(a, b)$ of (3.2) such that $\varphi^+(x) > 0$ and $\varphi^-(x) < 0$ in (a, b).*

The constants ν^+ and ν^- in the above theorem are called, respectively, the positive and negative principal eigenvalues of (3.2). The functions φ^+ and φ^- are called, respectively, positive and negative principal eigenfunctions of (3.2). Similarly, the pairs (ν^+, φ^+) and (ν^-, φ^-) are called, respectively, positive and negative principal eigenpairs of (3.2).

Let $f \in L^q(a, b)$, and we consider the Dirichlet problem

$$\begin{cases} F_\kappa\big(u'', u', u', x\big) + f = 0 & \text{in } (a, b), \\ u(a) = u(b) = 0. \end{cases} \tag{3.3}$$

Set $\tilde{F}(m, p, u, x) := F_\kappa(m, p, u, x) - f(x)$. Then it is easily seen that $\tilde{F}$ satisfies (F1), (F2) and $\tilde{F}[0] \in L^q(a, b)$. Hence, according to Theorem 2.7, there is a unique solution $u \in W^{2,q}(a, b)$ of (3.3). We introduce the solution mapping $T : L^q(a, b) \to W^{2,q}(a, b)$ by $Tf = u$.

Basic properties of the map T are stated in the following lemma.

Lemma 3.2.

(i) *The map T is positively homogeneous of degree one, i.e., $T(sf) = sTf$ for all $s \geqslant 0$ and $f \in L^q(a, b)$.*

(ii) *If $f \in L^q(a, b)$ and $f(x) \geqslant 0$ for a.a. $x \in (a, b)$, then $(Tf)(x) \geqslant 0$ in $[a, b]$. Furthermore, if $f \not\equiv 0$, then $(Tf)(x) > 0$ in (a, b), $(Tf)'(a) > 0$ and $(Tf)'(b) < 0$.*

(iii) *There is a constant $C > 0$, depending only on $b-a$, κ, λ, $\|\beta\|_{L^q(a,b)}$ and $\|\gamma\|_{L^q(a,b)}$, such that*

$$\|Tf - Tg\|_{W^{2,q}(a,b)} \leqslant C\|f-g\|_{L^q(a,b)} \quad \text{for all } f, g \in L^q(a,b). \tag{3.4}$$

Proof. Let $f \in L^q(a,b)$. By assumption (F3), we see that sTf, with $s \geqslant 0$, is a solution of (3.3) with f replaced by sf, which tells us that $sTf = T(sf)$, proving the homogeneity of T.

Suppose that f is a nonnegative function. We observe by (F3) that $v \equiv 0$ is a subsolution of $F_\kappa[v] + f = 0$ in (a,b). Theorem 2.4 tells us that $Tf(x) \geqslant 0$ in $[a,b]$. In the case where $f(x) \not\equiv 0$, we have $(Tf)(x) \not\equiv 0$. Hence, we find by Theorem 2.6 (or the uniqueness assertion of Theorem 2.2) that $u(x) > 0$ in (a,b), $u'(a) > 0$ and $u'(b) < 0$.

Let $f, g \in L^q(a,b)$ and set $u = Tf - Tg$. By Theorem 2.4 we have

$$\|u\|_{L^\infty(a,b)} \leqslant \frac{(b-a)\mathrm{e}^{\hat{B}}}{1-\sigma}\left\|\lambda^{-1}(f-g)\right\|_{L^1(a,b)} \leqslant \frac{(b-a)^{2-\frac{1}{q}}\mathrm{e}^{\hat{B}}}{1-\sigma}\left\|\lambda^{-1}(f-g)\right\|_{L^q(a,b)},$$

where $\hat{B} = \|\lambda^{-1}\beta\|_{L^1(a,b)}$. Both of the functions $\varphi = u$ and $\varphi = -u$ satisfy

$$\lambda\varphi'' + \beta\left|\varphi'\right| + (\gamma+\kappa)|\varphi| + |f-g| \geqslant 0 \quad \text{a.e. in } (a,b). \tag{3.5}$$

Hence, noting that $u'(c) = 0$ for some $c \in (a,b)$ and applying (2.4) and (2.5) of Lemma 2.3, we get

$$\left\|u'\right\|_{L^\infty(a,b)} \leqslant \mathrm{e}^{\hat{B}}\left\{\left\|\lambda^{-1}(\gamma+\kappa)\right\|_{L^1(a,b)}\|u\|_{L^\infty(a,b)} + \left\|\lambda^{-1}(f-g)\right\|_{L^1(a,b)}\right\}.$$

Finally, we observe by (3.5) that

$$\left\|u''\right\|_{L^q(a,b)} \leqslant \lambda^{-1}\left(\|\beta\|_{L^q(a,b)}\left\|u'\right\|_{L^\infty(a,b)} + \|\gamma+\kappa\|_{L^q(a,b)}\|u\|_{L^\infty(a,b)} + \|f-g\|_{L^q(a,b)}\right),$$

proving (3.4). □

Next we define

$$X := \left\{f \in C^1\left([a,b]\right):\ f(a) = f(b) = 0,\ f'(a) > 0,\ f'(b) < 0,\ f(x) > 0 \text{ in } (a,b)\right\},$$

and observe by Lemma 3.2 that $Tf \in X$ if $f \in X$. We introduce the mapping R from X to the functions on $[a,b]$ as follows:

$$Rf(x) := \begin{cases} \frac{Tf(x)}{f(x)} & \text{if } x \in (a,b), \\ \frac{(Tf)'(x)}{f'(x)} & \text{if } x = a, b. \end{cases}$$

It follows from the homogeneity of T that for each $t > 0$ and $f \in X$,

$$R(tf)(x) = Rf(x) \quad \text{for all } x \in [a,b]. \tag{3.6}$$

Lemma 3.3.

(i) *For any $f \in X$, we have $Rf \in C([a,b])$ and*

$$0 < \min_{x\in[a,b]} Rf(x) = \inf_{x\in(a,b)} \frac{Tf(x)}{f(x)} \leqslant \sup_{x\in(a,b)} \frac{Tf(x)}{f(x)} = \max_{x\in[a,b]} Rf(x) < \infty.$$

(ii) *The map $R : X \to C([a,b])$ is continuous, provided that X is equipped with the $C^1([a,b])$ topology.*

Proof. Since $f, Tf \in X$, l'Hôpital's rule tells us that Rf is continuous at a and b, and thus $Rf \in C([a,b])$. It is then clear that the other assertions of (i) hold.

Next we prove the continuity of R. Let ψ denote the function on (a,b) given by $\psi(x) = (x-a)^{-1}(b-x)^{-1}$. Note that $0 < \inf_{a<x<b} \psi(x)f(x) < \infty$ for any $f \in X$. Note also that for any function $f \in C^1([a,b])$ satisfying $f(a) = f(b) = 0$,

$$\left|\psi(x)f(x)\right| \leqslant \begin{cases} \psi(x)\int_a^x |f'(t)|\,\mathrm{d}t \leqslant \frac{2}{(b-a)}\|f'\|_{L^\infty(a,b)} & \text{for } a < x \leqslant (a+b)/2, \\ \psi(x)\int_x^b |f'(t)|\,\mathrm{d}t \leqslant \frac{2}{(b-a)}\|f'\|_{L^\infty(a,b)} & \text{for } (a+b)/2 \leqslant x < b. \end{cases}$$

Using these observations, we compute that for any $f, g \in X$ and $x \in (a, b)$,

$$\begin{aligned}\bigl|Rf(x) - Rg(x)\bigr| &= \frac{|g(x)(Tf(x) - Tg(x)) + (g(x) - f(x))Tg(x)|}{f(x)g(x)} \\ &\leqslant 4\frac{\|g'\|_{L^\infty(a,b)}\|(Tf - Tg)'\|_{L^\infty(a,b)} + \|(f - g)'\|_{L^\infty(a,b)}\|(Tg)'\|_{L^\infty(a,b)}}{(b-a)^2 \inf_{(a,b)} \psi^2 fg}.\end{aligned}$$

From this we see that $R : X \to C([a, b])$ is continuous. □

Lemma 3.4. *Let $f \in X$ and $u = Tf$. Then*

$$\min_{[a,b]} Rf \leqslant \min_{[a,b]} Ru \leqslant \max_{[a,b]} Ru \leqslant \max_{[a,b]} Rf.$$

Moreover, if $\min_{[a,b]} Rf = \min_{[a,b]} Ru$*, then*

$$Tu(x) = \Bigl(\min_{[a,b]} Rf\Bigr)u(x) \quad \textit{for every } x \in [a, b].$$

Proof. Set $v = Tu$ and $\theta = \min_{[a,b]} Rf$. Since $\theta f(x) \leqslant u(x)$ for all $x \in [a, b]$, the function v is a supersolution of (3.3), with f replaced by θf. By the homogeneity of F_κ, the function θu is a solution of (3.3), with f replaced by θf. By Theorem 2.4, we get $\theta u \leqslant v$ in $[a, b]$, which yields $\min_{[a,b]} Rf = \theta \leqslant \min_{[a,b]} Ru$. In a similar fashion one can prove that $\max_{[a,b]} Ru \leqslant \max_{[a,b]} Rf$.

Now, we assume that $\min_{[a,b]} Rf = \min_{[a,b]} Ru$. Setting $\theta = \min_{[a,b]} Rf$, we note that $\theta f \leqslant u$ in $[a, b]$ and $F_\kappa[v] = -u \leqslant -\theta f = F_\kappa[\theta u]$ a.e. in (a, b). By Theorem 2.6, we have either $\theta u(x) \equiv v(x)$ in $[a, b]$, or else $\theta u(x) < v(x)$ in (a, b), $v'(a) > \theta u'(a) > 0$ and $v'(b) < \theta u'(b) < 0$. If the latter is the case, then we have $\theta < \min_{[a,b]} Ru$, which is a contradiction. Thus we must have $\theta u = v$ in $[a, b]$. □

Proof of Theorem 3.1. Fix $f_0 \in X$ so that $\|f_0\|_{C([a,b])} = 1$, and define the sequences $(u_k)_{k\in\mathbb{N}}, (f_k)_{k\in\mathbb{N}} \subset X$ and $(M_k)_{k\in\mathbb{N}}$ by setting inductively $u_k := Tf_{k-1}$, $M_k := \max_{[a,b]} u_k$ and $f_k(x) := u_k(x)/M_k$ for $k \in \mathbb{N}$. Then set $\theta_k := \min_{[a,b]} Ru_k$ and $\Theta_k := \max_{[a,b]} Ru_k$. From (3.6) and Lemma 3.4, we obtain $\theta_k \leqslant \theta_{k+1} \leqslant \Theta_{k+1} \leqslant \Theta_k$. Hence, the sequence $(\theta_k)_{k\in\mathbb{N}}$ is convergent. We set $\theta := \lim_{k\to\infty} \theta_k$.

Since $\|f_k\|_{C([a,b])} = 1$, the sequence (u_k) is bounded in $W^{2,q}(a, b)$ thanks to (3.4). Hence, by the Ascoli–Arzela theorem, (u_k) has a subsequence (u_{k_j}) converging to a nonnegative function u in $C^1([a, b])$. Since $Rf_k(x) = Ru_k(x) = u_{k+1}(x)/f_k(x)$ for all $x \in (a, b)$, we have

$$\theta_k f_k(x) \leqslant u_{k+1}(x) \leqslant \Theta_k f_k(x) \quad \text{for all } x \in [a, b]. \tag{3.7}$$

Since $\|f_k\|_{C([a,b])} = 1$, we therefore get $\theta_k \leqslant \max_{[a,b]} u_{k+1} = M_{k+1} \leqslant \Theta_k$. Noting that $f_{k_j}(x) = M_{k_j}^{-1} u_{k_j}(x)$, we see that, as $j \to \infty$, $f_{k_j} \to f := (\max_{[a,b]} u)^{-1} u$ in $C^1([a, b])$. By Lemma 3.2, we see that the sequence (Tf_{k_j}) converges to Tf in $C^1([a, b])$, which reads that (u_{k_j+1}) converges to Tf in $C^1([a, b])$. Setting $v := Tf$, by Lemma 3.3, we thus obtain

$$\min_{[a,b]} Rv = \lim_{j\to\infty} \min_{[a,b]} Ru_{k_j+1} = \lim_{j\to\infty} \theta_{k_j+1} = \theta. \tag{3.8}$$

Since $RTu_{k_j+1} = RTf_{k_j+1} = Ru_{k_j+2}$, we obtain as above

$$\min_{[a,b]} RTv = \lim_{j\to\infty} \min_{[a,b]} Ru_{k_j+2} = \theta. \tag{3.9}$$

Consequently, by Lemma 3.4, we get $Tv(x) \equiv \theta v(x)$ in $[a, b]$, which implies that v is a solution of (3.2) with $\nu = \theta^{-1}$. The pair $(\nu^+, \varphi^+) = (\theta^{-1}, v)$ is an eigenpair of (3.2) satisfying $\varphi^+(x) > 0$ for all $x \in (a, b)$.

Note that the function $G(m, p, u, x) := -F(-m, -p, -u, x)$ satisfies (F1)–(F3), with the same constants λ, $\Lambda = \infty$ and functions β, γ. If we define the function G_κ by the formula $G_\kappa(m, p, u, x) = G(m, p, u, x) - \kappa u$, then we have $G_\kappa(m, p, u, x) = -F_\kappa(-m, -p, -u, x)$. Observe also that $u \in W^{2,q}(a, b)$ satisfies $F_\kappa[u] + \nu u = 0$ a.e. in (a, b) if and only if $v := -u$ satisfies $G_\kappa[v] + \nu v = 0$ a.e. in (a, b). We apply the previous observation on the existence of an eigenpair of (3.2) to the eigenvalue problem (3.2), with G_κ in place of F_κ, to find an eigenpair (ν^-, ψ^-) of (3.2), with G_κ in place of F_κ, such that $\psi^-(x) > 0$ for all $x \in (a, b)$. If we put $\varphi^-(x) = -\psi^-(x)$, then (μ^-, φ^-) is an eigenpair of (3.2) such that $\varphi^-(x) < 0$ for all $x \in (a, b)$. □

Remark 3.5. The above proof is based on the so-called inverse power method. Indeed, combining the above proof with the uniqueness result of the principal eigenpairs, Theorem 4.1, we see easily that the sequences (θ_k) and (Θ_k) converge to the constant θ and (f_k) converges to the function f in $C(\bar{\Omega})$. Moreover, it is not hard to see that the positive principal eigenvalue is given by the formula $\min_{f\in X}\sup_{x\in(a,b)} f(x)/Tf(x)$.

4. Basic properties of principal eigenpairs in one dimension

In this section we study basic properties, like uniqueness and dependence on intervals Ω, of principal eigenpairs of (1.1) in one dimension.

As in the previous section, we assume throughout this section that $N=1$, $\Omega=(a,b)$ for some $-\infty<a<b<\infty$, and (F1)–(F3) hold with $\Lambda=\infty$.

Let (μ^+,φ^+) and (μ^-,φ^-) denote eigenpairs of (1.1) such that $\varphi^+(x)>0$ and $\varphi^-(x)<0$ for all $x\in(a,b)$. The existence of such eigenpairs has been established in Theorem 3.1.

Theorem 4.1. *If $(\mu,\varphi)\in\mathbb{R}\times W^{2,q}(a,b)$ is an eigenpair of (1.1) such that $\varphi(x)\geqslant 0$ (resp., $\varphi(x)\leqslant 0$) for all $x\in(a,b)$, then there exists a constant $\theta>0$ such that $(\mu,\varphi)=(\mu^+,\theta\varphi^+)$ (resp. $(\mu,\varphi)=(\mu^-,\theta\varphi^-)$).*

The above theorem says that the principal eigenvalues μ^+ and μ^- are unique and "half simple".

Proof. Let $(\mu,\varphi)\in\mathbb{R}\times W^{2,q}(a,b)$ be an eigenpair of (1.1) such that either $\varphi\geqslant 0$ or $\varphi\leqslant 0$ in (a,b). The assertion with a nonpositive φ can be reduced to that of with a nonnegative φ by replacing the functions φ^-, φ and F by the functions $-\varphi^-$, $-\varphi$ and $-F(-m,-p,-u,x)$, respectively. We may thus assume that $\varphi\geqslant 0$ in (a,b).

Using Theorem 2.6, we compare the functions φ^+ and φ with the constant function zero, to find that $\varphi^+(x)>0$ and $\varphi(x)>0$ in (a,b), $(\varphi^+)'(a)>0$, $\varphi'(a)>0$, $(\varphi^+)'(b)<0$ and $\varphi'(b)<0$.

To prove that $\mu^+=\mu$, we suppose that $\mu^+\neq\mu$, and obtain a contradiction. By symmetry, we may assume that $\mu^+<\mu$. Now, if we set $\theta=\inf_{(a,b)}\varphi/\varphi^+$, then $0<\theta<\infty$ and $\varphi\geqslant\theta\varphi^+$ in $[a,b]$. Since $\mu^+\varphi<\mu\varphi$ in (a,b), we have

$$F[\varphi]+\mu^+\varphi<F[\varphi]+\mu\varphi=0=F\big[\theta\varphi^+\big]+\mu^+\big(\theta\varphi^+\big)\quad\text{a.e. in } (a,b).$$

In particular, we have $\varphi(x)\not\equiv\theta\varphi^+(x)$ in $[a,b]$. Applying Theorem 2.6 again, we see that $\varphi(x)>\theta\varphi^+(x)$ for all $x\in(a,b)$, $\varphi'(a)>\theta(\varphi^+)'(a)$ and $\varphi'(b)<\theta(\varphi^+)'(b)$. But this tells us that $\theta<\inf_{(a,b)}\varphi/\varphi^+$, which contradicts the definition of θ.

Having shown that $\mu^+=\mu$, if we suppose that $\varphi\neq\theta\varphi^+$ and repeat the same argument as above, then we get a contradiction, which guarantees that $\varphi=\theta\varphi^+$. □

For any nonempty subinterval $[s,t]\subset[a,b]$, we denote by $\mu^+(s,t)$ and $\mu^-(s,t)$, respectively, the positive and negative principal eigenvalues of the eigenvalue problem (1.1), with $\Omega=(s,t)$. Such positive and negative principal eigenvalues $\mu^+(s,t)$, $\mu^-(s,t)$ exist and are unique due to Theorems 3.1 and 4.1.

Theorem 4.2.

(i) *Let $[s_1,t_1]$ and $[s_2,t_2]$ be nonempty subintervals of $[a,b]$ such that $[s_2,t_2]\subsetneq[s_1,t_1]$. Then $\mu^+(s_1,t_1)<\mu^+(s_2,t_2)$ and $\mu^-(s_1,t_1)<\mu^-(s_2,t_2)$.*

(ii) *The functions $\mu^+(s,t)$ and $\mu^-(s,t)$ are continuous in $\{(s,t)\in\mathbb{R}^2\colon a\leqslant s<t\leqslant b\}$.*

(iii) *The functions $\mu^+(s,t)$ and $\mu^-(s,t)$ diverge to infinity uniformly as $t-s\to 0$, that is,*

$$\lim_{\varepsilon\to 0+}\inf\big\{\mu^+(s,t),\mu^-(s,t)\colon a\leqslant s<t\leqslant b,\ t-s<\varepsilon\big\}=\infty. \tag{4.1}$$

Proof. As before, we only prove the assertion for $\mu^+(s,t)$.

We first prove the assertion (i). Let $[s_1,t_1]$ and $[s_2,t_2]$ be two intervals such that $[a,b]\supset[s_1,t_1]\supsetneq[s_2,t_2]\neq\emptyset$. Set $\mu_1=\mu^+(s_1,t_1)$ and $\mu_2=\mu^+(s_2,t_2)$. Let $\varphi_1\in W^{2,q}(s_1,t_1)$ and $\varphi_2\in W^{2,q}(s_2,t_2)$ be eigenfunctions corresponding to μ_1 and μ_2, respectively, such that $\varphi_i(x)>0$ for $x\in(s_i,t_i)$ and $i=1,2$. Setting $\theta=\inf_{(s_2,t_2)}\varphi_1/\varphi_2$, we observe that

$\varphi_1 \geqslant \theta\varphi_2$ in $[s_2, t_2]$. Observe also by the definition of θ that if we set $u(x) := \varphi_1(x) - \theta\varphi_2(x)$ for $x \in [s_2, t_2]$, then we have either $u(x_0) = 0$ for some $x_0 \in (s_2, t_2)$, or $u'(s_2) = 0$, or $u'(t_2) = 0$. Suppose by contradiction that $\mu_2 \leqslant \mu_1$. As in the proof of Theorem 4.1, since $\mu_2\varphi_1 \leqslant \mu_1\varphi_1$ in (s_2, t_2), we have $F[\varphi_1] + \mu_2\varphi_1 \leqslant 0 = F[\theta\varphi_2] + \mu_2\theta\varphi_2$ a.e. in (s_2, t_2). By Theorem 2.6, we deduce that $\varphi_1(x) \equiv \theta\varphi_2(x)$ in (s_2, t_2), but this is impossible since we have either $\varphi_1(s_2) > \varphi_2(s_2) = 0$ or $\varphi_1(t_2) > \varphi(t_2) = 0$. We therefore conclude that $\mu^+(s_2, t_2) > \mu^+(s_1, t_1)$.

Next we turn to (ii). Consider two sequences $(s_j)_{j\in\mathbb{N}}, (t_j)_{j\in\mathbb{N}} \subset [a, b]$ such that $s_j < t_j$ and $s_0 := \lim_{j\to\infty} s_j < t_0 := \lim_{j\to\infty} t_j$. For each $j \in \mathbb{N}$, let (μ_j, φ_j) be an eigenpair associated with the interval (s_j, t_j) satisfying $\varphi_j > 0$ in (s_j, t_j). Moreover, we may suppose that $\max_{[s_j,t_j]} \varphi_j = 1$. Let $\varphi_0 \in W^{2,q}(s_0, t_0)$ be the eigenfunction associated with the interval (s_0, t_0) and the eigenvalue $\mu_0 := \mu^+(s_0, t_0)$ such that $\varphi_0(x) > 0$ for all $x \in (s_0, t_0)$ and $\max_{[s_0,t_0]} \varphi_0 = 1$.

We intend to show that the eigenpairs (μ_j, φ_j) converge to the eigenpair (μ_0, φ_0) in the sense that, as $j \to \infty$, $\max_{I_j} |\varphi_j - \varphi_0| + |\mu_j - \mu_0| \to 0$, where

$$I_j := [s_0, t_0] \cap [s_j, t_j] = \bigl[\max\{s_0, s_j\}, \min\{t_0, t_j\}\bigr].$$

To this end, we argue by contradiction and assume that this is not the case. We may choose a subsequence of $(\mu_j, \varphi_j)_{j\in\mathbb{N}}$ so that the infimum over the subsequence of the quantities $\max_{I_j} |\varphi_j - \varphi_0| + |\mu_j - \mu_0|$ is positive. For notational simplicity, we denote this subsequence by the same symbol.

Fix constants ζ and η so that $s_0 < \zeta < \eta < t_0$. We may assume by focusing our attention to sufficiently large j that $s_j < \zeta < \eta < t_j$. In particular, we have $[\zeta, \eta] \subset [s_j, t_j] \subset [a, b]$ and $\mu^+(\zeta, \eta) \geqslant \mu_j \geqslant \mu^+(a, b)$, which shows that the sequence (μ_j) is bounded. We may therefore assume by passing again to a subsequence if necessary that (μ_j) converges to a constant μ.

We fix $\kappa \geqslant 0$ as in Section 3 so that (3.1) holds. If we define F_κ as in Section 3, then we have $F_\kappa[\varphi_j] + (\kappa + \mu_j)\varphi_j = 0$ a.e. in (s_j, t_j). According to (iii) of Lemma 3.2, there is a constant $C_0 > 0$, independent of j, such that

$$\|\varphi_j\|_{W^{2,q}(s_j,t_j)} \leqslant C_0\bigl(\kappa + |\mu_j|\bigr)\|\varphi_k\|_{L^\infty(s_k,t_k)} = C_0\bigl(\kappa + \bigl|\mu^+(a, b)\bigr| + \bigl|\mu^+(\zeta, \eta)\bigr|\bigr).$$

Using the Ascoli–Arzela theorem, we may assume that (φ_j) converges to a nonnegative function $\varphi \in C^1([s_0, t_0])$ in the sense that $\max_{I_j} |\varphi_j - \varphi| \to 0$ as $j \to \infty$. Moreover, it is easily seen that $\max_{[s_0,t_0]} \varphi = 1$.

Now, in view of Theorem 2.7, let $\psi \in W^{2,q}(s_0, t_0)$ be the solution of the Dirichlet problem $F_\kappa[\psi] + (\kappa + \mu)\varphi = 0$ a.e. in (s_0, t_0) and $\psi(s_0) = \psi(t_0) = 0$. Set $d_j = \max_{\partial I_j} \psi$ and $e_j = \max_{\partial I_j} \varphi_j$. Note here that ∂I_j consists of exactly two points $\max\{s_0, s_j\}$ and $\min\{t_0, t_j\}$ for $j \in \mathbb{N}$. Observe that for each j, the function $u(x) := \psi(x) - d_j$ satisfies $u|_{\partial I_j} \leqslant 0$ and

$$\begin{aligned} 0 &= F_\kappa[\psi] + (\kappa + \mu)\varphi = F_\kappa[u + d_j] + (\kappa + \mu)\varphi \\ &\leqslant F_\kappa[u] + d_j\gamma + (\kappa + \mu)\varphi \quad \text{a.e. in } I_j. \end{aligned}$$

Apply Theorem 2.4 to the functions u and φ_j, to find a constant $C_1 > 0$, independent of j, such that

$$\max_{I_j}(\psi - \varphi_j) \leqslant d_j + C_1\bigl\|d_j\gamma + \kappa|\varphi - \varphi_j| + |\mu\varphi - \mu_j\varphi_j|\bigr\|_{L^q(I_j)}.$$

Similarly, we obtain

$$\max_{I_j}(\varphi_j - \psi) \leqslant e_j + C_1\bigl\|e_j\gamma + \kappa|\varphi - \varphi_j| + |\mu\varphi - \mu_j\varphi_j|\bigr\|_{L^q(I_j)}.$$

These inequalities show in the limit as $j \to \infty$ that $\psi = \varphi$ in $[s_0, t_0]$. Thus, the pair (μ, φ) is an eigenpair of (1.1), $\varphi \geqslant 0$ in $[s_0, t_0]$ and $\max_{[s_0,t_0]} \varphi = 1$. Theorem 4.1 ensures that $(\mu, \varphi) = (\mu_0, \varphi_0)$. This is a contradiction, which proves that the eigenpairs (μ_j, φ_j) converge to the eigenpair (μ_0, φ_0) in the sense that, as $j \to \infty$, $\max_{I_j} |\varphi_j - \varphi_0| + |\mu_j - \mu_0| \to 0$. In particular, we see that $\mu_j \to \mu_0$ as $j \to \infty$, proving the continuity of $(s, t) \mapsto \mu^+(s, t)$.

Finally we prove the assertion (iii). Let (μ, φ) be an eigenpair of (1.1) with $\Omega = (s, t)$, where $a \leqslant s < t \leqslant b$, satisfying $\varphi(x) > 0$ in (s, t). Applying Theorem 2.4 yields

$$\max_{[s,t]} \varphi \leqslant (t - s)C_2(\kappa + \mu)_+ \max_{[s,t]} \varphi,$$

where C_2 is a positive constant independent of s, t, μ and φ. Hence, we have $1 \leqslant C_2(\kappa + \mu)_+(t - s)$, which shows that

$$\lim_{\varepsilon\to 0+} \inf\bigl\{\mu^+(s, t)\colon a \leqslant s < t \leqslant b,\ t - s < \varepsilon\bigr\} = \infty. \qquad \square$$

5. General eigenvalues in one dimension

In this section, we complete the proof of Theorem 1.1. We thus establish the existence of general eigenpairs of (1.1) and their uniqueness and "half simplicity" in one dimension under hypotheses (F1)–(F3).

Throughout this section we assume as in the previous section that $N = 1$, $\Omega = (a, b)$ for some $-\infty < a < b < \infty$, and (F1)–(F3) hold with $\Lambda = \infty$.

Before going into the detail of the proof of Theorem 1.1, we illustrate very briefly how the proof goes.

The case $n = 1$ of Theorem 1.1 is a direct consequence of Theorems 3.1 and 4.1. As before, let $\mu^+(s,t)$ and $\mu^-(s,t)$ denote, respectively, the positive and negative principal eigenvalues of (1.1) with $\Omega = (s,t)$, where $a \leqslant s < t \leqslant b$. For the proof in the case $n = 2$, we consider the function $g_2(s) = \mu^+(a,s) - \mu^-(s,b)$ on the interval (a,b) and observe by Theorem 4.2 that g_2 is continuous and decreasing in (a,b),

$$\lim_{s\to a+0} g_2(s) = \infty \quad \text{and} \quad \lim_{s\to b-0} g_2(s) = -\infty.$$

Hence there is a unique $\tau_2 \in (a,b)$ such that $g_2(\tau_2) = 0$, i.e., $\mu^+(a,\tau_2) = \mu^-(\tau_2,b)$. Set $\mu_2^+ := \mu^+(a,\tau_2) = \mu^-(\tau_2,b)$. We then choose a positive eigenfunction φ^+ on (a,τ_2) and a negative eigenfunction φ^- on (τ_2,b) corresponding to the eigenvalue μ_2^+. Here, by multiplying φ^- by a positive constant if needed, we may assume that $(\varphi^+)'(\tau_2 - 0) = (\varphi^-)'(\tau_2 + 0)$. Setting $\varphi_2^+(x) = \varphi^+(x)$ for $x \in [a,\tau_2]$ and $\varphi_2^+(x) = \varphi^-(x)$ for $x \in [\tau_2,b]$, we obtain an eigenpair (μ_2^+,φ_2^+) of (1.1), which proves one half of assertion (i) of Theorem 1.1. The other half is proved similarly.

Regarding assertion (i), the next step is to show that the second eigenvalues $\mu_2^\pm(s,t)$ corresponding to the interval $[s,t] \subset [a,b]$ have the monotonicity, continuity and unboundedness properties as (i), (ii) and (iii) of Theorem 4.2. Then, we proceed to introduce the function $g_3(s) = \mu_2^+(a,s) - \mu^+(s,b)$ on (a,b), to choose a constant $\tau_3 \in (a,b)$ so that $g_3(\tau_3) = 0$, to define $\mu_3^+ := \mu_2^+(a,\tau_3) = \mu^+(\tau_3,b)$, and so on. To make it logically precise, we will employ the induction argument. The completeness assertion (ii) will follow from the strong maximum principle for (1.1), which is a consequence of Theorem 2.6 or the uniqueness assertion of Theorem 2.2.

We begin the detail with two lemmas.

Lemma 5.1. *Let $(\mu,\nu) = (\mu^-,\mu^+)$ or $(\mu,\nu) = (\mu^+,\mu^-)$. Let $h:(a,b) \to (a,b)$ be a nondecreasing continuous function such that $h(s) \leqslant s$ in (a,b). Then there exists a unique function $\tau:(a,b] \to (a,b)$ such that $\tau(t) < t$ and $\mu(a,h(\tau(t))) = \nu(\tau(t),t)$ for each $t \in (a,b]$. Moreover, the function τ is continuous and (strictly) increasing in $(a,b]$.*

Proof. According to Theorem 4.2, the functions $\mu(s,t)$ and $\nu(s,t)$ are continuous on $\{(s,t)\colon a \leqslant s < t \leqslant b\}$, increasing as functions of s in (a,t) and decreasing as functions of t in (s,b). We define the continuous function g on $\{(s,t) \in \mathbb{R}^2\colon a < s < t \leqslant b\}$ by $g(s,t) = \mu(a,h(s)) - \nu(s,t)$. Observe that the function $g(s,t)$ is decreasing as a function of s in (a,t) and increasing as a function of t in (s,b).

Using Theorem 4.2, we deduce that

$$\lim_{s\to a+} g(s,t) = \infty \quad \text{and} \quad \lim_{s\to t-} g(s,t) = -\infty.$$

It is now obvious that for each $t \in (a,b]$ there exists a unique $\tau(t) \in (a,t)$ such that $g(\tau(t),t) = 0$. It is easily seen by the monotonicity of $g(s,t)$ in s and in t that the function $\tau:(a,b] \to (a,b)$ is increasing.

Finally, to check the continuity of τ, we fix a sequence $(t_k)_{k\in\mathbb{N}} \subset (a,b]$ converging to a point $t_0 \in (a,b]$ and prove that $\lim_{k\to\infty} \tau(t_k) = \tau(t_0)$. We may assume that $t_k > c$ for all k and some $c \in (a,b)$. By the monotonicity of τ, we have $b > \tau(b) > \tau(t_k) \geqslant \tau(c) > a$ for all k. If we set $s^+ := \limsup_{k\to\infty} \tau(t_k)$ and $s^- := \liminf_{k\to\infty} \tau(t_k)$, then $a < s^- \leqslant s^+ < t_0$, by Theorem 4.2(iii), and $g(s^+,t_0) = g(s^-,t_0) = 0$ by the continuity of g. Hence, we must have $\lim_{k\to\infty} \tau(t_k) = \tau(t_0)$. □

Lemma 5.2. *Let $n \in \mathbb{N}$ and $(x_j)_{j=0}^n, (y_j)_{j=0}^n \subset [a,b]$ be increasing finite sequences such that $[x_0,x_n] \subset [y_0,y_n]$. Then there exists an index $j \in \{1,\dots,n\}$ such that $[x_{j-1},x_j] \subset [y_{j-1},y_j]$ and moreover, if $[x_0,x_n] \neq [y_0,y_n]$, then $[x_{j-1},x_j] \neq [y_{j-1},y_j]$.*

Proof. First we consider the case $[x_0, x_n] \subsetneq [y_0, y_n]$. If $y_0 < x_0$, then we set $k := \max\{j\colon 0 \leqslant j \leqslant n-1,\ y_j < x_j\}$ and observe that $[x_k, x_{k+1}] \subsetneq [y_k, y_{k+1}]$. Otherwise, we have $x_n < y_n$. If we set $\ell := \min\{j\colon 1 \leqslant j \leqslant n,\ x_j < y_j\}$, then $[x_{\ell-1}, x_\ell] \subsetneq [y_{\ell-1}, y_\ell]$.

Next we consider the case $[x_0, x_n] = [y_0, y_n]$. If either $x_1 \leqslant y_1$ or $y_{n-1} \leqslant x_{n-1}$ hold, then our claim follows. Otherwise, we find $[x_1, x_{n-1}] \subsetneq [y_1, y_{n-1}]$ and our claim follows from the above argument. □

Henceforth we use this notation: we denote by s_j the symbols $+$, if j is odd, and $-$ if j is even. For instance, $\psi^{\mathrm{s}_2} = \psi^-$, $\psi^{\mathrm{s}_3} = \psi^+$ and so on.

Proof of Theorem 1.1. We here prove the assertion for (μ_n^+, φ_n^+) since this assertion is easily converted to that for (μ_n^-, φ_n^-) by replacing the function $F(m, p, u, x)$ by $-F(-m, -p, -u, x)$.

We treat the existence assertion (i). As noted above, the case $n = 1$ has already been shown in Theorem 3.1. We are thus concerned with the case where $n \geqslant 2$.

We show by induction that for any $n \in \mathbb{N}$, there exists a sequence $(x_{n,j})_{j=1}^n$ of functions on $(a, b]$ such that

$$a < x_{n,1}(t) < x_{n,2}(t) < \cdots < x_{n,n}(t) = t \quad \text{for every } t \in (a, b], \tag{5.1}$$

$$x_{n,j}(t) \text{ is a (strictly) increasing continuous function on } (a, b] \quad \text{for all } j, \tag{5.2}$$

$$\mu^{\mathrm{s}_j}\big(x_{n,j-1}(t), x_{n,j}(t)\big) = \mu^{\mathrm{s}_1}\big(a, x_{n,1}(t)\big) \quad \text{for all } t \in (a, b] \text{ and } j \geqslant 2. \tag{5.3}$$

In the case where $n = 1$, the function $x_{1,1}(t) = t$ trivially satisfies (5.1)–(5.3).

Now, suppose that we are given a finite sequence $(x_{n,j})_{j=1}^n$ satisfying (5.1)–(5.3) for some $n \in \mathbb{N}$. We apply Lemma 5.1, to find an increasing continuous function τ on $(a, b]$ such that $\tau(t) < t$ and $\mu^{\mathrm{s}_1}(a, x_{n,1}(\tau(t))) = \mu^{\mathrm{s}_{n+1}}(\tau(t), t)$ for all $t \in (a, b]$. From (5.3), we get $\mu^{\mathrm{s}_j}(x_{n,j-1}(\tau(t)), x_{n,j}(\tau(t))) = \mu^{\mathrm{s}_1}(a, x_{n,1}(\tau(t)))$ for all $t \in (a, b]$ and $j = 2, \ldots, n$. We define the finite sequence $(x_{n+1,j})_{j=1}^{n+1}$ by setting $x_{n+1,j} = x_{n,j} \circ \tau$ if $1 \leqslant j \leqslant n$ and $x_{n+1,n+1}(t) = t$. It is clear that $(x_{n+1,j})_{j=1}^{n+1}$ satisfies (5.1)–(5.3) with $n + 1$ in place of n. This completes our induction argument.

Next, fix $n \geqslant 2$ and set $x_0^+ = a$, $x_j^+ = x_{n,j}(b)$ for $j = 1, \ldots, n$, and $\mu_n^+ = \mu^{\mathrm{s}_1}(a, x_1^+)$. It follows from (5.3) that $\mu^{\mathrm{s}_j}(x_{j-1}^+, x_j^+) = \mu_n^+$ for $j = 1, \ldots, n$. We choose functions $\varphi_{n,j} \in W^{2,q}(x_{j-1}^+, x_j^+)$, with $j = 1, \ldots, n$, so that if j is odd (resp., even), then the function $\varphi_{n,j}$ is a positive (resp., negative) principal eigenfunction corresponding to $\mu^+(x_{j-1}^+, x_j^+)$ (resp., $\mu^-(x_{j-1}^+, x_j^+)$). From Theorem 2.6, we see that for all $j = 1, \ldots, n-1$,

$$(-1)^j \varphi_{n,j}'\big(x_j^+ - 0\big) > 0 \quad \text{and} \quad (-1)^j \varphi_{n,j+1}'\big(x_j^+ + 0\big) > 0.$$

Hence we can choose a finite sequence $(\theta_j)_{j=1}^n$ of positive numbers so that $\theta_1 = 1$ and

$$\theta_j \varphi_{n,j}'\big(x_j^+ - 0\big) = \theta_{j+1} \varphi_{n,j+1}'\big(x_j^+ + 0\big) \quad \text{for all } j = 1, \ldots, n-1.$$

Set

$$\varphi_n^+(x) = \theta_j \varphi_{n,j}(x) \quad \text{if } x \in \big[x_{j-1}^+, x_j^+\big] \text{ and } 1 \leqslant j \leqslant n,$$

and observe that $\varphi_n^+ \in W^{2,q}(a, b)$ and (μ_n^+, φ_n^+) is an eigenpair of (1.1) having the property that $(-1)^{j-1}\varphi_n(x) > 0$ in (x_{j-1}^+, x_j^+) for $j = 1, \ldots, n$.

Now, we deal with the assertion (ii). Fix an $n \in \mathbb{N}$ and let $(\mu_n^+, \varphi_n^+) \in \mathbb{R} \times W^{2,q}(a, b)$ be an eigenpair obtained in the above. Let $(x_j^+)_{j=0}^n$ be the increasing finite sequence of the zeroes in $[a, b]$ of φ_n^+. Let $(\mu, \varphi) \in \mathbb{R} \times W^{2,q}(a, b)$ be any eigenpair of (1.1) such that the function φ vanishes exactly at $n + 1$ distinct points in $[a, b]$. Let $(y_j)_{j=0}^n$ be the increasing finite sequence of zeroes of φ so that $y_0 = a$ and $b = y_n$.

To proceed, we may focus on the case where $\varphi(x) > 0$ in (y_0, y_1). We intend to show that $\mu_n^+ = \mu$ and there is a constant $\theta > 0$ such that $\varphi = \theta\varphi_n^+$. If $n = 1$, then this is a consequence of Theorem 4.1. We may therefore assume that $n \geqslant 2$.

From Theorem 2.2 or 2.6, we see that $(-1)^j \varphi'(y_j) > 0$ for all $j = 0, 1, \ldots, n$ and accordingly, $(-1)^{j-1}\varphi(x) > 0$ in (y_{j-1}, y_j) for $j = 1, \ldots, n$. By Theorem 4.1, we have $\mu_n^+ = \mu^{s_j}(x_{j-1}^+, x_j^+)$ and $\mu = \mu^{s_j}(y_{j-1}, y_j)$ for $1 \leqslant j \leqslant n$.

Applying Lemma 5.2, we find $j, k \in \{1, \dots, n\}$ satisfying $[x_{j-1}^+, x_j^+] \subset [y_{j-1}, y_j]$ and $[y_{k-1}, y_k] \subset [x_{k-1}^+, x_k^+]$. In view of Theorem 4.2, we obtain

$$\mu_n^+ = \mu^{s_j}\big(x_{j-1}^+, x_j^+\big) \geqslant \mu^{s_j}(y_{j-1}, y_j) = \mu = \mu^{s_k}(y_{k-1}, y_k) \geqslant \mu^{s_k}\big(x_{k-1}^+, x_k^+\big) = \mu_n^+,$$

which yields $\mu = \mu_n^+$.

By Theorem 4.2(i) and the fact that $\mu = \mu_+^n$, we infer that $y_j = x_j^+$ for all $1 \leqslant j \leqslant n-1$. Furthermore, by Theorem 4.1, we see that there is a finite sequence $(\theta_j)_{j=1}^n$ of positive numbers so that $\varphi = \theta_j \varphi_n^+$ in $[x_{j-1}^+, x_j^+]$ for $1 \leqslant j \leqslant n$. But, since φ and φ_n^+ are both C^1 functions on $[a,b]$, we see that the constants θ_j are all the same. Thus, $\varphi = \theta\varphi_n^+$ in $[a,b]$ for some constant $\theta > 0$.

What remains is to show that every eigenfunction of (1.1) has a finite number of zeroes. To this end, we suppose by contradiction that there is an eigenpair (μ, φ) of (1.1) such that φ has infinitely many zeroes. This means that there exists an accumulation point $c \in [a,b]$ of zeroes of φ. We see immediately that $\varphi(c) = 0$, and moreover by using Rolle's theorem that $\varphi'(c) = 0$. Theorem 2.2 now allows us to conclude that $\varphi(x) \equiv 0$ in $[a,b]$, which is a contradiction. This proves that every eigenfunction of (1.1) has a finite number of zeroes. □

Next, we give basic properties of the sequence $(\mu_n^\pm)_{n\in\mathbb{N}}$.

Proposition 5.3. *Let (μ_n^+) and (μ_n^-) be sequences of eigenvalues given by Theorem 1.1. Then*

$$\lim_{n\to\infty} \min\{\mu_n^+, \mu_n^-\} = \infty, \tag{5.4}$$

$$\max\{\mu_n^+, \mu_n^-\} < \min\{\mu_{n+1}^+, \mu_{n+1}^-\} \quad \textit{for each } n \in \mathbb{N}. \tag{5.5}$$

Proof. Let φ be an eigenfunction corresponding to μ_n^+ and $(x_j)_{j=0}^n$ the finite sequence of zeroes of φ. Since $\mu_n^+ = \mu^{s_j}(x_{j-1}, x_j)$ for $1 \leqslant j \leqslant n$ and $\min_{1\leqslant j\leqslant n}(x_j - x_{j-1}) \leqslant (b-a)/n$, we see that

$$\mu_n^+ \geqslant \inf\{\mu^+(s,t), \mu^-(s,t)\colon a \leqslant s < t \leqslant b,\ t-s \leqslant (b-a)/n\}.$$

Similarly, we get

$$\mu_n^- \geqslant \inf\{\mu^+(s,t), \mu^-(s,t)\colon a \leqslant s < t \leqslant b,\ t-s \leqslant (b-a)/n\}.$$

Thus, by Theorem 4.2(iii), (5.4) holds.

Next let φ_n^+, φ_n^- and φ_{n+1}^+ be eigenfunctions corresponding to the eigenvalues μ_n^+, μ_n^- and μ_{n+1}^+, respectively. Also let $(x_j^+)_{j=0}^n$, $(y_j^-)_{j=0}^n$ and $(z_j^+)_{j=0}^{n+1}$ be the finite sequences of the zeroes of φ_n^+, φ_n^- and φ_{n+1}^+, respectively. By Lemma 5.2, there is a $k \in \{1, \dots, n\}$ such that $[z_{k-1}^+, z_k^+] \subsetneq [x_{k-1}^+, x_k^+]$. Using Theorem 4.2, we have

$$\mu_{n+1}^+ = \mu_n^{s_k}\big(z_{k-1}^+, z_k^+\big) > \mu_n^{s_k}\big(x_{k-1}^+, x_k^+\big) = \mu_n^+.$$

Similarly, we deduce that there is an integer $\ell \in \{2, \dots, n+1\}$ satisfying $[z_{\ell-1}^+, z_\ell^+] \subsetneq [y_{\ell-2}^-, y_{\ell-1}^-]$ and $[z_{\ell-1}^+, z_\ell^+] \neq [y_{\ell-2}^-, y_{\ell-1}^-]$ and that

$$\mu_{n+1}^+ = \mu^{s_\ell}\big(z_{\ell-1}^+, z_\ell^+\big) > \mu^{s_\ell}\big(y_{\ell-2}^-, y_{\ell-1}^-\big) = \mu_n^-.$$

Thus we have $\mu_{n+1}^+ > \max\{\mu_n^+, \mu_n^-\}$. Similarly, we obtain $\mu_{n+1}^- > \max\{\mu_n^+, \mu_n^-\}$, which completes the proof. □

Finally, by reviewing the proof of Theorem 1.1, we note that the eigenvalues μ_n^+ and μ_n^-, with any $n \in \mathbb{N}$, are continuous as functions of (a,b) on the set $\{(x,y) \in \mathbb{R}^2\colon x < y\}$.

6. Radially symmetric solutions

In the rest of this paper, we assume that $N \geqslant 2$ and study radially symmetric solutions of PDE of the form

$$F\big(D^2u, Du, u, x\big) = 0 \quad \text{in } B_R, \tag{6.1}$$

where $0 < R < \infty$.

Let u be a smooth function on $\bar{B}_R$. Assume that u is radially symmetric, i.e., $u(x) = g(|x|)$ in B_R for some function g on $[0, R]$. Note that for $1 \leqslant q < \infty$,

$$\int_{B_R} \left|u(x)\right|^q \,\mathrm{d}x = \alpha_N \int_0^R \left|g(r)\right|^q r^{N-1}\,\mathrm{d}r, \tag{6.2}$$

where α_N is the surface measure of the unit sphere S^{N-1}, and that if $u \in C^2(B_R)$, then

$$Du(x) = g'\big(|x|\big)\frac{x}{|x|} \quad \text{and} \quad D^2u(x) = g''\big(|x|\big)P_x + \frac{g'(|x|)}{|x|}(I_N - P_x) \quad \text{for } x \neq 0, \tag{6.3}$$

where P_x denotes the matrix $x \otimes x/|x|^2 = (x_i x_j/|x|^2)$ which represents the orthogonal projection in $\mathbb{R}^N$ onto the one-dimensional space spanned by the vector x. In the above situation, we have

$$\left|D^2u(x)\right| := \left(\sum_{i,j}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}(x)\right|^2\right)^{1/2} = \left(\left|g''\big(|x|\big)\right|^2 + (N-1)\frac{|g'(|x|)|^2}{|x|^2}\right)^{1/2}. \tag{6.4}$$

With these observations at hand, we introduce the function spaces $L^q_{\mathrm{r}}(a, R)$ and $W^{2,q}_{\mathrm{r}}(a, R)$, where $0 \leqslant a < R$ and $q \in [1, \infty]$, as follows: if $q < \infty$, $L^q_{\mathrm{r}}(a, R)$ denotes the space of all measurable functions g on (a, R) such that $r \mapsto |g(r)|^q r^{N-1}$ is integrable on (a, R), with norm given by

$$\|g\|_{L^q_{\mathrm{r}}(a,R)} = \left(\int_a^R \left|g(r)\right|^q r^{N-1}\,\mathrm{d}r\right)^{1/q},$$

and $W^{2,q}_{\mathrm{r}}(a, R)$ denotes the space of all functions $g \in L^q_{\mathrm{r}}(a, R)$ such that the functions $r \mapsto (|g'(r)|/r)^q r^{N-1}$ and $r \mapsto |g''(r)|^q r^{N-1}$ are integrable on (a, R), with norm given by

$$\|g\|_{W^{2,q}_{\mathrm{r}}(a,R)} = \|g\|_{L^q_{\mathrm{r}}(a,R)} + \left\|g'/r\right\|_{L^q_{\mathrm{r}}(a,R)} + \left\|g''\right\|_{L^q_{\mathrm{r}}(a,R)},$$

where g'/r denotes conveniently the function $r \mapsto g'(r)/r$. In the case where $q = \infty$, we set $L^\infty_{\mathrm{r}}(a, R) = L^\infty(a, R)$ and $W^{2,\infty}_{\mathrm{r}}(a, R) = \{g \in W^{2,\infty}(a, R)\colon\ g'(0) = 0\}$ if $a = 0$ and $= W^{2,\infty}(a, R)$ otherwise.

We remark that $L^q_{\mathrm{r}}(a, R) \subset L^p_{\mathrm{r}}(a, R)$ and $W^{2,q}_{\mathrm{r}}(a, R) \subset W^{2,p}_{\mathrm{r}}(a, R)$, if $p \leqslant q$, by Hölder's inequality and that $L^q(a, R) = L^q_{\mathrm{r}}(a, R)$ and $W^{2,q}(a, R) = W^{2,q}_{\mathrm{r}}(a, R)$, if $a > 0$, together with the equivalence of their respective norms.

We recall that $W^{2,q}_{\mathrm{r}}(B_R)$ is the subspace of the usual Sobolev space $W^{2,q}(B_R)$ consisting of all radially symmetric functions $u \in W^{2,q}(B_R)$, with norm

$$\|u\|_{W^{2,q}(B_R)} = \|u\|_{L^q(B_R)} + \|Du\|_{L^q(B_R)} + \left\|\left|D^2u\right|\right\|_{L^q(B_R)}.$$

The following lemma says that $W^{2,q}_{\mathrm{r}}(B_R)$ can be identified with $W^{2,q}_{\mathrm{r}}(0, R)$.

Lemma 6.1. *Let $q \in [1, \infty]$ and u and g be measurable functions on B_R and $(0, R)$, respectively. Assume that $u(x) = g(|x|)$ a.e. in B_R. Then, $u \in W^{2,q}_{\mathrm{r}}(B_R)$ if and only if $g \in W^{2,q}_{\mathrm{r}}(0, R)$. Furthermore, in this case we have*

$$Du(x) = g'\big(|x|\big)\frac{x}{|x|} \quad \textit{and} \quad D^2u(x) = g''\big(|x|\big)P_x + \frac{g'(|x|)}{|x|}(I_N - P_x) \quad \textit{a.e.}$$

Proof. We treat here only the case where $q < \infty$, and leave it to the reader to prove the assertion in the case where $q = \infty$.

First, we assume that $u \in W^{2,q}(B_R)$, and show that $g \in W^{2,q}_{\mathrm{r}}(0, R)$. Choose a sequence $(u_k)_{k\in\mathbb{N}}$ of smooth radial functions on $\bar{B}_R$ so that $\lim_{k\to\infty} \|u_k - u\|_{W^{2,q}(B_R)} = 0$. The existence of such a sequence (u_k) can be shown by the combination of the mollification technique and scaling of functions by multiplying the independent variables by a positive constant less than one. For more detail on this, we note first that one can approximate u in $W^{2,q}(B_R)$ by the family of functions $u_\eta(x) := u(\eta x)$, where $0 < \eta < 1$, as $\eta \to 1 - 0$ and secondly that if ρ_ε denotes the standard mollification kernel with support in B_ε, then, for each $0 < \eta < 1$, the convolution $\rho_\varepsilon * u_\eta$ belongs in $C^\infty(\bar{B}_R)$ for every $\varepsilon > 0$ sufficiently small and $\rho_\varepsilon * u_\eta \to u_\eta$ in $W^{2,q}(B_R)$ as $\varepsilon \to 0+$.

Define the function g_k on $[0, R]$ by setting $g_k(r) = u_k(x)$ if $|x| = r$. Combining (6.2)–(6.4) applied to (u_k, g_k) yields

$$\alpha_N^{1/q} \|g_k\|_{W_{\mathrm{r}}^{2,q}(0,R)} \leqslant 2\|u_k\|_{W^{2,q}(B_R)},$$

which is still valid if one replaces (u_k, g_k) by $(u_k - u_j, g_k - g_j)$. Accordingly, the sequence (g_k) is a Cauchy sequence in $W_{\mathrm{r}}^{2,q}(0, R)$, which implies that $g \in W_{\mathrm{r}}^{2,q}(0, R)$ and moreover, $\alpha_N^{1/q}\|g\|_{W_{\mathrm{r}}^{2,q}(0,R)} \leqslant 2\|u\|_{W^{2,q}(B_R)}$.

Next, we assume that $g \in W_{\mathrm{r}}^{2,q}(0, R)$, and prove that $u \in W^{2,q}(B_R)$. Note that $g \in W^{2,q}(a, R) \subset C^1([a, R])$ for any $a \in (0, R)$. We calculate for $0 < a < R$,

$$\left|g(R) - g(a)\right| a^{N-2} \leqslant a^{N-2} \int_a^R \left|g'(t)\right| \mathrm{d}t \leqslant \int_a^R \left|\frac{g'(t)}{t}\right| t^{N-1}\, \mathrm{d}t \leqslant \left\|g'/r\right\|_{L_{\mathrm{r}}^1(0,R)},$$

and

$$a^{N-1}\left|g(a)\right| \leqslant a^{N-1}\left|g(R)\right| + a\left\|g'/r\right\|_{L_{\mathrm{r}}^1(0,R)}. \tag{6.5}$$

Now, let $\psi \in C_0^1(B_R)$ and $0 < a < R$. Using the divergence theorem, we get

$$-\int_{B_R\setminus B_a} \psi_{x_i} u(x)\, \mathrm{d}x = \int_{\partial B_a} \psi(x) g(a) \frac{x_i}{|x|}\, \mathrm{d}S + \int_{B_R\setminus B_a} \psi(x) g'\big(|x|\big) \frac{x_i}{|x|}\, \mathrm{d}x,$$

where $\mathrm{d}S$ denotes the surface measure. Noting by (6.5) that

$$\left|\int_{\partial B_a} \psi(x) g(a) \frac{x_i}{|x|}\, \mathrm{d}S\right| \leqslant \left|g(a)\right| \alpha_N a^{N-1} \|\psi\|_{L^\infty(B_R)} \to 0 \quad \text{as } a \to 0,$$

we get

$$-\int_{B_R} \psi_{x_i} u(x)\, \mathrm{d}x = \int_{B_R} \psi(x) g'\big(|x|\big) \frac{x_i}{|x|}\, \mathrm{d}x.$$

Thus, we have $Du(x) = g'(|x|)x/|x|$ a.e. in B_R.

Let $0 < a < b < R$, and compute that

$$\left|g'(b) - g'(a)\right| a^{N-1} \leqslant \int_a^b \left|g''(t)\right| t^{N-1}\, \mathrm{d}t \leqslant \left\|g''\right\|_{L_{\mathrm{r}}^1(0,b)},$$

and

$$a^{N-1}\left|g'(a)\right| \leqslant a^{N-1}\left|g(b)\right| + \left\|g''\right\|_{L_{\mathrm{r}}^1(0,b)}. \tag{6.6}$$

Note here that the right-hand side converges to $\|g''\|_{L_{\mathrm{r}}^1(0,b)}$ as $a \to 0$ and $\|g''\|_{L_{\mathrm{r}}^1(0,b)} \to 0$ as $b \to 0$. As before, let $\psi \in C_0^1(B_R)$ and $0 < a < b < R$. By the divergence theorem, we get

$$\begin{aligned} -\int_{B_R\setminus B_a} \psi_{x_i}(x) u_{x_j}(x)\, \mathrm{d}x &= -\int_{B_R\setminus B_a} \psi_{x_i}(x) g'\big(|x|\big) \frac{x_j}{|x|}\, \mathrm{d}x \\ &= \int_{\partial B_a} \psi(x) g'(a) \frac{x_i x_j}{|x|^2}\, \mathrm{d}S + \int_{B_R\setminus B_a} \psi(x) \left(g''\big(|x|\big) \frac{x_i x_j}{|x|^2} + g'\big(|x|\big) \frac{\delta_{ij}|x|^2 - x_i x_j}{|x|^3}\right) \mathrm{d}x, \end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and $= 0$ if $i \neq j$. The last equality is clearly valid when g is smooth. In general it may need a justification, which can be done by approximating g by smooth functions. By (6.6), we get

$$\lim_{a\to 0+}\left| \int_{\partial B_a} \psi(x) g'(a) \frac{x_i x_j}{|x|^2}\, \mathrm{d}S \right| = 0,$$

and accordingly,

$$-\int_{B_R} \psi_{x_i}(x) u_{x_j}(x)\, \mathrm{d}x = \int_{B_R} \psi(x) \left(g''\big(|x|\big) \frac{x_i x_j}{|x|^2} + g'\big(|x|\big) \frac{\delta_{ij}|x|^2 - x_i x_j}{|x|^3} \right) \mathrm{d}x.$$

Thus, we have

$$D^2 u(x) = g''\big(|x|\big) P_x + \frac{g'(|x|)}{|x|} (I_N - P_x) \quad \text{a.e. in } B_R.$$

Finally, a simple calculation shows that

$$\|u\|_{W^{2,q}(B_R)} \leqslant \alpha_N^{1/q} (R + \sqrt{N-1}) \|g\|_{W_{\mathrm{r}}^{2,q}(0,R)}.$$

We therefore conclude that $u \in W^{2,q}(B_R)$. □

We assume in the rest of this section that F satisfies (F1), (F2) with $\Lambda < \infty$ and (F4). Let $u \in W_{\mathrm{r}}^{2,q}(B_R)$ and $g \in W_{\mathrm{r}}^{2,q}(0,R)$ satisfy $u(x) = g(|x|)$ a.e. in B_R. In view of Lemma 6.1, we see that u is a solution of (6.1) if and only if for a.a. $(r,\omega) \in (0,R) \times S^{N-1}$,

$$F\left(g''(r)\omega\otimes\omega + \frac{g'(r)}{r}(I_N - \omega\otimes\omega), g'(r)\omega, g(r), r\omega \right) = 0.$$

Thanks to (F4), this last condition is equivalent to the condition: for any fixed $\omega \in S^{N-1}$,

$$F\left(g''(r)\omega\otimes\omega + \frac{g'(r)}{r}(I_N - \omega\otimes\omega), g'(r)\omega, g(r), r\omega \right) = 0 \quad \text{a.e. } r \in (0,R).$$

We fix a point $\omega_0 \in S^{N-1}$ and define the function $\mathcal{F} : \mathbb{R}^4 \times (0,R) \to \mathbb{R}$ by

$$\mathcal{F}(m,l,p,u,r) = F\big(m\omega_0\otimes\omega_0 + l(I_N - \omega_0\otimes\omega_0), p\omega_0, u, r\omega_0\big).$$

Also, we introduce radial versions $\mathcal{P}^+, \mathcal{P}^- : \mathbb{R}^2 \to \mathbb{R}$ of the Pucci operators adapted to this circumstance by

$$\mathcal{P}^+(m,l) = P^+\big(m\omega_0\otimes\omega_0 + l(I_N - \omega_0\otimes\omega_0)\big),$$

and $\mathcal{P}^-(m,l) = -\mathcal{P}^+(-m,-l)$. By (F2), we have

$$\begin{aligned} &\mathcal{F}(m_1,l_1,p_1,u_1,r) - \mathcal{F}(m_2,l_2,p_2,u_2,r) \\ &\quad \leqslant \mathcal{P}^+(m_1-m_2, l_1-l_2) + \beta(r\omega)|p_1-p_2| + \gamma(r\omega)|u_1-u_2| \end{aligned} \tag{6.7}$$

for all $(m_i,l_i,p_i,u_i,r) \in \mathbb{R}^4$, $i = 1,2$, and a.a. $(r,\omega) \in (0,R) \times S^{N-1}$. In view of Fubini's theorem in the polar coordinates, there is a choice of $\omega \in S^{N-1}$ having the properties that the inequality (6.7), with this ω, holds for all $(m_i,l_i,p_i,u_i) \in \mathbb{R}^4$, $i = 1,2$, and a.a. $r \in (0,R)$ and that the functions $r \mapsto \beta(r\omega)$ and $r \mapsto \gamma(r\omega)$ belong to $L_{\mathrm{r}}^q(0,R)$. We fix such an ω, call it ω_1, and, with abuse of notation, we write β and γ the functions $r \mapsto \beta(r\omega_1)$ and $r \mapsto \gamma(r\omega_1)$, respectively. In other words, under the assumptions (F1), (F2) and (F4), we conclude the following:

(F5) There exist functions $\beta, \gamma \in L_{\mathrm{r}}^q(0,R)$ such that

$$\mathcal{F}(m_1,l_1,p_1,u_1,r) - \mathcal{F}(m_2,l_2,p_2,u_2,r) \leqslant \mathcal{P}^+(m_1-m_2, l_1-l_2) + \beta(r)|p_1-p_2| + \gamma(r)|u_1-u_2|$$

for all $(m_i,l_i,p_i,u_i) \in \mathbb{R}^4$, $i = 1,2$, and a.a. $r \in (0,R)$.

7. Estimates on radial functions

We establish a priori type estimates on functions in $W^{2,q}_{\rm r}(a,R)$, motivated by the boundary value problem for the ODE $\mathcal{F}(u'', u'/r, u', u, r) = 0$ in (a,R), where $a \in [0,R)$, with the boundary condition

$$u'(a) = 0 \quad \text{if } a > 0, \quad \text{and} \quad u(R) = 0.$$

Throughout this section we assume that $N \geqslant 2$, fix two constants $0 < \lambda \leqslant \Lambda < \infty$, and set $\lambda_* = \lambda/\Lambda$ and $q_* = N/(1 + \lambda_*(N-1)) = N/(\lambda_* N + (1-\lambda_*))$.

Lemma 7.1. *Let $a \in [0,R)$, $q \in (q_*, \infty]$, $g \in L^N_{\rm r}(0,R)$ and $f \in L^q_{\rm r}(a,R)$. Let v be a measurable function on $[a,R]$ such that for each $b > 0$ v is absolutely continuous on $[a,R] \cap [b,R]$. Assume that $f \geqslant 0$ a.e. in (a,R), $v/r \in L^q_{\rm r}(a,R)$, $v \geqslant 0$ in $[a,R]$, $v(a) = 0$ if $a > 0$ and*

$$v'(r) + \lambda_*(N-1)\frac{v(r)}{r} \leqslant g(r)v(r) + f(r) \quad \text{for a.a. } r \in (a,R).$$

Then there exists a constant $C > 0$, depending only on λ_*, q, $\|g\|_{L^N_{\rm r}(0,R)}$ and N, such that

$$\|v/r\|_{L^q_{\rm r}(a,R)} \leqslant C\|f\|_{L^q_{\rm r}(a,R)}. \tag{7.1}$$

An important point of the above estimate is that the constant C can be chosen independently of the parameter a.

Proof. Set $\varepsilon = \lambda_*(N-1)$, so that $v' + \varepsilon v \leqslant gv + f$ a.e. in (a,R). Note that $(r^\varepsilon v)' \leqslant gvr^\varepsilon + fr^\varepsilon$ a.e. in (a,R). Accordingly, if $b \in (a,R)$, then we have for all $r \in [b,R]$,

$$r^\varepsilon v(r) \leqslant b^\varepsilon v(b) \exp\left(\int_b^r g(s)\,\mathrm{d}s\right) + \int_a^r f(t)t^\varepsilon \mathrm{e}^{\int_t^r g\,\mathrm{d}s}\,\mathrm{d}t. \tag{7.2}$$

Since $q > N/(1+\varepsilon)$, we have $1 + \varepsilon - N/q > 0$. We fix

$$\delta = \frac{1}{2}\left(1 + \varepsilon - \frac{N}{q}\right),$$

so that $\delta > 0$. By Hölder's inequality, for $a < t < r \leqslant R$, we have

$$\int_t^r g\,\mathrm{d}s \leqslant \left(\int_t^r g(s)^N s^{N-1}\,\mathrm{d}s\right)^{1/N} \left(\int_t^r s^{-1}\,\mathrm{d}s\right)^{1-1/N} \leqslant \left(\log\frac{r}{t}\right)^{1-1/N} \|g\|_{L^N_{\rm r}(0,R)}.$$

By Young's inequality, we get

$$\int_t^r g\,\mathrm{d}s \leqslant \|g\|_{L^N_{\rm r}(0,R)}\left(\log\frac{r}{t}\right)^{1-1/N} \leqslant \delta\log\frac{r}{t} + \frac{(N-1)^{N-1}}{N^N\delta^{N-1}}\|g\|^N_{L^N_{\rm r}(0,R)}.$$

Setting

$$B = \frac{(N-1)^{N-1}}{N^N\delta^{N-1}}\|g\|^N_{L^N_{\rm r}(0,R)},$$

we obtain

$$\exp\left(\int_t^r g\,\mathrm{d}s\right) \leqslant \left(\frac{r}{t}\right)^\delta \mathrm{e}^B. \tag{7.3}$$

Consider the case where $a = 0$. By comparison of the integrable function $r \to (v(r)/r)^q r^{N-1}$ on $(0,R)$ and the nonintegrable function $r \to 1/r$, we deduce that there is a sequence $(b_k)_{k\in\mathbb{N}} \subset (0,R)$ converging to zero such that

$$\left(\frac{v(b_k)}{b_k}\right)^q b_k^{N-1} \leqslant \frac{1}{b_k} \quad \text{for all } k,$$

that is, $b_k^{\varepsilon} v(b_k) \leqslant b_k^{\varepsilon+1-N/q} = b_k^{2\delta}$ for all k. This together with (7.3) yields

$$b_k^{\varepsilon} v(b_k) \exp\left(\int_{b_k}^{r} g \, ds\right) \leqslant (b_k r)^{\delta} e^{B}.$$

Thus, sending $b \to a$ in (7.2) (along the sequence $b = b_k$ if $a = 0$), we obtain

$$r^{\varepsilon} v(r) \leqslant \int_a^r f(t) t^{\varepsilon} e^{\int_t^r g \, ds} \, dt \quad \text{for all } r \in [a, R]. \tag{7.4}$$

Combining (7.4) and (7.3), we get

$$v(r) \leqslant e^{B} r^{\delta-\varepsilon} \int_a^r f(t) t^{\varepsilon-\delta} \, dt \quad \text{for } r \in [a, R]. \tag{7.5}$$

Now, if $q = \infty$, we note that $\varepsilon - \delta = \delta - 1$ and get from (7.5)

$$v(r) \leqslant \frac{e^{B} r^{1-\delta} \|f\|_{L^\infty(a,R)}}{\delta} \left(r^{\delta} - a^{\delta}\right) \leqslant \frac{e^{B} r \|f\|_{L^\infty(a,R)}}{\delta} \quad \text{for all } r \in [a, R],$$

which gives the desired estimate (7.1) in the case $q = \infty$.

Next, let $q < \infty$ and note that $\varepsilon - \delta = (N - 1 + \delta)/q + (-1 + \delta)(q - 1)/q$ and

$$r^{N-1}\left(\frac{v}{r}\right)^{q} \leqslant e^{qB} r^{N-1-q+(\delta-\varepsilon)q} \left(\int_a^r f(t) t^{\varepsilon-\delta} \, dt\right)^{q} = e^{qB} r^{-1-\delta q} \left(\int_a^r f(t) t^{\varepsilon-\delta} \, dt\right)^{q}.$$

By Hölder's inequality we get

$$\int_a^r f(t) t^{\varepsilon-\delta} \, dt \leqslant \left(\int_a^r f(t)^q t^{N-1+\delta} \, dt\right)^{1/q} \left(\int_a^r t^{-1+\delta} \, dt\right)^{1-1/q} \leqslant \left(\int_a^r f(t)^q t^{N-1+\delta} \, dt\right)^{1/q} \left(\frac{r^{\delta}}{\delta}\right)^{1-1/q},$$

and hence,

$$\int_a^R r^{N-1}\left(\frac{v(r)}{r}\right)^{q} dr \leqslant \frac{e^{qB}}{\delta^{q-1}} \int_a^R f(t)^q t^{N-1+\delta} \, dt \int_t^b r^{-1-\delta} \, dr \leqslant \frac{e^{qB}}{\delta^{q}} \int_a^R f(t)^q t^{N-1} \, dt,$$

from which we get the estimate (7.1) with e^{B}/δ. □

Lemma 7.2. *Let $q \in (N/2, \infty]$ and $a \in [0, R)$. Let u be a function on $[a, R]$ such that for each $b \in (a, R]$, the function u is absolutely continuous on $[b, R]$, $u(R) \leqslant 0$ and $\|(u')_-/r\|_{L_r^q(a,R)} < \infty$. Then there exists a constant $C > 0$, depending only on q and N, such that*

$$\sup_{(a,R]} u \leqslant C\left(R^{\frac{2q-N}{q-1}} - a^{\frac{2q-N}{q-1}}\right)^{\frac{q-1}{q}} \left\|(u')_-/r\right\|_{L_r^q(a,R)}.$$

As a consequence of the Sobolev embedding theorem, we have $W_r^{2,q}(B_R) \subset C([0, R])$. This inclusion can be deduced by the above lemma as follows. Let $u \in W_r^{2,q}(0, R)$. By the above lemma, we get

$$\|u\|_{L^\infty(0,R)} \leqslant |u(R)| + C\|u'/r\|_{L_r^q(0,R)}.$$

But, this inequality tells us that if we select a sequence (u_k) of smooth functions which approximates u in $W_r^{2,q}(0, R)$, then it also approximates u in $C([0, R])$.

Proof of Lemma 7.2. Fix any $r \in (a, R]$. We have

$$u(R) - u(r) = \int_r^R u'(t)\,\mathrm{d}t.$$

Accordingly, if $q < \infty$, we get

$$\begin{aligned} u(r) &\leqslant \int_a^R \frac{(u')_-(t)}{t} t\,\mathrm{d}t \leqslant \big\| (u')_-/r \big\|_{L^q_\mathrm{r}(a,R)} \left(\int_a^R t^{\frac{q-N+1}{q-1}}\,\mathrm{d}t \right)^{(q-1)/q} \\ &\leqslant \left(\frac{q-1}{2q-N} \right)^{(q-1)/q} \big(R^{\frac{2q-N}{q-1}} - a^{\frac{2q-N}{q-1}} \big)^{\frac{q-1}{q}} \big\| (u')_-/r \big\|_{L^q_\mathrm{r}(a,R)}. \end{aligned}$$

If $q = \infty$, we get

$$u(r) \leqslant \int_a^R \frac{(u')_-(t)}{t} t\,\mathrm{d}t \leqslant \frac{(R^2 - a^2)}{2} \big\| (u')_-/r \big\|_{L^\infty(a,R)}. \qquad \square$$

Lemma 7.3. *Let $a \in [0, R)$ and $u \in W^{2,N}_\mathrm{r}(a, R)$. Assume in addition that $u'(a) = 0$ if $a > 0$. Then*

$$\|u'\|_{L^\infty(a,R)} \leqslant N^{1/N} \|u'/r\|^{1-1/N}_{L^N_\mathrm{r}(a,R)} \|u''\|^{1/N}_{L^N_\mathrm{r}(a,R)}.$$

We remark that the above lemma implies that $W^N_\mathrm{r}(0, R) \subset C^1([0, R])$.

Proof. Note that any function $v \in W^{2,N}_\mathrm{r}(B_R)$ can be approximated by a sequence of smooth radial functions in $W^{2,N}_\mathrm{r}(B_R)$. Thus, even in the case where $a = 0$, we may assume by approximation that u is smooth and $u'(a) = 0$.

For any $a \leqslant r \leqslant R$, we have

$$\begin{aligned} \big| u'(r)^N \big| &\leqslant \int_a^r N \big| u'(t)^{N-1} u''(t) \big|\,\mathrm{d}t = N \int_a^r \big| \big(u'(t)/t\big)^{N-1} u''(t) \big| t^{N-1}\,\mathrm{d}t \\ &\leqslant N \|u'/r\|^{N-1}_{L^N_\mathrm{r}(a,R)} \|u''\|_{L^N_\mathrm{r}(a,R)}, \end{aligned}$$

and hence the conclusion follows. $\square$

A simple consequence of the above lemma is that if $g \in L^q_\mathrm{r}(0, R)$ and $u \in W^{2,q}_\mathrm{r}(0, R)$ for some $q \geqslant N$, then $gu' \in L^q_\mathrm{r}(0, R)$. The next lemma shows that a similar regularity result holds for $q < N$ under the assumption that $g \in L^N_\mathrm{r}(0, R)$.

Lemma 7.4. *Let $a \in [0, R)$, $q \in (1, N)$ and $u \in W^{2,q}_\mathrm{r}(a, R)$. Assume that $u'(a) = 0$ if $a > 0$ and that $g \in L^N_\mathrm{r}(0, R)$. Then there exists a constant $C > 0$, depending only on q and N, such that*

$$\|gu'\|_{L^q_\mathrm{r}(a,R)} \leqslant C \|g\|_{L^N_\mathrm{r}(a,R)} \big(\|u'/r\|^{(q-1)/q}_{L^q_\mathrm{r}(a,R)} \|u''\|^{1/q}_{L^q_\mathrm{r}(a,R)} + \|u'/r\|_{L^q_\mathrm{r}(a,R)} \big).$$

Proof. We may assume by approximation that u is smooth and $u'(a) = 0$.

Fix any $\varepsilon > 0$, and note that for $r \in (a, R)$,

$$r^{N-q+\varepsilon} |u'|^q = (N - q + \varepsilon) \int_a^r t^{N-1-q+\varepsilon} \big| u'(t) \big|^q\,\mathrm{d}t + q \int_a^r t^{N-q+\varepsilon} |u'|^{q-2} u'(t) u''(t)\,\mathrm{d}t.$$

Observe that

$$\|gu'\|^q_{L^q_\mathrm{r}(a,R)} \leqslant (N - q + \varepsilon) A + qB, \tag{7.6}$$

where

$$A := \int_a^R t^{N-1-q+\varepsilon} \left|u'(t)\right|^q \,\mathrm{d}t \int_t^R \left|g(r)\right|^q r^{-N+q-\varepsilon} r^{N-1} \,\mathrm{d}r, \tag{7.7}$$

and

$$B := \int_a^R t^{N-q+\varepsilon} \left|u'\right|^{q-1} \left|u''(t)\right| \,\mathrm{d}t \int_t^R \left|g(r)\right|^q r^{-N+q-\varepsilon} r^{N-1} \,\mathrm{d}r. \tag{7.8}$$

Now, noting that $q/N + (N-q)/N = 1$, we compute that for $t \in (a, R)$,

$$\begin{aligned}\int_t^R \left|g(r)\right|^q r^{-N+q-\varepsilon} r^{N-1} \,\mathrm{d}r &\leqslant \|g\|^q_{L^N_{\mathrm{r}}(a,R)} \left(\int_t^R r^{N(q-N-\varepsilon)/(N-q)} r^{N-1} \,\mathrm{d}r \right)^{(N-q)/N} \\ &\leqslant \|g\|^q_{L^N_{\mathrm{r}}(a,R)} \left(\frac{N-q}{N\varepsilon} t^{-\frac{N\varepsilon}{N-q}} \right)^{(N-q)/N}.\end{aligned}$$

Combining this with (7.7) and (7.8) yields

$$\begin{aligned}A &\leqslant \left(\frac{N-q}{N\varepsilon}\right)^{(N-q)/N} \|g\|^q_{L^N_{\mathrm{r}}(a,R)} \int_a^R \left|u'(t)\right|^q t^{N-1-q} \,\mathrm{d}t \\ &= \left(\frac{N-q}{N\varepsilon}\right)^{(N-q)/N} \|g\|^q_{L^N_{\mathrm{r}}(a,R)} \left\|u'/r\right\|^q_{L^q_{\mathrm{r}}(a,R)},\end{aligned}$$

and

$$\begin{aligned}B &\leqslant \left(\frac{N-q}{N\varepsilon}\right)^{(N-q)/N} \|g\|^q_{L^N_{\mathrm{r}}(a,R)} \int_a^R \left|u'/t\right|^{q-1} \left|u''(t)\right| t^{N-1} \,\mathrm{d}t \\ &\leqslant \left(\frac{N-q}{N\varepsilon}\right)^{(N-q)/N} \|g\|^q_{L^N_{\mathrm{r}}(a,R)} \left\|u'/r\right\|^{q-1}_{L^q_{\mathrm{r}}(a,R)} \left\|u''\right\|_{L^q_{\mathrm{r}}(a,R)}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\left\|gu'\right\|^q_{L^q_{\mathrm{r}}(a,R)} \leqslant{}& \left(\frac{N-q}{N\varepsilon}\right)^{(N-q)/N} \|g\|^q_{L^N_{\mathrm{r}}(a,R)} \\ &\times \left((N-q+\varepsilon)\left\|u'/r\right\|^q_{L^q_{\mathrm{r}}(a,R)} + q\left\|u'/r\right\|^{q-1}_{L^q_{\mathrm{r}}(a,R)} \left\|u''\right\|_{L^q_{\mathrm{r}}(a,R)}\right). \qquad \square\end{aligned}$$

Theorem 7.5. *Let $a \in [0, R)$, $q \in (\max\{N/2, q_*\}, \infty]$, $\beta \in L^N_{\mathrm{r}}(0, R) \cap L^q_{\mathrm{r}}(0, R)$, $f^1, f^2 \in L^q_{\mathrm{r}}(a, R)$ and $u \in W^{2,q}_{\mathrm{r}}(a, R)$. Assume that $\beta \geqslant 0$ a.e. in (a, R) and that*

$$\begin{cases} \mathcal{P}^+\left(u'', u'/r\right) + \beta\left|u'\right| + f^1 \geqslant 0 & \text{a.e. in } (a, R), \\ \mathcal{P}^-\left(u'', u'/r\right) - \beta\left|u'\right| - f^2 \leqslant 0 & \text{a.e. in } (a, R), \\ u'(a) = 0 \quad \text{if } a > 0, \quad \text{and} & u(R) = 0. \end{cases}$$

Then there exists a constant $C > 0$, depending only on q, λ, Λ, N, R, $\|\beta\|_{L^N_{\mathrm{r}}(0,R)}$ and $\|\beta\|_{L^q_{\mathrm{r}}(0,R)}$, such that

$$\|u\|_{W^{2,q}_{\mathrm{r}}(a,R)} \leqslant C\left(\left\|f^1_+\right\|_{L^q_{\mathrm{r}}(a,R)} + \left\|f^2_+\right\|_{L^q_{\mathrm{r}}(a,R)}\right).$$

The above theorem gives the $W^{2,q}$ estimates on the radial solutions of (6.1). Although these estimates apply only to radial solutions, in comparison with known results (see [19,9,6,14]), they are relatively sharp in the exponent q and the requirement on β that $\beta \in L^N_{\mathrm{r}}(0, R) \cap L^q_{\mathrm{r}}(0, R)$.

Proof of Theorem 7.5. Fix any $(m, l, d) \in \mathbb{R}^3$ such that $\mathcal{P}^+(m, l) + d \geqslant 0$ and $d \geqslant 0$. Assume that $l \leqslant 0$. We have $0 \leqslant \lambda m + \lambda(N-1)l + d$ if $m \leqslant 0$ and $0 \leqslant \Lambda m + \lambda(N-1)l + d$ if $m > 0$. Dividing the former and latter inequalities, respectively, by λ and Λ, after some manipulations, we get $0 \leqslant m + \lambda_*(N-1)l + \lambda^{-1}d$. That is, we have

$$m + \lambda_*(N-1)l + \lambda^{-1}d \geqslant 0 \quad \text{if } l \leqslant 0. \tag{7.9}$$

Similarly, we have $0 \leqslant \lambda m + \Lambda(N-1)|l| + d$ if $m < 0$, and hence

$$m + \lambda_*^{-1}(N-1)|l| + \lambda^{-1}d \geqslant 0. \tag{7.10}$$

If we set $v = (u')_-$, then we have $v(r) = -u'(r)$ and $v'(r) = -u''(r)$ a.e. if $v(r) > 0$, and $v(r) = 0$ and $v'(r) = 0$ a.e. if $v(r) \leqslant 0$. Using (7.9), we get

$$-v' - \lambda_*(N-1)\frac{v}{r} + \lambda^{-1}\beta v + \lambda^{-1} f_+^1(r) \geqslant 0 \quad \text{a.e. in } (a, R).$$

By Lemma 7.1, there exists a constant $C_1 > 0$, depending only on λ_*, q, N and $\|\lambda^{-1}\beta\|_{L_r^N(0,R)}$, such that

$$\|(u')_-/r\|_{L_r^q(a,R)} \leqslant C_1 \|\lambda^{-1} f_+^1\|_{L_r^q(a,R)}.$$

Similarly, since

$$\mathcal{P}^+\bigl(-u'', -u'/r\bigr) + \beta|u'| + f^2 \geqslant 0 \quad \text{a.e. in } (a, R), \tag{7.11}$$

we get

$$\|(u')_+/r\|_{L_r^q(a,R)} \leqslant C_1 \|\lambda^{-1} f_+^2\|_{L_r^q(a,R)}.$$

Thus, setting $M = \|\lambda^{-1} f_+^1\|_{L_r^q(a,R)} + \|\lambda^{-1} f_+^2\|_{L_r^q(a,R)}$, we have

$$\|u'/r\|_{L_r^q(a,R)} \leqslant C_1 M. \tag{7.12}$$

Using (7.10) and (7.11), we observe that

$$|u''| \leqslant \lambda_*^{-1}(N-1)\frac{|u'|}{r} + \lambda^{-1}\beta|u'| + \lambda^{-1}\bigl(f_+^1 + f_+^2\bigr) \quad \text{a.e. in } (a, R). \tag{7.13}$$

By Lemma 7.2 and (7.12), we can choose a constant $C_2 > 0$, depending only on q, R and N, for which we have

$$\|u\|_{L^\infty(a,R)} \leqslant C_1 C_2 M. \tag{7.14}$$

Also, by Lemmas 7.3 and 7.4 with $g = \lambda^{-1}\beta$, and by Young's inequality, for each $\varepsilon > 0$, we find a constant $C_3 > 0$, depending only on ε, q, N, R, $\|\lambda^{-1}\beta\|_{L_r^N(0,R)}$ and $\|\lambda^{-1}\beta\|_{L_r^q(0,R)}$, for which we have

$$\|\lambda^{-1}\beta u'\|_{L_r^q(a,R)} \leqslant \varepsilon\|u''\|_{L_r^q(a,R)} + C_1 C_3 M. \tag{7.15}$$

Combining this, with $\varepsilon = 1/2$, and (7.13), we get

$$\begin{aligned}\frac{1}{2}\|u''\|_{L_r^q(a,R)} &\leqslant \lambda_*^{-1}(N-1)\|u'/r\|_{L_r^q(a,R)} + C_1 C_3 M + \|\lambda^{-1}(f_+ + g_+)\|_{L_r^q(a,R)} \\ &\leqslant \bigl(\lambda_*^{-1}(N-1)C_1 + C_1 C_3 + 1\bigr)M.\end{aligned}$$

This inequality together with (7.14) and (7.15) yields an estimate on $\|u\|_{W_r^{2,q}(a,R)}$ with the desired properties. □

A weak maximum principle is stated as follows.

Theorem 7.6. *Let $q \in (\max\{N/2, q_*\}, \infty]$, $a \in [0, R)$, $u \in W_r^{2,q}(a, R)$ and $f \in L_r^q(a, R)$. Assume that $\beta \in L_r^N(0, R)$, $\beta \geqslant 0$ a.e. in (a, R), $u(R) = 0$, $u'(a) = 0$ if $a > 0$, and u satisfies*

$$\mathcal{P}^+\bigl(u'', u'/r\bigr) + \beta|u'| + f \geqslant 0 \quad \text{a.e. in } (a, R).$$

Then there exists a constant $C > 0$, depending only on λ, Λ, q, N and $\|\beta\|_{L_r^N(0,R)}$, such that

$$\max_{[a,R]} u \leqslant C\bigl(R^{\frac{2q-N}{q-1}} - a^{\frac{2q-N}{q-1}}\bigr)^{\frac{q-1}{q}} \|f_+\|_{L_r^q(a,R)}.$$

Proof. As in the previous proof, by Lemma 7.1, there exists a constant $C_1 > 0$, depending only on λ_*, q, N and $\|\lambda^{-1}\beta\|_{L^N_{\mathrm{r}}(0,R)}$, such that

$$\left\|(u')_-/r\right\|_{L^q_{\mathrm{r}}(a,R)} \leqslant C_1\left\|\lambda^{-1} f_+\right\|_{L^q_{\mathrm{r}}(a,R)}.$$

Next, by Lemma 7.2, there is a constant $C_2 > 0$, depending only on q and N, such that

$$\max_{[a,R]} u \leqslant C_2\left(R^{\frac{2q-N}{q-1}} - a^{\frac{2q-N}{q-1}}\right)^{\frac{q-1}{q}} \left\|(u')_-/r\right\|_{L^q_{\mathrm{r}}(a,R)}.$$

We combine these two inequalities, to obtain the desired estimate. □

The next theorem is a version for radial functions of the strong maximum principle.

Theorem 7.7. *Let $q \in (\max\{N/2, q_*\}, \infty]$, $u \in W^{2,q}_{\mathrm{r}}(0,R)$, $\beta \in L^N_{\mathrm{r}}(0,R)$ and $\gamma \in L^q_{\mathrm{r}}(0,R)$. Assume that $u \geqslant 0$ in $[0,R]$ and*

$$\mathcal{P}^-\left(u'', u'/r\right) - \beta\left|u'\right| - \gamma u \leqslant 0 \quad \text{a.e. in } (0,R).$$

Then either $u(r) \equiv 0$ in $[0,R]$ or $u(r) > 0$ for all $r \in [0,R)$.

It should be noticed that the second possibility in the last statement includes the inequality $u(0) > 0$.

Proof. Note that for any fixed $\varepsilon \in (0,R)$, the function $(m,p,u,r) \mapsto \mathcal{P}^-(m, p/r) - \beta(r)|p| - \gamma(r)u$ on $\mathbb{R}^3 \times (\varepsilon, R)$ satisfies (F1)–(F3), with $\Omega = (\varepsilon, R)$. In view of Theorem 2.6, it is enough to show that if $u(0) = 0$, then $u(r) \equiv 0$ in $[0,a]$ for some $0 < a < R$.

To this end, we suppose that $u(0) = 0$. Let $a \in (0,R)$ be a constant to be fixed later on. We may assume by replacing q by $\min\{q,N\}$ if needed that $q \leqslant N$. As in the previous proof, if we set $v = (u')_+$, then we have

$$v' + \lambda_*(N-1)\frac{v}{r} \leqslant \lambda^{-1}(\beta v + \gamma u) \quad \text{a.e. in } (0,R).$$

Hence, by Lemma 7.1, we get

$$\left\|(u')_+/r\right\|_{L^q_{\mathrm{r}}(0,a)} \leqslant C_1\|\gamma u\|_{L^q_{\mathrm{r}}(0,a)} \leqslant C_1\|\gamma\|_{L^q_{\mathrm{r}}(0,a)} \max_{[0,a]} u,$$

where $C_1 > 0$ is a constant independent of the choice of a. Applying Lemma 7.2 to the function $r \mapsto u(c) - u(r)$, with $0 < c \leqslant a$, we get

$$\max_{0 \leqslant r \leqslant c}\left(u(c) - u(r)\right) \leqslant C_2 c^{\frac{2q-N}{q}} \left\|(u')_+/r\right\|_{L^q_{\mathrm{r}}(0,c)},$$

where $C_2 > 0$ is a constant independent of c and a. In particular, since $u(0) = 0$, we have

$$\max_{0 \leqslant c \leqslant a} u(c) \leqslant C_2 a^{\frac{2q-N}{q}} \left\|(u')_+/r\right\|_{L^q_{\mathrm{r}}(0,a)}.$$

Thus, we get

$$\max_{[0,a]} u \leqslant C_1 C_2 a^{\frac{2q-N}{q}} \|\gamma\|_{L^q_{\mathrm{r}}(0,a)} \max_{[0,a]} u.$$

We now fix $a \in (0,R)$ small enough so that

$$C_1 C_2 a^{\frac{2q-N}{q}} \|\gamma\|_{L^q_{\mathrm{r}}(0,a)} < 1,$$

and find that $\max_{[0,a]} u = 0$. □

8. Existence and uniqueness of eigenpairs in the radial case

This section is devoted to the proof of Theorem 1.2. Throughout this section we assume that $N \geqslant 2$ and (F1)–(F4) hold with $\Lambda < \infty$. Let β and γ be the functions from (F5), and we assume throughout this section that $\beta \in L^q_{\mathrm{r}}(0,R) \cap L^N_{\mathrm{r}}(0,R)$ and $\gamma \in L^q_{\mathrm{r}}(0,R)$ for some $q \in (\max\{N/2, q_*\}, \infty]$.

As discussed in Section 6, the Dirichlet problem (1.1) for radial solutions is equivalent to the following problem for functions $u \in W^{2,q}_{\mathrm{r}}(0,R)$,

$$\begin{cases} \mathcal{F}(u'', u'/r, u', u, r) + \mu u = 0 & \text{in } (0,R), \\ u(R) = 0. \end{cases} \tag{8.1}$$

For notational simplicity, we write $\mathcal{F}[u](r)$ and $\mathcal{P}^{\pm}[u](r)$ for $\mathcal{F}(u''(r), u'(r)/r, u'(r), u(r), r)$ and $\mathcal{P}^{\pm}(u''(r), u'(r)/r)$, respectively.

Proof of Theorem1.2(i). As usual, we are concerned only with (μ_n^+, φ_n^+). In view of the argument in Section 6, we may work in the framework of the space $W^{2,q}_{\mathrm{r}}(0,R)$, but not in that of $W^{2,q}_{\mathrm{r}}(B_R)$.

Let $\varepsilon \in (0, R/4)$, and define the function $\mathcal{F}_\varepsilon$ on $\mathbb{R}^3 \times [2\varepsilon - R, R]$ by

$$\mathcal{F}_\varepsilon(m,p,u,r) := \begin{cases} \mathcal{F}(m, p/r, p, u, r) & \text{if } \varepsilon \leqslant r \leqslant R, \\ \mathcal{F}(m, -p/(2\varepsilon - r), -p, u, 2\varepsilon - r) & \text{if } 2\varepsilon - R \leqslant r \leqslant \varepsilon. \end{cases}$$

Next set $I_\varepsilon = (2\varepsilon - R, R)$, and note that for all $(m,p,u,r) \in \mathbb{R}^3 \times I_\varepsilon$,

$$\mathcal{F}_\varepsilon(m,p,u,r) = \mathcal{F}_\varepsilon(m,-p,u,2\varepsilon - r) \quad \text{if } r \neq \varepsilon, \tag{8.2}$$

and $\mathcal{F}_\varepsilon$ satisfies hypotheses (F1)–(F4) with $\Omega = I_\varepsilon$ and an appropriate choice of β and γ. The identity (8.2) is a manifestation of the symmetry in our problem with respect to the reflection at $r = \varepsilon$. Indeed, using (8.2), we easily see that if $u \in W^{2,q}(I_\varepsilon)$ and $v(r) := u(2\varepsilon - r)$, then $\mathcal{F}_\varepsilon[v](r) = \mathcal{F}_\varepsilon[u](2\varepsilon - r)$ for a.e. $r \in I_\varepsilon$. Thus, for any constant $\mu \in \mathbb{R}$ we have $\mathcal{F}_\varepsilon[u](r) + \mu u(r) = 0$ a.e. $r \in I_\varepsilon$ if and only if $\mathcal{F}_\varepsilon[v](r) + \mu v(r) = 0$ a.e. $r \in I_\varepsilon$.

Now, let $n \in \mathbb{N}$. By Theorem 1.1, there exist an eigenpair $(\mu_\varepsilon, \varphi_\varepsilon) \in \mathbb{R} \times W^{2,q}_{\mathrm{r}}(I_\varepsilon)$ and a finite sequence $2\varepsilon - R = a_{\varepsilon,n} < a_{\varepsilon,n-1} < \cdots < a_{\varepsilon,1} < b_{\varepsilon,1} < \cdots < b_{\varepsilon,n} = R$ such that

$$\begin{cases} \mathcal{F}_\varepsilon[\varphi_\varepsilon] + \mu_\varepsilon \varphi_\varepsilon = 0 & \text{a.e. in } I_\varepsilon, \\ \varphi_\varepsilon(r) > 0 & \text{in } (a_{\varepsilon,1}, b_{\varepsilon,1}), \\ (-1)^j \varphi_\varepsilon(r) > 0 & \text{in } (a_{\varepsilon,j+1}, a_{\varepsilon,j}) \cup (b_{\varepsilon,j}, b_{\varepsilon,j+1}) \text{ for } 1 \leqslant j \leqslant n-1. \end{cases}$$

Observe by the symmetry with respect to the reflection at $r = \varepsilon$ that the function $r \mapsto \varphi_\varepsilon(2\varepsilon - r)$ is an eigenfunction of (1.1), with $\Omega = (2\varepsilon - R, R)$ and F replaced by $\mathcal{F}_\varepsilon$, corresponding to μ_ε. By the half simplicity of the eigenvalues (Theorem 1.1(ii)), we may deduce that $\varphi_\varepsilon(r) = \varphi_\varepsilon(2\varepsilon - r)$ for all $r \in \bar{I}_\varepsilon$. In particular, we have $\varphi'_\varepsilon(\varepsilon) = 0$ and $(a_{\varepsilon,j} + b_{\varepsilon,j})/2 = \varepsilon$ for all $j = 1, \ldots, n$.

Next, we show that $(\mu_\varepsilon)_{0<\varepsilon<R/4}$ is bounded. To give an upper bound of $(\mu_\varepsilon)_{0<\varepsilon<R/4}$, we divide the interval $(R/4, R)$ into n intervals, $J_1 := (R/4, R/4 + h_n), \ldots, J_n := (R - h_n, R)$, where $h_n := 3R/4n$. For each $j = 1, \ldots, n$, let ν_j^+ and ν_j^- be the positive and negative principal eigenvalues of (1.1), with $\mathcal{F} = \mathcal{F}_\varepsilon$, in place of F, and $\Omega = J_j$. Since there are at most $n-1$ zeroes of the function φ_ε in the interval $(R/4, R)$, we may choose an interval J_j, with $j \in \{1, \ldots, n\}$, in which φ_ε does not vanish. This means that either $J_j \subset (a_{\varepsilon,1}, b_{\varepsilon,1})$ or $J_j \subset (b_{\varepsilon,k-1}, b_{\varepsilon,k})$ for some $k \in \{2, \ldots, n\}$. By the monotonicity (Theorem 4.2(i)) on the domains of the principal eigenvalues, we infer that

$$\mu_\varepsilon \leqslant \max\{\nu_j^+, \nu_j^-\} \leqslant \max\{\nu_i^+, \nu_i^- \colon i = 1, \ldots, n\}, \tag{8.3}$$

the right-hand side of which gives an upper bound of $(\mu_\varepsilon)_{0<\varepsilon<R/4}$ independent of ε.

To see that (μ_ε) is bounded from below, we set $m_\varepsilon := \max_{r \in [\varepsilon, b_{\varepsilon,1}]} \varphi_\varepsilon(r)$ and note that $\varphi'(\varepsilon) = 0$, $\varphi_\varepsilon(b_{\varepsilon,1}) = 0$ and

$$\mathcal{P}^+(\varphi''_\varepsilon, \varphi'_\varepsilon/r) + \beta|\varphi'_\varepsilon| + (\gamma + \mu_\varepsilon)\varphi_\varepsilon(r) \geqslant 0 \quad \text{a.e. in } (\varepsilon, b_{\varepsilon,1}).$$

By Theorem 7.6, there is a constant $C_1 > 0$, independent of ε, such that

$$m_\varepsilon \leqslant m_\varepsilon C_1 \|(\gamma + \mu_\varepsilon)_+\|_{L^q_{\mathrm{r}}(0,R)}. \tag{8.4}$$

Since $\lim_{t\to-\infty}\|(\gamma+t)_+\|_{L^q_r(0,R)}=0$, we may choose $\sigma_0\in\mathbb{R}$ such that $C_1\|(\gamma+t)_+\|_{L^q_r(0,R)}<1$ if $t\leqslant\sigma_0$. Thus, from (8.4), we deduce that the inequality $\sigma_0<\mu_\varepsilon$ holds, and conclude that (μ_ε) is bounded.

Now, we prove that there exists a constant $\delta_0>0$, independent of ε, such that $b_{\varepsilon,1}-\varepsilon\geqslant\delta_0$ and $b_{\varepsilon,j}-b_{\varepsilon,j-1}\geqslant\delta_0$ for all $j=2,\dots,n$. To this end, we set $b_{\varepsilon,0}:=\varepsilon$, $m_{\varepsilon,j}:=\max_{[b_{\varepsilon,j-1},b_{\varepsilon,j}]}|\varphi_\varepsilon|$ for $1\leqslant j\leqslant n$. Also set $u=|\varphi_\varepsilon|$ temporarily, and observe that, depending on the parity of j, we have two possibilities: either $u(r)=\varphi_\varepsilon(r)$ for all $r\in(b_{\varepsilon,j-1},b_{\varepsilon,j})$, or $u(r)=-\varphi_\varepsilon(r)$ for all $r\in(b_{\varepsilon,j-1},b_{\varepsilon,j})$. In either cases, we have $\mathcal{P}^+[u]+\beta|u'|+(\gamma+\mu_\varepsilon)_+u\geqslant0$ a.e. in $(b_{\varepsilon,j-1},b_{\varepsilon,j})$. Hence, as a consequence of Theorem 7.6, we have

$$m_{\varepsilon,j}\leqslant m_{\varepsilon,j}C_2\big(b_{\varepsilon,j}^{\frac{2q-N}{q-1}}-b_{\varepsilon,j-1}^{\frac{2q-N}{q-1}}\big)^{\frac{q-1}{q}}\big\|(\gamma+\mu_\varepsilon)_+\big\|_{L^q_r(0,R)},\tag{8.5}$$

for some constant C_2 independent of ε. Since $m_{\varepsilon,j}>0$ for all $j=1,\dots,n$, we see from the above inequality that

$$1\leqslant C_2\big(b_{\varepsilon,j}^{\frac{2q-N}{q-1}}-b_{\varepsilon,j-1}^{\frac{2q-N}{q-1}}\big)^{\frac{q-1}{q}}\big\|(\gamma+\mu_\varepsilon)_+\big\|_{L^q_r(0,R)},$$

which, together with the boundedness of (μ_ε), gives a lower bound $\delta_0>0$, independent of ε, of $b_{\varepsilon,j}-b_{\varepsilon,j-1}$, with $j=1,\dots,n$.

We next note that $u:=\varphi_\varepsilon$ satisfies a.e. in (ε,R),

$$\mathcal{P}^+[u]+\beta\big|u'\big|+\big(\gamma+|\mu_\varepsilon|\big)|u|\geqslant0\geqslant\mathcal{P}^-[u]-\beta\big|u'\big|-\big(\gamma+|\mu_\varepsilon|\big)|u|.$$

We may assume without loss of generality that $\|\varphi_\varepsilon\|_{L^\infty(\varepsilon,R)}=1$ for all ε. By Theorem 7.5, there exists a constant $C_3>0$, independent of ε, such that

$$\|\varphi_\varepsilon\|_{W^{2,q}_r(\varepsilon,R)}\leqslant C_3\big\|\big(\gamma+|\mu_\varepsilon|\big)\varphi_\varepsilon\big\|_{L^q_r(\varepsilon,R)}\leqslant C_3\big\|\gamma+|\mu_\varepsilon|\big\|_{L^q_r(\varepsilon,R)}.\tag{8.6}$$

We extend the domain of definition of φ_ε to $[0,R]$ by setting $\hat\varphi_\varepsilon(r)=\varphi_\varepsilon(r)$, if $\varepsilon\leqslant r\leqslant R$, and $=\varphi_\varepsilon(\varepsilon)$ otherwise. We note that $\hat\varphi_\varepsilon\in W^{2,q}_r(0,R)$ and that, by (8.6), $(\hat\varphi_\varepsilon)$ is bounded in $W^{2,q}_r(0,R)$. Hence there exist a sequence $(\varepsilon_k)_{k=1}^\infty$ converging to zero, a constant $\mu\in\mathbb{R}$, a sequence $0=r_0\leqslant r_1\leqslant\cdots\leqslant r_n=R$ and a function $\varphi\in W^{2,q}_r(0,R)$ such that, as $k\to\infty$, $\mu_{\varepsilon_k}\to\mu$, $b_{\varepsilon_k,j}\to r_j$ for all $j=1,\dots,n-1$, $\|\hat\varphi_{\varepsilon_k}-\varphi\|_{L^\infty(0,R)}\to0$ and $\|\hat\varphi'_{\varepsilon_k}-\varphi'\|_{L^\infty(a,R)}\to0$ for any $a\in(0,R)$. It is obvious that $r_j-r_{j-1}\geqslant\delta_0$, $\varphi(r_j)=0$ and $(-1)^{j-1}\varphi(r)\geqslant0$ in (r_{j-1},r_j) for all $1\leqslant j\leqslant n$.

We show that φ is a solution of $\mathcal{F}[\varphi]+\mu\varphi=0$ in $(0,R)$. Fix any $a\in(0,R)$, and observe that the function $\mathcal{F}_\varepsilon=\mathcal{F}$ on $\mathbb{R}^3\times[a,R]$ satisfies (F2) with β replaced by the function $\beta+\Lambda(N-1)/a$. We choose a constant $\kappa>0$ so large as in Section 3 that

$$(R-a)\exp\big(\big\|\lambda^{-1}\beta+\Lambda(N-1)/a\big\|_{L^1(a,R)}\big)\big\|\lambda^{-1}(\gamma-\kappa)_+\big\|_{L^1(a,R)}<1,$$

and set $\mathcal{F}_\kappa(m,l,p,u,r)=\mathcal{F}(m,l,p,u,r)-\kappa u$ for $(m,l,p,u,r)\in\mathbb{R}^4\times[a,R]$. Note that $\mathcal{F}_\kappa[\varphi_\varepsilon]+(\mu_\varepsilon+\kappa)\varphi_\varepsilon=0$ a.e. in (a,R). Let $\psi\in L^q(a,R)$ be the unique solution of $\mathcal{F}_\kappa[\psi]+(\mu+\kappa)\varphi=0$ with the boundary condition $\psi(a)=\varphi(a)$ and $\psi(R)=0$. We define the functions $\psi_\varepsilon^+,\psi_\varepsilon^-$ by putting $\psi_\varepsilon^\pm(r)=\psi(r)\pm|(\varphi_\varepsilon-\varphi)(a)|$. Observe that for a.e. $r\in(a,R)$,

$$\mathcal{F}_\kappa\big[\psi_\varepsilon^+\big](r)=\mathcal{F}\big[\psi_\varepsilon^+\big](r)-\kappa\psi_\varepsilon^+(r)\leqslant\mathcal{F}_\kappa[\psi](r)+\big(\gamma(r)-\kappa\big)\big|(\varphi_\varepsilon-\varphi)(a)\big|,$$

and hence, $\mathcal{F}_\kappa[\psi_\varepsilon^+](r)+(\mu+\kappa)\varphi(r)-\gamma(r)|(\varphi_\varepsilon-\varphi)(a)|\leqslant0$. Similarly, we get $\mathcal{F}_\kappa[\psi_\varepsilon^-]+(\mu+\kappa)\varphi(r)+\gamma(r)|(\varphi_\varepsilon-\varphi)(a)|\geqslant0$ for a.e. $r\in(a,R)$. We apply Theorem 2.4 to the pairs $(\varphi_\varepsilon,\psi_\varepsilon^+)$ and $(\psi_\varepsilon^-,\varphi_\varepsilon)$, to find that

$$\|\varphi_\varepsilon-\psi\|_{L^\infty(a,R)}\leqslant C\big(\|\varphi_\varepsilon-\varphi\|_{L^\infty(a,R)}+|\mu_\varepsilon-\mu|\big)$$

for some constant C independent of ε. This guarantees that $\psi=\varphi$ in $[a,R]$ and hence φ is a solution of $\mathcal{F}[\varphi]+\mu\varphi=0$ in (a,R). It is now clear that φ is a solution of $\mathcal{F}[\varphi]+\mu\varphi=0$ in $(0,R)$. Thus, the pair of μ and the function φ is an eigenpair of (8.1).

To complete the proof, we show that $\varphi(r)>0$ in $[0,r_1)$ and $(-1)^{j-1}\varphi(r)>0$ in (r_{j-1},r_j) for all $j=2,\dots,n$. We suppose by contradiction that either $\varphi(0)=0$, or else $\varphi(b)=0$ for some $b\in(r_{j-1},r_j)$ and $j\in\{1,\dots,n\}$. By Theorem 7.7, if $\varphi(0)=0$, then $\varphi(r)\equiv0$ in $[0,r_1]$, and if the latter is the case, then $\varphi(b)=\varphi'(b)=0$. Then, by the uniqueness of solution of the Cauchy problem (Theorem 2.2), we see that $\varphi(r)\equiv0$ in $[0,R]$, which is a contradiction. The function φ has therefore the right sign property. □

The next lemma states analogues in the radial case of Theorems 4.1 and 4.2(i).

Lemma 8.1.

(i) *Let* $(\mu, \varphi) \in W_{\mathrm{r}}^{2,q}(0, R)$ *be an eigenpair of* (8.1). *Assume that the function* φ *is nonnegative* (*resp., nonpositive*) *on* $[0, R]$. *Then we have* $\varphi > 0$ (*resp.,* $\varphi < 0$) *in* $[0, R)$.

(ii) *If* $(\mu, \varphi), (\nu, \psi) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(0, R)$ *are eigenpairs of* (8.1) *and either* $\varphi > 0$ *and* $\psi > 0$ *in* $[0, R)$, *or else* $\varphi < 0$ *and* $\psi < 0$ *in* $[0, R)$, *then* $\mu = \nu$ *and* $\varphi = \theta\psi$ *in* $(0, R)$ *for some constant* $\theta > 0$.

(iii) *Let* $0 < a < b \leqslant R$. *Let* $(\mu, \varphi) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(0, a)$ *and* $(\nu, \psi) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(0, b)$ *be eigenpairs of* (8.1) *in* $(0, a)$ *and in* $(0, b)$, *respectively. Assume that either* $\varphi > 0$ *in* $[0, a)$ *and* $\psi > 0$ *in* $[0, b)$ *or else* $\varphi < 0$ *in* $[0, a)$ *and* $\psi < 0$ *in* $[0, b)$. *Then we have* $\mu > \nu$.

Proof. The assertion (i) is a direct consequence of Theorem 7.7.

To check (ii), we may assume by symmetry that $\mu \leqslant \nu$. We treat only the case where $\varphi > 0$ and $\psi > 0$ in $[0, R)$; the other case can be treated similarly. Set $\theta = \inf_{[0,R)} \psi/\varphi$. We have either $\theta = \psi(s)/\varphi(s)$ for some $s \in [0, R)$ or else, in view of l'Hôpital's rule and the strong maximum principle (Theorem 2.6), $\theta = \psi'(R)/\varphi'(R)$. As in the proof of Theorem 4.1, we see that the function $u := \psi - \theta\varphi$ satisfies

$$0 \geqslant \mathcal{F}[\psi] + \mu\psi - \mathcal{F}[\theta\varphi] - \mu\theta\varphi \geqslant \mathcal{P}^-[u] - \beta|u'| - (\gamma + |\mu|)u \quad \text{a.e. in } (0, R),$$

and that either $u(s) = 0$ for some $s \in [0, R)$ or $u'(R) = 0$. Applying Theorems 7.7 and 2.6 to the function u, we find that $u(r) \equiv 0$ in $[0, R]$, that is, $\psi = \theta\varphi$. Furthermore, if $\mu < \nu$, then $\nu\psi = -\mathcal{F}[\psi] = -\mathcal{F}[\theta\varphi] = \mu\theta\varphi = \mu\psi$ in $(0, R)$, which is impossible. That is, we have $\mu = \nu$.

We prove that (iii) holds. Again, we treat only the case where both φ and ψ are positive in $[0, R)$. Suppose by contradiction that $\mu \leqslant \nu$. Set $\theta = \inf_{[0,a)} \psi/\varphi$. Clearly, we have $\psi(s) = \theta\varphi(s)$ for some $s \in [0, a)$. If we set $u := \psi - \theta\varphi$, then u satisfies $\mathcal{P}^-[u] - \beta|u'| - (\gamma + |\mu|)u \leqslant 0$ a.e. in $(0, a)$. Hence, we deduce as above that $u(r) \equiv 0$ in $[0, a]$, while we have $u(a) > 0$. This contradiction shows that $\mu > \nu$. □

Proof of Theorem 1.2(ii). Let $(\mu, \varphi) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(0, R)$ be an eigenpair of (8.1). We treat only the case where $\varphi(0) \geqslant 0$, since the other case can be dealt with in a parallel way.

We first prove that φ has at most a finite number of zeroes. For this, we suppose by contradiction that it has infinitely many zeroes. As a result, the set of zeroes of φ has an accumulation point a in $[0, R]$. We first suppose that $a > 0$. Clearly we have $\varphi(a) = 0$. Moreover, by Rolle's theorem, we see that $\varphi'(a) = 0$. By the uniqueness result (Theorem 2.2) for the Cauchy problem for ODE, we find that $\varphi(r) \equiv 0$ in $[0, R]$, which is a contradiction. We next suppose that $a = 0$. By the above argument, we have $(\varphi(r), \varphi'(r)) \neq (0, 0)$ for all $r \in (0, R]$. Because of the choice of a, there are sequences $(a_k), (b_k) \subset (0, R)$ such that $0 < a_k < b_k$ for all k, $b_k \to 0$ as $k \to \infty$, $\varphi(a_k) = \varphi(b_k) = 0$ for all k and $\varphi(r) > 0$ for all $r \in (a_k, b_k)$ and all k. Since $b_k - a_k \to 0$ as $k \to \infty$, following the argument which led to (8.5), we get a contradiction. Thus, φ has at most a finite number of zeroes.

We note here by Theorem 7.7 that $\varphi(0) > 0$. Let $(r_k)_{k=1}^n$ be the finite sequence of all zeroes of φ such that $r_0 := 0 < r_1 < \cdots < r_n = R$. If $n = 1$, then our claim is a consequence of Lemma 8.1(i) and (ii).

We may thus assume that $n \geqslant 2$. Fix any eigenpair $(\nu, \psi) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(0, R)$ having exactly n zeroes in $[0, R]$ such that $\psi(0) > 0$. It is enough to show that $\mu = \nu$ and that there is a constant $\theta > 0$ such that $\psi = \theta\varphi$ in $[0, R]$.

Let $(s_k)_{k=1}^n$ be the finite sequence of all zeroes of ψ such that $s_0 := 0 < s_1 < \cdots < s_n = R$. By Lemma 5.2, there are two indices $k, j \in \{1, \ldots, n\}$ such that $[r_{k-1}, r_k] \subset [s_{k-1}, s_k]$ and $[s_{j-1}, s_j] \subset [r_{j-1}, r_j]$. Hence, Theorem 4.2(i) and Lemma 8.1 together imply that $\mu = \nu$ and $[r_{j-1}, r_j] = [s_{j-1}, s_j]$ for some $j \in \{1, \ldots, n\}$. Applying the same argument repeatedly for complementary intervals, we infer that $[r_{k-1}, r_k] = [s_{k-1}, s_k]$ for all $k = 1, \ldots, n$. Now, by Lemma 8.1, we may choose a constant $\theta > 0$ so that $\psi = \theta\varphi$ in $[0, r_1]$. Theorem 2.2 (a uniqueness result for the Cauchy problem for ODE) allows us to conclude that $\psi = \theta\varphi$ in $[0, R]$. □

The following proposition is analogous to Proposition 5.3.

Proposition 8.2. *Let* (μ_n^+) *and* (μ_n^-) *be the sequences of eigenvalues given by Theorem* 1.2. *Then*

$$\lim_{n\to\infty} \min\{\mu_n^+, \mu_n^-\} = \infty, \tag{8.7}$$

$$\max\{\mu_n^+, \mu_n^-\} < \min\{\mu_{n+1}^+, \mu_{n+1}^-\} \quad \text{for every } n \in \mathbb{N}. \tag{8.8}$$

Proof. Fix $n \in \mathbb{N}$, and let $\varphi \in W^{2,q}_{\mathrm{r}}(0,R)$ be an eigenfunction corresponding to μ_n^+. Let $(r_j)_{j=1}^n$ be the increasing finite sequences of zeroes of the eigenfunction φ. Set $r_0 = 0$. Obviously, there exists $j \in \{1,\dots,n\}$ such that $r_j - r_{j-1} \leqslant R/n$. Fix $j \in \{1,\dots,n\}$ so that $r_j - r_{j-1} \leqslant R/n$. Set $m = \max_{[r_{j-1},r_j]} |\varphi|$ and

$$\varepsilon_n = \max_{0 \leqslant r \leqslant R} \left(\left(r + \frac{R}{n} \right)^{\frac{2q-N}{q-1}} - r^{\frac{2q-N}{q-1}} \right)^{\frac{q-1}{q}}.$$

Similarly to how we have obtained (8.5), we get $m \leqslant C\varepsilon_n \|(\gamma + \mu_n^+)_+\|_{L^q_{\mathrm{r}}(0,R)} m$ for some constant $C > 0$ independent of n. It is then easily seen that $\mu_n^+ \to \infty$ as $n \to \infty$. Similarly, we find that $\mu_n^- \to \infty$ as $n \to \infty$. Thus, (8.7) is valid.

The inequality (8.8) is proved in the same way as the proof of (5.5) in Proposition 5.3, and we do not give here the detail. □

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Appendix A. An example of *F*

Let $N \geqslant 3$, and we give an example of $F : \mathbb{S}^N \to \mathbb{R}$ which satisfies (F1)–(F4) but is not invariant under conjugation of the orthogonal matrices $Q \in O(N)$.

For any matrix $M \in \mathbb{S}^N$, let $\lambda_1(M) \leqslant \lambda_2(M) \leqslant \cdots \leqslant \lambda_N(M)$ denote the eigenvalues of M. Let $\|M\|$ denote the norm of the matrix M given by $\|M\| = \max_{\omega \in S^{N-1}} |M\omega \cdot \omega|$. It is well known and easily seen that $\|M\| = \max\{|\lambda_1(M)|, |\lambda_N(M)|\}$. In view of the Courant minimax theorem, which states that

$$\lambda_i(M) = \min_V \max_{\omega \in S^{N-1} \cap V} M\omega \cdot \omega,$$

where the minimum is taken over all i-dimensional subspaces V of $\mathbb{R}^N$, we see that the functions $\lambda_i(M)$ of M are Lipschitz continuous on $\mathbb{S}^N$. In fact, as is easily seen, we have

$$\left|\lambda_i(X) - \lambda_i(Y)\right| \leqslant \|X - Y\| \quad \text{for all } X, Y \in \mathbb{S}^N \text{ and } i = 1,\dots,N.$$

Define the function $g : \mathbb{S}^N \to \mathbb{R}$ by

$$g(M) = \min\left\{\left|\lambda_1(M) - \lambda_2(M)\right|, \left|\lambda_N(M) - \lambda_2(M)\right|, |Me_1 \cdot e_1|\right\},$$

where $e_1 = (1,0,\dots,0) \in \mathbb{R}^N$. Note that g is Lipschitz continuous on $\mathbb{S}^N$. More precisely, we have

$$\left|g(X) - g(Y)\right| \leqslant 2\|X - Y\| \quad \text{for all } X, Y \in \mathbb{S}^N \text{ and } i = 1,\dots,N. \tag{A.1}$$

Now, we define the function $F : \mathbb{S}^N \to \mathbb{R}$ by

$$F(M) = \operatorname{tr} M + \frac{1}{4} g(M).$$

Proposition A.1. *The function F given by the above formula satisfies* (F1)–(F4)*, but it is not invariant under conjugation of the matrices $Q \in O(N)$.*

Proof. It is clear that F satisfies (F1) and (F3).

For any $X, Y \in \mathbb{S}^N$ we have

$$F(X) - F(Y) \leqslant \operatorname{tr}(X - Y) + \frac{1}{2}\|X - Y\|.$$

Moreover, choosing a vector $\omega \in S^{N-1}$ so that

$$\|X - Y\| = \left|(X - Y)\omega \cdot \omega\right|,$$

we observe that if $(X - Y)\omega \cdot \omega \geqslant 0$, then

$$F(X) - F(Y) \leqslant \operatorname{tr}(X - Y) + \frac{1}{2}(X - Y)\omega \cdot \omega = \operatorname{tr}\left(I_N + \frac{1}{2}\omega \otimes \omega\right)(X - Y),$$

and if $(X - Y)\omega \cdot \omega < 0$, then

$$F(X) - F(Y) \leqslant \operatorname{tr}\left(I_N - \frac{1}{2}\omega \otimes \omega\right)(X - Y).$$

Hence, if we set $\lambda = 1/2$ and $\Lambda = 3/2$, then we have

$$F(X) - F(Y) \leqslant P^+(X - Y) \quad \text{for all } X, Y \in \mathbb{S}^N.$$

(Recall that $P^+(M) = \max\{\operatorname{tr} AM\colon \lambda I_N \leqslant A \leqslant \Lambda I_N\}$.) This shows that F satisfies condition (F2).

Now, we show that F satisfies (F4). For any $m, l \in \mathbb{R}$ and $\omega \in S^{N-1}$, we set

$$M = m\omega \otimes \omega + l(I_N - \omega \otimes \omega).$$

If $m < l$, then we have

$$m = \lambda_1(M) < l = \lambda_2(M) = \cdots = \lambda_N(M),$$

which shows that $g(M) = |\lambda_N(M) - \lambda_2(M)| = 0$ and $F(M) = \operatorname{tr} M = m + (N - 1)l$. If $m > l$, then

$$l = \lambda_1(M) = \cdots = \lambda_{N-1}(M) < m = \lambda_N(M),$$

from which we see that $g(M) = 0$ and $F(M) = m + (N - 1)l$. Similarly, if $m = l$, then $F(M) = Nm = Nl$. Thus, $F(M)$ is a function of m and l and does not depend on ω. This proves that F satisfies (F4).

Next we show that F is not invariant under conjugation of the matrices $Q \in O(N)$. Let $M \in \mathbb{S}^N$ be the diagonal matrix given by

$$M = \operatorname{diag}(0, 1, \ldots, N - 1).$$

We have

$$F(M) = \operatorname{tr} M + \frac{1}{4}g(M) = \operatorname{tr} M + \frac{1}{4}\min\{1, N - 2, 0\} = \operatorname{tr} M.$$

Let $Q \in O(N)$ be the matrix given by

$$Q = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{N-2} & \\ 0 & 0 & & & \end{pmatrix}.$$

Then we have $Q^{-1}MQ = \operatorname{diag}(1, 0, 2, \ldots, N - 1)$ and

$$F\left(Q^{-1}MQ\right) = \operatorname{tr} M + \frac{1}{4}g(M) = \operatorname{tr} M + \frac{1}{4}\min\{1, N - 2, 1\} \geqslant \operatorname{tr} M + \frac{1}{4}.$$

Thus, $F(Q^{-1}MQ) > F(M)$ and F is not invariant under conjugation of the matrices $Q \in O(N)$. □

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