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Nonequilibrium central limit theorem for a tagged particle in symmetric simple exclusion

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Abstract

We prove a nonequilibrium central limit theorem for the position of a tagged particle in the one-dimensional nearest neighbor symmetric simple exclusion process under diffusive scaling starting from a Bernoulli product measure associated to a smooth profile $\rho_0:\mathbb{R}\to[0,1]$.

Résumé

Nous démontrons un théorème central limite hors d'équilibre pour la position de la particule marquée dans l'exclusion simple symétriqué unidimensionelle à plus proche voisin en partant d'un produit de mesures de Bernoulli produit associée à un profil $\rho_0:\mathbb{R}\to[0,1]$ lisse.

MSC: 60G17

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1. Introduction

The asymptotic behavior of a tagged particle appears as one of the central problems in the theory of interacting particle systems and remains mostly unsolved.

The first important result on the position of a tagged particle in the diffusive scaling is due to Kipnis and Varadhan [4]. By proving an invariance principle for additive functionals of reversible Markov processes, Kipnis and Varadhan deduced an equilibrium central limit theorem for the position of a tagged particle in symmetric simple exclusion processes. This result was extended by Varadhan [11] for mean-zero asymmetric exclusion processes, through an invariance principle for Markov processes with generator satisfying a sector condition; and by Sethuraman, Varadhan

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and Yau [9] to asymmetric exclusion processes in dimension $d \geqslant 3$, relaxing the sector condition by a graded sector condition. In these three contexts the authors prove that

$$\frac{X_{tN^2} - E[X_{tN^2}]}{N}$$

converges in law, as $N \uparrow \infty$, to a Brownian motion with diffusion coefficient given by a variational formula. Here X_t stands for the position of the tagged particle at time t.

The nonequilibrium picture is much less clear. Even a law of large numbers for a tagged particle starting from a Bernoulli product measure with slowly varying parameter seems still out of reach. Rezakhanlou [7] proved a propagation of chaos result which states that the average behavior of tagged particles is described by diffusion process. A large deviations from this diffusive limit in dimension $d \geqslant 3$ was obtained by Quastel, Rezakhanlou and Varadhan [5].

We prove in this article the first nonequilibrium central limit theorem for a tagged particle. Consider the one-dimensional nearest neighbor symmetric situation. In this context, as already observed by Arratia [1], the scaling changes dramatically since to displace the tagged particle from the origin to a site $N > 0$, all particles between the origin and N need to move to the right of N. This observation relates the asymptotic behavior of the tagged particle to the hydrodynamic behavior of the system. The correct scaling for the law of large numbers should therefore be X_{tN^2}/N and we expect $(X_{tN^2} - E[X_{tN^2}])/\sqrt{N}$ to converge to a Gaussian variable.

The central limit theorem in equilibrium was obtained by Rost and Vares [8] for a slightly different model. They proved that for each fixed $t > 0$, $X_{tN^2}/\sqrt{N}$ converges to a fractional Brownian motion W_t with variance given by $E[W_t^2] = \alpha t^{1/2}$. We extend their result to the nonequilibrium case.

The idea of the proof is to relate the position of the tagged particle to the well known hydrodynamic behavior of the symmetric exclusion process. Since particles cannot jump over other particles, the position of the tagged particle is determined by the current over one bond and the density profile of particles. Therefore, a nonequilibrium central limit theorem for the position of the tagged particle follows from a joint central limit theorem for the current and the density profile. Since the current over a bond can itself, at least formally, be written as the difference between the mass at the right of the bond at time t and the mass at time 0, a central limit theorem for the position of the tagged particle should follow from a nonequilibrium central limit theorem for the density field. This is the content of the article.

This general method permits to deduce a nonequilibrium central limit theorem for the tagged particle for one-dimensional nearest neighbor systems from the nonequilibrium fluctuations of the current and sharp estimates on the two point space-time correlation functions.

There are three main ingredients in the proof. In Section 3 we present a nonequilibrium central limit theorem for the current over a bond and show how it relates to the fluctuations of the density field. In Section 5 we obtain a formula which relates the position of the tagged particle to the current over one bond and the density field. Finally, in Appendix A we present a sharp estimate on the difference of the solution of the hydrodynamic equation and the solution of a discretized version of the hydrodynamic equation.

2. Notation and results

The nearest neighbor one-dimensional symmetric exclusion process is a Markov process on $\{0,1\}^{\mathbb{Z}}$ which can be described as follows. Particles are initially distributed over $\mathbb{Z}$ in such a way that each site is occupied by at most one particle. A particle at a site x waits for an exponential time and then jumps to $x \pm 1$ provided the site is vacant. Otherwise the jump is suppressed and the process starts again.

The state space of this Markov process is denoted by $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ and the configurations by the Greek letter η, so that $\eta(x) = 1$ if site x is occupied for the configuration η and 0 otherwise. The generator L_N of the process speeded up by N^2 is given by

$$(L_N f)(\eta) = N^2 \sum_{x \in \mathbb{Z}} \left[f\left(\sigma^{x,x+1}\eta\right) - f(\eta) \right],$$

where $\sigma^{x,x+1}\eta$ is the configuration obtained from η by interchanging the occupation variables $\eta(x)$ and $\eta(x+1)$:

$$\left(\sigma^{x,x+1}\eta\right)(z) = \begin{cases} \eta(x+1) & \text{if } z = x, \\ \eta(x) & \text{if } z = x+1, \\ \eta(z) & \text{otherwise.} \end{cases}$$

For each configuration η, denote by $\pi(\eta)$ the positive measure on $\mathbb{R}$ obtained by assigning mass N^{-1} to each particle:

$$\pi(\eta) = N^{-1} \sum_{x \in \mathbb{Z}} \eta(x) \delta_{x/N}$$

and let $\pi_t = \pi(\eta_t)$.

Fix a profile $\rho_0 : \mathbb{R} \to [0,1]$ with the first four derivatives limited. Denote by $\nu^N_{\rho_0(\cdot)}$ the product measure on $\mathcal{X}$ associated to ρ_0:

$$\nu^N_{\rho_0(\cdot)}\{\eta, \eta(x) = 1\} = \rho_0(x/N)$$

for x in $\mathbb{Z}$. For each $N \geqslant 1$ and each measure μ on $\mathcal{X}$, denote by $\mathbb{P}_\mu$ the probability on the path space $D(\mathbb{R}_+, \mathcal{X})$ induced by the measure μ and the Markov process with generator L_N. Expectation with respect to $\mathbb{P}_\mu$ is denoted by $\mathbb{E}_\mu$. Note that we omitted the dependence of the probability $\mathbb{P}_\mu$ on N to keep notation simple. This convention is adopted below for several other quantities which also depend on N. The hydrodynamic behavior of the symmetric simple exclusion process is well known and described by the heat equation.

Theorem 2.1. *Fix a profile $\rho_0 : \mathbb{R} \to [0,1]$. Then, for all time $t \geqslant 0$, under $\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}$ the sequence of random measures π_t converges in probability to the absolutely continuous measure $\rho(t,u)\,\mathrm{d}u$ whose density ρ is the solution of the heat equation with initial condition ρ_0:*

$$\begin{cases} \partial_t \rho = \Delta \rho, \\ \rho(0,\cdot) = \rho_0(\cdot). \end{cases} \tag{2.1}$$

Here and below, Δ stands for the Laplacian.

This theorem establishes a law of large numbers for the empirical measure. To state the central limit theorem some notation is required. For $k \geqslant 0$, denote by $\mathcal{H}_k$ the Hilbert space induced by smooth rapidly decreasing functions and the scalar product $\langle \cdot, \cdot \rangle_k$ defined by

$$\langle f, g \rangle_k = \langle f, (x^2 - \Delta)^k g \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $\mathbb{R}^d$. Notice that $\mathcal{H}_0 = L^2(\mathbb{R}^d)$ and denote by $\mathcal{H}_{-k}$ the dual of $\mathcal{H}_k$.

Let $\rho^N_t(x) = \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[\eta_t(x)]$. A trivial computation shows that $\rho^N_t(x)$ is the solution of the discrete heat equation:

$$\begin{cases} \partial_t \rho^N_t(x) = \Delta_N \rho^N_t(x), \\ \rho^N_0(x) = \rho_0(x/N), \end{cases} \tag{2.2}$$

where $(\Delta_N h)(x) = N^2 \sum_{y, |y-x|=1} [h(y) - h(x)]$.

Fix $k \geqslant 4$ and denote by $\{Y^N_t,\ t \geqslant 0\}$ the so-called density field, a $\mathcal{H}_{-k}$-valued process given by

$$Y^N_t(G) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} G(x/N) \{\eta_t(x) - \rho^N_t(x)\}$$

for G in $\mathcal{H}_k$. Denote by Q_N the probability measure on the path space $D(\mathbb{R}_+, \mathcal{H}_{-k})$ induced by the process Y^N_t and the measure $\nu^N_{\rho_0(\cdot)}$. Next result is due to Galves, Kipnis and Spohn [3] in dimension 1 and to Ravishankar [6] in dimension $d \geqslant 2$.

Theorem 2.2. *The sequence Q_N converges to Q, the probability measure concentrated on $C(\mathbb{R}_+, \mathcal{H}_{-k})$ corresponding to the Ornstein–Uhlenbeck process Y_t with mean zero and covariance given by*

$$\mathbb{E}[Y_t(H) Y_s(G)] = \int_{\mathbb{R}} (T_{t-s} H) G \chi_s - \int_0^s \mathrm{d}r \int_{\mathbb{R}} (T_{t-r} H)(T_{s-r} G) \{\partial_r \chi_r - \Delta \chi_r\}$$

for $0 \leqslant s < t$ and $G, H \in \mathcal{H}_k$. In this formula, $\{T_t : t \geqslant 0\}$ stands for the semigroup associated to the Laplacian and χ_s for the function $\chi(s,u) = \rho(s,u)[1 - \rho(s,u)]$.

Note that in the case of the heat equation, $\partial_r \chi_r - \Delta \chi_r = 2(\partial_x \rho)^2$. Also, in the equilibrium case, χ is constant in space and time so that the second term vanishes and we recover the equilibrium covariances. Finally, integrating by parts twice the expression with $\Delta \chi_r$, we rewrite the limiting covariances as

$$\mathbb{E}\big[Y_t(H)Y_s(G)\big] = \int_{\mathbb{R}} (T_t H)(T_s G)\chi_0 + 2\int_0^s \mathrm{d}r \int_{\mathbb{R}} (\nabla T_{t-r} H)(\nabla T_{s-r} G)\chi_r, \tag{2.3}$$

where ∇f is the space derivative of f.

We examine in this article nonequilibrium central limit theorems for the current through a bond and the position of a tagged particle. For a bond $(x, x+1)$, denote by $J_{x,x+1}(t)$ the current over this bond. This is the total number of jumps from site x to site $x+1$ in the time interval $[0, t]$ minus the total number of jumps from site $x+1$ to site x in the same time interval.

Theorem 2.3. *Fix u in $\mathbb{R}$ and let*

$$Z_t^N = \frac{1}{\sqrt{N}}\big\{J_{x_N,x_N+1}(t) - E_{\nu^N_{\rho_0(\cdot)}}\big[J_{x_N,x_N+1}(t)\big]\big\},$$

where $x_N = [uN]$. Then, for every $k \geqslant 1$ and every $0 \leqslant t_1 < \cdots < t_k$, $(Z_{t_1}^N, \ldots, Z_{t_k}^N)$ converges in law to a Gaussian vector $(Z_{t_1}, \ldots, Z_{t_k})$ with covariance given by

$$E[Z_s Z_t] = \int_{-\infty}^{0} \mathrm{d}v\, P[B_s \leqslant v]P[B_t \leqslant v]\chi_0(v) + \int_0^{\infty} \mathrm{d}v\, P[B_s \geqslant v]P[B_t \geqslant v]\chi_0(v)$$
$$+ 2\int_0^s \mathrm{d}r \int_{-\infty}^{\infty} \mathrm{d}v\, p_{t-r}(0, v)p_{s-r}(0, v)\chi_r(v)$$

provided $s \leqslant t$ and $u = 0$. In this formula, B_t is a standard Brownian motion starting from the origin and $p_t(v, w)$ is the Gaussian kernel.

By translation invariance, in the case $u \neq 0$, we just need to translate χ by $-u$ in the covariance.

Let $H_0 = \mathbf{1}\{[0, \infty)\}$. The covariance appearing in the previous theorem is easy to understand. Formally the current $N^{-1/2}J_{-1,0}(t)$ centered by its mean corresponds to $Y_t^N(H_0) - Y_0^N(H_0)$ since both processes increase (resp. decrease) by $N^{-1/2}$ whenever a particle jumps from -1 to 0 (resp. 0 to -1). The limiting covariance $E[Z_s Z_t]$ corresponds to the formal covariance

$$E\big[\big\{Y_t(H_0) - Y_0(H_0)\big\}\big\{Y_s(H_0) - Y_0(H_0)\big\}\big].$$

Denote by $\nu^{N,*}_{\rho_0(\cdot)}$ the measure $\nu^N_{\rho_0(\cdot)}$ conditioned to have a particle at the origin.

Remark 2.4. The law of large numbers and the central limit theorem for the empirical measure and for the current starting from $\nu^{N,*}_{\rho_0(\cdot)}$ follow from the law of large numbers and the central limit theorem for the empirical measure and the current starting from the measure $\nu^N_{\rho_0(\cdot)}$ since we may couple both processes in such a way that they differ at most at one site at any given time.

Fix a profile ρ_0 with the first four derivatives limited, and consider the product measure $\nu^{N,*}_{\rho_0(\cdot)}$. Denote by X_t the position at time $t \geqslant 0$ of the particle initially at the origin. A law of large numbers for X_t follows from the hydrodynamic behavior of the process:

Theorem 2.5. *Fix $t \geqslant 0$. X_t/N converges in $\mathbb{P}_{\nu^{N,*}_{\rho_0(\cdot)}}$-probability to u_t, the solution of*

$$\dot{u}_t = -\frac{(\partial_u \rho)(t, u_t)}{\rho(t, u_t)}.$$

Note that the solution of the previous equation is given by

$$\int_0^{u_t} \mathrm{d}u\, \rho(t,u) = -\int_0^t \mathrm{d}s (\partial_u \rho)(s,0).$$

Theorem 2.6. *Assume that ρ_0 has a bounded fourth derivative. Let $W_t = N^{-1/2}\ (X_t - Nu_t)$. Under $\mathbb{P}_{\nu^{N,*}_{\rho_0(\cdot)}}$, for every $k \geqslant 1$ and every $0 \leqslant t_1 < \cdots < t_k$, $(W^N_{t_1}, \ldots, W^N_{t_k})$ converges in law to a Gaussian vector $(W_{t_1}, \ldots, W_{t_k})$ with covariance given by*

$$\rho(s,u_s)\rho(t,u_t)E[W_s W_t] = \int_{-\infty}^0 \mathrm{d}v P_{u_s}[B_s \leqslant v] P_{u_t}[B_t \leqslant v]\chi_0(v) + \int_0^\infty \mathrm{d}v P_{u_s}[B_s \geqslant v] P_{u_t}[B_t \geqslant v]\chi_0(v)$$
$$+ 2\int_0^s \mathrm{d}r \int_{-\infty}^\infty \mathrm{d}v\, p_{t-r}(u_t, v) p_{s-r}(u_s, v)\chi_r(v).$$

In this formula, P_u stands for the probability corresponding to a standard Brownian motion starting from u.

The assumption made on the smoothness of ρ_0 appears because in the proof of Theorem 2.6 we need a sharp estimate on the difference of the discrete approximation of the heat equation (2.2) and the heat equation (2.1). In Appendix A we show that there exists a finite constant C_0 for which $|\rho^N_t(x) - \rho(t, x/N)| \leqslant C_0 t N^{-2}$ for all $N \geqslant 1$, x in $\mathbb{Z}$ and $t \geqslant 0$ under the assumption that ρ_0 has a bounded fourth derivative.

3. Nonequilibrium fluctuations of the current

Suppose for a moment that the profile ρ_0 has a compact support. Then, η_0 is almost surely a configuration with a finite number of particles, and it is easy to see that we have a simple formula for the current $J_{-1,0}(t)$:

$$J_{-1,0}(t) = \sum_{x \geqslant 0} \eta_t(x) - \eta_0(x). \tag{3.1}$$

In particular, we can write $J_{-1,0}(t)$ in terms of the fluctuation field:

$$\frac{1}{\sqrt{N}}\left\{J_{-1,0}(t) - E_{\nu^N_{\rho_0(\cdot)}}\left[J_{-1,0}(t)\right]\right\} = Y^N_t(H_0) - Y^N_0(H_0),$$

where H_a is the indicator function of the interval $[a, \infty)$:

$$H_a(u) = \mathbf{1}\left\{[a,\infty)\right\}(u).$$

Since the profile has compact support, it is possible to define $Y_t(H_0)$ as the limit $Y_t(G_n)$ for some sequence G_n of compact supported function converging to H_0 on compact subsets of $\mathbb{R}$ and to prove that $Y^N_t(H_0)$, defined in a similar way, converges to $Y_t(H_0)$.

In the general case, however, when ρ_0 is an arbitrary profile, neither formula (3.1) makes sense, nor the fluctuation field $Y^N_t(H_0)$ is well defined. Nevertheless, there is a way to calculate the fluctuations of the current by appropriated approximations of the function G, as made by Rost and Vares [8] in the equilibrium case.

Define the sequence $\{G_n\colon n \geqslant 1\}$ of approximating functions of H_0 by

$$G_n(u) = \left\{1 - (u/n)\right\}^+ \mathbf{1}\{u \geqslant 0\}.$$

From here we use the next convention: if X is a random variable, we denote by $\overline{X}$ the centered variable $X - E_{\nu^N_{\rho_0(\cdot)}}[X]$.

Proposition 3.1. *For every $t \geqslant 0$,*

$$\lim_{n\to\infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\left[N^{-1/2}\bar{J}_{-1,0}(t) - Y^N_t(G_n) + Y^N_0(G_n)\right]^2 = 0$$

uniformly in N.

Proof. Clearly,

$$M_{x,x+1}(t) := J_{x,x+1}(t) - N^2 \int_0^t \mathrm{d}s \{\eta_s(x) - \eta_s(x+1)\}$$

is a martingale with quadratic variation given by

$$\langle M_{x,x+1}\rangle_t = N^2 \int_0^t \mathrm{d}s \{\eta_s(x) - \eta_s(x+1)\}^2.$$

The goal is to express the difference $Y_t^N(G_n) - Y_0^N(G_n)$ in terms of the martingales $M_{x,x+1}(t)$ and to notice that these martingales are orthogonal, since they have no common jumps.

Since

$$J_{x-1,x}(t) - J_{x,x+1}(t) = \eta_t(x) - \eta_0(x)$$

for all x in $\mathbb{Z}^d$, $t \geqslant 0$,

$$Y_t^N(G_n) - Y_0^N(G_n) = N^{-1/2} \sum_{x\in\mathbb{Z}} G_n(x/N)\{\bar{J}_{x-1,x}(t) - \bar{J}_{x,x+1}(t)\}.$$

A summation by parts and the explicit form of G_n permits to rewrite this expression as

$$N^{-1/2} \bar{J}_{-1,0}(t) - N^{-1/2} \sum_{x=1}^{nN} \frac{1}{nN} \bar{J}_{x-1,x}(t).$$

Representing the currents $J_{x,x+1}(t)$ in terms of the martingales $M_{x,x+1}(t)$, we obtain that

$$N^{-1/2}\bar{J}_0(t) - [Y_t^N(G_n) - Y_0^N(G_n)] = \frac{1}{\sqrt{N}} \sum_{x=1}^{nN} \frac{1}{nN} M_{x-1,x}(t) + \frac{1}{\sqrt{N}} \int_0^t \mathrm{d}s \frac{N}{n} [\bar{\eta}_s(0) - \bar{\eta}_s(nN)].$$

We claim that the martingale and the integral term converge to 0 in $L^2(\mathbb{P}_{\nu^N_{\rho_0(\cdot)}})$. In fact, since the martingales are orthogonal, estimating their quadratic variations by tN^2, an elementary computation shows that

$$\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\left[\frac{1}{\sqrt{N}} \sum_{x=1}^{nN} \frac{1}{nN} M_{x-1,x}(t)\right]^2 \leqslant \frac{t}{n}.$$

The integral term is more demanding, because in nonequilibrium the two-point correlations are not easy to estimate. Expanding the square we have that

$$\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\left[\frac{1}{\sqrt{N}} \int_0^t \mathrm{d}s \frac{N}{n}[\bar{\eta}_s(0) - \bar{\eta}_s(nN)]\,\mathrm{d}s\right]^2 = \frac{2N}{n^2} \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[\big(\bar{\eta}_s(0) - \bar{\eta}_s(nN)\big)\big(\bar{\eta}_r(0) - \bar{\eta}_r(nN)\big)\big].$$

By Lemma 3.2 the previous expression is less than or equal to $C_0 t^{5/2} n^{-2}$ for some finite constant C_0 depending only on ρ_0. This concludes the proof of the proposition. □

A central limit theorem for the current $\bar{J}_{-1,0}(t)$ is a consequence of this proposition.

Proof of Theorem 2.3. Fix $t \geqslant 0$ and $n \geqslant 1$. By approximating G_n in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ by a sequence $\{H_{n,k}\colon k \geqslant 1\}$ of smooth functions with compact support, recalling Theorem 2.2, we show that $Y_t^N(G_n)$ converges in law to a Gaussian variable denoted by $Y_t(G_n)$.

By Proposition 3.1, $\{Y_t^N(G_n) - Y_0^N(G_n)\colon n \geqslant 1\}$ is a Cauchy sequence uniformly in N. In particular, $Y_t(G_n) - Y_0(G_n)$ is a Cauchy sequence and converges to a Gaussian limit denoted by $Y_t(H_0) - Y_0(H_0)$. Therefore, by Proposition 3.1, $N^{-1/2}\bar{J}_{-1,0}(t)$ converges in law to $Y_t(H_0) - Y_0(H_0)$.

The same argument show that any vector $(\bar{J}_{-1,0}(t_1), \ldots, \bar{J}_{-1,0}(t_k))$ converges in law to $(Y_{t_1}(H_0) - Y_0(H_0), \ldots, Y_{t_k}(H_0) - Y_0(H_0))$. The covariances can be computed since by (2.3)

$$E\big[\big\{Y_t(H_0) - Y_0(H_0)\big\}\big\{Y_s(H_0) - Y_0(H_0)\big\}\big] = \lim_{n\to\infty} E\big[\big\{Y_t(G_n) - Y_0(G_n)\big\}\big\{Y_s(G_n) - Y_0(G_n)\big\}\big]$$
$$= \lim_{n\to\infty}\left\{\int_{\mathbb{R}} \big\{(T_tG_n)(T_sG_n) + G_n^2 - (T_tG_n)G_n - (T_sG_n)G_n\big\}\chi_0 + 2\int_0^s \mathrm{d}r \int_{\mathbb{R}} (\nabla T_{t-r}G_n)(\nabla T_{s-r}G_n)\chi_r\right\}. \tag{3.2}$$

A long but elementary computation permits to recover the expression presented in the statement of the theorem. Indeed, the first term in the previous integral can be written as

$$\int_{\mathbb{R}} (T_tG_n - G_n)(T_sG_n - G_n)\chi_0.$$

By definition of G_n, for $u \leqslant 0$, $T_tG_n(u) - G_n(u)$ is absolutely bounded by and converges to $P_u[B_t \geqslant 0]$, as $n \uparrow \infty$, which is integrable in $\mathbb{R}_-$. Here, P_u stands for the probability corresponding to a Brownian motion starting from u. In particular, the integral over $\mathbb{R}_-$ of the first term in (3.2) converges to

$$\int_{\mathbb{R}_-} P_u[B_t \geqslant 0]P_u[B_s \geqslant 0]\chi_0(u) = \int_{\mathbb{R}_-} P_0[B_t \leqslant u]P_0[B_s \leqslant u]\chi_0(u).$$

It is easy to show that the integral over $[n, \infty)$ vanishes since in this interval $T_tG_n(u) - G_n(u)$ vanishes pointwisely as $n \uparrow \infty$ and is bounded by the integrable function $P_u[B_t \leqslant n]$.

Finally, on the interval $[0, n]$, $T_tG_n(u) - G_n(u)$ is equal to

$$-E_u\big[\mathbf{1}\{B_t \leqslant 0\}(1 - B_t/n)\big] - E_u\big[\mathbf{1}\{B_t \geqslant n\}(1 - B_t/n)\big].$$

By Schwarz inequality, $E_u[\mathbf{1}\{B_t \leqslant 0\}B_t/n]$ and $E_u[\mathbf{1}\{B_t \geqslant n\}(1 - B_t/n)]$ vanish in $L^2(\mathbb{R}_+)$ as $n \uparrow \infty$. Therefore, the integral on $[0, n]$ of the first term on the right-hand side of (3.2) is equal to a negligible term in n plus

$$\int_0^n P_u[B_t \leqslant 0]P_u[B_s \leqslant 0]\chi_0(u) \to \int_{\mathbb{R}_+} P_0[B_t \geqslant u]P_0[B_s \geqslant u]\chi_0(u).$$

To compute the second term, note that

$$\nabla T_tG_n = p_t(0, u) + \frac{1}{n}\int_{-u}^{n-u} p_t(0, v)\,\mathrm{d}v,$$

where p_t is the Gaussian kernel. It follows from Schwarz inequality that the second term vanishes in $L^2(\mathbb{R})$. Therefore, the second term on the right-hand side of (3.2) converges, as $n \uparrow \infty$, to

$$2\int_0^s \mathrm{d}r \int_{\mathbb{R}} p_{t-r}(0, u)p_{s-r}(0, u)\chi_r(u).$$

This concludes the proof of the theorem. □

We conclude this section with some elementary estimates on two points correlation functions. For $0 \leqslant s \leqslant t$ and $x \neq y$ in $\mathbb{Z}$, let

$$\phi(t; x, y) = E_{\nu^N_{\rho_0(\cdot)}}\big[\eta_t(x); \eta_t(y)\big], \qquad \phi(s, t; x, y) = E_{\nu^N_{\rho_0(\cdot)}}\big[\eta_s(x); \eta_t(y)\big].$$

In this formula and below, $E_\mu[f; g]$ stands for the covariance of f and g with respect to μ.

Lemma 3.2. *There exists a finite constant $C_0 = C_0(\rho_0)$ depending only on the initial profile ρ_0 such that*

$$\sup_{x,y\in\mathbb{Z}} \left|\phi(t;x,y)\right| \leqslant \frac{C_0\sqrt{t}}{N}, \qquad \sup_{x,y\in\mathbb{Z}} \left|\phi(s,t;x,y)\right| \leqslant \frac{C_0}{N}\left\{\sqrt{s} + \frac{1}{\sqrt{t-s}}\right\}.$$

The first statement is a particular case of an estimate proved in [2]. In sake of completeness, we present an elementary proof of this lemma.

Proof. Consider a symmetric simple exclusion process with only two particles on $\mathbb{Z}$ and denote by L_2 the generator of this process. An elementary computation shows that $\phi(t;x,y)$ satisfies the difference equation

$$\begin{cases} (\partial_t\phi)(t;x,y) = N^2(L_2\phi)(t;x,y) - \mathbf{1}\left\{|x-y|=1\right\}N^2\left[\rho^N(t,x)-\rho^N(t,y)\right]^2, \\ \phi(0;x,y) = 0. \end{cases}$$

This equation has an explicit solution which is (negative and) absolutely bounded by

$$C_0(\rho_0)\int_0^t \mathrm{d}s\,\mathbb{P}_{x,y}\left[|X_s-Y_s|=1\right]$$

for $C_0 = \|\partial\rho_0\|_\infty^2$. In this formula, (X_s, Y_s) represent the position of the symmetric exclusion process speeded up by N^2 and starting from $\{x,y\}$. A coupling argument shows that $\mathbb{P}_{x,y}[|X_s-Y_s|=1] \leqslant \mathbb{P}^0_{x,y}[|X_s-Y_s|=1]$ where in the second probability particles are evolving independently. Since $\mathbb{P}^0_{x,y}[|X_s-Y_s|=1] \leqslant C(sN^2)^{-1/2}$, the first part of the lemma is proved.

To prove the second statement, recall that we denote by Δ_N the discrete Laplacian in $\mathbb{Z}$. $\phi(t;y) = \phi(s,t;x,y)$ satisfies the difference equation

$$\begin{cases} (\partial_t\phi)(t;y) = (\Delta_N\phi)(t;y), \\ \phi(s;y) = \phi(s;x,y) \quad \text{if } y \neq x, \\ \phi(s;y) = \rho^N(s,x)\left[1-\rho^N(s,x)\right] \quad \text{for } y = x. \end{cases}$$

This equation has an explicit solution

$$\phi(s;y) = \sum_{z\neq x} p_{t-s}(y,z)\phi(s;x,z) + p_{t-s}(y,x)\rho^N(s,x)\left[1-\rho^N(s,x)\right],$$

where $p_s(x,y)$ stands for the transition probability of a nearest neighbor symmetric random walk speeded up by N^2. The first part of the lemma together with well known estimates on p_s permit to conclude. □

4. Law of large numbers for the tagged particle

We prove in this section Theorem 2.5. We assume the initial measure to be $\nu^{N,*}_{\rho_0(\cdot)}$, the product measure $\nu^N_{\rho_0(\cdot)}$ conditioned to have a particle at the origin. Keep in mind Remark 2.4.

Fix a positive integer n. The tagged particle is at the right of n at time t if and only if the total number of particles in the interval $\{0,\dots,n-1\}$ is less than or equal to the current $J_{-1,0}(t)$:

$$\{X_t \geqslant n\} = \left\{J_{-1,0}(t) \geqslant \sum_{x=0}^{n-1}\eta_t(x)\right\}. \tag{4.1}$$

This equation indicates that a law of large numbers and a central limit theorem for the position of the tagged particle are intimately connected to the joint asymptotic behavior of the current and the empirical measure. We prove in this section the law of large numbers.

Denote by $\lceil a\rceil$ the smallest integer larger than or equal to a. Fix $u>0$ and set $n = \lceil uN\rceil$ in (4.1) to obtain that

$$\{X_t \geqslant uN\} = \left\{N^{-1}J_{-1,0}(t) \geqslant \left\langle\pi^N_t, \mathbf{1}\left\{[0,u)\right\}\right\rangle\right\}. \tag{4.2}$$

By Theorem 2.1, $\langle\pi^N_t, \mathbf{1}\{[0,u]\}\rangle$ converges in probability to $\int_0^u \rho(t,w)\,\mathrm{d}w$, where ρ is the solution of the heat equation (2.1).

On the other hand, the law of large numbers for $J_{-1,0}(t)$ under $\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}$ is an elementary consequence of the central limit theorem proved in the last section and the convergence of the expectation of $N^{-1}J_{-1,0}(t)$. By the martingale decomposition of the current and by Theorem A.1,

$$\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[N^{-1}J_{-1,0}(t)\big] = \int_0^t \mathrm{d}s\, N\big[\rho^N_s(-1) - \rho^N_s(0)\big] = -\int_0^t \partial_u\rho(s,0)\,\mathrm{d}s + \mathrm{O}\big(N^{-1}\big).$$

Hence, $N^{-1}J_{-1,0}(t)$ converges in probability to $-\int_0^t \partial_u\rho(s,0)\,\mathrm{d}s$.

In view of (4.2) and the law of large numbers for the current and the empirical measure,

$$\lim_{N\to\infty} \mathbb{P}_{\nu^{N,*}_{\rho_0(\cdot)}}\big[N^{-1}X_t \geqslant u\big] = \begin{cases} 0 & \text{if } -\int_0^t \partial_u\rho(s,0)\,\mathrm{d}s < \int_0^u \rho(t,w)\,\mathrm{d}w, \\ 1 & \text{if } -\int_0^t \partial_u\rho(s,0)\,\mathrm{d}s > \int_0^u \rho(t,w)\,\mathrm{d}w. \end{cases}$$

By symmetry around the origin, a similar statement holds for $u<0$. Thus, X^N_t/N converges to u_t in probability, where u_t is the solution of the implicit equation

$$\int_0^{u_t} \rho(t,w)\,\mathrm{d}w = -\int_0^t \partial_u\rho_s(0)\,\mathrm{d}s.$$

5. Central limit theorem for the tagged particle

In this section we prove Theorem 2.6 developing the ideas of the previous section. Assume first that $u_t>0$ and fix a in $\mathbb{R}$. By Eq. (4.1), the set $\{X_t \geqslant Nu_t + a\sqrt{N}\}$ is equal to the set in which

$$\bar{J}_{-1,0}(t) \geqslant \sum_{x=0}^{Nu_t} \bar{\eta}_t(x) + \sum_{x=1}^{a\sqrt{N}-1} \eta_t(x+Nu_t) - \left\{\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[J_{-1,0}(t)\big] - \sum_{x=0}^{Nu_t}\rho^N_t(x)\right\}, \tag{5.1}$$

where $\rho^N_t(x)$ is the solution of the discrete heat equation (2.2).

We claim that second term on the right-hand side of (5.1) divided by $\sqrt{N}$ converges to its mean in L^2. Indeed, by Lemma 3.2, its variance is bounded by $C_0aN^{-1/2}$ for some finite constant C_0. Since by Theorem A.1,

$$\frac{1}{\sqrt{N}}\sum_{x=1}^{a\sqrt{N}-1}\rho^N_t(x+Nu_t)$$

converges to $a\rho(t,u_t)$, the second term on the right-hand side of (5.1) converges in probability to $a\rho(t,u_t)$.

An elementary computation based on the definition of u_t and on Theorem A.1 shows that the third term on the right-hand side of (5.1) divided by $\sqrt{N}$ is of order $N^{-1/2}$.

Finally, by Proposition 3.1, for fixed t, $N^{-1/2}\{\bar{J}_{-1,0}(t) - \sum_{x=0}^{Nu_t}\bar{\eta}_t(x)\}$ behaves as $Y^N_t(G_n) - Y^N_0(G_n) - Y^N_t(\mathbf{1}\{[0,u_t]\})$, as $N\uparrow\infty$, $n\uparrow\infty$. Repeating the arguments presented at the beginning of the proof of Theorem 2.3, we show that this latter variable converges in law to a centered Gaussian variable, denoted by W_t, and which is formally equal to $Y_t(H_{u_t}) - Y_0(H_0)$.

Up to this point we proved that

$$\lim_{N\to\infty}\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}\left[\frac{X_t - u_tN}{\sqrt{N}} \geqslant a\right] = P\big[W_t \geqslant a\rho(t,u_t)\big]$$

provided $u_t>0$. The same arguments permit to prove the same statement in the case $u_t=0$, $a>0$. By symmetry around the origin, we can recover the other cases: $u_t<0$ and a in $\mathbb{R}$, $u_t=0$ and $a<0$.

Putting all these facts together, we conclude that for each fixed t, $(X_t - Nu_t)/\sqrt{N}$ converges in distribution to the Gaussian $W_t/\rho(t,u_t) = [Y_t(H_{u_t}) - Y_0(H_0)]/\rho(t,u_t)$. The same arguments show that any vector $(N^{-1/2}[X_{t_1} - Nu_{t_1}],\dots,N^{-1/2}[X_{t_k} - Nu_{t_k}])$ converges to the corresponding centered Gaussian vector.

It remains to compute the covariances, which can be derived as in the proof of Theorem 2.3. We leave the details to the reader.

Appendix A

In sake of completeness, we present in this section a result on the approximation of the heat equation by solutions of discrete heat equations.

Fix a profile $\rho_0 : \mathbb{R} \to \mathbb{R}$ with a bounded fourth derivative. Let $\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be the solution of the heat equation with initial profile ρ_0:

$$\begin{cases} \partial_t \rho(t,x) = \partial_x^2 \rho(t,x), \\ \rho(0,x) = \rho_0(x). \end{cases}$$

Recall that we denote by Δ_N the discrete Laplacian. For each $N \in \mathbb{N}$, define $\rho_t^N(x)$ as the solution of the system of ordinary differential equations

$$\begin{cases} (\mathrm{d}/\mathrm{d}t)\rho_t^N(x) = \big(\Delta_N \rho_t^N\big)(x), \\ \rho_0^N(x) = \rho_0(x/N). \end{cases} \tag{A.1}$$

The main result of this section asserts that ρ^N approximates ρ up to order N^{-2}:

Theorem A.1. *Assume that $\rho_0 : \mathbb{R} \to [0,1]$ is a function with a bounded fourth derivative. There exists a finite constant C_0 such that*

$$\left|\rho_t^N(x) - \rho\left(t, \frac{x}{N}\right)\right| \leqslant \frac{C_0 t}{N^2}$$

for all $N \geqslant 1$, $t \geqslant 0$, $x \in \mathbb{Z}$.

An easy way to prove this statement is to introduce a time discrete approximation of the heat equation. For each N in $\mathbb{N}$ and each $\delta > 0$, we define $\rho_l^{\delta,N}(k)$, k in $\mathbb{Z}$, $l \geqslant 0$ by the recurrence formula

$$\begin{cases} \rho_{l+1}^{\delta,N}(k) = \rho_l^{\delta,N}(k) + \delta N^2\big[\rho_l^{\delta,N}(k+1) + \rho_l^{\delta,N}(k-1) - 2\rho_l^{\delta,N}(k)\big], \\ \rho_0^{\delta,N}(k) = \rho_0(k/N). \end{cases} \tag{A.2}$$

We now recall two well known propositions whose combination leads to the proof of Theorem A.1. The first one states that the solution of (A.2) converges as $\delta \downarrow 0$ to the solution of (A.1) uniformly on compact sets. The second one furnishes a bound on the distance between the solution of the discrete equation (A.2) and the solution of the heat equation.

For a in $\mathbb{R}$, denote by $\lfloor a \rfloor$ the largest integer smaller or equal to a.

Proposition A.2. *For each $N \geqslant 1$,*

$$\lim_{\delta \to 0} \rho_{\lfloor t/\delta \rfloor}^{N,\delta}(k) = \rho_t^N(k)$$

uniformly on compacts of $\mathbb{R}_+ \times \mathbb{Z}$.

Proposition A.3. *Suppose that $\delta N^2 < 1/2$. Then, there exist a finite constant $C_0 = C_0(\rho_0)$ such that*

$$\left|\rho_l^{\delta,N}(k) - \rho(\delta l, k/N)\right| \leqslant C_0 \left\{\delta^2 l + \frac{\delta l}{N^2}\right\}$$

for all $l \geqslant 0$, $k \in \mathbb{Z}$.

Clearly, Theorem A.1 is an immediate consequence of Propositions A.2 and A.3. Proposition A.2 is a consequence of Proposition A.3 and the Cauchy–Peano existence theorem for ordinary differential equations. Proposition A.3 is a standard result on numerical analysis (see [10] for instance).

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