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# Existence results for mean field equations

by

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ABSTRACT. – Let $\Omega$ be an annulus. We prove that the mean field equation

$$-\Delta\psi = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}} \quad \text{in } \Omega$$
$$\psi = 0 \quad \text{on } \partial\Omega$$

admits a solution for $\beta \in (-16\pi, -8\pi)$. This is a supercritical case for the Moser-Trudinger inequality. © Elsevier, Paris

RÉSUMÉ. – On montre que l'équation de champ moyen

$$-\Delta\psi = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}} \quad \text{dans } \Omega$$
$$\psi = 0 \quad \text{sur } \partial\Omega,$$

pour $\Omega$ étant un anneau, admet une solution pour $\beta \in (-16\pi, -8\pi)$. Celà represente un cas supercritique pour l'inegalité de Moser-Trudinger. © Elsevier, Paris

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## 1. INTRODUCTION

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$. In this paper, we consider the following mean field equation

$$(1.1)\qquad \begin{aligned} -\Delta\psi &= \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, && \text{in } \Omega,\\ \psi &= 0, && \text{on } \partial\Omega, \end{aligned}$$

for $\beta \in (-\infty, +\infty)$. (1.1) is the Euler-Lagrange equation of the following functional

$$(1.2)\qquad J_\beta(\psi) = \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \frac{1}{\beta}\log\int_\Omega e^{-\beta\psi}$$

in $H_0^{1,2}(\Omega)$. This variational problem arises from Onsager's vortex model for turbulent Euler flows. In that interpretation, $\psi$ is the stream function in the infinite vortex limit, see [12,p256ff]. The corresponding canonical Gibbs measure and partition function are finite precisely if $\beta > -8\pi$. In that situation, Caglioti et al. [4] and Kiessling [9] showed the existence of a minimizer of $J_\beta$. This is based on the Moser-Trudinger inequality

$$(1.3)\qquad \frac{1}{2}\int_\Omega |\nabla\psi|^2 \geq \frac{1}{8\pi}\log\int_\Omega e^{-8\pi\psi}, \qquad \text{for any } \psi \in H_0^{1,2}(\Omega),$$

which implies the relevant compactness and coercivity condition for $J_\beta$ in case $\beta > -8\pi$. For $\beta \leq -8\pi$, the situation becomes different as described in [4]. On the unit disk, solutions blow up if one approaches $\beta = -8\pi$ -the critical case for (1.3)-(see also [5] and [19]), and more generally, on starshaped domains, the Pohozaev identity yields a lower bound on the possible values of $\beta$ for which solutions exist. On the other hand, for an annulus, [4] constructed radially symmetric solutions for any $\beta$, and the construction of Bahri-Coron [2] makes it plausible that solutions on domains with non-trivial topology exist below $-8\pi$. Thus, for $\beta \leq -8\pi$, $J_\beta$ is no longer compact and coercive in general, and the existence of solution depends on the geometry of the domain.

In the present paper, we thus consider the supercritical case $\beta < -8\pi$ on domains with non-trivial topology.

THEOREM 1.1. – *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain whose complement contains a bounded region, e.g. $\Omega$ an annulus. Then (1.1) has a solution for all $\beta \in (-16\pi, -8\pi)$.*

The solutions we find, however, are not minimizers of $J_\beta$-those do not exist in case $\beta < 8\pi$, since $J_\beta$ has no lower bound-but unstable critical points. Thus, these solutions might not be relevant to the turbulence problem that was at the basis of [4] and [9].

Certainly we can generalize Theorem 1.1 to the following equation

$$-\Delta\psi = \frac{Ke^{-\beta\psi}}{\int_\Omega Ke^{-\beta\psi}}, \quad \text{in } \Omega,$$
$$\psi = 0, \qquad \text{on } \partial\Omega,$$

which was studied in [5]. Here $K$ is a positive function on $\bar{\Omega}$.

With the same method, we may also handle the equation

$$\Delta u - c + cKe^u = 0, \qquad \text{for } 0 \le c < \infty \tag{1.4}$$

on a compact Riemann surface $\Sigma$ of genus at least 1, where $K$ is a positive function. (1.4) can also be considered as a mean field equation because it is the Euler-Lagrange equation of the functional

$$J_c(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 + c\int_\Sigma u - c\log\int_\Sigma Ke^u. \tag{1.5}$$

Because of the term $c\int_\Sigma u$, $J_c$ remains invariant under adding a constant to $u$, and therefore we may normalize $u$ by the condition

$$\int_\Sigma Ke^u = 1$$

which explains the absence of the factor $(\int Ke^u)^{-1}$ in (1.4). $c < 8\pi$ again is a subcritical case that can easily be handled with the Moser-Trudinger inequality. The critical case $c = 8\pi$ yields the so-called Kazdan-Warner equation [8] and was treated in [7] and [14] by giving sufficient conditions for the existence of a minimizer of $J_{8\pi}$. Here, we construct again saddle point type critical points to show

THEOREM 1.2. – *Let $\Sigma$ be a compact Riemann surface of positive genus. Then (1.4) admits a non-minimal solution for* $8\pi < c < 16\pi$.

Now we give a outline of the proof of the Theorems. First from the non-trivial topology of the domain, we can define a minimax value $\alpha_\beta$, which is bounded below by an improved Moser-Trudinger inequality, for $\beta \in (-16\pi, -8\pi)$. Using a trick introduced by Struwe in [16] and [17], for a certain dense subset $\Lambda \subset (-16\pi, -8\pi)$ we can overcome the lack of a coercivity condition and show that $\alpha_\beta$ is achieved by some $u_\beta$ for $\beta \in \Lambda$. Next, for any fixed $\bar{\beta} \in (-16\pi, -8\pi)$, considering a sequence $\beta_k \subset \Lambda$ tending to $\bar{\beta}$, with the help of results in [3] and [11] we show that $u_{\beta_k}$ subconverges strongly to some $u_{\bar{\beta}}$ which achieves $\alpha_{\bar{\beta}}$.

After completing our paper, we were informed that Struwe and Tarantello [18] obtained a non-constant solution of (1.4), when $\Sigma$ is a flat torus with fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, $K \equiv 1$ and $c \in (8\pi, 4\pi^2)$. In this case, it is easy to check that our solution obtained in Theorem 1.2 is non-constant.

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## 2. MINIMAX VALUES

Let $\rho = -\beta$ and $u = -\beta\psi$. We rewrite (1.1) as

$$\begin{aligned} -\Delta u &= \rho \frac{e^u}{\int_\Omega e^u}, && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

and (1.2) as

$$J_\rho(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \rho \log \int_\Omega e^u \tag{2.2}$$

for $u \in H_0^{1,2}(\Omega)$.

It is easy to see that $J_\rho$ has no lower bound for $\rho \in (8\pi, 16\pi)$. Hence, to get a solution of (1.1) for $\rho \in (8\pi, 16\pi)$, we have to use a minimax method. First, we define a center of mass of $u$ by

$$m_c(u) = \frac{\int_\Omega x e^u}{\int_\Omega e^u}.$$

Let $B$ be the bounded component of $\mathbb{R}^2 \setminus \Omega$. For simplicity, we assume that $B$ is the unit disk centered at the origin. Then we define a family of functions

$$h : D \to H_0^{1,2}(\Omega)$$

satisfying

$$\lim_{r\to 1} J_\rho(h(r,\theta)) \to -\infty \tag{2.3}$$

and

$$\lim_{r\to 1} m_c(h(r,\theta)) \text{ is a continuous curve enclosing } B. \tag{2.4}$$

Here $D = \{(r,\theta)|0 \le r < 1, \theta \in [0,2\pi)\}$ is the open unit disk. We denote the set of all such families by $\mathcal{D}_\rho$. It is easy to check that $\mathcal{D}_\rho \neq \emptyset$. Now we can define a minimax value

$$\alpha_\rho := \inf_{h\in\mathcal{D}_\rho} \sup_{u\in h(D)} J_\rho(u).$$

The following lemma will make crucial use of the non-trivial topology of $\Omega$, more precisely of the fact that the complement of $\Omega$ has a bounded component.

LEMMA 2.1. – *For any $\rho \in (8\pi, 16\pi)$ $\alpha_\rho > -\infty$.*

*Remark.* – It is an interesting question weather $\alpha_{16\pi} = -\infty$.

To prove Lemma 2.1, we use the improved Moser-Trudinger inequality of [6] (see also [1]). Here we have to modify a little bit.

LEMMA 2.2. –*Let $S_1$ and $S_2$ be two subsets of $\bar\Omega$ satisfying $dist(S_1, S_2) \ge \delta_0 > 0$ and $\gamma_0 \in (0, 1/2)$. For any $\epsilon > 0$, there exists a constant $c = c(\epsilon, \delta_0, \gamma_0) > 0$ such that*

$$\int_\Omega e^u \le c\exp\{\frac{1}{32\pi-\epsilon}\int_\Omega |\nabla u|^2 + c\}$$

*holds for all $u \in H_0^{1,2}(\Omega)$ satisfying*

$$\frac{\int_{S_1} e^u}{\int_\Omega e^u} \ge \gamma_0 \quad \text{and} \quad \frac{\int_{S_2} e^u}{\int_\Omega e^u} \ge \gamma_0. \tag{2.5}$$

*Proof.* – The Lemma follows from the argument in [6] and the following Moser-Trudinger inequality

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - 8\pi \log \int_\Omega e^u \ge c \tag{$*$}$$

for any $u \in H_0^{1,2}(\Omega)$, where $c$ is a constant independent of $u \in H_0^{1,2}(\Omega)$.□

We will discuss the inequality $(*)$ and its application in another paper.

*Proof of Lemma* 2.1. – For fixed $\rho \in (8\pi, 16\pi)$ we claim that there exists a constant $c_\rho$ such that

$$\sup_{u\in h(D)} J_\rho(u) \geq c_\rho, \qquad \text{for any } h \in \mathcal{D}_\rho. \tag{2.6}$$

Clearly (2.6) implies the Lemma. By the definition of $h$, for any $h \in \mathcal{D}_\rho$, there exists $u \in h(D)$ such that

$$m_c(u) = 0.$$

We choose $\epsilon > 0$ so small that $\rho < 16\pi - 2\epsilon$. Assume (2.6) does not hold. Then we have sequences $\{h_i\} \subset \mathcal{D}_\rho$ and $\{u_i\} \subset H_0^{1,2}(\Omega)$ such that $u_i \in h_i(D)$ and

$$m_c(u_i) = 0 \tag{2.7}$$

$$\lim_{i\to\infty} J(u_i) = -\infty. \tag{2.8}$$

We have the following Lemma.

LEMMA 2.3. –*There exists* $x_0 \in \bar{\Omega}$ *such that*

$$\lim_{i\to\infty} \frac{\int_{B_{1/2}(x_0)\cap\Omega} e^{u_i}}{\int_\Omega e^{u_i}} \to 1. \tag{2.9}$$

*Proof.* – Set

$$A(x) := \lim_{i\to\infty} \frac{\int_{B_{1/4}(x)\cap\Omega} e^{u_i}}{\int_\Omega e^{u_i}}.$$

Assume that the Lemma were false, then there exists $x_0 \in \bar{\Omega}$ such that

$$A(x_0) < 1 \quad \text{and} \quad A(x_0) \geq A(x) \quad \text{for any } x \in \Omega.$$

It is easy to check $A(x_0) > 0$, since $\Omega$ can be covered by finite many balls of radius $1/4$. Let $\gamma_0 = A(x_0)/2$. Recalling (2.8) and applying lemma 2.2, we obtain

$$\frac{\int_{\Omega\setminus B_{1/2}(x_0)} e^{u_i}}{\int_\Omega e^{u_i}} \to 0 \tag{2.10}$$

as $i \to \infty$, which implies (2.9). □

Now we continue to prove Lemma 2.1. (2.9) implies

$$\frac{\int_\Omega x e^{u_i}}{\int_\Omega e^{u_i}} - x_0 = \frac{\int_\Omega (x - x_0) e^{u_i}}{\int_\Omega e^{u_i}}$$
$$= \frac{\int_{B_{1/2}(x_0}(x - x_0) e^{u_i}}{\int_\Omega e^{u_i}} + o(1)$$

which, in turn, implies that $|m_c(u_i) - x_0| < 2/3$. This contradicts (2.7). □

LEMMA 2.4. – $\alpha_\rho/\rho$ *is non-increasing in* $(8\pi, 16\pi)$.

*Proof.* – We first observe that if $J(u) \leq 0$, then $\log \int_\Omega e^u > 0$ which implies that

$$J_\rho(u) \geq J_{\rho'}(u) \qquad \text{for } \rho' \geq \rho.$$

Hence $\mathcal{D}_\rho \subset \mathcal{D}_{\rho'}$ for any $16\pi > \rho' \geq \rho > 8\pi$. On the other hand, it is clear that

$$\frac{J_\rho}{\rho} - \frac{J_{\rho'}}{\rho'} = \frac{1}{2}(\frac{1}{\rho} - \frac{1}{\rho'}) \int_\Omega |\nabla u|^2 \geq 0,$$

if $\rho' \geq \rho$. Hence we have

$$\frac{\alpha_\rho}{\rho} \geq \frac{\alpha_{\rho'}}{\rho'}$$

for $16\pi > \rho' \geq \rho > 8\pi$. □

## 3. EXISTENCE FOR A DENSE SET

In this section we show that $\alpha_\rho$ is achieved if $\rho$ belongs to a certain dense subset of $(8\pi, 16\pi)$ defined below.

The crucial problem for our functional is the lack of a coercivity condition, *i.e.* for a Palais-Smale sequence $u_i$ for $J_\rho$, we do not know whether $\int_\Omega |\nabla u_i|^2$ is bounded.

We first have the following lemma.

LEMMA 3.1. –*Let* $u_i$ *be a Palais-Smale sequence for* $J_\rho$, *i.e.* $u_i$ *satisfies*

$$|J_\rho(u_i)| \leq c < \infty \tag{3.1}$$

*and*

$$(3.2)\qquad dJ_\rho(u_i) \to 0 \text{ strongly in } H^{-1,2}(\Omega).$$

*If, in addition, we have*

$$(3.3)\qquad \int_\Omega |\nabla u_i|^2 \le c_0, \qquad \text{for } i = 1, 2, \cdots$$

*for a constant $c_0$ independent of $i$, then $u_i$ subconverges to a critical point $u_0$ for $J_\rho$ strongly in $H_0^{1,2}(\Omega)$.*

*Proof.* – The proof is standard, but we provide it here for convenience of the reader.

Since $\int_\Omega |\nabla u_i|^2$ is bounded, there exists $u_0 \in H_0^{1,2}(\Omega)$ such that

(i) $u_i$ converges to $u_0$ weakly in $H_0^{1,2}(\Omega)$,

(ii) $u_i$ converges to $u_0$ strongly in $L^p(\Omega)$ for any $p > 1$ and almost everywhere,

(iii) $e^{u_i}$ converges to $e^{u_0}$ strongly in $L^p(\Omega)$ for any $p \ge 1$.

From (i)-(iii), we can show that $dJ(u_0) = 0$, *i.e.* $u_0$ satisfies

$$-\Delta u_0 = \rho \frac{e^{u_0}}{\int_\Omega e^{u_0}}.$$

Testing $dJ_\rho$ with $u_i - u_0$, we obtain

$$\begin{aligned} o(1) &= \langle dJ_\rho(u_i) - dJ_\rho(u), u_i - u_0 \rangle \\ &= \int_\Omega |\nabla(u_i - u_0)|^2 - \rho \int_\Omega \Big( \frac{e^{u_i}}{\int_\Omega e^{u_i}} - \frac{e^{u_0}}{\int_\Omega e^{u_0}} \Big)(u_i - u_0) \\ &= \int_\Omega |\nabla(u_i - u_0)|^2 + o(1), \end{aligned}$$

by (i)-(iii). Hence $u_i$ converges to $u_0$ strongly in $H_0^{1,2}(\Omega)$. □

Since by Lemma 2.4 $\rho \to \alpha_\rho/\rho$ is non-increasing in $(8\pi, 16\pi)$, $\rho \to \alpha_\rho/\rho$ is a.e. differentiable. Set

$$(3.4)\qquad \Lambda := \{\rho \in (8\pi, 16\pi) | \alpha_\rho/\rho \text{ is differentiable at } \rho\}.$$

$\bar{\Lambda} = [8\pi, 16\pi]$, see [16]. Let $\rho \in \Lambda$ and choose $\rho_k \nearrow \rho$ such that

$$(3.5)\qquad 0 \le \lim_{k\to\infty} -\frac{1}{(\rho - \rho_k)} \Big( \frac{\alpha_\rho}{\rho} - \frac{\alpha_{\rho_k}}{\rho_k} \Big) \le c_1$$

for some constant $c_1$ independent of $k$.

LEMMA 3.2. – *$\alpha_\rho$ is achieved by a critical point $u_\rho$ for $J_\rho$ provided that $\rho \in \Lambda$.*

*Proof.* – Assume, by contradiction, that the Lemma were false. From Lemma 3.1, there exists $\delta > 0$ such that

$$\|dJ_\rho(u)\|_{H^{-1,2}(\Omega)} \geq 2\delta \tag{3.6}$$

in

$$N_\delta := \{u \in H_0^{1,2}(\Omega) | \int_\Omega |\nabla u|^2 \leq c_2, |J_\rho(u) - \alpha_\rho| < \delta\}.$$

Here, $c_2$ is any fixed constant such that $N_\delta \neq \emptyset$. Let $X_\rho : N_\delta \to H_0^{1,2}(\Omega)$ be a pseudo-gradient vector field for $J_\rho$ in $N_\delta$, *i.e.* a locally Lipschitz vector field of norm $\|X_\rho\|_{H_0^{1,2}} \leq 1$ with

$$\langle dJ_\rho(u), X_\rho(u)\rangle < -\delta. \tag{3.7}$$

See [15] for the construction of $X_\rho$.

Since

$$\begin{aligned}\|dJ_\rho(u) - dJ_{\rho_k}(u)\| &= \|dJ_\rho - \frac{\rho}{\rho_k} dJ_{\rho_k}(u)\| + \|(1 - \frac{\rho}{\rho_k}) dJ_{\rho_k}(u)\| \\ &\leq \frac{1}{2}(1 - \frac{\rho}{\rho_k}) \int |\nabla u|^2 + c(1 - \frac{\rho}{\rho_k}) \int_\Omega |\nabla u|^2 \to 0\end{aligned}$$

uniformly in $\{u| \int_\Omega |\nabla u|^2 \leq c_2\}$, $X_\rho$ is also a pseudo-gradient vector field for $J_{\rho_k}$ in $N_\delta$ with

$$\langle dJ_{\rho_k}(u), X_\rho(u)\rangle < -\delta/2, \tag{3.8}$$

for $u \in N_\delta$, provided that $k$ is sufficiently large.

For any sequence $\{h_k\}$, $h_k \in \mathcal{D}_{\rho_k} \subset \mathcal{D}_\rho$ such that

$$\sup_{u \in h_k(D)} J_{\rho_k}(u) \leq \alpha_{\rho_k} + \rho - \rho_k \tag{3.9}$$

and all $u \in h_k(D)$ such that

$$J_\rho(u) \geq \alpha_\rho - (\rho - \rho_k), \tag{3.10}$$

we have the following estimate

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u|^2 &= \rho \cdot \rho_k \frac{\frac{J_{\rho_k}(u)}{\rho_k} - \frac{J_\rho(u)}{\rho}}{\rho - \rho_k} \\ &\leq \rho \cdot \rho_k \frac{\frac{\alpha_{\rho_k}}{\rho_k} - \frac{\alpha_\rho}{\rho}}{\rho - \rho_k} + (\rho + \rho_k) \\ &\leq C\end{aligned} \tag{3.11}$$

by (3.5), (3.9) and (3.10), where $C = (16\pi)^2 c_1 + 32\pi$.

Now we consider in $N_\delta$ the following pseudo-gradient flow for $J_\rho$. First choose a Lipschitz continuous cut-off function $\eta$ such that $0 \le \eta \le 1$, $\eta = 0$ outside $N_\delta$, $\eta = 1$ in $N_{\delta/2}$. Then consider the following flow in $H_0^{1,2}(\Omega)$ generated by $\eta X_\rho$

$$\frac{\partial \phi}{\partial t}(u,t) = \eta(\phi(u,t))X_\rho(\phi(u,t))$$
$$\phi(u,0) = u.$$

By (3.7) and (3.8), for $u \in N_{\delta/2}$, we have

$$\frac{d}{dt}J_\rho(\phi(u,t))_{|t=0} \le -\delta \tag{3.12}$$

and

$$\frac{d}{dt}J_{\rho_k}(\phi(u,t))_{|t=0} \le -\delta/2 \tag{3.13}$$

for large $k$.

It is clear that for any $h \in \mathcal{D}_{\rho_k}$ $h(r,\theta) \notin N_\delta$ for $r$ close to 1. Hence $\phi(h,t) \in \mathcal{D}_{\rho_k}$ for any $t > 0$. In particular, $\phi(\cdot,t)$ preserves the class of $h_k \in \mathcal{D}_{\rho_k}$ with condition (3.9). On the other hand, for any $h \in \mathcal{D}_\rho$ by definition

$$\sup_{u\in h(D)} J_\rho(u) \ge \alpha_\rho.$$

Hence for any $h_k \in \mathcal{D}_{\rho_k}$ with condition (3.9), $\sup_{u\in\phi(h(D),t)} J_\rho(u)$ is achieved in $N_{\delta/2}$, provided that $k$ is large enough. Consequently, by (3.12), we have

$$\frac{d}{dt}\sup\{J_\rho(u)|u \in \phi(h(D),t)\} \le -\delta$$

for all $t \ge 0$, which is a contradiction. □

## 4. PROOF OF THEOREM 1.1

From section 3, we know that for any $\bar\rho \in (8\pi, 16\pi)$ there exists a sequence $\rho_k \nearrow \bar\rho$ such that $\alpha_{\rho_k}$ is achieved by $u_k$. Consequently $u_k$ satisfies

$$\begin{aligned} -\Delta u_k &= \rho_k \frac{e^{u_k}}{\int_\Omega e^{u_i}}, \quad &&\text{in } \Omega,\\ u_k &= 0, &&\text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

From Lemma 2.4, we have

$$J_{\bar{\rho}}(u_k) = \alpha_{\rho_k} \leq c_0, \tag{4.2}$$

for some constant $c_0 > 0$ which is independent of $k$. Let $v_k = u_k - \log \int_\Omega e^{u_k}$. Then $v_k$ satisfies

$$-\Delta v_k = \rho_k e^{v_k} \tag{4.3}$$

with

$$\int_\Omega e^{v_k} = 1. \tag{4.4}$$

By results of Brezis-Merle [3] and Li-Shafrir [11] we have

LEMMA 4.1 ([3], [11]). *–There exists a subsequence (also denoted by $v_k$) satisfying one of the following alternatives:*

(i) *$\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$;*
(ii) *$v_k \to -\infty$ uniformly on any compact subset of $\Omega$;*
(iii) *there exists a finite blow-up set $\Sigma = \{a_1, \cdots, a_m\} \subset \Omega$ such that, for any $1 \leq i \leq m$, there exists $\{x_k\} \subset \Omega$, $x_k \to a_i$, $u_k(x_k) \to \infty$, and $v_k(x) \to -\infty$ uniformly on any compact subset of $\Omega \setminus \Sigma$. Moreover,*

$$\rho_k \int_\Omega e^{v_k} \to \sum_{i=1}^m 8\pi n_i \tag{4.5}$$

*where $n_i$ is positive integer.*

For our special functions $v_k$, we can improve Lemma 4.1 as follows

LEMMA 4.2. *–There exists a subsequence (also denoted by $v_k$) satisfying one of the following alternatives:*

(i) *$\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$;*
(ii) *$v_k \to -\infty$ uniformly on $\bar{\Omega}$;*
(iii) *there exists a finite blow-up set $\Sigma = \{a_1, \cdots, a_m\} \subset \bar{\Omega}$ such that, for any $1 \leq i \leq m$, there exists $\{x_k\} \subset \Omega$, $x_k \to a_i$, $u_k(x_k) \to \infty$, and $v_k(x) \to -\infty$ uniformly on any compact subset of $\bar{\Omega} \setminus \Sigma$. Moreover, (4.5) holds.*

*Proof.* – From Lemma 4.1, we only have to consider one more case in which blow-up points are in the boundary of $\Omega$. There are two possibilities: One is bubbling too fast such that after rescaling we obtain a solution of $-\Delta u = e^u$ in a half plane; Another is bubbling slow such that after rescaling we obtain a solution of $-\Delta u = e^u$ in $\mathbb{R}^2$. One can exclude the first case. In the second case, one can follow the idea in [11] to show that (4.5) holds. See also [10]. □

*Proof of Theorem* 1.1. – (4.4), (4.5) and $\bar\rho \in (8\pi, 16\pi)$ imply that cases (ii) and (iii) in Lemma 4.2 does not occur. Consequently $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$. Now we can again apply Lemma 2.2 as follows.

Let $S_1$ and $S_2$ be two disjoint compact subdomains of $\Omega$. Since $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega)$, we have

$$\frac{\int_{S_i} e^{u_k}}{\int_\Omega e^{u_k}} = \int_{S_i} e^{v_k} \geq c_0, \qquad i = 1, 2$$

for a constant $c_0 = c_0(S_1, S_2, \Omega) > 0$ independent of $k$. Choosing $\epsilon$ such that $16\pi - \bar\rho > 2\epsilon$ and applying Lemma 2.2, with the help of (4.2), we obtain

$$\begin{aligned} c \geq J_{\rho_k}(u_k) &= \frac{1}{2}\int_\Omega |\nabla u_k|^2 - \rho_k \log \int_\Omega e^{u_k} \\ &\geq \frac{1}{2}(1 - \frac{\rho_k}{16\pi - \epsilon/2}) \int_\Omega |\nabla u|^2 \\ &\geq \frac{1}{2}(1 - \frac{\bar\rho}{16\pi - \epsilon/2}) \int_\Omega |\nabla u|^2 \end{aligned}$$

which implies that $\int_\Omega |\nabla u_k|^2$ is bounded. Now by the same argument in the proof of Lemma 3.1, $u_k$ subconverges to $u_{\bar\rho}$ strongly in $H^{1,2}_0(\Omega)$ and $u_{\bar\rho}$ is a critical point of $J_{\bar\rho}$. Clearly, $u_{\bar\rho}$ achieves $\alpha_{\bar\rho}$. This finishes the proof of Theorem 1.1. □

*Proof of Theorem* 1.2. – Since the proof is very similar to one presented above, we only give a sketch of the proof of Theorem 1.2. Let $\Sigma$ be a Riemann surface of positive genus. We embed $X : \Sigma \to \mathbb{R}^N$ for some $N \geq 3$ and define the center of mass for a function $u \in H^{1,2}(\Sigma)$ by

$$m_c(u) = \frac{\int_\Sigma X e^u}{\int_\Sigma e^u}.$$

Since $\Sigma$ is of positive genus, we can choose a Jordan curve $\Gamma^1$ on $\Sigma$ and a closed curve $\Gamma^2$ in $\mathbb{R}^N \setminus \Sigma$ such that $\Gamma^1$ links $\Gamma^2$. We know that $\inf_{u \in H^{1,2}(\Sigma)} J_c(u)$ is finite if and only if $c \in [0, 8\pi]$ (see [7]). Now define a family of functions $h : D \to H^{1,2}(\Sigma)$ (as in section 2) satisfying

$$\lim_{r \to 1} J_\rho(h(r, \theta)) \to -\infty$$

and

$$\lim_{r \to 1} m_c(h(r,\theta)) \text{ as a map from } S^1 \to \Gamma^1 \text{ is of degree 1.}$$

Let $\mathcal{D}_c$ denote the set of all such families. It is also easy to check that $\mathcal{D}_c \neq \emptyset$. Set

$$\alpha_c := \inf_{h \in \mathcal{D}_c} \sup_{u \in h(D)} J_c(u).$$

We first have

$$\alpha_c > -\infty,$$

using the fact that $\Gamma^1$ links $\Gamma^2$ and Lemma 2.2. Then by the same method as presented above, we can prove that $\alpha_c$ is achieved by some $u_c \in H^{1,2}(\Sigma)$, which is a solution of (1.4), for $c \in (8\pi, 16\pi)$. □

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