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Asymptotic transversality and symmetry breaking bifurcation from boundary concentrating solutions

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Abstract

Let $A := \{a < |x| < 1 + a\} \subset \mathbb{R}^N$ and $p \geqslant 2$. We consider the Neumann problem

$$\varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } A, \qquad \partial_\nu u = 0 \quad \text{on } \partial A.$$

Let $\lambda = 1/\varepsilon^2$. When λ is large, we prove the existence of a smooth curve $\{(\lambda, u(\lambda))\}$ consisting of radially symmetric and radially decreasing solutions concentrating on $\{|x| = a\}$. Moreover, checking the transversality condition, we show that this curve has infinitely many symmetry breaking bifurcation points from which continua consisting of nonradially symmetric solutions emanate. If $N = 2$, then the closure of each bifurcating continuum is locally homeomorphic to a disk. When the domain is a rectangle $(0, 1) \times (0, a) \subset \mathbb{R}^2$, we show that a curve consisting of one-dimensional solutions concentrating on $\{0\} \times [0, a]$ has infinitely many symmetry breaking bifurcation points. Extending this solution with even reflection, we obtain a new entire solution.

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1. Introduction and main results

In this paper we are concerned with the Neumann problem

$$\varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega, \qquad u > 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p \geqslant 2$ and $\varepsilon > 0$ is small. Let $a > 0$. We consider the case $\Omega = A$ or R, where

$$A := \left\{x \in \mathbb{R}^N;\ a < |x| < a + 1\right\} \quad \text{and} \quad R := \left\{(x, y) \in \mathbb{R}^2;\ 0 < x < 1,\ 0 < y < a\right\}.$$

Note that a scaling argument shows that (1.1) with $\Omega = A$ (resp. $\Omega = R$) is equivalent to (1.1) for an arbitrary annulus (resp. rectangle) and that

$$1 < p < \frac{N + 2}{N - 2} \tag{1.2}$$

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is *not* assumed. Singularly perturbed elliptic equations arise in physical and biological models. In particular, the Neumann problem (1.1) on a bounded domain appears in the stationary problems of the Keller–Segel model for chemotaxis aggregation [12] and the shadow system of the Gierer–Meinhardt model for biological pattern formations [8]. For these two decades the problem (1.1) has considerable attention and solutions with various shapes have been found. See [30,31] for single-peak solutions, [10] for multi-peak solutions and [1,2] for solutions concentrating on a sphere of inhomogeneous equations. A boundary concentrating solution is one of the solutions of (1.1). For an arbitrary planar smooth bounded domain this solution was established by Malchiodi and Montenegro [17]. The aim of this paper is to study the solution structure of (1.1) from a viewpoint of the bifurcation theory when $\Omega = A$ or R. Since we mainly consider the solution concentrating not on a point but on a boundary, we do not need (1.2). In the proofs of the main results below we prove the monotonicity of a certain eigenvalue and obtain the asymptotic expansion. In order to prove these properties we need that $f \in C^2([0,\infty))$. Thus the assumption $p \geqslant 2$ is needed.

It is convenient for our aim to consider the equation of the form

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega, \qquad u > 0 \quad \text{in } \Omega. \tag{N_Ω}$$

Throughout the present article we define $f(u) := -u + u^p$ and $\lambda := \frac{1}{\varepsilon^2}$. Then (N_Ω) is equivalent to (1.1). When $\Omega = A$ (resp. $\Omega = R$), we show that (N_Ω) has a smooth curve $\{(\lambda, u(\lambda))\}$ of radially symmetric (resp. one-dimensional) solutions. We say that $(\lambda_*, u(\lambda_*))$ is a symmetry breaking bifurcation point if there is a sequence $\{(\tilde\lambda_j, \tilde u_j)\}_{j\geqslant 0}$ consisting of nonradially symmetric (resp. non-one-dimensional) solutions and converging to $(\lambda_*, u(\lambda_*))$, i.e., $(\tilde\lambda_j, \tilde u_j) \to (\lambda_*, u(\lambda_*))$ as $j \to \infty$. The first main result is

Theorem A. *Let $p \geqslant 2$. (N_A) has a smooth curve $\mathcal{C}_A := \{(\lambda, u(\lambda))\}_{\lambda>\lambda_0}$ consisting of radially symmetric and radially decreasing solutions concentrating on $\{|x| = a\}$. $\mathcal{C}_A$ has infinitely many symmetry breaking bifurcation points $\{(\lambda_k, u(\lambda_k))\}_{k>k_0}$, where $\lambda_k \to \infty$ as $k \to \infty$. When $N = 2$, for each $k > k_0$, the closure of the bifurcating solutions near $(\lambda_k, u(\lambda_k))$ is locally homeomorphic to a disk.*

The precise statements of Theorem A are in Lemma 4.1, Theorems 4.6 and 4.12 and Corollary 4.13.

The second main result is about (N_R).

Theorem B. *Let $p \geqslant 2$. (N_R) has a smooth curve $\mathcal{C}_R := \{(\lambda, u(\lambda))\}_{\lambda>\lambda_0}$ consisting of one-dimensional solutions concentrating on $\{0\} \times [0,a]$. The continuum including $\mathcal{C}_R$ bifurcates from the branch of constant solutions $\{(\lambda, 1)\}_{\lambda>0}$. $\mathcal{C}_R$ has infinitely many symmetry breaking bifurcation points $\{(\lambda_k, u(\lambda_k))\}_{k>k_0}$, where $\lambda_k \to \infty$ as $k \to \infty$. The closure of the bifurcating solutions near $(\lambda_k, u(\lambda_k))$ is locally homeomorphic to a curve.*

In particular, if $a > 0$ is small, then every symmetry breaking bifurcation point on the continuum including $\mathcal{C}_R$ can be obtained. See Theorem 3.5 and Corollary 3.6 for the precise statement. Extending this solution with even reflection, we obtain a new entire solution (Corollary 3.7).

Let us explain technical details. Let $\mathbb{B}_0$, $\mathbb{B}_1$ be two Banach spaces. We consider the abstract functional equation

$$E(\lambda, u) = 0, \tag{1.3}$$

where $E : \mathbb{R} \times \mathbb{B}_0 \to \mathbb{B}_1$ is a nonlinear smooth mapping. We assume that $\{(\lambda, 0)\}_{\lambda\in\mathbb{R}}$ are solutions of (1.3). We call $\{(\lambda, 0)\}$ the trivial branch. When the linearized eigenvalue problem

$$E_u(\lambda_*, 0)[\phi] = \mu\phi \tag{1.4}$$

has a simple zero eigenvalue, the Crandall–Rabinowitz bifurcation theorem [4] (Proposition 2.1 in the present paper) guarantees that a curve consisting of nontrivial solutions emanates from $(\lambda_*, 0)$ provided that

$$E_{\lambda u}(\lambda_*, 0)[\phi_*] \notin \operatorname{Ran} E_u(\lambda_*, 0), \tag{1.5}$$

where ϕ_* is an eigenfunction associated to the simple zero eigenvalue. This condition is called *the transversality condition* (or *the nondegeneracy condition*). See (b) in Proposition 2.1. In the proofs of Theorems A and B we consider the case where the trivial branch $\{(\lambda, v(\lambda))\}$ consists of nonconstant solutions. Specifically, we consider the equation $\tilde E(\lambda, u) = 0$ in the case where $\tilde E(\lambda, u) = 0$ has a smooth curve of nonconstant solutions $\{(\lambda, v(\lambda))\}$. Let $E(\lambda, u) := \tilde E(\lambda, u + v(\lambda))$. If $E_u(\lambda_*, 0)$ has a simple zero eigenvalue, then (1.5) becomes

$$\tilde E_{\lambda u}\big(\lambda_*, v(\lambda_*)\big)[\phi_*] + \tilde E_{uu}\big(\lambda_*, v(\lambda_*)\big)\left[v_\lambda(\lambda_*), \phi_*\right] \notin \operatorname{Ran} \tilde E_u\big(\lambda_*, v(\lambda_*)\big). \tag{1.6}$$

It is well known that (1.6) is equivalent to

$$\left.\frac{d\mu}{d\lambda}\right|_{\lambda=\lambda_*} \neq 0, \tag{1.7}$$

where μ is a unique near-zero eigenvalue of the eigenvalue problem

$$E_u(\lambda, 0)[\phi] = \mu\phi \tag{1.8}$$

provided that λ is near λ_*. When E_u is defined in a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and E_u is self-adjoint, we briefly show this equivalence. We can show that the eigenpair (μ, ϕ) is continuously differentiable in λ. Differentiating $E_u(\lambda, 0)[\phi] = \mu\phi$ in λ, we have

$$E_{\lambda u}(\lambda, 0)[\phi] + E_u(\lambda, 0)[\phi_\lambda] = \mu_\lambda\phi + \mu\phi_\lambda. \tag{1.9}$$

Calculating $\langle(1.9), \phi\rangle - \langle(1.8), \phi_\lambda\rangle$ and evaluating it at $\lambda = \lambda_*$, we have

$$\big\langle E_{\lambda u}(\lambda_*, 0)[\phi_*], \phi_*\big\rangle = \mu_\lambda(\lambda_*)\langle\phi_*, \phi_*\rangle. \tag{1.10}$$

Since $E_{\lambda u}(\lambda_*, 0)[\phi_*] = \tilde{E}_{\lambda u}(\lambda_*, v(\lambda_*))[\phi_*] + \tilde{E}_{uu}(\lambda_*, v(\lambda_*))[v_\lambda(\lambda_*), \phi_*]$, by (1.10) we have

$$\left.\frac{d\mu}{d\lambda}\right|_{\lambda=\lambda_*} = \frac{\langle \tilde{E}_{\lambda u}(\lambda_*, v(\lambda_*))[\phi_*] + \tilde{E}_{uu}(\lambda_*, v(\lambda_*))[v_\lambda(\lambda_*), \phi_*], \phi_*\rangle}{\langle\phi_*, \phi_*\rangle}, \tag{1.11}$$

which indicates the equivalence between (1.6) and (1.7).

When the trivial solution $v(\lambda)$ depends on λ, i.e., $v_\lambda \not\equiv 0$, it is hard to check (1.7), because it is almost impossible to obtain exact expressions of v_λ and ϕ_* and it is difficult to determine the sign of the RHS of (1.11). Shi [24] studied the same bifurcation problem as (N_R) for a general nonlinear term f. However, he assumed the transversality condition in [24, Proposition 4.2]. (There are several exceptional cases where the transversality condition can be checked. Lin [13] considered the Dirichlet problem of the Liouville–Gel'fand equation $\Delta u + \lambda e^u = 0$ on an annulus. He showed that there is a radial branch having infinitely many symmetry breaking bifurcation points, checking the transversality condition. In this problem the radial solutions and eigenfunctions associated to a zero eigenvalue can be written explicitly, hence the situation seems rare.) In order to avoid checking the transversality condition, topological methods using the degree theory have been developed [23,26] and applied to many problems. In [9,14,15,22,27,6] symmetry breaking bifurcations of Dirichlet problems in annuli were studied with topological methods. If topological methods are used, then we cannot obtain information on the shape of bifurcating solutions. The shape can be used for the study of the global property of the bifurcating branch. (However, the global property is beyond the scope of this article. In [18–20] one can prove the existence of unbounded continua of nonradially symmetric solutions, using the nodal structure of bifurcating solutions.)

In the proofs of Theorems A and B we directly check (1.7) when λ is large (asymptotic transversality). In the case $\Omega = A$ the radially symmetric and radially decreasing solution $u(r)$ is close to a decreasing solution in a finite interval (Lemma 4.1). Using this closeness, we obtain an apriori estimate of a certain eigenfunction (4.25). The boundedness of the solution (4.3) and this apriori estimate enable us to use the dominated convergence theorem in Lemmas 4.15 and 4.16. Then we can calculate the RHS of (1.11) and obtain the asymptotic behavior of a certain eigenvalue (Lemma 4.9) which indicates (1.7). Using the transversality condition, we can make detailed studies on not only the shape of solutions but also the shape of bifurcating branches. See Corollary 4.13.

This work was motivated by results on symmetry breaking bifurcations of Srikanth [27] and Gladiali et al. [9]. The transversality property has been first proved by Bartsch et al. [3]. The authors of [9] studied symmetry breaking bifurcations of (N_A) with fixed λ for expanding annuli $a \to \infty$. They showed the monotonicity of a certain eigenvalue, which is corresponding to $\hat{\nu}_0$ in Lemma 4.9. However, the singularly perturbed problem is not considered. In their problem the term including v_a tends to 0 as $a \to \infty$, where $\{(a, v(a))\}$ is a (nonconstant) trivial branch. In our problem the corresponding term does not tend to 0, hence a detailed analysis is needed. See Lemma 4.16.

This article consists of four sections. In Section 2 we recall known results about a bifurcation theorem and useful properties of the one-dimensional problem $(\mathrm{N}_{(0,1)})$. In Sections 3 and 4 we prove Theorems B and A, respectively.

Notations.

- $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$.
- $L^p(\Omega)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_p$.
- $H^k(\Omega)$ denotes the usual Sobolev space with the norm $\|\cdot\|_{H^k}$.
- $H^2_N(\Omega)$ denotes the Banach space consisting of the functions $u \in H^2(\Omega)$ that satisfy the Neumann boundary condition on $\partial\Omega$.
- $\mathcal{B}(\mathbb{B}_0, \mathbb{B}_1)$ denotes the Banach space of the bounded linear operators from $\mathbb{B}_0$ to $\mathbb{B}_1$ equipped with the operator norm $\|\cdot\|_{\mathcal{B}}$, where $\mathbb{B}_0$ and $\mathbb{B}_1$ are two Banach spaces.
- $\langle f_0(x), f_1(x)\rangle := \int f_0(x) f_1(x)\, dx$.

2. Known results

2.1. Bifurcation from a simple eigenvalue

Let $\mathbb{B}_0$, $\mathbb{B}_1$ be two Banach spaces. We consider the abstract functional equation (1.3), where $E : \mathbb{R} \times \mathbb{B}_0 \to \mathbb{B}_1$ is a nonlinear smooth mapping. We assume that $E(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$. Crandall and Rabinowitz [4] studied nontrivial solutions near the trivial branch $\{(\lambda, 0)\}$ and gave a sufficient condition for bifurcation. The celebrated Crandall–Rabinowitz bifurcation theorem [4] is the following:

Proposition 2.1. *(See [4, Theorems 1 and 1.7].) Let E be as defined above. If the following conditions hold*:

(a) *There are ϕ_* and λ_* such that* $\dim\ker E_u(\lambda_*, 0) = \operatorname{codim} \operatorname{Ran} E_u(\lambda_*, 0) = 1$ *and* $\ker E_u(\lambda_*, 0) = \operatorname{span}\langle\phi_*\rangle$,
(b) $E_{\lambda u}(\lambda_*, 0)[\phi_*] \notin \operatorname{Ran} E_u(\lambda_*, 0)$.

Then there are a neighborhood $\mathcal{U}$ of $(\lambda_*, 0) \in \mathbb{R} \times \mathbb{B}_0$, an interval $(-\varepsilon_0, \varepsilon_0)$ and continuous functions $\varphi : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, $\psi : (-\varepsilon_0, \varepsilon_0) \to \mathbb{B}_0$ such that $\varphi(0) = \lambda_*$, $\psi(0) = 0$ and

$$E^{-1}(0) \cap \mathcal{U} = \big\{\big(\varphi(\tau), \tau\phi_* + \tau\psi(\tau)\big);\ |\tau| < \varepsilon_0\big\} \cup \big\{(t, 0);\ (t, 0) \in \mathcal{U}\big\}.$$

2.2. One-dimensional problem

We consider the problem

$$\varepsilon^2 u_{xx} + f(u) = 0 \quad \text{in } (0, 1), \qquad u_x = 0 \quad \text{at } x = 0, 1, \tag{2.1}$$

where $f(u) = -u + u^p$ and $p > 1$. Let $\tilde{u}(\xi) := u(x)$ and $\xi := \frac{x}{\varepsilon}$. We also consider the stretched problem

$$\tilde{u}_{\xi\xi} + f(\tilde{u}) = 0 \quad \text{in } (0, d_\varepsilon), \qquad \tilde{u}_\xi = 0 \quad \text{at } \xi = 0, d_\varepsilon, \tag{2.2}$$

where $d_\varepsilon = \frac{1}{\varepsilon}$.

2.2.1. Homoclinic orbit

For $p > 1$ the system of equations for $(\tilde{u}, \tilde{v})$ $(\tilde{v} := \tilde{u}_\xi)$ in the phase plane

$$\begin{cases} \tilde{u}_\xi = \tilde{v}, \\ \tilde{v}_\xi = -f(\tilde{u}) \end{cases} \tag{2.3}$$

has a saddle point at $(0, 0)$ and a center $(1, 0)$. There is a unique homoclinic solution around the center connecting the saddle to itself. This homoclinic solution can be written explicitly as

$$w(\xi) := \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left(\cosh\left(\frac{p-1}{2}\xi\right)\right)^{-\frac{2}{p-1}}. \tag{2.4}$$

2.2.2. Continuum of monotone solutions

From the phase portrait for $(\tilde{u}, \tilde{v})$ it is clear that all the orbits on $\{\tilde{u} > 0\}$ satisfying the Neumann boundary conditions are inside of the homoclinic orbit which is tear-shaped and that every orbit in this region is periodic one. Hence $\tilde{u}$ is a solution of (2.2) if and only if an integral multiple of its half period is equal to the interval length d_ε. Now we will find a decreasing solution. Let $\tilde{u}(\xi)$ be a decreasing solution that has maximum α and minimum β. Then

$$0 < \beta < 1 < \alpha < \bar{\alpha} := \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}. \tag{2.5}$$

Multiplying (2.2) by $\tilde{u}_\xi$ and integrating it, we have

$$\tilde{u}_\xi^2 = F(\tilde{u}) - F(\beta), \qquad F(\alpha) = F(\beta) \quad \text{and} \quad F(\tilde{u}) = \tilde{u}^2 - \frac{2}{p+1}\tilde{u}^{p+1}. \tag{2.6}$$

The half period is given by the integral

$$T(\alpha) := \int_\beta^\alpha \frac{d\tilde{u}}{\sqrt{F(\tilde{u}) - F(\beta)}}. \tag{2.7}$$

Thus $\tilde{u}$ is a decreasing solution of (2.2) if and only if $T(\alpha) = d_\varepsilon$. (2.7) was studied by De Groen and Karadzhov [7]. Among other things, they obtained

Proposition 2.2. *There is a small $\varepsilon_0 > 0$ such that the problem* (2.1) *has a smooth curve of decreasing solutions $\{u(x;\varepsilon)\}_{0<\varepsilon<\varepsilon_0}$, which can be described as a graph of ε, satisfying the following*: *For $\delta > 0$, there exists $\varepsilon_1 > 0$ such that, if $0 < \varepsilon < \varepsilon_1$, then*

$$\left|\tilde{u}(\xi;\varepsilon) - w(\xi)\right| < \delta \quad \textit{for } \xi \in [0, d_\varepsilon], \tag{2.8}$$

where $\tilde{u}(\xi;\varepsilon) := u(x;\varepsilon)$ $(\xi = \frac{x}{\varepsilon})$. Moreover, the first two eigenvalues of the eigenvalue problem

$$\varepsilon^2\phi_{xx} + f'(u)\phi = \eta\phi \quad \textit{in } (0,1), \qquad \phi_x = 0 \quad \textit{at } x = 0, 1 \tag{2.9}$$

are

$$\begin{aligned} &\eta_0(\varepsilon) = \frac{(p-1)(p+3)}{4} + O\left(e^{-\frac{2}{\varepsilon}}\right), \\ &\eta_1(\varepsilon) \begin{cases} = -\frac{(p-1)(5-p)}{4} + O(e^{-\frac{3-p}{\varepsilon}}) & (1 < p < 3), \\ \leqslant -1 + O(e^{-\frac{2}{\varepsilon}}) & (3 \leqslant p). \end{cases} \end{aligned} \tag{2.10}$$

In particular, if $\varepsilon > 0$ is small, then $\eta_1(\varepsilon) < 0 < \eta_0(\varepsilon)$ and $u(x;\varepsilon)$ is nondegenerate.

Let $u(x;\varepsilon)$ be the decreasing solution obtained in Proposition 2.2. Let $L := \varepsilon^2\partial_{xx} + f'(u(x;\varepsilon)) \in \mathcal{B}(H_N^2(0,1), L^2(0,1))$. In Section 4 we construct a boundary concentrating solution of (N_A), perturbing $u(x;\varepsilon)$. Hence we need a property of L.

Proposition 2.3. *Let L be as defined above. If $\varepsilon > 0$ is small, then there is $C_0 > 0$ such that, for $\mu \in [-C_0, C_0]$, $(L-\mu)^{-1} \in \mathcal{B}(L^2, H_N^2)$ exists and there is $C_1 > 0$ independent of ε and $\mu \in [-C_0, C_0]$ such that $\|(L-\mu)^{-1}\|_{\mathcal{B}(L^2, H_N^2)} < C_1$.*

This proposition immediately follows from (2.10) in Proposition 2.2, because every eigenvalue of L is uniformly away from 0 when $\varepsilon > 0$ is small.

2.2.3. Limit problem

Proposition 2.2 shows that $\tilde{u}(\xi) \xrightarrow{\varepsilon\downarrow 0} w(\xi)$ in the sense of (2.8). In this subsection we recall some known result of the "limiting" operator $\widetilde{L} := \partial_{\xi\xi} + f'(w) \in \mathcal{B}(H_N^2(0,\infty), L^2(0,\infty))$. The operator $\widetilde{L}$ has a continuous spectrum $(-\infty, -1]$ and may have discrete eigenvalues outside $(-\infty, -1]$ [11, p. 140]. In our study the first eigenpair is important.

Proposition 2.4. *The eigenvalue problem*

$$\widetilde{L}\tilde{\phi} = \tilde{\eta}\tilde{\phi} \quad \text{in } (0,\infty), \qquad \tilde{\phi}_\xi(0) = 0, \quad \tilde{\phi} > 0,\ \tilde{\phi} \in L^2(0,\infty)$$

has a unique (up to multiples) solution

$$\tilde{\phi} = cw^{\frac{p+1}{2}} \quad (c \in \mathbb{R}), \qquad \tilde{\eta} = \frac{(p-1)(p+3)}{4}.$$

We set

$$\alpha_0 := \left(\frac{p+1}{2}\right)^{\frac{p+1}{2(p-1)}}. \tag{2.11}$$

Then $\|w^{\frac{p+1}{2}}\|_\infty = \alpha_0$.

By direct calculation we see that $\tilde{\phi}_1 := w^{\frac{3-p}{2}} - \frac{1}{2}\frac{p+3}{p+1}w^{\frac{p+1}{2}}$ and $\tilde{\eta}_1 := \frac{(p-1)(5-p)}{4}$ satisfy

$$\widetilde{L}\tilde{\phi}_1 = \tilde{\eta}_1\tilde{\phi}_1 \quad \text{in } (0,\infty), \qquad \partial_\xi\tilde{\phi}_1(0) = 0, \quad \tilde{\phi}_1 \in L^2(0,\infty).$$

It is known that if $1 < p < 3$, then $\tilde{\eta}_1$ is the second eigenvalue and that if $p \geqslant 3$, then $\widetilde{L}$ has only one eigenvalue above -1. In particular, 0 is not an eigenvalue and $\widetilde{L}$ is invertible.

In the proofs of Theorems A and B we use (2.12) below. The validity of the transversality condition follows from (2.12).

Proposition 2.5. *Let w be as defined by* (2.4). *Then*

$$\int_0^\infty \left(f'(w) + \frac{1}{2}f''(w)\xi w_\xi\right)w^{p+1}\,d\xi = \frac{(p-1)(p+3)}{4}\int_0^\infty w^{p+1}\,d\xi. \tag{2.12}$$

We briefly prove this equality. We have

$$\left(f'(w) + \frac{1}{2}f''(w)\xi w_\xi\right)w^{p+1} = -w^{p+1} + pw^{2p} + \frac{p(p-1)}{2}\xi w_\xi w^{2p-1}.$$

By integration by parts we have

$$\int_0^\infty \xi w_\xi w^{2p-1}\,d\xi = -\frac{1}{2p}\int_0^\infty w^{2p}\,d\xi.$$

Therefore,

$$\text{LHS of (2.12)} = \int \left(-w^{p+1} + pw^{2p} - \frac{p-1}{4}w^{2p}\right)d\xi = \int \left(-w^{p+1} + \frac{3p+1}{4}w^{2p}\right)d\xi. \tag{2.13}$$

Since w is the homoclinic orbit, w satisfies $w_\xi^2 - w^2 + \frac{2}{p+1}w^{p+1} = 0$, hence

$$\int \left(w_\xi^2 w^{p-1} - w^{p+1} + \frac{2}{p+1}w^{2p}\right)d\xi = 0. \tag{2.14}$$

Multiplying $w_{\xi\xi} - w + w^p = 0$ by $\frac{w^p}{p}$ and integrating it, we have

$$0 = \int \left(\frac{1}{p}w_{\xi\xi}w^p - \frac{1}{p}w^{p+1} + \frac{1}{p}w^{2p}\right)d\xi = \int \left(-w_\xi^2 w^{p-1} - \frac{1}{p}w^{p+1} + \frac{1}{p}w^{2p}\right)d\xi. \tag{2.15}$$

Adding (2.14) and (2.15), we have

$$\int \left[-\left(1 + \frac{1}{p}\right)w^{p+1} + \left(\frac{1}{p} + \frac{2}{p+1}\right)w^{2p}\right]d\xi = 0,$$

which indicates

$$\int\left(-w^{p+1}+\frac{3p+1}{4}w^{2p}\right)d\xi=\frac{(p-1)(p+3)}{4}\int w^{p+1}\,d\xi. \tag{2.16}$$

Substituting (2.16) into (2.13), we obtain (2.12).

Remark 2.6. We can prove (2.12) by direct calculation. The integral $\int_0^\infty w^q\,d\xi$ can be written in terms of the Gamma function $\Gamma(\cdot)$,

$$\int\limits_0^\infty w^q\,d\xi=\left(\frac{p+1}{2}\right)^{\frac{q}{p-1}}\frac{1}{p-1}\frac{\sqrt{\pi}\,\Gamma(\frac{q}{p-1})}{\Gamma(\frac{q}{p-1}+\frac{1}{2})}.$$

We have

$$\begin{aligned}&\int\left(-w^{p+1}+\frac{3p+1}{4}w^{2p}\right)d\xi\\&\quad=\left(\frac{p+1}{2}\right)^{\frac{p+1}{p-1}}\frac{\sqrt{\pi}}{p-1}\left(\frac{(3p+1)(p+1)}{8}\frac{\Gamma(\frac{2p}{p-1})}{\Gamma(\frac{2p}{p-1}+\frac{1}{2})}-\frac{\Gamma(\frac{p+1}{p-1})}{\Gamma(\frac{p+1}{p-1}+\frac{1}{2})}\right).\end{aligned} \tag{2.17}$$

It follows from a property of the Gamma function that

$$\Gamma\left(\frac{2p}{p-1}\right)=\frac{p+1}{p-1}\Gamma\left(\frac{p+1}{p-1}\right),\qquad \Gamma\left(\frac{2p}{p-1}+\frac{1}{2}\right)=\left(\frac{p+1}{p-1}+\frac{1}{2}\right)\Gamma\left(\frac{p+1}{p-1}+\frac{1}{2}\right).$$

Using these equalities, we have

$$\text{RHS of (2.17)}=\frac{(p-1)(p+3)}{4}\left(\frac{p+1}{2}\right)^{\frac{p+1}{p-1}}\frac{1}{p-1}\frac{\sqrt{\pi}\,\Gamma(\frac{p+1}{p-1})}{\Gamma(\frac{p+1}{p-1}+\frac{1}{2})}=\frac{(p-1)(p+3)}{4}\int w^{p+1}\,d\xi.$$

Thus (2.12) follows from this equality and (2.13).

In Sections 3 and 4 we need a solution of

$$\widetilde{L}\phi=-f(w)\quad\text{in }(0,\infty),\qquad \phi_\xi(0)=0,\quad \phi\in L^2(0,\infty). \tag{2.18}$$

Proposition 2.7. *There is a unique solution $\phi(\xi):=\frac{1}{2}\xi w_\xi(\xi)$ of* (2.18).

A direct calculation shows that $\frac{1}{2}\xi w_\xi$ is a solution of (2.18). Since $\widetilde{L}$ is invertible, the uniqueness follows from this invertibility.

2.3. Apriori estimate

We use the following apriori estimate in order to use the dominated convergence theorem.

Proposition 2.8. *Let Ω be a bounded domain. Let ϕ be a C^2 function satisfying the equation*

$$\varepsilon^2\big(a(x)\phi_{xx}+b(x)\phi_x\big)-c(x)\phi=0\quad\text{in }\Omega,$$

where the coefficients $a(x)$ and $b(x)$ are bounded and $c(x)\geqslant c_0>0$ for all $x\in\Omega$. Then there is a constant $C_0>0$ depending only on $a(x)$, $b(x)$, c_0 such that

$$\big|\phi(x)\big|\leqslant 2\big(\sup\big|\phi(x)\big|\big)\exp\left(-\frac{C_0\delta(x)}{\varepsilon}\right),$$

where $\delta(x)$ is the distance from x to $\partial\Omega$.

See [21, p. 840] for details of this proposition.

Using this proposition, we have

Lemma 2.9. *Let $u(x)$ be a decreasing solution of* (2.1). *Then there are $C_0 > 0$, $C_1 > 0$ such that*

$$\left|u(x)\right| \leqslant 2\|u\|_\infty e^{C_0C_1} e^{-\frac{C_0 x}{\varepsilon}} \quad \text{for } x \in (C_1\varepsilon, 1). \tag{2.19}$$

Proof. By (2.8) we see that there are a small $\delta_0 > 0$ and $C_1 > 0$ such that for $\varepsilon > 0$, $|u(x)| < \delta_0$ $(C_1\varepsilon < x < 1)$. Because $-1 + u^{p-1} \leqslant -1 + \delta_0^{p-1} < 0$, Proposition 2.8 is applicable. We extends $u(x)$ with even reflection at $x = 1$. We define $\Omega = (C_1\varepsilon, 2 - C_1\varepsilon)$. Then we see that $\delta(x) = x - C_1\varepsilon$ $(C_1\varepsilon \leqslant x \leqslant 1)$, where $\delta(x)$ is a function defined in Proposition 2.8. Since u is a solution on Ω, we apply Proposition 2.8 and obtain

$$\left|u(x)\right| \leqslant 2\|u\|_\infty \exp\left(-\frac{C_0(x - C_1\varepsilon)}{\varepsilon}\right) = \text{RHS of (2.19)}. \qquad \square$$

3. $\Omega = R$

3.1. Preliminaries

We consider (N_R). An immediate extension of a solution of the one-dimensional problem (2.1) is a solution of (N_R). We identify the decreasing solution on $[0, 1]$ with the solution of (N_R) and denote them by the same u. By Proposition 2.2 we obtain a smooth curve of solutions of (N_R), $\mathcal{C}_R := \{(\lambda, u(\lambda))\}_{\lambda > \lambda_0}$, concentrating on $\{0\} \times [0, a]$ which can be described as a smooth graph of λ. First we obtain degenerate solutions on $\mathcal{C}_R$. Here a degenerate solution is a solution having a zero eigenvalue. Let $\overline{L} := \Delta + \lambda f'(u) \in \mathcal{B}(H^2_N(R), L^2(R))$. The linearized eigenvalue problem is

$$\overline{L}\Phi = \mu\Phi \quad \text{in } R, \qquad \partial_\nu\Phi = 0 \quad \text{on } \partial R. \tag{3.1}$$

Now u is a solution of

$$u_{xx} + \lambda f(u) = 0 \quad \text{in } (0, 1), \qquad u_x = 0 \quad \text{at } x = 0, 1. \tag{3.2}$$

Let $\widehat{L} := \partial_{xx} + \lambda f'(u) \in \mathcal{B}(H^2_N(0, 1), L^2(0, 1))$ and let $\partial_{yy} \in \mathcal{B}(H^2_N(0, a), L^2(0, a))$. Let $\{\hat{\eta}_j(\lambda)\}_{j \geqslant 0}$ denote the eigenvalues of $\widehat{L}$. In this section we mainly study the first eigenvalue of $\widehat{L}$. Multiplying (2.9) by λ, we obtain the relation

$$\hat{\eta}_j(\lambda) = \lambda\eta_j\left(\frac{1}{\sqrt{\lambda}}\right). \tag{3.3}$$

In particular,

$$\begin{aligned} &\hat{\eta}_0(\lambda) = \frac{(p-1)(p+3)}{4}\lambda + O\left(\lambda e^{-2\sqrt{\lambda}}\right), \\ &\hat{\eta}_1(\lambda) \begin{cases} = -\frac{(p-1)(5-p)}{4}\lambda + O(\lambda e^{-(3-p)\sqrt{\lambda}}) & (1 < p < 3), \\ \leqslant -\lambda + O(\lambda e^{-2\sqrt{\lambda}}) & (3 \leqslant p). \end{cases} \end{aligned} \tag{3.4}$$

Let $\zeta_k := \frac{(\pi k)^2}{a^2}$ $(k \in \mathbb{N}_0)$. Let $\sigma(\cdot)$ denote the spectrum of a linear operator. Then it is clear that $\sigma(\partial_{yy}) = \{-\zeta_k\}_{k \geqslant 0}$.

The next proposition shows that every eigenvalue of $\overline{L}$ can be described by eigenvalues of $\widehat{L}$ and ∂_{yy}.

Proposition 3.1. *The following holds*:

$$\sigma(\overline{L}) = \sigma(\widehat{L}) + \sigma(\partial_{yy}).$$

Moreover, each eigenfunction of (3.1) *can be written as*

$$\Phi_{j,k}(x, y) := \phi_j(x)\cos\left(\frac{k\pi y}{a}\right) \quad (j, k \in \mathbb{N}_0), \tag{3.5}$$

where $\phi_j(x)$ is an eigenfunction of $\widehat{L}$.

Using this proposition, we obtain a degenerate solution on $\mathcal{C}_R$ which is a candidate of a bifurcation point.

Lemma 3.2. *Let $\varepsilon > 0$ be small.* (3.1) *has a zero eigenvalue if and only if there exists $k \geqslant 1$ such that $\hat{\eta}_0(\lambda) - \zeta_k = 0$.*

Proof. Because of Proposition 3.1, each eigenvalue can be written $\hat{\eta}_j - \zeta_k$ $(j, k \in \mathbb{N}_0)$. When $\varepsilon > 0$ is small, $\hat{\eta}_1(\lambda) < 0$ (3.4). Since $\zeta_k \geqslant 0$ $(k \geqslant 0)$, $\hat{\eta}_j - \zeta_k = 0$ if and only if $j = 0$. □

From Lemma 3.2 it is important to study the behavior of $\hat{\eta}_0(\lambda)$ as $\lambda \to \infty$. In the next lemma we show that $\hat{\eta}_0(\lambda) \in C^1$. In general it is difficult to determine the sign of $\frac{d\hat{\eta}_0(\lambda)}{d\lambda}$. However, the following lemma, which is the main technical result of this section, shows that this sign is positive when λ is large.

Lemma 3.3. *Let $p \geqslant 2$. $\hat{\eta}_0(\lambda)$ is continuously differentiable in λ and the following holds*:

$$\frac{d\hat{\eta}_0(\lambda)}{d\lambda} = \frac{(p-1)(p+3)}{4} + o(1) \quad (\lambda \to \infty). \tag{3.6}$$

We postpone the proof of Lemma 3.3. We prove this lemma in Section 3.2 below.

Remark 3.4. (3.6) follows from a formal differentiation of $\hat{\eta}_0(\lambda)$ in (3.4).

Assuming Lemma 3.3, we obtain

Theorem 3.5. *Let $p \geqslant 2$. On the continuum $\mathcal{C}_R = \{(\lambda, u(\lambda))\}_{\lambda > \lambda_0}$ there are infinitely many symmetry breaking bifurcation points $\{(\lambda_k, u(\lambda_k))\}_{k > k_0}$, where $\lambda_k \to \infty$ as $k \to \infty$. Specifically, the closure of the non-one-dimensional solutions near $(\lambda_k, u(\lambda_k))$ is a curve, namely,*

$$\begin{aligned}
&\text{there are continuous functions } \lambda(\tau) : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R} \text{ and} \\
&\Psi(\tau) : (-\varepsilon_0, \varepsilon_0) \to L^2 \text{ such that } \lambda(0) = \lambda_k,\ \Psi(0) = 0 \text{ and the curve can be written as} \\
&\bigl\{\bigl(\lambda(\tau), u\bigl(\lambda(\tau)\bigr) + \tau\Phi_{0,k} + \tau\Psi(\tau)\bigr);\ \lambda(0) = \lambda_k,\ |\tau| \text{ is small}\bigr\}. \\
&\text{Here } \Phi_{0,k} \text{ is the eigenfunction defined by (3.5).}
\end{aligned} \tag{3.7}$$

Proof. Because of Lemma 3.3(i), there is a large $\tilde{\lambda} (> \lambda_0)$ such that $\frac{d\hat{\eta}_0(\lambda)}{d\lambda} > C_0 > 0$ for $\lambda > \tilde{\lambda}$. Because of this monotonicity of $\hat{\eta}_0$ and Lemma 3.2, there are infinitely many degenerate solutions on $\mathcal{C}_R$. We show that these degenerate solutions are symmetry breaking bifurcation points. Suppose that (3.1) has a zero eigenvalue at $\lambda_k > \tilde{\lambda}$. By Lemma 3.2 we can assume that $\hat{\eta}_0(\lambda_k) - \zeta_k = 0$. From the expression of each eigenfunction (3.5) it is clear that the zero eigenvalue is simple. There is a simple near-zero eigenvalue, which is $\hat{\eta}_0(\lambda) - \zeta_k$, if λ is close to λ_k. The transversality condition (1.7) holds, since

$$\left.\frac{d(\hat{\eta}_0(\lambda) - \zeta_k)}{d\lambda}\right|_{\lambda=\lambda_k} = \left.\frac{d\hat{\eta}_0(\lambda)}{d\lambda}\right|_{\lambda=\lambda_k} > 0.$$

It is clear that $\lambda_k \to \infty$ as $k \to \infty$. Let $E(\lambda, v) := \Delta(v + u(\lambda)) + \lambda f(v + u(\lambda))$ be the mapping from $H^2_N(R)$ to $L^2(R)$. We can apply the Crandall–Rabinowitz bifurcation theorem (Proposition 2.1) to $E(\lambda, v) = 0$ and obtain (3.7). □

Let us consider the curve of decreasing solutions of (3.2). By a phase plane analysis we see that this continuum is a curve and it emanates from the branch of constant solutions $\{(\lambda, 1)\}$. This curve may have a turning point. However another continuum does not bifurcate from the curve. It connects to the curve obtained in Proposition 2.2 and it can be described as a graph of λ provided that λ is large. If $a > 0$ is small, then ζ_1 is large, hence $\hat{\eta}_j(\lambda) - \zeta_1$ $(j = 0, 1, \dots, j_0)$ does not become 0 on a bounded portion of the curve. Therefore all the bifurcation points are on an unbounded portion which can be described by Proposition 2.2. Thus we have

Corollary 3.6. *Let $a > 0$ be small. Then every bifurcation point on the curve including $\mathcal{C}_R$ is on $\mathcal{C}_R$. Moreover, for each $k > 1$, there is a unique symmetry breaking bifurcation point $(\lambda_k, u(\lambda_k)) \in \mathcal{C}_R$ and at each bifurcation point $(\lambda_k, u(\lambda_k))$,* (3.7) *holds.*

Let $a > 0$ be small. Extending the bifurcating solution from $(\lambda_1, u(\lambda_1))$ with even reflection, we obtain an entire solution.

Corollary 3.7. *Let $a > 0$ be small and $p \geqslant 2$. Then $(\mathrm{N}_{\mathbb{R}^2})$ has a positive entire solution $u(x, y)$ such that u is periodic in x (resp. y) with period 2 (resp. $2a$) and u concentrates on $\{x = 2n;\ n \in \mathbb{Z}\}$.*

By (3.4) we obtain the Morse index of the concentrating solution on $\mathcal{C}_R$.

Corollary 3.8. *Let $\{(\lambda, u(\lambda))\}_{\lambda>\lambda_0} := \mathcal{C}_R$ and let $M(u(\lambda))$ denote the Morse index of $u(\lambda)$. Then*

$$\lim_{\lambda\to\infty} \frac{M(u(\lambda))^2}{\lambda} = \frac{(p-1)(p+3)a^2}{4\pi^2}.$$

In particular, $M(u(\lambda))$ diverges as $\lambda \to \infty$.

Let $D \in \mathbb{R}^{N-1}$ be a bounded domain and let $\Omega := D \times (0, 1)$. By Δ_D we denote the Neumann Laplacian on D. (N_Ω) has a branch $\mathcal{C}_\Omega$ consisting of one-dimensional solutions. If Δ_D has infinitely many simple eigenvalues, then by the same way we can show that $\mathcal{C}_\Omega$ has infinitely many symmetry breaking bifurcation points. For example, we let $D := (0, R_2) \times (0, R_3) \times \cdots \times (0, R_N)$, where $R_2^{-2}, R_3^{-2}, \ldots, R_N^{-2}$ are independent over $\mathbb{Q}$. Then every eigenvalue of Δ_D is simple. Moreover, extending this solution with even reflection, we obtain an entire solution on $\mathbb{R}^N$ that concentrates on $\{x_1 = 2n;\ n \in \mathbb{Z}\}$ and that is periodic in x_j $(j \in \{2, 3, \ldots, N\})$ with period $2R_j$, respectively when one of $R_2, \ldots, R_N$ is small.

Remark 3.9. In our study the monotonicity of the eigenvalue $\hat{\eta}_0(\lambda)$ plays a crucial rule. However, it seems that this monotonicity can be obtained in few cases. Wakasa [28] obtained this monotonicity for the decreasing solution of (3.2) for $f(u) = u - u^3$. Moreover, Wakasa and Yotsutani [29] obtained an exact expression of all eigenvalues of all solutions to (3.2) for $f(u) = \sin u$.

3.2. Proof of Lemma 3.3

We need two lemmas to prove Lemma 3.3.

Lemma 3.10. *Let ϕ $(\|\phi\|_\infty = \alpha_0,\ \phi > 0)$ be the first eigenfunction of $\widehat{L}$. Then*

$$\frac{1}{\varepsilon}\int_0^1 \phi^2\,dx \to \int_0^\infty w^{p+1}(\xi)\,d\xi \quad (\varepsilon \downarrow 0). \tag{3.8}$$

Proof. First we see that ϕ is also a first eigenfunction of (2.9). Because of (2.8) in Proposition 2.2, for an arbitrary small $\varepsilon > 0$, there are $C_0 > 0$ and a small $\delta_0 > 0$ independent of ε such that $0 < u(x) < \delta_0$ $(C_0\varepsilon \leqslant x \leqslant 1)$. Since $\eta_0(\varepsilon) \to \frac{(p-1)(p+3)}{4}$ $(\varepsilon \downarrow 0)$ and $f' < 0$ in $(0, \delta_0)$, there is $\delta_1 > 0$ independent of ε such that $f'(u(x)) - \eta_0 < -\delta_1$ $(C_0\varepsilon \leqslant x \leqslant 1)$. Since ϕ satisfies $\varepsilon^2\phi_{xx} + (f'(u) - \eta_0)\phi = 0$ and $0 < \phi \leqslant \alpha_0$, we can apply Proposition 2.8 to ϕ on $(C_0\varepsilon, 1]$. Then

$$0 < \phi(x) \leqslant \begin{cases} \alpha_0 & (0 \leqslant x \leqslant C_0\varepsilon), \\ C_1 e^{-C_2\frac{x}{\varepsilon}} & (C_0\varepsilon \leqslant x \leqslant 1). \end{cases}$$

Let $\tilde{\phi}(\xi) := \phi(x)$ $(\xi = \frac{x}{\varepsilon})$. There is $C_3 > 0$ such that

$$0 < \tilde{\phi}(\xi) < C_3 e^{-C_2\xi} \quad (0 \leqslant \xi \leqslant d_\varepsilon) \quad \text{and} \quad C_3 e^{-C_2\xi} \in L^2(0, \infty). \tag{3.9}$$

Let $\tilde{u}(\xi) := u(x)$. Since $\tilde{\phi}$ satisfies

$$\tilde{\phi}_{\xi\xi} + f'\big(\tilde{u}(\xi)\big)\tilde{\phi} = \eta_0\tilde{\phi} \quad \text{in } (0, d_\varepsilon), \qquad \tilde{\phi}_\xi = 0 \quad \text{at } x = 0, d_\varepsilon,\ \tilde{\phi} > 0,\ \|\tilde{\phi}\|_\infty = \alpha_0,$$

there is $C_4 > 0$ such that

$$\int_0^{d_\varepsilon} \tilde{\phi}_\xi^2\,d\xi = \int_0^{d_\varepsilon} \big(f'(\tilde{u}) - \eta_0\big)\tilde{\phi}^2\,d\xi \leqslant C_4 \int_0^{d_\varepsilon} \tilde{\phi}^2\,d\xi.$$

Combining the above inequality and (3.9), we see that there is $C_5 > 0$ independent of ε such that $\|\tilde{\phi}\|_{H^1(0,d_\varepsilon)} < C_5$. Since for any bounded interval I, $H^1(I) \subset C^\gamma(I)$ $(0 < \gamma < 1/2)$ is a continuous inclusion, there is $C_6 > 0$ independent of ε such that $\|\tilde{\phi}\|_{C^\gamma(I)} < C_6$. By the Ascoli–Arzelá's theorem we have that as $\varepsilon \downarrow 0$, $\tilde{\phi} \to \tilde{\phi}_*$ in $C^0_{\mathrm{loc}}[0,\infty)$, where $\tilde{\phi}_*$ is an eigenfunction of

$$\partial_{\xi\xi}\tilde{\phi}_* + f'(w)\tilde{\phi}_* = \eta_*\tilde{\phi}_* \quad \text{in } (0,\infty), \qquad \partial_\xi\tilde{\phi}_*(0) = 0, \quad \tilde{\phi}_* > 0,\ \|\tilde{\phi}_*\|_\infty = \alpha_0. \tag{3.10}$$

Here $\eta_* = \frac{(p-1)(p+3)}{4}$, because of (2.10). By Proposition 2.4 we have that $\tilde{\phi}_*(\xi) \equiv w^{\frac{p+2}{2}}(\xi)$. Thus

$$\tilde{\phi}(\xi) \xrightarrow{\varepsilon\downarrow 0} w^{\frac{p+1}{2}}(\xi) \quad \text{pointwisely in } [0,\infty). \tag{3.11}$$

Here we need not choose a subsequence in using Ascoli–Arzelá's theorem, because $\tilde{\phi}_*$ is uniquely determined by (3.10). Because of (3.9) and (3.11), the dominated convergence theorem tells us that

$$\frac{1}{\varepsilon}\int_0^1 \phi^2\,dx = \int_0^{d_\varepsilon} \tilde{\phi}^2(\xi)\,d\xi \to \int_0^\infty w^{p+1}(\xi)\,d\xi \quad (\varepsilon \downarrow 0). \qquad \square$$

Since $\mathcal{C}_R = \{(\lambda, u(\lambda))\}_{\lambda>\lambda_0}$ is a smooth curve, $u(\lambda)$ is differentiable for $\lambda > \lambda_0$. Let $u_\lambda(\lambda) := \frac{du(\lambda)}{d\lambda}$. Differentiating (3.2) in λ, we have $Lu_\lambda = -f(u)$. Since L is invertible, $u_\lambda = L^{-1}[-f(u)]$. In general it is difficult to estimate the term including u_λ. However, using Proposition 2.7, we can integrate that term for a small $\varepsilon > 0$ in the following lemma:

Lemma 3.11. *Let ϕ $(\|\phi\|_\infty = \alpha_0,\ \phi > 0)$ be the first eigenfunction of $\widehat{L}$. Then*

$$\frac{1}{\varepsilon}\int_0^1 \big(f'(u) + f''(u)\lambda u_\lambda\big)\phi^2\,dx \to \int_0^\infty \left(f'(w) + \frac{1}{2}f''(w)\xi w_\xi\right) w^{p+1}\,d\xi \quad (\varepsilon \downarrow 0). \tag{3.12}$$

Proof. Let $\tilde{\phi}(\xi) := \phi(x)$, $\tilde{u}(\xi) := u(x)$, $\tilde{u}_\lambda(\xi) := u_\lambda(x)$ $(\xi = \frac{x}{\varepsilon})$. In the proof of Lemma 3.10 we already see that (3.11) holds. It follows from Proposition 2.2 that there is $C_0 > 0$ independent of ε such that

$$0 < \tilde{u}(\xi) < C_0 \quad \text{and} \quad \tilde{u} \xrightarrow{\varepsilon\downarrow 0} w(\xi) \quad \text{pointwisely in } [0,\infty). \tag{3.13}$$

Therefore there is $C_1 > 0$ independent of ε such that

$$\big|f'(\tilde{u})\big| < C_1 \quad \text{and} \quad \big|f''(\tilde{u})\big| < C_1. \tag{3.14}$$

Moreover,

$$\begin{aligned} &f'\big(\tilde{u}(\xi)\big) \xrightarrow{\varepsilon\downarrow 0} f'\big(w(\xi)\big) \quad \text{pointwisely in } [0,\infty),\\ &f''\big(\tilde{u}(\xi)\big) \xrightarrow{\varepsilon\downarrow 0} f''\big(w(\xi)\big) \quad \text{pointwisely in } [0,\infty). \end{aligned} \tag{3.15}$$

We will show that λu_λ is bounded and that

$$\lambda\tilde{u}_\lambda(\xi) \xrightarrow{\varepsilon\downarrow 0} \frac{1}{2}\xi w_\xi(\xi) \quad \text{pointwisely in } [0,\infty). \tag{3.16}$$

Differentiating (3.2) in λ, we easily see that λu_λ satisfies

$$\big(\varepsilon^2\partial_{xx} + f'(u)\big)[\lambda u_\lambda] = -f(u) \quad \text{in } (0,1), \qquad \partial_x(\lambda u_\lambda) = 0 \quad \text{at } x = 0, 1.$$

Since L^{-1} exists and $\|L^{-1}\|_{\mathcal{B}(L^2,H^2_N)} < C_2$ (Proposition 2.3),

$$\|\lambda u_\lambda\|_\infty \leqslant C_3\|\lambda u_\lambda\|_{H^2} \leqslant C_3\big\|L^{-1}\big[f(u)\big]\big\|_{H^2} \leqslant C_3\big\|L^{-1}\big\|_{\mathcal{B}}\big\|f(u)\big\|_2 \leqslant C_2C_3\big\|f(u)\big\|_2.$$

It is clear that $\|f(u)\|_2$ is bounded uniformly in ε, hence

$$\|\lambda u_\lambda\|_\infty \quad \text{and} \quad \|\lambda\tilde{u}_\lambda\|_\infty \quad \text{are bounded uniformly in } \varepsilon. \tag{3.17}$$

It follows from Proposition 2.9 that there are $C_4 > 0$, $C_5 > 0$ such that

$$0 < \tilde{u}(\xi) < C_4e^{-C_5\xi} \quad (0 \leqslant \xi \leqslant \infty) \quad \text{and} \quad \tilde{u}(\xi) \xrightarrow{\varepsilon\downarrow 0} w(\xi) \quad \text{pointwisely in } [0,\infty). \tag{3.18}$$

Now $\lambda\tilde{u}_\lambda(\xi)$ satisfies

$$\big(\partial_{\xi\xi} + f'(u)\big)[\lambda\tilde{u}_\lambda] = -f(\tilde{u}) \quad \text{in } (0,d_\varepsilon), \qquad \partial_\xi(\lambda\tilde{u}_\lambda) = 0 \quad \text{at } \xi = 0, d_\varepsilon.$$

Multiplying the equation by $\lambda\tilde{u}_\lambda$ and integrating it, we have

$$\int_0^{d_\varepsilon}\big(\big\{\partial_\xi(\lambda\tilde{u}_\lambda)\big\}^2 + (\lambda\tilde{u}_\lambda)^2\big)\,d\xi = \int_0^{d_\varepsilon}\big(f(\tilde{u})\lambda\tilde{u}_\lambda + p\tilde{u}^{p-1}(\lambda\tilde{u}_\lambda)^2\big)\,d\xi.$$

Since $\|\lambda\tilde{u}_\lambda\|_\infty$ is bounded, by (3.18) we see that $f(\tilde{u})\lambda\tilde{u}_\lambda + p\tilde{u}^{p-1}(\lambda\tilde{u}_\lambda)^2$ is dominated by some integrable function independent of ε. The dominated convergence theorem tells us that there is $C_6 > 0$ independent of ε such that

$$\left|\int_0^{d_\varepsilon}\big(f(\tilde{u})\lambda\tilde{u}_\lambda + p\tilde{u}^{p-1}(\lambda\tilde{u}_\lambda)^2\big)\,d\xi\right| < C_6.$$

Using the continuous inclusion $H^1(I) \subset C^\gamma(I)$ $(0 < \gamma < 1/2)$, we see that for an arbitrary interval $I \subset [0,\infty)$, there is $C_7 > 0$ independent of ε such that $\|\lambda u_\lambda\|_{C^\gamma(I)} < C_7$. By Ascoli–Arzelá's theorem we have that $\lambda\tilde{u}_\lambda \xrightarrow{\varepsilon\downarrow 0} u_*$ in $C^0_{\mathrm{loc}}[0,\infty)$, where u_* satisfies

$$\big(\partial_{\xi\xi} + f'(w)\big)[u_*] = -f(w) \quad \text{in } (0,\infty), \qquad \partial_\xi u_*(0) = 0.$$

By Proposition 2.7 we see that $u_*(\xi) = \frac{1}{2}\xi w_\xi$. By (3.9), (3.11), (3.13), (3.14) and (3.17) we see that $|(f'(\tilde{u}) + f''(\tilde{u})\lambda\tilde{u}_\lambda)\tilde{\phi}^2|$ is dominated by some integrable function independent of ε. Using the dominated convergence theorem with (3.11), (3.15) and (3.16), we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_0^1\big(f'(u) + f''(u)\lambda u_\lambda\big)\phi^2\,dx \\
&\quad = \int_0^{d_\varepsilon}\big(f'(\tilde{u}) + f''(\tilde{u})\lambda\tilde{u}_\lambda\big)\tilde{\phi}^2\,d\xi \to \int_0^\infty\left(f'(w) + \frac{1}{2}f''(w)\xi w_\xi\right)w^{p+1}\,d\xi \quad (\varepsilon\downarrow 0).
\end{aligned}$$

The limits of $\tilde{u}$ and $\lambda\tilde{u}_\lambda$ are uniquely determined, respectively. We need not choose a subsequence. □

Proof of Lemma 3.3. Let $\phi_0 = \phi_0(\lambda)$ $(\|\phi_0\|_2 = 1)$ be an eigenfunction associated to $\hat{\eta}_0(\lambda)$. First we show that $\hat{\eta}_0(\lambda)$ and $\phi_0(\lambda)$ are continuously differentiable in λ. We define a mapping $\mathcal{G}: H^2_N \times \mathbb{R} \times \mathbb{R} \to L^2 \times \mathbb{R}$ by

$$\mathcal{G}(\phi,\hat{\eta},\lambda) := \begin{pmatrix} \widehat{L}\phi - \hat{\eta}\phi \\ \langle\phi,\phi\rangle - 1 \end{pmatrix}.$$

Differentiating $\mathcal{G}$ in $(\phi,\hat{\eta})$, we have

$$D_{(\phi,\hat{\eta})}\mathcal{G}(\phi_0,\hat{\eta}_0,\lambda_0) = \begin{pmatrix} \widehat{L} - \hat{\eta}_0 & -\phi_0 \\ 2\langle\phi_0,\cdot\rangle & 0 \end{pmatrix}.$$

For arbitrary $(\Psi, \tau) \in L^2 \times \mathbb{R}$, we consider

$$\begin{pmatrix} \widehat{L} - \hat{\eta}_0 & -\phi_0 \\ 2\langle \phi_0, \cdot \rangle & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \rho \end{pmatrix} = \begin{pmatrix} \Psi \\ \tau \end{pmatrix}.$$

Then we can easily check that $(\psi, \rho) = ((\widehat{L} - \hat{\eta}_0)^{-1}[\Psi - \langle \Psi, \phi_0 \rangle \phi_0] + \tau\phi_0/2, -\langle \Psi, \phi_0 \rangle)$ is the unique solution, using the fact that $\hat{\eta}_0$ is a simple eigenvalue. Hence $D_{(\phi,\hat{\eta})}\mathcal{G}(\phi_0, \hat{\eta}_0, \lambda_0)$ is invertible. By the implicit function theorem we see that $\hat{\eta}_0$ and ϕ_0 are continuously differentiable in λ.

Second, we prove (3.6). Hereafter by (η, ϕ) we denote the first eigenpair of $\widehat{L}$. Differentiating $\phi_{xx} + \lambda f'(u)\phi = \eta\phi$ in λ, we have

$$\phi_{xx\lambda} + f'(u)\phi + \lambda f''(u)u_\lambda\phi + \lambda f'(u)\phi_\lambda = \eta_\lambda\phi + \eta\phi_\lambda. \tag{3.19}$$

Multiplying (3.19) by ϕ and integrating it over [0, 1], we have

$$\int \big(\phi\phi_{xx\lambda} + f'(u)\phi^2 + \lambda f''(u)u_\lambda\phi^2 + \lambda f'(u)\phi\phi_\lambda\big)\,dx = \int \big(\eta_\lambda\phi^2 + \eta\phi\phi_\lambda\big)\,dx.$$

By integration by parts we have

$$\int \big(\phi_{xx}\phi_\lambda + f'(u)\phi^2 + \lambda f''(u)u_\lambda\phi^2 + \lambda f'(u)\phi\phi_\lambda\big)\,dx = \int \big(\eta_\lambda\phi^2 + \eta\phi\phi_\lambda\big)\,dx. \tag{3.20}$$

Multiplying $\phi_{xx} + \lambda f'(u)\phi = \eta\phi$ by ϕ_λ and integrating it, we have

$$\int \big(\phi_{xx}\phi_\lambda + \lambda f'(u)\phi\phi_\lambda\big)\,dx = \int \eta\phi\phi_\lambda\,dx. \tag{3.21}$$

Subtracting (3.21) from (3.20) and dividing it by ε, we have

$$\frac{\eta_\lambda}{\varepsilon}\int \phi^2\,dx = \frac{1}{\varepsilon}\int \big(f'(u) + f''(u)\lambda u_\lambda\big)\phi^2\,dx. \tag{3.22}$$

Because of Lemma 3.10, we have

$$\frac{1}{\varepsilon}\int_0^1 \phi^2\,dx = \int_0^\infty w^{p+1}\,d\xi + o(1) \quad (\varepsilon \downarrow 0). \tag{3.23}$$

By Lemma 3.11 we have

$$\frac{1}{\varepsilon}\int_0^1 \big(f'(u) + f''(u)\lambda u_\lambda\big)\phi^2\,dx = \int_0^\infty \left(f'(w) + \frac{1}{2}f''(w)\xi w_\xi\right)w^{p+1}\,d\xi + o(1) \quad (\varepsilon \downarrow 0). \tag{3.24}$$

Combining (3.24) and Proposition 2.5, we have

$$\frac{1}{\varepsilon}\int_0^1 \big(f'(u) + f''(u)\lambda u_\lambda\big)\phi^2\,dx = \frac{(p-1)(p+3)}{4}\int_0^\infty w^{p+1}\,d\xi + o(1) \quad (\varepsilon \downarrow 0). \tag{3.25}$$

Substituting (3.23) and (3.25) into (3.22), we have

$$\eta_\lambda\left(\int_0^\infty w^{p+1}\,d\xi + o(1)\right) = \frac{(p-1)(p+3)}{4}\int_0^\infty w^{p+1}\,d\xi + o(1) \quad (\varepsilon \downarrow 0). \tag{3.26}$$

For any sequence $\{\lambda_n\}$ ($\lambda_n \to \infty$), (3.26) holds, because we need not choose a subsequence in the proofs of Lemmas 3.10 and 3.11. Thus (3.6) holds. The proof is complete. □

The proof of Theorem 3.5 completes. □

4. $\Omega = A$

In Section 4 we consider (N_A). By $G : \mathbb{R} \times H^2_N(a, a+1) \to L^2(a, a+1)$ we define

$$G(\varepsilon, u) := \varepsilon^2\left(u_{rr} + \frac{N-1}{r}u_r\right) + f(u).$$

Then a radially symmetric solution to (N_A) is given by a solution to

$$G(\varepsilon, u) = 0. \tag{4.1}$$

Let $U(r;\varepsilon)$ denote the decreasing solution to

$$\varepsilon^2 U_{rr} - U + U^p = 0 \quad \text{in } (a, a+1), \qquad U_r = 0 \quad \text{at } r = a, a+1, \qquad U > 0,\ U_r < 0 \quad \text{in } (a, a+1). \tag{4.2}$$

Since the interval length is 1, (4.2) is equivalent to (2.1). The existence of $U(r;\varepsilon)$ is studied in Proposition 2.3 when $\varepsilon > 0$ is small. In Section 4.1 we will find a solution to (4.1) near $U(r;\varepsilon)$, using the contraction mapping theorem with properties studied in Section 2.2.

4.1. Construction of $\mathcal{C}_A$

Lemma 4.1. *Let $p \geqslant 2$. There is $\varepsilon_0 > 0$ such that* (4.1) *has a family of solutions $\{z(r;\varepsilon)\}_{0<\varepsilon<\varepsilon_0}$ consisting of solutions concentrating at $r = a$. Specifically, when $0 < \varepsilon < \varepsilon_0$, $z(r;\varepsilon)$ satisfies* (4.1) *and*

$$\left\| z(\cdot,\varepsilon) - U(\cdot,\varepsilon)\right\|_{H^2} < \varepsilon. \tag{4.3}$$

Proof. Let $\varepsilon > 0$ be small. Let G be as defined above. We will find a unique decreasing solution in a neighborhood of $U(r;\varepsilon)$. Specifically, we solve the equations $G(\varepsilon, U+v) = 0$, i.e.,

$$\varepsilon^2\left\{\partial_{rr}(U+v) + \frac{N-1}{r}\partial_r(U+v)\right\} - (U+v) + (U+v)^p = 0.$$

Then we have

$$\left(\varepsilon^2 v_{rr} - v + pU^{p-1}v\right) + \left\{(U+v)^p - U^p - pU^{p-1}v\right\} + \varepsilon^2\frac{N-1}{r}\partial_r(U+v) = 0. \tag{4.4}$$

Let $L := \varepsilon^2\partial_{rr} - 1 + pU^{p-1} \in \mathcal{B}(H^2_N(a,a+1), L^2(a,a+1))$. By Proposition 2.3 we see that L has the inverse and that $\|L^{-1}\|_{\mathcal{B}(L^2, H^2_N)}$ is uniformly bounded for $\varepsilon > 0$ small. Thus we set

$$\mathcal{F}(\varepsilon, v) := -L^{-1}\left[(U+v)^p - U^p - pU^{p-1}v + \varepsilon^2\frac{N-1}{r}\partial_r(U+v)\right].$$

Solving (4.4) is equivalent to finding the solutions of $v = \mathcal{F}(\varepsilon, v)$. We will solve this equation with the contraction mapping theorem. We let

$$B_\varepsilon := \left\{v \in H^2_N(a, a+1);\ v_r = 0 \text{ at } a, a+1,\ \|v\|_{H^2} \leqslant \varepsilon\right\}. \tag{4.5}$$

Note that for $v \in B_\varepsilon$, $\|v\|_\infty \leqslant C_0\varepsilon$, because of the continuous inclusion $H^2(a, a+1) \hookrightarrow L^\infty(a, a+1)$.

First, we show that if $\varepsilon > 0$ is small, then

$$\mathcal{F}(\varepsilon, \cdot) : B_\varepsilon \to B_\varepsilon. \tag{4.6}$$

Since $\|v\|_\infty < C_0\varepsilon$ and $p \geqslant 2$, for $|v|$ small,

$$\left|(U+v)^p - U^p - pU^{p-1}v\right| \leqslant C_1|v|^2.$$

Using this inequality, we have that if $v \in B_\varepsilon$, then

$$\begin{aligned}\left\|L^{-1}\left[(U+v)^p - U^p - pU^{p-1}v\right]\right\|_{H^2} &\leqslant \left\|L^{-1}\right\|_{\mathcal{B}}\left\|(U+v)^p - U^p - pU^{p-1}v\right\|_2 \\ &\leqslant C_2\left\|(U+v)^p - U^p - pU^{p-1}v\right\|_\infty \\ &\leqslant C_3\|v\|^2_\infty \\ &\leqslant C_4\|v\|^2_{H^2}.\end{aligned} \tag{4.7}$$

Since $U \xrightarrow{\varepsilon\downarrow 0} 0$ pointwisely on $(a, a+1]$ (Proposition 2.2) and $\|v\|_\infty \leqslant C_0\varepsilon$, the dominated convergence theorem says

$$\int_a^{a+1} \left(U^{p+1} - U^2\right) dr = o(1) \quad (\varepsilon \downarrow 0).$$

Since $\varepsilon^2 \int U_r^2\, dr = \int (U^{p+1} - U^2)\, dr = o(1)$,

$$\left\| \varepsilon^2 \frac{N-1}{r} U_r \right\|_2 \leqslant C_5 \varepsilon \|\varepsilon U_r\|_2 = o(\varepsilon) \quad (\varepsilon \downarrow 0). \tag{4.8}$$

Using (4.5), (4.7) and (4.8), we have that, for $v \in B_\varepsilon$,

$$\begin{aligned}
\left\|\mathcal{F}(\varepsilon, v)\right\|_{H^2} &\leqslant \left\|-L^{-1}\left[(U+v)^p - U^p - pU^{p-1}v\right]\right\|_{H^2} + \left\|-L^{-1}\left[\varepsilon^2 \frac{N-1}{r} \partial_r (U+v)\right]\right\|_{H^2} \\
&\leqslant C_4 \|v\|_{H^2}^2 + \left\|L^{-1}\right\|_{\mathcal{B}} \left\| \varepsilon^2 \frac{N-1}{r} \partial_r (U+v) \right\|_2 \\
&\leqslant C_4 \|v\|_{H^2}^2 + C_5 \left\| \varepsilon^2 \frac{N-1}{r} U_r \right\|_2 + C_5 \left\| \varepsilon^2 \frac{N-1}{r} v_r \right\|_2 \\
&\leqslant C_4 \varepsilon^2 + o(\varepsilon) + C_6 \varepsilon^2 \|v\|_{H^2} \\
&\leqslant \left(C_4 \varepsilon + o(1) + C_6 \varepsilon^2\right)\varepsilon.
\end{aligned}$$

Thus if $\varepsilon > 0$ is small, then (4.6) holds.

Second, we show that there is $K \in (-1, 1)$ such that for $v_0, v_1 \in B_\varepsilon$,

$$\left\|\mathcal{F}(\varepsilon, v_0) - \mathcal{F}(\varepsilon, v_1)\right\|_{H^2} \leqslant K \|v_0 - v_1\|_{H^2}. \tag{4.9}$$

For $v_0, v_1 \in B_\varepsilon$,

$$\begin{aligned}
&\left\| \left\{(U+v_0)^p - U^p - pU^{p-1}v_0\right\} - \left\{(U+v_1)^p - U^p - pU^{p-1}v_1\right\} \right\|_\infty \\
&\quad \leqslant \left\| p(U+v_1)^{p-1}(v_0 - v_1) + o(v_0 - v_1) - pU^{p-1}(v_0 - v_1) \right\|_\infty \\
&\quad \leqslant \left\| p\left\{(U+v_1)^{p-1} - U^{p-1}\right\} + o(1) \right\|_\infty \|v_0 - v_1\|_\infty.
\end{aligned} \tag{4.10}$$

Since $p \geqslant 2$,

$$\left\| p\left\{(U+v_1)^{p-1} - U^{p-1}\right\} + o(1) \right\|_\infty \to 0 \quad (\varepsilon \downarrow 0). \tag{4.11}$$

Using (4.10) and (4.11), we have

$$\begin{aligned}
&\left\|\mathcal{F}(\varepsilon, v_0) - \mathcal{F}(\varepsilon, v_1)\right\|_{H^2} \\
&\quad \leqslant \left\|-L^{-1}\right\|_{\mathcal{B}} \left\| \left\{(U+v_0)^p - U^p - pU^{p-1}v_0\right\} - \left\{(U+v_1)^p - U^p - pU^{p-1}v_1\right\} \right\|_2 \\
&\qquad + \left\|L^{-1}\right\|_{\mathcal{B}} \left\| \varepsilon^2 \frac{N-1}{r} \partial_r (U+v_0) - \varepsilon^2 \frac{N-1}{r} \partial_r (U+v_1) \right\|_2 \\
&\quad \leqslant C_7 \left\| \left\{(U+v_0)^p - U^p - pU^{p-1}v_0\right\} - \left\{(U+v_1)^p - U^p - pU^{p-1}v_1\right\} \right\|_\infty + C_7 \left\| \varepsilon^2 \frac{N-1}{r} \partial_r (v_0 - v_1) \right\|_2 \\
&\quad \leqslant o(1)\|v_0 - v_1\|_\infty + C_8 \varepsilon^2 \|v_0 - v_1\|_{H^2} \\
&\quad \leqslant o(1)\|v_0 - v_1\|_{H^2}.
\end{aligned}$$

Thus if $\varepsilon > 0$ is small, (4.9) holds for $v_0, v_1 \in B_\varepsilon$. By (4.6) and (4.9) we see that $\mathcal{F}$ is a contraction mapping on B_ε. The contraction mapping theorem tells us that there is $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $\mathcal{F}$ has a unique fixed point in B_ε. By $v(r; \varepsilon)$ we denote this solution of $\mathcal{F}(\varepsilon; U + v) = 0$. We define $z(r; \varepsilon) := U(r; \varepsilon) + v(r; \varepsilon)$. Then $\{z(r; \varepsilon)\}_{0<\varepsilon<\varepsilon_0}$ is a desired family of solutions concentrating at $r = a$. Since $v \in B_\varepsilon$, (4.3) holds. □

Let z_ε be a concentrating solution of (4.2) obtained in Lemma 4.1. By $\mathcal{L} \in \mathcal{B}(H^2_N(a,a+1), L^2(a,a+1))$ we define

$$\mathcal{L} := \varepsilon^2\left(\partial_{rr} + \frac{N-1}{r}\partial_r\right) + f'(z_\varepsilon). \tag{4.12}$$

In the next lemma we show that z_ε is nondegenerate in the space of radial functions, studying eigenvalues of $\mathcal{L}$.

Lemma 4.2. *There is $C_0 > 0$ such that the eigenvalue problem*

$$\mathcal{L}\phi = \kappa\phi \quad \text{in } (a,a+1), \qquad \phi_r = 0 \quad \text{at } r = a, a+1$$

does not have an eigenvalue in $[-C_0, C_0]$.

Proof. Let $C_0 > 0$ be small and let $\kappa \in [-C_0, C_0]$. Let $L := \varepsilon^2\partial_{rr} - 1 + pU^{p-1}$. We set

$$A := \varepsilon^2\frac{N-1}{r}\partial_r - p\left(z_\varepsilon^{p-1} - U^{p-1}\right).$$

The spectrum of the operator $\mathcal{L} = L + A \in \mathcal{B}(H^2_N, L^2)$ consists only of eigenvalues. We prove the lemma by contradiction. Let $\phi \in H^2_N$ ($\|\phi\|_{H^2} = 1$) be an eigenfunction. Since $(L + A - \kappa)\phi = 0$, we have $\phi = (L-\kappa)^{-1}[-A\phi]$. Since $\|(L-\kappa)^{-1}\|_{\mathcal{B}}$ is bounded (Proposition 2.3), we have

$$\|\phi\|_{H^2} = \left\|(L-\kappa)^{-1}[-A\phi]\right\|_{H^2} \leqslant \left\|(L-\kappa)^{-1}\right\|_{\mathcal{B}(L^2, H^2_N)}\|A\phi\|_2 \to 0 \quad (\varepsilon \downarrow 0).$$

This convergence is uniform in $\kappa \in [-C_0, C_0]$. We obtain a contradiction. □

By $u(r;\lambda)$ we define

$$u(r;\lambda) := z_\varepsilon(r) \quad \left(\lambda = \frac{1}{\varepsilon^2}\right). \tag{4.13}$$

Then there is a large $\lambda_0 > 0$ such that $\{u(r;\lambda)\}_{\lambda>\lambda_0}$ are concentrating solutions to (4.1).

Let $\mathcal{L}$ be defined in (4.12). By $\widehat{\mathcal{L}} \in \mathcal{B}(H^2_N, L^2)$ we define $\widehat{\mathcal{L}} := \lambda\mathcal{L}$, i.e.,

$$\widehat{\mathcal{L}} := \partial_{rr} + \frac{N-1}{r}\partial_r + \lambda f'(u).$$

Then κ is an eigenvalue of $\mathcal{L}$ if and only if $\frac{\kappa}{\varepsilon^2}$ is an eigenvalue of $\widehat{\mathcal{L}}$. Therefore $\widehat{\mathcal{L}}$ does not have an eigenvalue in $[-\frac{C_0}{\varepsilon^2}, \frac{C_0}{\varepsilon^2}]$, because of Lemma 4.2. $u(r;\lambda)$ is nondegenerate. Since u is unique in B_ε which is defined by (4.5), there is $\lambda_0 > 0$ such that $\{(\lambda, u(r;\lambda))\}_{\lambda>\lambda_0}$ is a smooth curve. We have

Corollary 4.3. *Let $p \geqslant 2$. There is a large $\lambda_0 > 0$ such that $\{(\lambda, u(r;\lambda))\}_{\lambda>\lambda_0}$ is a smooth curve.*

Lemma 4.4. *Let $u(r;\lambda)$ be a concentrating solution obtained in Lemma 4.1. Then $u(r;\lambda)$ is decreasing in r.*

Proof. First we show that $u_r \neq 0$ on $\{u = 1\}$. If $u_r = 0$ at some $r \in (a, a+1)$, then $u \equiv 1$ on $[a, a+1]$, because of the uniqueness of the solution to the ODE. It is a contradiction.

We see that if u has a critical point on $\{u > 1\}$ (resp. $\{u < 1\}$), then that point is a local maximum (resp. minimum) point of u, because $\varepsilon^2 u_{rr} = u - u^p < 0$ (resp. > 0). Therefore the critical points are only on the boundary $r = a, a+1$. If there is a critical point at an interior point, then since u satisfies the Neumann boundary condition, $\{u > 1\}$ or $\{u < 1\}$ has at least two connected components, which contradicts to (4.3). □

Let z be a solution obtained in Lemma 4.1 and let u be as defined by (4.13). Let $\mathcal{L}$ be as defined in (4.12). By $\{\kappa_j(\varepsilon)\}_{j\geqslant 0}$ we denote the eigenvalues of $\mathcal{L}$.

We show that the Morse index of z is one in the space of radial functions.

Lemma 4.5. *Let $\varepsilon > 0$ be small. Let $\kappa_0(\varepsilon)$, $\kappa_1(\varepsilon)$ denote the first and second eigenvalues of $\mathcal{L}$, respectively. If $\varepsilon > 0$ is small, then $\kappa_1(\varepsilon) < 0 < \kappa_0(\varepsilon)$.*

Proof. Let $\mathcal{L}_\delta := \varepsilon^2\partial_{rr} - 1 + pU^{p-1} + \delta\{\varepsilon^2\frac{N-1}{r}\partial_r + p(z^{p-1} - U^{p-1})\}$ $(0 \leqslant \delta \leqslant 1)$. Note that $\mathcal{L}_0 = L(= \varepsilon^2\partial_{rr} - 1 + pU^{p-1})$ and $\mathcal{L}_1 = \mathcal{L}$. We consider the eigenvalue problem

$$\mathcal{L}_\delta\phi = \zeta\phi \quad \text{in } (a, a+1), \qquad \phi_r = 0 \quad \text{at } r = a, a+1. \tag{4.14}$$

First we show that

$$\text{each eigenvalue continuously depends on } \delta. \tag{4.15}$$

By $\phi(r; \delta, \zeta)$ we denote the solution of

$$\mathcal{L}_\delta\phi = \zeta\phi \quad \text{in } (a, a+1), \qquad \phi(a) = 1, \quad \phi_r(a) = 0.$$

In particular, $\phi_{r\zeta}(a; \delta, \zeta) = 0$. It is clear that ζ is an eigenvalue if and only if $\phi_r(a+1; \delta, \zeta) = 0$. We show by contradiction that $\partial_\zeta\phi_r(a+1; \delta, \zeta) \neq 0$. Suppose that $\partial_\zeta\phi_r(a+1; \delta, \zeta) = 0$. Differentiating the equation in ζ, we have $\phi = \mathcal{L}_\delta\phi - \zeta\phi_\zeta$. Since $\partial_\zeta\phi_r(a; \delta, \zeta) = \partial_\zeta\phi_r(a+1; \delta, \zeta) = 0$ and $\int \mathcal{L}_\delta[\phi_0]\phi_1 r^{\delta(N-1)}\,dr = \int \psi_0\mathcal{L}_\delta[\phi_1]r^{\delta(N-1)}\,dr$ for $\phi_0, \phi_1 \in H^2_N$, we have

$$(0 \neq) \int_a^{a+1} \phi^2 r^{\delta(N-1)}\,dr = \int_a^{a+1} (\mathcal{L}_\delta - \zeta)[\phi_\zeta]\phi r^{\delta(N-1)}\,dr$$
$$= \int_a^{a+1} \phi_\zeta(\mathcal{L}_\delta - \zeta)[\phi]r^{\delta(N-1)}\,dx = 0,$$

which is a contradiction. Hence $\partial_\zeta\phi_r(a+1; \delta, \zeta) \neq 0$. The implicit function theorem tells us that $\zeta = \zeta(\delta)$ continuously depends on δ. Since $\phi_r(a+1; \delta, \zeta(\delta)) = 0$, we have proven (4.15).

Second, we show that

$$0 \text{ is not an eigenvalue of (4.14) for every } \delta \in [0, 1]. \tag{4.16}$$

Let $A_\delta := -\delta\{\varepsilon^2\frac{N-1}{r}\partial_r + p(z^{p-1} - U^{p-1})\}$. We can show by the same argument used in the proof of Lemma 4.2 that $L + A_\delta$ does not have an eigenvalue near zero. Thus $\mathcal{L}_\delta$ is invertible for every $\delta \in [0, 1]$. Therefore (4.16) holds.

Third, we prove the conclusion of the lemma. Let $\zeta_0(\delta)$, $\zeta_1(\delta)$ denote the first and second eigenvalues of (4.14), respectively. Note that every eigenvalue is simple, because of the one-dimensional problem. Since $\mathcal{L}_0 = L$, we see by Proposition 2.3 that $\zeta_1(0) = \eta_1(\varepsilon) < 0 < \eta_0(\varepsilon) = \zeta_0(0)$. Because of (4.15) and (4.16), $\zeta_1(\delta) < 0 < \zeta_0(\delta)$ for every $\delta \in [0, 1]$. $\square$

Combining Corollary 4.3 and Lemmas 4.4 and 4.5, we obtain

Theorem 4.6. *There is a large $\lambda_0 > 0$ such that (N_A) has a smooth curve $\mathcal{C}_A := \{(\lambda, u(\lambda))\}_{\lambda>\lambda_0}$ consisting of radially symmetric and radially decreasing solutions concentrating on the boundary $\{|x| = a\}$. Moreover, in the space of radial functions each solution u is nondegenerate and it has the Morse index* 1.

4.2. Symmetry breaking bifurcation

Let $\mathcal{C}_A$ be a smooth curve obtained in Theorem 4.6 and let $(\lambda, u(\lambda)) \in \mathcal{C}_A$. We consider the eigenvalue problem

$$\Delta\Phi + \lambda f'(u)\Phi = \mu\Phi \quad \text{in } A, \qquad \partial_\nu\Phi = 0 \quad \text{on } \partial A. \tag{4.17}$$

We find a degenerate solution on $\mathcal{C}_A$. In order to study the zero eigenvalue, we introduce the operator $\overline{\mathcal{L}} \in \mathcal{B}(H^2_N(A), L^2(A))$, $\overline{\mathcal{L}} := r^2(\Delta + \lambda f'(u))$, i.e.,

$$\overline{\mathcal{L}} := r^2\left(\partial_{rr} + \frac{N-1}{r}\partial_r + \frac{\Delta_{S^{N-1}}}{r^2} + \lambda f'(u)\right),$$

and $\widetilde{\mathcal{L}} \in \mathcal{B}(H^2_N(a, a+1), L^2(a, a+1))$, $\widetilde{\mathcal{L}} := \lambda r^2\mathcal{L}$, i.e.,

$$\widetilde{\mathcal{L}} := r^2\left(\partial_{rr} + \frac{N-1}{r}\partial_r + \lambda f'(u)\right).$$

Here $\Delta_{S^{N-1}}$ denotes the Laplace–Beltrami operator on the unit sphere S^{N-1}. We study the eigenvalues of $\overline{\mathcal{L}}$. The eigenvalues of $\overline{\mathcal{L}}$ can be described by the eigenvalues of $\widetilde{\mathcal{L}}$ and $\Delta_{S^{N-1}}$.

Proposition 4.7. *The following holds*:

$$\sigma(\overline{\mathcal{L}}) = \sigma(\widetilde{\mathcal{L}}) + \sigma(\Delta_{S^{N-1}}).$$

This proposition was observed by [9]. See also [16].

Let $\{\rho_k\}_{k\geqslant 0}$ denote the eigenvalues of $-\Delta_{S^{N-1}}$. It is well known that $\rho_k = k(k+N-2)$.

Let $\{\tilde{\nu}_j\}_{j\geqslant 0}$ denote the eigenvalues of $\widetilde{\mathcal{L}}$. Then we have

Proposition 4.8. (4.17) *has a zero eigenvalue if and only if there exists $k \geqslant 1$ such that*

$$\tilde{\nu}_0 - \rho_k = 0. \tag{4.18}$$

Moreover, the solution Φ can be written as

$$\Phi(x) = \tilde{\phi}_0\big(|x|\big) H_k\left(\frac{x}{|x|}\right),$$

where $\tilde{\phi}_0$ is the first positive eigenfunction of $\widetilde{\mathcal{L}}$ and $H_k(\theta)$ ($\theta \in S^{N-1}$) is the eigenfunction of $\Delta_{S^{N-1}}$ associated to the eigenvalue $-\rho_k$.

Proof. We easily see that (4.17) has a zero eigenvalue if and only if $\overline{\mathcal{L}}$ has a zero eigenvalue. It follows from Proposition 4.7 that $\overline{\mathcal{L}}$ has a zero eigenvalue if and only if $\tilde{\nu}_j - \rho_k = 0$ for some $j \geqslant 0$.

When $\varepsilon > 0$ is small, the second eigenvalue of $\widehat{\mathcal{L}}$ is negative (Lemma 4.5), since $\widehat{\mathcal{L}} = \lambda\mathcal{L}$. Comparing second eigenfunctions of $\widetilde{\mathcal{L}}$ and $\widehat{\mathcal{L}}$ with Sturm's comparison theorem, we see that the second eigenvalue of $\widetilde{\mathcal{L}}$ is negative, hence, for every $j \geqslant 1$, $\tilde{\nu}_j < 0$. Since $\rho_k > 0$, (4.18) holds. We omit the proof of the rest of the statements. □

The next lemma is the main technical lemma of this article.

Lemma 4.9. *Let $\tilde{\nu}_0(\lambda)$ be the first eigenvalue of $\widetilde{\mathcal{L}}$. Then $\tilde{\nu}_0(\lambda) \in C^1$ and*

$$\frac{d\tilde{\nu}_0(\lambda)}{d\lambda} = a^2\frac{(p-1)(p+3)}{4} + o(1) \quad (\lambda \to \infty). \tag{4.19}$$

We postpone the proof of Lemma 4.9.

Because of (4.19), $\tilde{\nu}_0(\lambda)$ diverges as $\lambda \to +\infty$. By (4.18) we see that eigenvalues pass 0 infinitely many times as $\lambda \to +\infty$. If we assume that Lemma 4.9 holds, then we can show that the transversality condition holds, using the Sturm's comparison theorem.

Lemma 4.10. *There is a large $\lambda_0 > 0$ such that the following holds*: *If there is $\lambda_* > \lambda_0$ such that $\tilde{\nu}_0(\lambda_*) - \rho_k = 0$ for some $k \geqslant 0$, then when λ is near λ_*, the eigenvalue problem*

$$\left(\widehat{\mathcal{L}} - \frac{\rho_k}{r^2}\right)\phi = \hat{\mu}\phi \quad \text{in } (a, a+1), \qquad \phi_r = 0 \quad \text{at } r = a, a+1$$

has a simple near-zero eigenvalue, $\hat{\mu}(\lambda)$, such that $\hat{\mu}(\lambda) \in C^1$ and $\hat{\mu}(\lambda_*) = 0$. Moreover,

$$\left.\frac{d\hat{\mu}}{d\lambda}\right|_{\lambda=\lambda_*} > 0. \tag{4.20}$$

Proof. By $\hat{\phi}(r;\lambda)$ ($\int_a^{a+1}\hat{\phi}^2 r^{N-1}\,dr = 1$, $\hat{\phi}(a;\lambda) > 0$) we denote the eigenfunction associated to $\hat{\mu}(\lambda)$. When $\lambda = \lambda_*$, both $\widehat{\mathcal{L}} - \frac{\rho_k}{r^2}$ and $\widetilde{\mathcal{L}} - \rho_k$ have a simple zero eigenvalue, hence $\hat{\mu}(\lambda_*) = 0$. By the same argument as in the proof of Lemma 3.3 we easily see that $\hat{\mu}(\lambda)$ and $\hat{\phi}(r;\lambda)$ are continuously differentiable in λ. Thus if λ is close to λ_*, then

$\hat{\phi}(r;\lambda)$ is positive and $\hat{\mu}(\lambda)$ is a simple near-zero eigenvalue. Next, we consider the first eigenvalue $\tilde{\mu}_0(\lambda)$ of the problem

$$(\widetilde{\mathcal{L}}-\rho_k)\phi=\tilde{\mu}\phi \quad \text{in } (a,a+1), \qquad \phi_r=0 \quad \text{at } r=a,a+1. \tag{4.21}$$

It is clear that $\tilde{\mu}_0(\lambda_*)=0$. Since λ is near λ_*, (4.21) has a simple near-zero eigenvalue which is $\tilde{\mu}_0(\lambda)=\tilde{\nu}_0(\lambda)-\rho_k$. By a standard argument using Sturm's comparison theorem we have, for $\lambda>\lambda_*$,

$$\frac{\tilde{\mu}_0(\lambda)}{(a+1)^2}\leqslant\hat{\mu}(\lambda)\leqslant\frac{\tilde{\mu}_0(\lambda)}{a^2}.$$

Therefore, for $\lambda>\lambda_*$,

$$\frac{1}{(a+1)^2}\frac{\tilde{\nu}_0(\lambda)-\rho_k}{\lambda-\lambda_*}\leqslant\frac{\hat{\mu}(\lambda)}{\lambda-\lambda_*}\leqslant\frac{1}{a^2}\frac{\tilde{\nu}_0(\lambda)-\rho_k}{\lambda-\lambda_*}.$$

Since $\hat{\mu}(\lambda)$ and $\tilde{\nu}_0(\lambda)$ are of class C^1, taking the limit $\lambda\downarrow 0$ yields

$$\frac{1}{(a+1)^2}\frac{d\tilde{\nu}_0}{d\lambda}\leqslant\left.\frac{d\hat{\mu}}{d\lambda}\right|_{\lambda=\lambda_*}\leqslant\frac{1}{a^2}\frac{d\tilde{\nu}_0}{d\lambda}. \tag{4.22}$$

We obtain (4.20). □

Remark 4.11. Let $(\lambda_*,u(\lambda_*))\in\mathcal{C}_A$ be a degenerate solution. If zero is a simple eigenvalue, then there is a near-zero eigenvalue, $\hat{\mu}(\lambda)$, when λ is near λ_*. By (4.22) and (4.19) we obtain the bound

$$\left(\frac{a}{a+1}\right)^2\frac{(p-1)(p+3)}{4}+o(1)\leqslant\left.\frac{d\hat{\mu}}{d\lambda}\right|_{\lambda=\lambda_*}\leqslant\frac{(p-1)(p+3)}{4}+o(1).$$

Smoller and Wasserman [25] showed that for arbitrary $k\geqslant 0$ the eigenspace of $-\Delta_{S^{N-1}}$ associated to ρ_k has a unique (up to multiples) eigenfunction that is $O(N-1)$ invariant. Thus each zero eigenvalue of (4.17) is simple in the space of $O(N-1)$ invariant functions. Using this simplicity, Lemma 4.10 and the same argument as in the proof of Theorem 3.5, we obtain

Theorem 4.12. *Let $p\geqslant 2$. On the continuum $\mathcal{C}_A=\{(\lambda,u(\lambda))\}_{\lambda>\lambda_0}$ there are infinitely many symmetry breaking bifurcation points $\{(\lambda_k,u(\lambda_k)\}_{\lambda>\lambda_0}$ from which continua consisting of nonradially symmetric solutions emanate.*

We restrict the functional space. Let

$$\mathcal{G}_h:=O(h)\times O(N-h), \quad 1\leqslant h\leqslant\left[\frac{N}{2}\right].$$

Here $[\frac{N}{2}]$ is the largest integer that is not greater than $\frac{N}{2}$. In [26] it was shown that if k is even then the eigenspace associated to ρ_k in the space of $\mathcal{G}_h$ invariant functions is one-dimensional. Moreover, the $\mathcal{G}_{h_1}$ invariant function and the $\mathcal{G}_{h_2}$ invariant function are distinct if $h_1\neq h_2$ and if two functions are nonradially symmetric. Thus a continuum obtained in Theorem 4.12 has $[\frac{N}{2}]$ distinct solutions provided that k is even.

It seems difficult to obtain all the nonradially symmetric solutions even locally. However, if $N=2$, we can obtain more information of the continuum.

Corollary 4.13. *Let $p\geqslant 2$ and $N=2$. Let $(\lambda_k,u(\lambda_k))$ be a symmetry breaking bifurcation point obtained in Theorem* 4.12*. The closure of the nonradially symmetric solution near $(\lambda_k,u(\lambda_k))$ is locally homeomorphic to a disk and it can be described as*

$$\mathcal{C}_k:=\bigcup_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}\big\{\big(\lambda,R_\theta(u)\big);\ (\lambda,u)\in\mathcal{C}_{k,e}\big\}, \tag{4.23}$$

where $R_\theta(u)$ is a counter-clockwise rotation of u by θ and $\mathcal{C}_{k,e}$ is a curve of nonradially symmetric solutions in the space of even functions with respect to x.

This is an immediate consequence of an abstract result [5]. However, we give an alternative proof, because results of [5] needs definitions and notations of Lie groups and manifolds.

Proof. We work on the space $X := \{u \in L^2;\ \langle u, \tilde{\phi}_0(|x|)\sin(k\frac{x}{|x|})\rangle = 0\}$. Then every eigenvalue of $-\Delta_{S^1}$ does not vanish and it becomes simple. Hence we can construct a curve $\mathcal{C}_{k,e}$, using Proposition 2.1. We can easily see that each solution in $\mathcal{C}_{k,e}$ is even in x, because $\mathcal{C}_{k,e}$ is the same curve obtained in the space of even functions with respect to x. The set $\mathcal{C}_k$ is in the solution set. We show that all nonradially symmetric solutions near $(\lambda_k, u(\lambda_k))$ are in $\mathcal{C}_k$. Suppose the contrary, there is a sequence of nonradially symmetric solutions $\{(\tilde{\lambda}_j, \tilde{u}_j)\}_{j\geqslant 0} \notin \mathcal{C}_k$ such that $(\tilde{\lambda}_j, \tilde{u}_j) \to (\lambda_k, u(\lambda_k))$ $(j \to \infty)$. Then for each $j > 0$ there is θ_j such that $R_{\theta_j}(\tilde{u}_j) \in X$. The local uniqueness of $\mathcal{C}_{k,e}$ indicates that $(\tilde{\lambda}_j, \tilde{u}_j) \in \mathcal{C}_{k,e}$ if j is large. This contradicts that $(\tilde{\lambda}_j, \tilde{u}_j) \notin \mathcal{C}_k$. It is clear from the shape of u that $\mathcal{C}_k$ is locally homeomorphic to a disk. $\square$

Remark 4.14. When $N = 2$, we easily see that the bifurcating solution $u(r,\theta)$ near the bifurcation point $(\lambda_k, u(\lambda_k))$ is periodic in θ with period $\frac{2\pi}{k}$.

4.3. Proof of Lemma 4.9

We need two lemmas to prove Lemma 4.9.

Lemma 4.15. *Let ϕ ($\|\phi\|_\infty = \alpha_0$, $\phi > 0$) be the first eigenfunction of $\widetilde{\mathcal{L}}$. Then*

$$\frac{1}{\varepsilon}\int_a^{a+1} \phi^2 r^{N-3}\,dr \to a^{N-3}\int_0^\infty w^{p+1}\,d\xi \quad (\varepsilon \downarrow 0). \tag{4.24}$$

Proof. The proof is almost the same as one of Lemma 3.10. We briefly prove (4.24). ϕ is also a first eigenfunction of $r^2\mathcal{L}$, i.e.,

$$r^2\mathcal{L}\phi = \kappa\phi \quad \text{in } (a, a+1), \qquad \phi_r = 0 \quad \text{at } r = a, a+1,\ \phi > 0.$$

Because of (4.3) and the positivity of κ (Lemma 4.5), there is $\delta_1 > 0$ such that $f'(u(r)) - \frac{\kappa}{r^2} < \delta_1$ $(a + C_0\varepsilon \leqslant r \leqslant a+1)$. Applying Proposition 2.8, we have

$$0 < \phi \leqslant \begin{cases} \alpha_0 & (a \leqslant r \leqslant a + C_0\varepsilon), \\ C_1 e^{-C_2\frac{r-a}{\varepsilon}} & (a + C_0\varepsilon \leqslant r \leqslant a+1). \end{cases}$$

Let $\tilde{\phi}(\xi) := \phi(r)$ ($\xi := \frac{r-a}{\varepsilon}$). Then there is $C_3 > 0$ such that

$$0 < \tilde{\phi}(\xi) < C_3 e^{-C_2\xi} \quad (0 \leqslant \xi \leqslant d_\varepsilon) \quad \text{and} \quad C_3 e^{-C_2\xi} \in L^2(0,\infty). \tag{4.25}$$

Let $\tilde{u}(\xi) := u(r)$. Since $\tilde{\phi}(\xi)$ satisfies

$$\tilde{\phi}_{\xi\xi} + \varepsilon\frac{N-1}{a+\varepsilon\xi}\tilde{\phi}_\xi + f'(\tilde{u})\tilde{\phi} = \frac{\kappa}{(a+\varepsilon\xi)^2}\tilde{\phi} \quad \text{in } (0, d_\varepsilon), \qquad \tilde{\phi}_r = 0 \quad \text{at } \xi = 0, d_\varepsilon, \quad \tilde{\phi} > 0,\ \|\tilde{\phi}\|_\infty = \alpha_0.$$

Multiplying the equation by $\tilde{\phi}\cdot(a+\varepsilon\xi)^{N-1}$ and integrating it, we have

$$\begin{aligned} a^{N-1}\int_0^{d_\varepsilon} \tilde{\phi}_\xi^2\,d\xi &< \int_0^{d_\varepsilon} \tilde{\phi}_\xi^2 (a+\varepsilon\xi)^{N-1}\,d\xi \\ &= \int_0^{d_\varepsilon} \left(f'(\tilde{u}) - \frac{\kappa}{(a+\varepsilon\xi)^2}\right)\tilde{\phi}^2 (a+\varepsilon\xi)^{N-1}\,d\xi \leqslant C_4\int_0^{d_\varepsilon} \tilde{\phi}^2\,d\xi. \end{aligned}$$

Since $\|\tilde{\phi}\|_{L^2(0,d_\varepsilon)} \leqslant \|C_3 e^{-C_2\xi}\|_{L^2(0,\infty)} < \infty$, by the above inequality we have that $\|\tilde{\phi}\|_{H^1(0,d_\varepsilon)} < C_5$. By the same argument as in the proof of Lemma 3.10 we see that as $\varepsilon \downarrow 0$, $\tilde{\phi} \to \tilde{\phi}_*$ in $C^0_{\mathrm{loc}}[0,\infty)$, where $\tilde{\phi}_*$ satisfies

$$\partial_{\xi\xi}\tilde{\phi}_* + f'(w)\tilde{\phi}_* = \frac{\kappa}{a^2}\tilde{\phi}_* \quad \text{in } (0,\infty), \qquad \partial_\xi\tilde{\phi}_*(0) = 0, \quad \tilde{\phi} > 0, \quad \|\tilde{\phi}\|_\infty = \alpha_0.$$

The rest of the proof is similar to Lemma 3.10. We omit the details. □

Since $\mathcal{C}_A = \{(\lambda, u(\lambda))\}_{\lambda>\lambda_0}$ is a smooth curve, $u(\lambda)$ is differentiable for $\lambda > \lambda_0$. Let $u_\lambda := \frac{du(\lambda)}{d\lambda}$.

Lemma 4.16. *Let ϕ ($\|\phi\|_\infty = \alpha_0$, $\phi > 0$) be the first eigenfunction of $\widetilde{\mathcal{L}}$. Then*

$$\frac{1}{\varepsilon}\int_a^{a+1} \big(f'(u) + \lambda f''(u)u_\lambda\big)\phi^2 r^{N-1}\,dr \to a^{N-1}\int_0^\infty \left(f'(w) + \frac{1}{2}f''(w)\xi w_\xi\right) w^{p+1}\,d\xi \quad (\varepsilon \downarrow 0).$$

Modifying the argument used in the proof of Lemma 3.11, we can show that the above limit is valid. We omit the proof.

Proof of Lemma 4.9. Let ϕ_0 ($\int_a^{a+1}\phi_0^2 r^{N-1}\,dr = 1, \phi_0 > 0$) be the eigenfunction associated to ν_0. By the same method used in Lemma 3.3 we can show that ϕ_0 and ν_0 are continuously differentiable in λ.

We prove (4.19). Hereafter by (ϕ, ν) we denote the first eigenpair for ease of notation. Differentiating $\widetilde{\mathcal{L}}\phi = \nu\phi$ in λ, we have

$$r^2\left(\phi_{rr\lambda} + \frac{N-1}{r}\phi_{r\lambda} + f'(u)\phi + \lambda f''(u)u_\lambda\phi + \lambda f'(u)\phi_\lambda\right) = \nu_\lambda\phi + \nu\phi_\lambda,$$

where we use the fact that u is differentiable in λ. Multiplying both sides by ϕr^{N-3} and integrating it over $[a, a+1]$, we have

$$\int\left(\phi_{rr\lambda}\phi + \frac{N-1}{r}\phi_{r\lambda}\phi + f'(u)\phi^2 + \lambda f''(u)u_\lambda\phi^2 + \lambda f'(u)\phi_\lambda\phi\right)r^{N-1}\,dr$$
$$= \int\big(\nu_\lambda\phi^2 + \nu\phi_\lambda\phi\big)r^{N-3}\,dr. \tag{4.26}$$

Multiplying $\widetilde{\mathcal{L}}\phi = \nu\phi$ by $\phi_\lambda r^{N-3}$ and integrating it over $[a, b]$, we have

$$\int\left(\phi_{rr}\phi_\lambda + \frac{N-1}{r}\phi_r\phi_\lambda + \lambda f'(u)\phi\phi_\lambda\right)r^{N-1}\,dr = \int \nu\phi\phi_\lambda r^{N-3}\,dr.$$

By integration by parts we have

$$\int\left(\phi_{rr\lambda}\phi + \frac{N-1}{r}\phi_{r\lambda}\phi + \lambda f'(u)\phi\phi_\lambda\right)r^{N-1}\,dr = \int \nu\phi\phi_\lambda r^{N-3}\,dr. \tag{4.27}$$

Subtracting (4.27) from (4.26) and dividing it by ε, we have

$$\frac{\nu_\lambda}{\varepsilon}\int \phi^2 r^{N-3}\,dr = \frac{1}{\varepsilon}\int\big(f'(u)\phi^2 + \lambda f''(u)u_\lambda\phi^2\big)r^{N-1}\,dr. \tag{4.28}$$

Because of Lemma 4.15, it follows from the dominated convergence theorem that

$$\frac{1}{\varepsilon}\int_a^{a+1} \phi^2 r^{N-3}\,dr = a^{N-3}\int_0^\infty w^{p+1}\,d\xi + o(1) \quad (\varepsilon \downarrow 0). \tag{4.29}$$

Because of Lemma 4.16, we have

$$\frac{1}{\varepsilon}\int_a^{a+1} \big(f'(u) + \lambda f''(u)u_\lambda\big)\phi^2 r^{N-1}\,dr$$
$$= a^{N-1}\int_0^\infty \left(f'(w) + \frac{1}{2}f''(w)\xi w_\xi\right)w^{p+1}\,d\xi + o(1) \quad (\varepsilon \downarrow 0). \tag{4.30}$$

Using Proposition 2.5, we have

$$\frac{1}{\varepsilon}\int_a^{a+1}\bigl(f'(u)\phi^2+\lambda f''(u)u_\lambda\phi^2\bigr)r^{N-1}\,dr=\frac{(p-1)(p+3)}{4}a^{N-1}\int_0^\infty w^{p+1}\,d\xi+o(1)\quad(\varepsilon\downarrow 0).\tag{4.31}$$

Substituting (4.29) and (4.31) into (4.16), we have

$$\nu_\lambda\left(a^{N-3}\int_0^\infty w^{p+1}\,d\xi+o(1)\right)=\frac{(p-1)(p+3)}{4}a^{N-1}\int_0^\infty w^{p+1}\,d\xi+o(1)\quad(\varepsilon\downarrow 0).\tag{4.32}$$

For any sequence $\{\lambda_n\}$ ($\lambda_n\to 0$), (4.32) holds, because we need not choose a subsequence in the proof of Lemmas 4.15 and 4.16. Thus (4.19) holds. The proof is complete. □

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