Available online at www.sciencedirect.com

SciVerse ScienceDirect

ELSEVIER

Ann. I. H. Poincaré – AN 29 (2012) 401–411

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Dimension of images of subspaces under Sobolev mappings ☆

Stanislav Hencl *, Petr Honzík

Department of Mathematical Analysis, Charles University, Sokolovská 83, CZ – 186 00 Prague 8, Czech Republic

Received 16 June 2011; received in revised form 24 November 2011; accepted 3 January 2012

Available online 20 January 2012

Abstract

Let $m < \alpha < p \leqslant n$ and let $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ be p-quasicontinuous. We find an optimal value of $\beta(n, m, p, \alpha)$ such that for $\mathcal{H}^\beta$ a.e. $y \in (0,1)^{n-m}$ the Hausdorff dimension of $f((0,1)^m \times \{y\})$ is at most α. We construct an example to show that the value of the optimal β does not increase once p goes below the critical case $p < \alpha$.

MSC: 46E35; 28A78

Keywords: Sobolev mapping; Hausdorff dimension

1. Introduction

It is well known that each Sobolev function satisfies the ACL condition, i.e., the function is absolutely continuous when restricted to almost all lines parallel to coordinate axes. It follows that images of $\mathcal{H}^{n-1}$ almost all segments are rectifiable curves and thus have Hausdorff dimension at most one. We would like to study how often it can happen that the images of m-dimensional subspaces have bigger Hausdorff dimension. Such a result was studied for quasiconformal mapping by Gehring and Väisälä [2] and for supercritical Sobolev mappings (i.e. $f \in W^{1,p}$, $p > n$) by Kaufmann [3] and recently by Balogh, Monti and Tyson [1].

Let $\Omega \subset \mathbb{R}^n$ be a domain, $p > n$ and let $f \in W^{1,p}(\Omega, \mathbb{R}^k)$ be continuous. It was shown by Kaufmann that images of m-dimensional subspaces have zero $\mathcal{H}^{\frac{pm}{p-n+m}}$ measure. Let us point out that naive application of $(1 - \frac{n}{p})$ Hölder continuity would give the worse exponent $\frac{pm}{p-n}$. He also gave a probabilistic construction to show that the value $\frac{pm}{p-n+m}$ is optimal. This was later generalized in a nice paper of Balogh, Monti and Tyson [1] where they showed that for any $m < \alpha < \frac{pm}{p-n+m}$ it is true that the image of $\mathcal{H}^\beta$ a.e. m-dimensional subspace has dimension at most α where $\beta = n - m - (1 - \frac{m}{\alpha})p$ (see Theorem 1.1 below for exact formulation). By a similar construction as Kaufmann they

☆ The first author was supported by the RSJ algorithmic trading grant and the second author was supported by the Institutional Research Plan No. AV0Z10190503 of the Academy of Sciences of the Czech Republic (AS CR) and the grant KJB100190901 GAAV.

* Corresponding author.
E-mail addresses: hencl@karlin.mff.cuni.cz (S. Hencl), honzik@gmail.com (P. Honzík).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2012.01.002

also showed that this value of β is optimal for all $p > n$. The results of [1] are actually even more general and they deal also with mappings with values in metric spaces or with quasiconformal mappings and mappings in Sobolev–Lorentz spaces. We have not pursued this direction.

The counterexample in [1] is constructed for all $p \geqslant 1$ and in Problem 6.4 the authors ask for any generalization of the positive statements also in the subcritical case $p < n$. We were able to show that basically the same statement holds if $\alpha < p$. Here $\dim_{\mathcal{H}}(A)$ denotes the Hausdorff dimension of a set A (see Section 2 for the definition).

Theorem 1.1. *Let $n, k \in \mathbb{N}$ and $m \in \{1, \dots, n-1\}$. Let $m < \alpha < p \leqslant n$ and set*

$$\beta = \beta(\alpha, p) := (n-m) - \left(1 - \frac{m}{\alpha}\right) p.$$

Suppose that $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ is a p-quasicontinuous representative. If we denote

$$E = \left\{ y \in (0,1)^{n-m} \colon \dim_{\mathcal{H}}\bigl(f\bigl((0,1)^m \times \{y\}\bigr)\bigr) \geqslant \alpha \right\},$$

then $\dim_{\mathcal{H}}(E) \leqslant \beta$.

Since the important things occur on a set of measure zero we need to have a good representative of our function. In the theorem, we choose the p-quasicontinuous representative, but in fact the only thing that we will need is that the value of the representative of f is equal to the limit of integral averages whenever such limit exists.

The statement of the similar and even slightly sharper theorem for $p > n$ was given already by Balogh, Monti and Tyson [1] and the proof there is simpler. It relies on the Sobolev embedding theorem into Hölder continuous functions which is not available for us. Instead we need to use some analogous estimate on possibly smaller balls (see Lemma 3.1 below) and some finer covering arguments.

Let us now recall the statement of the counterexample from [1] that shows that the value of β from the previous theorem is optimal at least for Minkowski dimension.

Theorem 1.2. *Let $p \geqslant 1$, let α satisfy $m < \alpha \leqslant \frac{pm}{p-n+m}$ for $p > n-m$ and $m < \alpha$ for $p \leqslant n-m$, and define*

$$\beta = \beta(\alpha, p) = (n-m) - \left(1 - \frac{m}{\alpha}\right) p.$$

Let $E \subset (0,1)^{n-m}$ be any Borel set for which

$$\limsup_{r \to 0+} r^{\beta} \mathbf{N}(E, r) < \infty.$$

Then, for any integer $k > \alpha$, there is a continuous map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ so that $f(\mathbb{R}^m \times \{a\})$ has Hausdorff dimension at least α, for $\mathcal{H}^{\beta}$-almost every $a \in E$.

The requirement that $\alpha < p$ in Theorem 1.1 is natural as Theorem 1.3 below indicates. We were able to improve the construction from [1] and to show that in the case $p < \alpha$, $p < n$ even better example exists. We have shown that we do not get any improvement on β once p goes below the critical value α. This degeneracy seems to be connected with the fact that p-quasicontinuous representatives of Sobolev function are well-defined and have Lebesgue points up to a set of dimension $n-p$ (see Theorem 2.3 below) and for $p < \alpha$ we have $\beta(\alpha, p) < n - p$.

Theorem 1.3. *Let $1 \leqslant p < n$, $m < p < \alpha$ and let*

$$\tilde{\beta} < n - p = \beta(p, p).$$

Let $E \subset (0,1)^{n-m}$ be any Borel set for which

$$\limsup_{r\to 0+} r^{\tilde\beta}\mathbf{N}(E,r) < \infty.$$

Then, for any integer $k > \alpha$, there is a continuous map $f \in W^{1,p}(\mathbb{R}^n,\mathbb{R}^k)$ so that $f(\mathbb{R}^m \times \{a\})$ has Hausdorff dimension at least α, for $\mathcal{H}^{\tilde\beta}$-almost every $a \in E$.

2. Preliminaries

We use the notation $\mathbf{N}(E,r)$ for the smallest number of balls of radius $r > 0$ that cover the set $E \subset \mathbb{R}^d$. For $t > 0$ we denote the integer part of t as $[t]$. By $Q(z,r)$ we denote the cube centered at $z \in \mathbb{R}^d$ with radius $r > 0$. The oscillation of a function f on a set A is denoted by $\operatorname{osc}_A f := \operatorname{diam} f(A)$.

We use the usual convention that C denotes a generic positive constant whose value may change from line to line.

In order to prove Theorem 1.3 we will use a probabilistic approach and we will need the following lemma (see [1, Lemma 4.3] for the proof).

Lemma 2.1. *Let $\{X_i\}_{i=1}^\infty$ be a countable sequence of independent random variables, identically distributed according to the uniform distribution on the unit ball B in $\mathbb{R}^k$. Let $c = \{c_i\} \in \ell_\infty$ and finally let $0 < \alpha' < k+1$. Then there is a constant C which depends only on k and α' so that*

$$\mathbb{E}_\xi\left(\left|\sum_{i=1}^\infty c_i X_i\right|^{-\alpha'}\right) \leqslant C\rho(c)^{-\alpha'}$$

where $\rho(c)$ denotes the second largest value, i.e.

$$\rho(c) = \begin{cases} \|c\|_\infty & \text{if } \|c\|_\infty = \sup_{i\in\mathbb{N}}|c_i| \text{ is not attained,}\\ \sup_{i\neq i_0}|c_i| & \text{if the supremum is attained at } i_0.\end{cases}$$

2.1. Hausdorff and capacitary dimension

Let $\alpha > 0$ and $\varepsilon > 0$. We use the usual Hausdorff measure of a set $E \subset \mathbb{R}^d$, i.e.

$$\mathcal{H}^\alpha_\varepsilon(E) = \inf\left\{\sum_{i=1}^\infty \operatorname{diam}^\alpha A_i\colon E \subset \bigcup_{i=1}^\infty A_i,\ \operatorname{diam} A_i < \varepsilon\right\} \quad \text{and} \quad \mathcal{H}^\alpha(E) = \lim_{\varepsilon\to 0+}\mathcal{H}^\alpha_\varepsilon(E).$$

The Hausdorff dimension of a set E is

$$\dim_{\mathcal{H}}(E) = \sup\bigl\{\alpha > 0\colon \mathcal{H}^\alpha(E) = \infty\bigr\} = \inf\bigl\{\alpha > 0\colon \mathcal{H}^\alpha(E) = 0\bigr\}.$$

For $\alpha > 0$ and $A \subset \mathbb{R}^k$, denote by

$$I_\alpha(\mu) := \int_A\int_A |x-y|^{-\alpha}\,d\mu(x)\,d\mu(y)$$

the α-energy of a nonzero finite Radon measure μ with compact support in A. The capacitary dimension of a set A is defined as

$$\dim_c(A) = \sup\bigl\{\alpha > 0\colon \exists\mu \text{ with } I_\alpha(\mu) < \infty\bigr\}.$$

We will use the well-known fact (see [4, Theorem 8.9]) that the Hausdorff dimension is equal to the capacitary dimension.

2.2. Sobolev spaces

For a ball B we denote

$$f_B = \frac{1}{|B|}\int_B f(x)\,dx.$$

Theorem 2.2 *(Poincaré inequality). Let $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ and let $B \subset \mathbb{R}^n$ be a ball of radius R. Then*

$$\int_B \left|f(x) - f_B\right| dx \leqslant C R^{1+n-\frac{n}{p}} \left(\int_B \left|Df(x)\right|^p dx\right)^{\frac{1}{p}}. \tag{2.1}$$

We will not need the exact definition of a p-quasicontinuous representative. We will need only the following result from [5, Theorem 3.3.3 and Theorem 2.6.16].

Theorem 2.3. *Let $1 \leqslant p \leqslant n$ and let $f \in W^{1,p}(\mathbb{R}^n)$ be a p-quasicontinuous representative and set*

$$E_p = \left\{x \in \mathbb{R}^n\text{: } x \text{ is not a Lebesgue point of } f\right\}.$$

Then $\dim_{\mathcal{H}}(E_p) = n - p$ and for $p = 1$ we moreover get $\mathcal{H}^{n-1}(E_1) = 0$.

3. Positive result in the subcritical case

For $p > n$ we can use Sobolev embedding theorem to obtain

$$\int_B \left|Df(x)\right|^p dx \geqslant C(\operatorname{osc}_B f)^p R^{n-p} \tag{3.1}$$

for every ball B of radius R. The following technical lemma will be essential for our proof. It tells us that for every $p \geqslant 1$ we have an analogy of (3.1) on some smaller ball if we add some correction term to power $\gamma > 0$. Note that γ can be chosen as small as we wish.

Lemma 3.1. *Suppose that a and b are Lebesgue points of $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$. Let us denote $R_0 = |a - b|$ and let $\gamma > 0$. Then there are $z \in \{a, b\}$ and $0 < R \leqslant 2R_0$ such that*

$$\int_{B(z,R)} \left|Df(x)\right|^p dx \geqslant C_\gamma \left|f(a) - f(b)\right|^p R^{n-p} \left(\frac{R}{R_0}\right)^\gamma \tag{3.2}$$

where the positive constant C_γ depends only on γ and dimension n.

Proof. Suppose for contradiction that (3.2) is not valid for each $0 < R \leqslant 2R_0$, $C_\gamma > 0$ and for both choices of z. Set

$$B_i = B\left(a, R_0 2^{-i+1}\right) \quad \text{for } i \in \mathbb{N} \cup \{0\} \quad \text{and} \quad B_i = B\left(b, R_0 2^{-|i|+1}\right) \quad \text{for } i \in -\mathbb{N}.$$

Since a and b are Lebesgue points we have $f_{B_i} \to f(a)$ as $i \to \infty$ and similarly $f_{B_i} \to f(b)$ as $i \to -\infty$. It follows that

$$\sum_{i \in \mathbb{Z}} |f_{B_i} - f_{B_{i+1}}| \geqslant \left|f(b) - f(a)\right|. \tag{3.3}$$

For each $i \in \mathbb{N} \cup \{0\}$ we have $B_{i+1} \subset B_i$ and for each $i \in -\mathbb{N}$ we have $B_i \subset B_{i+1}$. In the first case we can use (2.1) to obtain

$$\begin{aligned}
|f_{B_i} - f_{B_{i+1}}| &\leqslant \frac{1}{|B_{i+1}|} \int_{B_{i+1}} \left|f(x) - f_{B_i}\right| dx \leqslant \frac{C}{|B_i|} \int_{B_i} \left|f(x) - f_{B_i}\right| dx \\
&\leqslant C\left(R_0 2^{-i}\right)^{1-\frac{n}{p}} \left(\int_{B_i} \left|Df(x)\right|^p dx\right)^{\frac{1}{p}}
\end{aligned}$$

and we have a similar estimate also in the second case if we exchange the roles of i and $i + 1$. Together with (3.3) and the opposite inequality to (3.2) for each B_i this implies

$$\begin{aligned}\left|f(a)-f(b)\right| &\leqslant \sum_{i\in\mathbb{Z}}|f_{B_i}-f_{B_{i+1}}| \leqslant C\sum_{i\in\mathbb{Z}}\left(R_0 2^{-|i|}\right)^{1-\frac{n}{p}}\left(\int_{B_i}\left|Df(x)\right|^p\,dx\right)^{\frac{1}{p}}\\ &\leqslant C\sum_{i\in\mathbb{Z}}\left(R_0 2^{-|i|}\right)^{1-\frac{n}{p}}\left(C_\gamma\left|f(a)-f(b)\right|^p\left(R_0 2^{-|i|}\right)^{n-p}2^{-|i|\gamma}\right)^{\frac{1}{p}}\\ &\leqslant CC_\gamma^{\frac{1}{p}}\left|f(a)-f(b)\right|\sum_{i\in\mathbb{Z}}2^{-|i|\frac{\gamma}{p}} = CC_\gamma^{\frac{1}{p}}\left|f(a)-f(b)\right|.\end{aligned}$$

We see that this is not possible if C_γ is chosen small enough, a contradiction. □

Proof of Theorem 1.1. To get our conclusion it is enough to show that $\dim_{\mathcal{H}}(E) < \tilde{\beta}$ for each $\tilde{\beta} > \beta$. Let us fix $\tilde{\beta} > \beta(\alpha, p)$ and assume for contradiction that $\dim_{\mathcal{H}}(E) \geqslant \tilde{\beta}$. By Theorem 2.3 we know that the set

$$F = \left\{x \in [0,1]^n\colon x \text{ is not Lebesgue point of } f\right\}$$

has Hausdorff dimension at most $n-p$ and the same holds for its projections. From $p > \alpha$ we know that $\tilde{\beta} > \beta > n-p$ and hence this set is negligible and $\dim_{\mathcal{H}}(E \setminus P(F)) \geqslant \tilde{\beta}$, where P is the projection on the last $n-m$ variables. By [4, Lemma 3.1 and Theorem 8.13] there is a compact set $E_0 \subset E \setminus P(F)$ so that $0 < \mathcal{H}^{\tilde{\beta}}(E_0) < \infty$. By Frostman's lemma [4, Theorem 8.8] we can fix a measure μ supported in E_0 with $\|\mu\| = M > 0$, and such that

$$\mu\left(B(a,r)\right) \leqslant r^{\tilde{\beta}} \quad \text{for any } a \in \mathbb{R}^{n-m} \text{ and } r > 0. \tag{3.4}$$

We can fix $\tilde{\alpha} < \alpha$ such that $\tilde{\beta} > \beta(\tilde{\alpha}, p)$. It follows that

$$\mathcal{H}^{\tilde{\alpha}}\left(f\left((0,1)^m \times \{y\}\right)\right) = \infty \quad \text{for every } y \in E_0.$$

Now let us fix a huge constant $c_0 > 0$ and let us select ε such that

$$\mu(E_1) > \frac{M}{2} \quad \text{for } E_1 := \left\{y \in E_0\colon \mathcal{H}^{\tilde{\alpha}}_\varepsilon\left(f\left((0,1)^m \times \{y\}\right)\right) > c_0\right\}. \tag{3.5}$$

Fix a point $y \in E_1$ and let us estimate the size of $f((0,1)^m \times \{y\})$. Let us introduce dyadic cubes on $[0,1]^m$. We denote by $\mathcal{D}_0 = \{[0,1]^m\}$ the mother cube, and $\mathcal{D}_k = \{Q_i\}_{i=1}^{2^{km}}$ where Q_i are closed cubes with vertices in the points $2^{-k}\mathbb{Z}^n \cap [0,1]^m$ and with volume 2^{-km}. We need to show that the sum of diameters of images of these cubes is big enough and we will discuss three cases. Let us call a point $x \in [0,1]^m$ 'bad' if $\operatorname{diam} f((Q(x,r) \cap (0,1)^m) \times \{y\}) > \varepsilon$ for every $r > 0$. In the first case there are no 'bad' points, in the second case we assume that there are at most N 'bad' points and in the third case we assume that the number of bad points is infinite.

In the first case, we can find a k such that for every $Q_i \in \mathcal{D}_k$ we get $\operatorname{diam} f(Q_i \times \{y\}) \leqslant \varepsilon$. Let us denote $\varepsilon_i = \operatorname{diam} f(Q_i \times \{y\})$. From (3.5) we get

$$\sum_{i=1}^{2^{km}} \operatorname{diam}^{\tilde{\alpha}} f\left(Q_i \times \{y\}\right) = \sum_{i=1}^{2^{km}} \varepsilon_i^{\tilde{\alpha}} \geqslant \frac{c_0}{2} \tag{3.6}$$

and thus we have found essentially disjoint cubes in $(0,1)^m \times \{y\}$ where the above inequality holds. We would like to have a similar estimate in other cases as well.

In the second case, there is a natural number N such that each $\mathcal{D}_k$ contains at most N cubes Q_i such that $\operatorname{diam} f(Q_i \times \{y\}) > \varepsilon$. For each k we denote

$$I_k^B = \left\{i \in \left\{1, \ldots, 2^{km}\right\}\colon \operatorname{diam} f\left(Q_i \times \{y\}\right) > \varepsilon\right\} \quad \text{and} \quad S_k = \bigcup_{i \in I_k^B} Q_i$$

the union of these cubes. We observe that $S = \bigcap_k S_k$ contains at most N points. We may find a covering of the set $[0,1]^m \setminus S$ by infinitely many dyadic cubes $\{Q_i\}$ (that are smaller close to the points of S) such that $\varepsilon_i = \operatorname{diam} f(Q_i \times \{y\}) \leqslant \varepsilon$. Since

$$\mathcal{H}^{\tilde{\alpha}}_\varepsilon\left(f\left((0,1)^m \times \{y\}\right)\right) = \mathcal{H}^{\tilde{\alpha}}_\varepsilon\left(f\left(\left[(0,1)^m \setminus S\right] \times \{y\}\right)\right)$$

we may use (3.5) again to obtain

$$\sum_{i=1}^{\infty} \operatorname{diam}^{\tilde{\alpha}} f\big(Q_i \times \{y\}\big) = \sum_{i=1}^{\infty} \varepsilon_i^{\tilde{\alpha}} \geqslant \frac{c_0}{2}. \tag{3.7}$$

In the third case, each $\mathcal{D}_k$ contains N_k cubes Q_i such that $\operatorname{diam} f(Q_i \times \{y\}) > \varepsilon$, and $\limsup_{k\to\infty} N_k = \infty$. Therefore for k big enough we get that N_k is big enough and hence

$$\sum_{i=1}^{2^{km}} \operatorname{diam}^{\tilde{\alpha}} f\big(Q_i \times \{y\}\big) \geqslant \sum_{i\in I_k^B} \varepsilon^{\tilde{\alpha}} = N_k \varepsilon^{\tilde{\alpha}} \geqslant \frac{c_0}{2}. \tag{3.8}$$

Now, for each $y \in E_1$ we are in one of the cases (3.6), (3.7) or (3.8) and we may select a finite number K_y such that for the sequences of cubes $Q_{y,i}$ and $\varepsilon_{y,i} = \operatorname{diam} f(Q_{y,i} \times \{y\})$ defined as in those inequalities we get

$$\sum_{i=1}^{K_y} \varepsilon_{y,i}^{\tilde{\alpha}} \geqslant \frac{c_0}{4}. \tag{3.9}$$

Let $\gamma > 0$ be a fixed constant whose value we will specify later. For any $Q_{y,i}$ we can find $a, b \in Q_{y,i}$ such that $2|f(a) - f(b)| \geqslant \varepsilon_{y,i}$ and hence we can use Lemma 3.1 on each of the cubes $Q_{y,i}$, $i \in \{1, \ldots, K_y\}$, to obtain a sequence of balls $B_i^y = B(c_i^y, R_i^y)$ such that

$$\int_{B_i^y} \big|Df(x)\big|^p\,dx \geqslant CC_\gamma |\varepsilon_{y,i}|^p \big(R_i^y\big)^{n-p} \left(\frac{R_i^y}{\operatorname{diam}(Q_{y,i})}\right)^{\gamma}. \tag{3.10}$$

For $y \in E_1$ we take a ball $B_y = B(y, r)$ where $r = \min_{i\in\{1,\ldots,K_y\}} R_i^y$. The balls B_y cover the set E_1 and we can use Besicovitch covering Theorem to select a disjoint subset $\mathcal{B} = \{B_{y_j}\}$ from them such that

$$\mu\bigg(\bigcup_{B_{y_j}\in\mathcal{B}} B_{y_j}\bigg) \geqslant CM. \tag{3.11}$$

Now for each B_{y_j} we denote by $B_i^j = B(c_i^{y_j}, R_i^{y_j})$ and $R_i^j = R_i^{y_j}$ the related balls and their dimensions and ε_i^j the related oscillations. We define the index families

$$\mathcal{R}_k = \big\{(i,j)\colon 2^{-k-1} < R_i^j \leqslant 2^{-k}\big\}.$$

Now we observe that there is $D_\gamma > 0$ small enough such that for each y_j we can find k such that

$$\sum_{i\colon (i,j)\in\mathcal{R}_k} \big(\varepsilon_i^j\big)^{\tilde{\alpha}} = \sum_{i\colon 2^{-k-1}<R_i^j\leqslant 2^{-k}} \big(\varepsilon_i^j\big)^{\tilde{\alpha}} \geqslant D_\gamma 2^{-\gamma k}. \tag{3.12}$$

Otherwise we would obtain

$$\sum_{i=1}^{K_{y_j}} \big(\varepsilon_i^j\big)^{\tilde{\alpha}} < \sum_{k=1}^{\infty} D_\gamma 2^{-\gamma k} = D_\gamma C,$$

which contradicts (3.9). Next we claim that there is a constant $A_\gamma > 0$ such that we can find k with

$$\mu(F_k) \geqslant A_\gamma 2^{-\gamma k} \quad \text{where } F_k = \bigcup \bigg\{B_{y_j}\colon \sum_{i\colon (i,j)\in\mathcal{R}_k} \big(\varepsilon_i^j\big)^{\tilde{\alpha}} \geqslant D_\gamma 2^{-\gamma k}\bigg\}$$

because otherwise we would get a contradiction with (3.11). The constant D_γ depends on γ and the original constant c_0 and the dependence of D_γ on c_0 may be chosen as linear, while A_γ depends on γ and n. It follows that for a huge c_0 we can get a huge number D_γ.

We cover the set F_k by open balls of the diameter 2^{-k+3} centered in each point of F_k and use Besicovitch covering Theorem to select a disjoint subcovering $\mathcal{U}$ such that $\mu(\bigcup\mathcal{U}) \geqslant CA_\gamma 2^{-\gamma k}$. By (3.4), $\mathcal{U}$ contains at least N balls, where

$$N \geqslant 2^{\tilde{\beta}k} C A_\gamma 2^{-\gamma k}. \tag{3.13}$$

For a fixed ball U in $\mathcal{U}$, we take the j such that $y_j \in U$. Such j exists, since diameters of the balls B_{y_j} are smaller than 2^{-k}. Using (3.10) we compute

$$\int\limits_{\mathbb{R}^m \times U} \left|Df(x)\right|^p dx \geqslant C \sum_{i:\ (i,j)\in\mathcal{R}_k} C_\gamma \left(\varepsilon_i^j\right)^p 2^{-k(n-p)} 2^{-k\gamma}. \tag{3.14}$$

Note that for a fixed j at most $L_{n,m}$ balls B_i^j may intersect, where $L_{n,m}$ is a dimensional constant. To verify this, one observes that the diameters of the balls are all comparable to 2^{-k} and that their centers are in disjoint dyadic cubes of diameter at least 2^{-k-10}. The balls with different j are disjoint.

For each fixed y^j we get at most $C2^{km}$ balls in $(0,1)^m \times \{y^j\}$ of size 2^{-k}. Therefore we can use Hölder's inequality to obtain

$$\sum_{i:\ (i,j)\in\mathcal{R}_k} \left(\varepsilon_i^j\right)^{\tilde{\alpha}} \leqslant \left(\sum_{i:\ (i,j)\in\mathcal{R}_k} \left(\varepsilon_i^j\right)^p \right)^{\frac{\tilde{\alpha}}{p}} \left(C2^{km}\right)^{1-\frac{\tilde{\alpha}}{p}}. \tag{3.15}$$

Now we can use (3.14), (3.15), (3.12) and (3.13) to obtain

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \left|Df(x)\right|^p dx &\geqslant \sum_{U\in\mathcal{U}} \int\limits_{\mathbb{R}^m\times U} \left|Df(x)\right|^p dx \\
&\geqslant \sum_{U\in\mathcal{U}} C \sum_{i:(i,j_U)\in\mathcal{R}_k} C_\gamma \left(\varepsilon_i^{j_U}\right)^p 2^{-k(n-p)} 2^{-k\gamma} \\
&\geqslant N C 2^{-k(n-p)} 2^{-k\gamma} \left(D_\gamma 2^{-\gamma k}\right)^{p/\tilde{\alpha}} 2^{-km\frac{p-\tilde{\alpha}}{\tilde{\alpha}}} \\
&\geqslant C (D_\gamma)^{p/\tilde{\alpha}} 2^{-k(-\tilde{\beta}+n-p+m\frac{p-\tilde{\alpha}}{\tilde{\alpha}})} 2^{-k\gamma\tilde{C}}.
\end{aligned}$$

Since $\tilde{\beta} > n-m-p+\frac{pm}{\tilde{\alpha}}$ we may take γ so small that the cumulative exponent above becomes bounded from below by a constant independent of k. Since the constant D_γ may be chosen arbitrarily large if c_0 was chosen large at the beginning of the proof, we get that f is not in $W^{1,p}$, a contradiction. □

4. Counterexample in the degenerate case

In this section we prove Theorem 1.3. We use the approach that was developed in [1, Theorem 1.4] and [3, Theorem 3]. For the convenience of the reader we include the details.

In contrast with the construction in Theorem 1.2 from [1] we do not put some basic function into each subcube that intersects our set but only into some of them. In the proof it is necessary to construct a measure on the image of m-dimensional hyperplanes and then use the definition of capacitary dimension which equals the Hausdorff dimension. In [1] it was enough to use the push-forward of the m-dimensional Hausdorff measure on the hyperplane but we need to use the push-forward of the natural measure on the Cantor type set that is created as the intersection of the subcubes from our construction.

Proof of Theorem 1.3. Let us denote the orthogonal splitting of $\mathbb{R}^n$ by

$$V = \mathbb{R}^m \times \{0\}^{n-m} \quad \text{and} \quad V^\perp = \{0\}^m \times \mathbb{R}^{n-m}$$

and for $a \in \mathbb{R}^n$ we denote $V_a = V + a$. We assume that our set E satisfies

$$\mathbf{N}(E,r) \leqslant Cr^{-\tilde{\beta}}. \tag{4.1}$$

We will construct a map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ that satisfies

$$\mathcal{H}^{\alpha'}\left(f_\xi\left(V_a \cap [0,1]^n\right)\right) = \infty \tag{4.2}$$

for $\mathcal{H}^{\tilde{\beta}}$ almost every $a \in E$ and almost surely in ξ, for each $\alpha' < \alpha$.

Let us introduce the sets that will serve as a set of indices in our construction. Denote $W = \{1, \dots, 2^n\}$ and let W^j be the set of (ordered) j-tuples of elements of W and let

$$W^* = \bigcup_{j \geqslant 0} W^j.$$

We say that $w = (w_1, \dots, w_k)$ is a subword of $v = (v_1, \dots, v_j)$ if $j \geqslant k$ and $v_i = w_i$ for $i = 1, \dots, k$. The length of a word $w \in W^j$ is equal to j and we denote it as $|w|$. We use the set W^* to index the cubes in the standard dyadic decomposition

$$\mathcal{D} = \{Q_w\}_{w \in W^*}$$

of $Q = [0, 1]^n$. It follows that the side length of Q_w is equal to 2^{-j} if $w \in W^j$ and that $Q_v \subset Q_w$ if w is a subword of v. We project these cubes into the subspaces V and $V^\perp$ and we denote

$$Q_w^{V^\perp} = P_{V^\perp}(Q_w) \quad \text{and} \quad Q_w^V = P_V(Q_w)$$

where P_V and $P_{V^\perp}$ are the corresponding projections. Analogously to the definition of W^j we can define a system of 2^{jm} dyadic cubes in $[0, 1]^m$ and we denote this system as $\tilde{W}^j$.

In W^j we have $2^{jn} = 2^{jm} \times 2^{j(n-m)}$ cubes and we would like to define $W_G^j \subset W^j$ with $2^{[\sqrt{j}]m} \times 2^{j(n-m)}$ cubes Q_w for $w \in W_G^j$. We first choose $2^{[\sqrt{j}]m}$ cubes from $\tilde{W}^j$ and then we choose all cubes Q_w, $w \in W^j$, such that Q_w^V lies in this system $\tilde{W}^j$. Our only requirements for the position of these cubes are that

a) for each $w \in W_G^j$ there is $v \in W_G^{j-1}$ such that $Q_w \subset Q_v$,
b) for each $w \in W_G^j$ there are at most 2^m pairwise essentially disjoint cubes $Q_{u_i} \in W_G^{j+1}$ such that $Q_{u_i} \subset Q_w$,
c) number of different cubes in $\{Q^{V^\perp},\ w \in W_G^j\}$ is $2^{j(n-m)}$.

Let us briefly sketch how to construct such a system of cubes by induction. Set $\tilde{W}_G^0 = \tilde{W}^0$. Assume that $\tilde{W}_G^j \subset \tilde{W}^j$ is defined and contains $2^{[\sqrt{j}]m}$ cubes. If $[j+1] = [j]$ then for each $w \in \tilde{W}_G^j$ we choose one $v \in \tilde{W}^{j+1}$ such that w is a subword of v and we put this v into $\tilde{W}_G^{j+1}$. In this way we obtain a system of $2^{[\sqrt{j}]m} = 2^{[\sqrt{j+1}]m}$ cubes $\tilde{W}_G^{j+1} \subset \tilde{W}^{j+1}$. If $[j+1] = [j] + 1$ then for each $w \in \tilde{W}_G^j$ there are 2^m words $v \in \tilde{W}^{j+1}$ such that w is a subword of v and we put all those v into $\tilde{W}_G^{j+1}$. In this way we obtain a system of $2^{[\sqrt{j+1}]m} = 2^m 2^{[\sqrt{j}]m}$ cubes $\tilde{W}_G^{j+1} \subset \tilde{W}^{j+1}$. In both cases we can easily check analogy of properties a) and b). Now we can define $W_G^j = \{w \in W^j\colon Q_w^V \in \tilde{W}_G^j\}$ and it is not difficult to check properties a), b) and c). In this way we obviously obtain $2^{[\sqrt{j}]m} \times 2^{j(n-m)}$ cubes from W^j.

To simplify the notation we write

$$W_G^j(E) = \bigl\{w \in W_G^j\colon Q_w^{V^\perp} \cap E \neq \emptyset\bigr\}.$$

The cubes from W_G^j naturally form a Cantor type set in $\mathbb{R}^m$

$$G := \bigcap_{j=1}^{\infty} \bigcup_{w \in W_G^j} Q_w^V. \tag{4.3}$$

For each $w \in W^*$, let ψ_w be a function in $C_0^\infty(\mathbb{R}^n)$ such that

(i) $0 \leqslant \psi_w \leqslant 1$,
(ii) $\psi_w \equiv 1$ on Q_w,
(iii) $\psi_w \equiv 0$ on the complement of $2Q_w$,
(iv) $|\nabla \psi_w| \leqslant C 2^{|w|}$.

Set $W_G^* = \bigcup_{j\geqslant 0} W_G^j$. Let $\xi = \{\xi_w\}_{w\in W_G^*}$ be a countable sequence of elements from the unit ball in $\mathbb{R}^k$. For each $j \geqslant 1$ we define

$$f_{\xi,j} = \sum_{w\in W_G^j(E)} 2^{-\frac{m[\sqrt{j}]}{\alpha}} \psi_w(a,x)\xi_w, \quad \text{for } x \in V,\ a \in V^\perp$$

and finally we set

$$f_\xi = \sum_{j=1}^{\infty} f_{\xi,j}.$$

Since

$$\|f_{\xi,j}\|_{L^\infty} \leqslant C2^{-\frac{m[\sqrt{j}]}{\alpha}}$$

it is easy to see that f_ξ is continuous.

We have $2^{[\sqrt{j}]m} \times 2^{j(n-m)}$ cubes Q_w for $w \in W_G^j$ and we have to estimate the number of such a cubes whose projection intersects E. From the construction of W_G^j c) we know that the number of cubes projected to $V^\perp$ is $2^{j(n-m)}$, that is all dyadic cubes are available for our covering. By (4.1) we know that we can cover E by $C2^{j\tilde{\beta}}$ balls of radius $2^{-(j+1)}$ and each of these balls can be covered by at most 2^{n-m} dyadic cubes of side length 2^{-j}. It follows that the number of cubes Q_w for $w \in W_G^j(E)$ can be estimated from above by

$$C2^{[\sqrt{j}]m} \times 2^{j\tilde{\beta}}.$$

The cubes $2Q_w$ have bounded overlap and thus we may estimate

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f_{\xi,j}|^p &\leqslant C \int_{\mathbb{R}^n} \sum_{w\in W_G^j(E)} 2^{-\frac{m[\sqrt{j}]}{\alpha}p} \left|\nabla\psi_w(x)\right|^p dx \\ &\leqslant C \sum_{w\in W_G^j(E)} 2^{-\frac{m[\sqrt{j}]}{\alpha}p} 2^{-jn} 2^{jp} \\ &\leqslant C2^{[\sqrt{j}]m} 2^{j(\tilde{\beta}-n+p)} 2^{-\frac{m[\sqrt{j}]}{\alpha}p}.\end{aligned}$$

Since $\tilde{\beta} < n - p$ it is easy to see that

$$\left(\int_{\mathbb{R}^n} |\nabla f_\xi|^p\right)^{\frac{1}{p}} \leqslant \sum_j \left(\int_{\mathbb{R}^n} |\nabla f_{\xi,j}|^p\right)^{\frac{1}{p}} \leqslant C \sum_j 2^{j\frac{(\tilde{\beta}-n+p)}{p}} 2^{-[\sqrt{j}]m(\frac{p}{\alpha}-1)\frac{1}{p}} < \infty.$$

Since $f_{\xi,j}$ are uniformly bounded we obtain that $f_\xi \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$.

In the remaining part of the proof we would like to show that for a generic choice of ξ we obtain a map f_ξ with the desired property (4.2). Let us view $\xi = \{\xi_w\}_{w\in W_G^*}$ as a sequence of independent random variables, identically distributed according to the uniform probability distribution on the unit ball B in $\mathbb{R}^k$. Instead of the conclusion (4.2) we will even show that

$$\mathcal{H}^{\alpha'}\left(f_\xi\left(G_a \cap [0,1]^n\right)\right) = \infty \tag{4.4}$$

where G_a is a Cantor type set in V_a constructed as in (4.3). Since Hausdorff and capacitary dimension coincide (see Section 2) it is now enough to show that for each $\alpha' < \alpha$ we can find a measure μ on $f_\xi(G_a \cap [0,1]^n)$ with finite α'-energy.

On the Cantor type set G_a there is a natural measure $\mathcal{H}_{G_a}$ such that

$$\mathcal{H}_{G_a}(Q_w) = \frac{1}{\#W_G^j} = 2^{-m[\sqrt{j}]} \quad \text{for each } w \in W_G^j. \tag{4.5}$$

Indeed, consider a sequence of Radon measures μ_j whose density with respect to the Lebesgue measure is

$$2^{-m[\sqrt{j}]} \sum_{w \in W_G^j} \chi_{Q_w}(x), \quad \text{i.e. } \mu_j(A) = 2^{-m[\sqrt{j}]} \sum_{w \in W_G^j} |Q_w \cap A|.$$

It is easy to see that $\mu_j([0,1]^n) = 1$ and hence there is a subsequence which converges to some Radon measure in the weak star topology. We call this limit measure $\mathcal{H}_{G_a}$. For each fixed continuous function $h \in C([0,1]^n)$ it is not difficult to see that the sequence $\int_{[0,1]^n} h \, d\mu_j$ is Cauchy in $\mathbb{R}$ and hence its limit must be $\int_{[0,1]^n} h \, d\mathcal{H}_{G_a}$. This holds for each continuous function h and hence this Radon measure is uniquely defined. By choosing proper continuous functions such that $h \equiv 1$ on a fixed Q_w, $w \in W_G^j$, we may obtain (4.5).

For each $a \in E$ consider the measure $(f_\xi)_\#(\mathcal{H}_{G_a})$, i.e. the push-forward of the $\mathcal{H}_{G_a}$-measure on G_a via the map f_ξ. This measure is nonzero, because the set G_a is nonempty. We claim that the expectation

$$\mathbb{E}_\xi \left(\int_E I_{\alpha'}\big((f_\xi)_\#(\mathcal{H}_{G_a})\big) \, d\mathcal{H}^{\tilde{\beta}}(a) \right) \tag{4.6}$$

is finite for each $\alpha' < \alpha$. It follows that almost surely with respect to ξ we obtain that

$$I_{\alpha'}\big((f_\xi)_\#(\mathcal{H}_{G_a})\big) \text{ is finite for } \mathcal{H}^{\tilde{\beta}} \text{ a.e. } a \in E$$

and our conclusion follows once we prove the claim (4.6).

Using Fubini theorem we may transform the integral from (4.6) to

$$\int_{[0,1]^m} \int_{[0,1]^m} \int_E \mathbb{E}_\xi \big(\big| f_\xi(a,x) - f_\xi(a,y) \big|^{-\alpha'} \big) \, d\mathcal{H}^{\tilde{\beta}}(a) \, d\mathcal{H}_{G_a}(x) \, d\mathcal{H}_{G_a}(y).$$

We write

$$f_\xi(a,x) - f_\xi(a,y) = \sum_{w \in W_G^*(E)} c_w(a,x,y) \xi_w \tag{4.7}$$

where the coefficients are given by

$$c_w(a,x,y) = 2^{-\frac{m[\sqrt{|w|}]}{\alpha}} \big(\psi_w(a,x) - \psi_w(a,y) \big). \tag{4.8}$$

Let us fix $a \in E$ and $y \in G_a$. The sequence of the coefficients c clearly belongs to ℓ^∞ and thus we may use Lemma 2.1 and our task is reduced to the proof of

$$\int_{[0,1]^m} \rho\big(c(a,x,y)\big)^{-\alpha'} \, d\mathcal{H}_{G_a}(x) \leqslant C < \infty.$$

where the constant C is independent of a and y. For $x \in G_a$ let us denote by $j(x)$ the largest integer such that both x and y lie in the same $Q_w \ni x, y$ for $w \in W_G^{j(x)}$. It follows that they lie in different $Q_{u_1} \ni x$ and $Q_{u_2} \ni y$ for $u_1, u_2 \in W_G^{j(x)+1}$. It follows that most terms in (4.7) and (4.8) cancel and the first nonzero term corresponds to $j(x)+1$. Since $\psi_w(a,x) = 1$ on Q_{u_1} and $\psi_w(a,x) = 0$ on the complement of $2Q_{u_2}$ it is easy to see that the supremum norm of the difference of these two functions is 1. We can do similar observation for the term $j(x)+2$ which must be again nonzero and hence we obtain

$$\big\| c(a,x,y) \big\|_\infty = 2^{-\frac{m[\sqrt{j(x)+1}]}{\alpha}} \quad \text{and} \quad \rho\big(c(a,x,y)\big) = 2^{-\frac{m[\sqrt{j(x)+2}]}{\alpha}}.$$

From the construction of W_G^j part b) we know that for each $j = j(x)$ we have a fixed cube $Q_{u_2} \ni y$ and we can find at most $2^m - 1$ cubes Q_{u_1} such that $x \in Q_{u_1}$ and hence

$$\mathcal{H}_{G_a}\big(\big\{x \in G_a \colon j(x) = j\big\}\big) = \big(2^m - 1\big)\mathcal{H}_{G_a}(Q_{u_2}) = \big(2^m - 1\big)2^{-m[\sqrt{j+1}]}.$$

Now we can estimate

$$\int\limits_{[0,1]^m} \rho\big(c(a,x,y)\big)^{-\alpha'}\, d\mathcal{H}_{G_a}(x) \leqslant \sum_{j=0}^{\infty}\big(2^m-1\big)2^{-m[\sqrt{j+1}]}2^{m[\sqrt{j+2}]\frac{\alpha'}{\alpha}}.$$

Since $\alpha' < \alpha$ it is easy to see that the series converges which finishes our proof. □

Acknowledgements

The authors would like to thank Jan Malý for many stimulating conversations and valuable comments. They would also like to thank the anonymous referee for carefully reading the paper and for very constructive comments that help to improve the paper.

References

[1] Z.M. Balogh, R. Monti, J.T. Tyson, Frequency of Sobolev and quasiconformal dimension distortion, preprint, 2010.
[2] F.W. Gehring, J. Väisälä, Hausdorff dimension and quasiconformal mappings, J. London Math. Soc. 6 (2) (1973) 504–512.
[3] R. Kaufman, Sobolev spaces, dimension, and random series, Proc. Amer. Math. Soc. 128 (2) (2000) 427–431.
[4] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, 1995.
[5] W.P. Ziemer, Weakly Differentiable Functions, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, 1989.