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A REPRESENTATION RESULT FOR TIME-SPACE BROWNIAN CHAOS

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ABSTRACT. – Given a Brownian motion X, we say that a square-integrable functional F belongs to the nth time-space Brownian chaos if F is contained in the L^2-closed vector space $\overline{\Pi}_n$, generated by r.v.'s of the form $f_1(X_{t_1})\cdots f_n(X_{t_n})$, and F is orthogonal to $\overline{\Pi}_{n-1}$. We show that every element of the nth time-space Brownian chaos can be represented as a multiple time-space Wiener integral of the nth order, thus proving a new chaotic representation property for Brownian motion. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Soit X un mouvement Brownien réel : on dit qu'une fonctionnelle F, de carré intégrable, appartient au n-ième chaos Brownien d'espace-temps, si F est dans l'espace vectoriel fermé $\overline{\Pi}_n$, engendré par les variables aléatoires de la forme $f_1(X_{t_1})\cdots f_n(X_{t_n})$, et F est orthogonale à $\overline{\Pi}_{n-1}$. Nous montrons que chaque élément du n-ième chaos Brownien d'espace-temps peut être représenté comme une intégrale de Wiener multiple dans l'espace-temps, prouvant ainsi une nouvelle propriété de représentation chaotique pour le mouvement Brownien. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Consider the space $L^2(\mathbb{P}) := L^2(C_{[0,T]}, \mathcal{C}, \mathbb{P})$, where $T \in (0, +\infty)$, $C_{[0,T]}$ is the vector space of continuous functions on $[0, T]$, endowed with its Borel σ-field, and $\mathbb{P}$ is the law of a standard Brownian motion started from zero (throughout the following, X represents the canonical process).

The aim of this paper is essentially to find, and completely describe, a new orthogonal decomposition of $L^2(\mathbb{P})$.

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Indeed, define, for every integer k,

$$\Pi_k := \left\{ \prod_{i=1}^{k} f_i(X_{t_i}):\ t_1 < \cdots < t_k \leqslant T;\ f_i \text{ is bounded, Borel measurable} \right\}.$$

According to the analysis of Föllmer, Wu and Yor [1], for every k, Π_k is not total in $L^2(\mathbb{P})$: nonetheless, if one defines $\overline{\Pi}_k$ to be the smallest, closed vector space containing Π_k, then clearly $L^2(\mathbb{P})$ coincides with the vector space generated by the union of the $\overline{\Pi}_k$'s. More to the point, we can set

$$K_0 := \Pi_0 = \Re, \qquad K_{n+1} := \overline{\Pi}_{n+1} \cap \overline{\Pi}_n^{\perp}, \tag{1}$$

where $\perp$ denotes orthogonality relation in $L^2(\mathbb{P})$, and then obtain that

$$L^2(\mathbb{P}) = \bigoplus_{n=0}^{\infty} K_n,$$

i.e. every element of $L^2(\mathbb{P})$ can be expressed as an infinite orthogonal sum, with the nth addend an element of K_n.

The rest of the paper is devoted to the explicit characterization, in terms of stochastic integrals, of the generic element of K_n. Since such a characterization suggests an evident parallelism with the classical description of *Wiener chaoses* (Wiener [4]), we will call the set K_n the nth *time-space Brownian chaos* (this terminology will become clear after the exposition of the main theorem).

In the next section we will state our main result (Theorem 1) and introduce some of the notation which is used throughout the paper; the third section contains a proof of Theorem 1 in the case of K_1; in Section 4 some results about Brownian bridges are presented, whereas the extension to higher orders is achieved in the subsequent paragraph.

2. The main theorem

Throughout the sequel, we will denote by $\mathcal{F}_t$ the natural filtration of the process X. Since X is a standard Brownian motion, it is well known (see, e.g., [5, Ch. 1]) that for every $u \leqslant T$ one can enlarge $\mathcal{F}_t$ with the σ-field generated by X_u, say $\sigma(X_u)$, thus obtaining that the process

$$X_t^{(u)} := X_t - \int_0^t \frac{X_u - X_s}{u - s}\,\mathrm{d}s, \quad t \leqslant u \tag{2}$$

is a Brownian motion with respect to the filtration $\mathcal{F}_t^{(u)} := \sigma(X_u) \vee \mathcal{F}_t$, hence is independent of X_u. Moreover, one can enlarge $\mathcal{F}_t$ with $\sigma(X_u, X_{u_1}, \ldots, X_{u_n})$ for every $u < u_1 < \cdots < u_n \leqslant T$ and still obtain that $X^{(u)}$ is a Brownian motion on $[0, u]$ with respect to $\mathcal{F}_t^{(u,u_1,\ldots,u_n)} := \sigma(X_u, X_{u_1}, \ldots, X_{u_n}) \vee \mathcal{F}_t$, and that $X^{(u)}$ is independent of

$X_u, X_{u_1}, \ldots, X_{u_n}$.[1] As a consequence, we can and do write without ambiguity the stochastic integral

$$\int_0^t g(X_u, X_{u_1}, \ldots, X_{u_n}; s; X_h, h \leqslant s)\, \mathrm{d}X_s^{(u)}, \quad t \leqslant u < u_1 < \cdots < u_n \leqslant T, \tag{3}$$

whenever

$$\int_0^t \mathbb{E}\big[g^2(X_u, X_{u_1}, \ldots, X_{u_n}; s; X_h, h \leqslant s)\big]\, \mathrm{d}s < +\infty$$

by regarding $X^{(u)}$ as an $\mathcal{F}_t^{(u,u_1,\ldots,u_n)}$-Brownian motion on $[0,u]$. In particular we have, for g measurable and bounded and $h \in L^2(\mathrm{d}\mathbb{P} \otimes \mathrm{d}s)$,

$$\int_0^t g(X_u, X_{u_1}, \ldots, X_{u_n}) h(s; X_w, w \leqslant s)\, \mathrm{d}X_s^{(u)}$$
$$= g(X_u, X_{u_1}, \ldots, X_{u_n}) \int_0^t h(s; X_w, w \leqslant s)\, \mathrm{d}X_s^{(u)}.$$

Such a convention will be tacitly assumed throughout the paper, and especially in the statement of the following result – whose proof is the subject of the next three sections – and in the subsequent remark. As usual, we will adopt the symbol Δ^n for the simplex contained in $[0,T]^n$, i.e.

$$\Delta^n := \big\{(t_1, \ldots, t_n)\colon\ 0 \leqslant t_n < t_{n-1} < \cdots < t_1 \leqslant T\big\}.$$

THEOREM 1. – *Let K_n $(n \geqslant 1)$ be defined as in* (1)*. Then, a r.v. H is an element of K_n if, and only if, there exists a measurable, deterministic function $h(u_1, x_1; \ldots; u_n, x_n)$, defined on $\Delta^n \times \Re^n$, such that*

$$\int_0^T \int_0^{u_1} \cdots \int_0^{u_{n-1}} \mathbb{E}\big(h^2(u_1, X_{u_1}; \ldots; u_n, X_{u_n})\big)\, \mathrm{d}u_n\, \mathrm{d}u_{n-1} \cdots \mathrm{d}u_1 < +\infty \tag{4}$$

and

$$H = \int_0^T \int_0^{u_1} \cdots \int_0^{u_{n-1}} h(u_1, X_{u_1}; \ldots; u_n, X_{u_n})\, \mathrm{d}X_{u_n}^{(u_{n-1})}\, \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \cdots \mathrm{d}X_{u_1} \tag{5}$$

where the processes $\{X_{u_k}^{(u_{k-1})},\ 0 \leqslant u_k \leqslant u_{k-1}\}$ are defined as in (2)*.*

Now define, for $s_1 > \cdots > s_n$, $P_{s_1,\ldots,s_n}(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$ to be the measure on $\Re^n$ induced by the vector $(X_{s_1}, \ldots, X_{s_n})$ and introduce the measure on $\Delta^n \times \Re^n$

$$\mu_n(\mathrm{d}s_1, \ldots, \mathrm{d}s_n; \mathrm{d}x_1, \ldots, \mathrm{d}x_n) := \mathrm{Leb}_{\Delta^n}(\mathrm{d}s_1, \ldots, \mathrm{d}s_n) P_{s_1,\ldots,s_n}(\mathrm{d}x_1, \ldots, \mathrm{d}x_n),$$

[1] As a matter of fact, $X^{(u)}$ is a $\mathcal{F}_t \vee \sigma\{X_w,\ w \geqslant u\}$ Brownian motion.

where Leb_{Δ^n} indicates Lebesgue measure on Δ^n. We set

$$L^2(\mu_n) := L^2\big(\Delta^n \times \Re^n,\ \mathrm{d}\mu_n\big)$$

so that it is straightforward to state the following

COROLLARY 2. – *Let $H \in K_n$, and h be the function introduced in Theorem* 1. *Then,*

$$\|H\|_{L^2(\mathbb{P})} = \|h\|_{L^2(\mu_n)}$$

Remark (*On multiple time-space Wiener integrals*). – Multiple stochastic integrals as in (5) can be formally defined in the following way. Take, for a fixed $n \geqslant 2$, the set $\mathcal{H}_n$ of linear combinations of functions on $\Delta^n \times \Re^n$ of the type

$$h(u_1, x_1; \ldots; u_n, x_n) = \prod_{k=1}^{n} h_k(u_k, x_k) 1_{(t_{n-k}, t_{(n-k)+1} - \varepsilon_{(n-k)+1})}(u_k),$$

where $t_n > \cdots > t_1 > t_0 = 0$, $\varepsilon_k \in (0, t_k - t_{k-1})$ for every k and the h_k's are measurable and bounded. Such a set is dense in $L^2(\mu_n)$. Moreover, for a function h as above, one can define stochastic integrals of the type (5) in the usual sense (i.e. as iterated stochastic integrals of progressive processes) by simply observing that, for every $\mathcal{F}_s$-progressive process f_s such that

$$\int_0^T \mathbb{E}\big(f_s^2\big)\, \mathrm{d}s < +\infty,$$

for every $t < T$ and for every $\varepsilon \in (0, t)$, the application

$$u \mapsto 1_{(u \geqslant t)} \int_0^{u \wedge (t-\varepsilon)} f_s \, \mathrm{d}X_s^{(u)}$$

defines an $\mathcal{F}_u$-progressive process, which moreover satisfies

$$\int_0^T \mathbb{E}\left[1_{(u \geqslant t)} \left(\int_0^{u \wedge (t-\varepsilon)} f_s \, \mathrm{d}X_s^{(u)} \right)^2 \right] \mathrm{d}u = \int_t^T \int_0^{u \wedge (t-\varepsilon)} \mathbb{E}\big(f_s^2\big)\, \mathrm{d}s \, \mathrm{d}u < +\infty.$$

The extension to $L^2(\mu_n)$ is therefore achieved, by defining (5) as the L^2-limit of stochastic integrals of elements of $\mathcal{H}_n$.

It is also worth noting that throughout the sequel we will use the following identity: for a fixed $t < T$, and for a fixed set $I^{(t)} \subseteq [0, t]$,

$$F\big(X_s, s \in I^{(t)}\big) \int_0^T \cdots \int_0^{u_{n-1}} 1_{A^{(n,t)}} h(u_1, X_{u_1}; \ldots; u_n, X_{u_n}) \, \mathrm{d}X_{u_n}^{(u_{n-1})} \, \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \cdots \mathrm{d}X_{u_1}$$

$$= \int_0^T \cdots \int_0^{u_{n-1}} 1_{A^{(n,t)}} F\left(X_s, s \in I^{(t)}\right) \\ \times h(u_1, X_{u_1}; \ldots; u_n, X_{u_n})\, \mathrm{d}X_{u_n}^{(u_{n-1})}\, \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \cdots \mathrm{d}X_{u_1}, \tag{6}$$

where $A^{(n,t)} := \{(u_1, \ldots, u_n) \in \Delta^n\colon\ u_n > t\}$ and F is a bounded functional. Such an identity is straightforward in the case of $h \in \mathcal{H}_n$. In the general case, define the right term of (6) to be the L^2-limit of the sequence

$$\int_0^T \cdots \int_0^{u_{n-1}} 1_{A^{(n,t)}} F h^{(k)}\, \mathrm{d}X_{u_n}^{(u_{n-1})}\, \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \cdots \mathrm{d}X_{u_1},$$

where all dependences have been dropped and $h^{(k)}$ is a sequence of elements of $\mathcal{H}_n$ converging to h, so that (6) follows from the boundedness of F.

We eventually observe that one can define integrals of the type

$$\int_0^T \cdots \int_0^{u_{n-1}} 1_{A^{(n,t)}}(u_1, \ldots, u_n) \\ \times \phi\left(X_s, s \in I^{(t)}; u_1, X_{u_1}; \ldots; u_n, X_{u_n}\right) \mathrm{d}X_{u_n}^{(u_{n-1})}\, \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \cdots \mathrm{d}X_{u_1}$$

with

$$\int_0^T \cdots \int_0^{u_{n-1}} 1_{A^{(n,t)}} \mathbb{E}\left[\phi\left(X_s, s \in I^{(t)}; u_1, X_{u_1}; \ldots; u_n, X_{u_n}\right)^2\right] \mathrm{d}u_n\, \mathrm{d}u_{n-1} \cdots \mathrm{d}u_1 < +\infty$$

as the L^2-limit of r.v.'s in the form of the right member of (6).

A last and immediate consequence of Theorem 1 is also

COROLLARY 3. – *Let F be a real r.v. in $L^2(\mathbb{P})$, then there exists a sequence of measurable functions*

$$h_{(F,n)} : \Delta^n \times \Re^n \mapsto \Re$$

such that

$$\int_0^T \int_0^{u_1} \cdots \int_0^{u_{n-1}} \mathbb{E}\left(h_{(F,n)}^2(u_1, X_{u_1}; \ldots; u_n, X_{u_n})\right) \mathrm{d}u_n\, \mathrm{d}u_{n-1} \cdots \mathrm{d}u_1 < +\infty$$

and

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} \int_0^T \int_0^{u_1} \cdots \int_0^{u_{n-1}} h_{(F,n)}(u_1, X_{u_1}; \ldots; u_n, X_{u_n})\, \mathrm{d}X_{u_n}^{(u_{n-1})}\, \mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \cdots \mathrm{d}X_{u_1},$$

where the equality holds in the L^2 sense. Moreover, the sequence is unique in the sense that, if $h'_{(F,n)}$ is another sequence satisfying the above relations, then

$$h_{(F,n)}(u_1, X_{u_1}; \ldots; u_n, X_{u_n}) = h'_{(F,n)}(u_1, X_{u_1}; \ldots; u_n, X_{u_n})$$

a.e. $\mathrm{d}\mathbb{P} \otimes \mathrm{Leb}_{\Delta^n}(\mathrm{d}u_1, \ldots, \mathrm{d}u_n)$.

Remark. – Corollary 3 shows that Brownian motion enjoys a chaotic representation property (that we name *chaotic time-space representation property* (CTSRP)) which is analogous to the one described in Wiener [4]. More precisely, Wiener proves that for every square integrable functional F of X there exists a sequence of deterministic functions $f_{(F,n)} \in L^2(\Delta^n, \mathrm{Leb}_{\Delta^n})$ such that, in L^2,

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} \int_0^T \cdots \int_0^{u_{n-1}} f_{(F,n)}(u_1, \ldots, u_n)\, \mathrm{d}X_{u_n} \cdots \mathrm{d}X_{u_1}. \tag{7}$$

As we point out in the final paragraph, the relation between the two decompositions is not completely clear, in the sense that there does not exist (except in trivial cases) a general formula permitting to represent a generic element of the space K_n in the form (7) starting from its characterization as an iterate time-space multiple integral, and *viceversa*. On the other hand, for a given *smooth* functional F (see [2] for definitions) the relation between the integrands $h_{(F,n)}$ of Corollary 3 and $f_{(F,n)}$ in (7) is quite explicit. Stroock proves indeed in [3] that, for a given smooth functional F the integrands $f_{(F,n)}$ of Wiener's representation satisfy

$$f_{(F,n)}(u_1, \ldots, u_n) = \mathbb{E}(D_{u_n} \cdots D_{u_1} F),$$

where D_u indicates the usual Malliavin's derivative at u. This formula can be easily proved by first applying Clark's formula to F, and then to $\mathbb{E}(D_{u_1} F \mid \mathcal{F}^X_{u_2})$, $\mathbb{E}(D_{u_2} D_{u_1} F \mid \mathcal{F}^X_{u_2})$ and so on, i.e. by regarding such random variables as functionals of $(X_w,\ w \leqslant u)$. Our point here is that a similar characterization is valid for our case. In particular, one can show by arguments similar to Stroock's that the form of the $h_{(F,n)}$'s for a smooth F must be

$$h_{(F,n)}(u_1, X_{u_1}; \ldots; u_n, X_{u_n}) = \mathbb{E}\big(D^{(u_{n-1})}_{u_n} \cdots D^{(u_1)}_{u_2} D_{u_1} F \mid X_{u_1}, \ldots, X_{u_n}\big),$$

where $D^{(s)}_u Y$ denotes Malliavin's derivative at u ofa given r.v. Y, regarded as a functional of the Brownian motion

$$X^{(s)}_t := X_t - \int_0^{t \wedge s} \frac{X_s - X_h}{s - h}\, \mathrm{d}h, \quad t \leqslant T.$$

However, the formal proof of this claim requires some careful discussion, and is therefore deferred to a separate paper.

As anticipated, we will first prove Theorem 1 for K_1.

3. The space K_1

In this section we shall concentrate on the first time-space chaos, i.e. the set of zero mean r.v.'s contained in the subspace of $L^2(\mathbb{P})$ generated by r.v.'s of the form $F = f(X_t)$.

Throughout the sequel, we will denote with $(P_t)_{t\geqslant 0}$ the Brownian semigroup, whereas $p_t(x, y)$ indicates its density, i.e.

$$P_t(x, dy) = p_t(x, y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(x-y)^2}{2t}\right]\mathrm{d}y.$$

We can therefore state the following

PROPOSITION 4. – *A r.v. H is in $K_1 := \overline{\Pi}_1 \cap \overline{\Pi}_0^{\perp}$ if, and only if, H admits a representation of the form*

$$H = \int_0^T h(s, X_s)\,\mathrm{d}X_s, \tag{8}$$

where $h(s, x)$ is a measurable and deterministic function, defined on $[0, T] \times \Re$ and such that

$$\int_0^T \mathbb{E}\big(h^2(s, X_s)\big)\,\mathrm{d}s < +\infty. \tag{9}$$

Proof. – (i) We consider an element of K_1 of the form $f(X_t)$, where f is bounded and measurable. Without loss of generality, we can also assume $f \in C_c^{1,2}$, so that the following particular case of Clark's formula holds: for $u < t$,

$$\mathbb{E}\big(f(X_t) \mid \mathcal{F}_u\big) = \int_0^u P_{t-s} f'(X_s)\,\mathrm{d}X_s$$

(note that $f(X_t) \in K_1$ implies $\mathbb{E}(f(X_t)) = 0$) and therefore

$$f(X_t) = \int_0^T 1_{(s\leqslant t)} P_{t-s} f'(X_s)\,\mathrm{d}X_s,$$

so that representation (8) holds for every linear combination of functions of the form $f(X_t)$. That (8) is valid for every element of K_1 stems from the fact that r.v.'s of the type $H = \int_0^T h(s, X_s)\,\mathrm{d}X_s$ form an Hilbert space, with respect to the L^2-norm, as seen from the isometry property

$$\mathbb{E}\big(H^2\big) = \|h\|^2_{L^2(\mu_1)},$$

where $\mu_1(\mathrm{d}t, \mathrm{d}x) := \mathrm{d}t\, P_t(0, \mathrm{d}x)$.

(ii) To prove that if H and h satisfy (8) and (9) then $H \in K_1$, is equivalent to show that the functions of the type

$$(t, x) \mapsto P_{u-t} f(x) 1_{(t\leqslant u)} \tag{10}$$

are total in $L^2([0,T]\times\Re,\,\mathrm{d}t\,P_t(0,\,\mathrm{d}x)) := L^2(\mu_1)$.

To show this, consider an element g of $L^2(\mu_1)$ such that, for every f bounded and measurable, for every $u\leqslant T$

$$\begin{aligned}0 &= \int_0^u\int_\Re g(t,x)P_{u-t}f(x)\mu_1(\mathrm{d}t,\mathrm{d}x)\\ &= \int_0^u \mathbb{E}\big(g(t,X_t)P_{u-t}f(X_t)\big)\,\mathrm{d}t.\end{aligned}$$

Then, one can choose $f(x)=\exp(\mathrm{i}\lambda x+\frac{1}{2}\lambda^2 u)$, and therefore obtain that, for every $u\leqslant T$

$$\int_0^u \mathbb{E}(g(t,X_t)\exp\big(\mathrm{i}\lambda X_t)\big)\exp\left(\frac{1}{2}\lambda^2 t\right)\mathrm{d}t = 0$$

which implies

$$g\big(t,X_t(\omega)\big)=0,\quad \text{a.e.-}\,\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$$

i.e.

$$\int_0^T\int_\Re g^2(t,x)\mu_1(\mathrm{d}t,\,\mathrm{d}x) = \int_0^T \mathbb{E}\big(g^2(t,X_t)\big)\,\mathrm{d}t = 0. \qquad \square$$

We say that a function h on $[0,T]\times\Re$ is *time-space harmonic* whenever it is in $C^{1,2}$ and satisfies

$$\frac{\partial}{\partial t}h(t,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}h(t,x)=0$$

for every t and x. Since functions of the form (10) obtained from a regular f are of this kind, we can state the following

COROLLARY 5. – *Time-space harmonic functions are dense in* $L^2(\mu_1)$.

As we said, before proceeding to the proof of Theorem 1 for higher orders, we need some result about the characterization of certain exponential transformations of Brownian bridges.

4. An ancillary result about Brownian bridges

As will become evident in the next section, to prove Theorem 1 for any n we shall deal with r.v.'s of the form

$$\mathbb{E}\big(F(X_s,s\leqslant u)\mid X_u\big).$$

Clearly, one way to handle such variables is to use the properties of Brownian bridges. In what follows $\mathbb{P}^t_{0,x}$ denotes the law of a Brownian bridge started from zero and conditioned to be x in t. Then, one has the following equivalent of certain 'exponential martingales' results for Brownian motion: it will permit us to use – for a generic K_n – the same line of reasoning as in the proof of Proposition 4.

PROPOSITION 6. – *Let $t_1 < t$ be fixed: then, for every real λ, the process*

$$Y_s^{(\lambda,t_1)} := \exp\left[\mathrm{i}\lambda \frac{(t-t_1)X_s + (t_1-s)x}{t-s} - \frac{1}{2}\lambda^2 \frac{(t-t_1)(t_1-s)}{t-s}\right]$$

is a $(\mathbb{P}_{0,x}^t, \mathcal{F}_s)$-martingale on $[0, t_1]$.

Proof. – The result is achieved, once it is shown that

$$Y_s^{(\lambda,t_1)} = \mathbb{E}_{0,x}^t\big(\exp(\mathrm{i}\lambda X_{t_1}) \mid X_s\big).$$

To this end, we recall that, for $s < t$, $\mathbb{P}_{0,x}^t|_{\mathcal{F}_s}$ is equivalent to $\mathbb{P}|_{\mathcal{F}_s}$, and that the Radon-Nikodym density is given by

$$\left.\frac{\mathrm{d}\mathbb{P}_{0,x}^t}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_s} = \frac{p_{t-s}(X_s,x)}{p_t(0,x)}.$$

By using this property, one can explicitly evaluate

$$\begin{aligned}
\mathbb{E}_{0,x}^t\big(\exp(\mathrm{i}\lambda X_{t_1}) \mid X_s\big) &= \frac{\mathbb{E}(\exp(\mathrm{i}\lambda X_{t_1}) p_{t-t_1}(X_{t_1},x) \mid X_s)}{p_{t-s}(X_s,x)} \\
&= \frac{\mathbb{E}_{X_s}(\exp(\mathrm{i}\lambda X_{t_1-s}) p_{t-t_1}(X_{t_1-s},x))}{p_{t-s}(X_s,x)} \\
&=: \frac{A(X_s,\lambda,t_1)}{p_{t-s}(X_s,x)}.
\end{aligned}$$

Moreover, one gets from simple calculations

$$\begin{aligned}
A(X_s,\lambda,t_1) &= (2\pi)^{-1/2}\int \big[2\pi(t-t_1)(t_1-s)\big]^{-1/2} \mathrm{e}^{\mathrm{i}\lambda y} \mathrm{e}^{-\frac{1}{2(t-t_1)}(y-x)^2} \mathrm{e}^{-\frac{1}{2(t_1-s)}(y-X_s)^2}\, \mathrm{d}y \\
&= p_{t-s}(X_s,x)\sqrt{t-s}\int \big[2\pi(t-t_1)(t_1-s)\big]^{-1/2} \mathrm{e}^{\mathrm{i}\lambda y} \\
&\quad \times \exp\left(-\frac{1}{2(t-t_1)(t_1-s)}\left(y\sqrt{t-s} - \frac{(t-t_1)X_s + (t_1-s)x}{\sqrt{t-s}}\right)^2\right) \mathrm{d}y \\
&= p_{t-s}(X_s,x)\exp\left[\mathrm{i}\lambda \frac{(t-t_1)X_s + (t_1-s)x}{t-s} - \frac{1}{2}\lambda^2 \frac{(t-t_1)(t_1-s)}{t-s}\right]
\end{aligned}$$

and the result is therefore achieved. □

Remark. – One can alternatively prove Proposition 6 by writing

$$X_s = X_s^{(u,x)} + \int_0^s \frac{x - X_h}{u-h}\, \mathrm{d}h,$$

where $X^{(u,x)}$ is a $(\mathbb{P}_{0,x}^u, \mathcal{F}_s)$ – Brownian motion, and then by using Itô's formula.

5. The extension to higher orders

We want to prove Theorem 1 in a recursive manner: to do this, we introduce the following recursive assumption

$$(\mathbf{H}_n) : \text{Theorem 1 is true for } K_1, \ldots, K_n$$

and recall that, in the previous sections, we have verified $(\mathbf{H}_1)$.

We start by observing that, if one defines a process $\{X_s^{(u)}, 0 \leqslant s \leqslant u\}$ as in (2), one has that, under $(\mathbf{H}_n)$, for every $F \in L^2(\mathbb{P})$ and for any $k \leqslant n+1$, there exists a measurable application

$$(u_1, \ldots, u_k; \omega) \longmapsto \phi_{u_1,\ldots,u_k}^{(F,k)}\big(X_{u_1}(\omega), \ldots, X_{u_{k-1}}(\omega); X_s(\omega), s \leqslant u_k\big), \tag{11}$$

where $u_0 := 0$, such that

$$\begin{aligned} F = \mathbb{E}(F) &+ \int_0^T h_{(F,1)}(u, X_u)\,\mathrm{d}X_u \\ &+ \int_0^T \int_0^u h_{(F,2)}(u, X_u; s, X_s)\,\mathrm{d}X_s^{(u)}\,\mathrm{d}X_u + \cdots \\ &+ \int_0^T \cdots \int_0^{u_{k-2}} h_{(F,k-1)}(u_1, X_{u_1}; \ldots; u_{k-1}, X_{u_{k-1}})\,\mathrm{d}X_{u_{k-1}}^{(u_{k-2})}\,\mathrm{d}X_{u_{k-2}}^{(u_{k-3})} \cdots \mathrm{d}X_{u_1} \\ &+ \int_0^T \cdots \int_0^{u_{k-1}} \phi_{u_1,\ldots,u_k}^{(F,k)}(X_{u_1}, \ldots, X_{u_{k-1}}; X_s, s \leqslant u_k)\,\mathrm{d}X_{u_k}^{(u_{k-1})}\,\mathrm{d}X_{u_{k-1}}^{(u_{k-2})} \cdots \mathrm{d}X_{u_1}, \end{aligned} \tag{12}$$

where we adopt the conventions discussed at the beginning of the second paragraph, and the notation is the same as in Corollary 3.

For $k = 1$, formula (12) corresponds to Itô's theorem, stating that for every $F \in L^2(\mathbb{P})$ there exists a unique progressive process $f_u(\omega)$ such that

$$\int_0^T \mathbb{E}\big(f_u^2\big)\,\mathrm{d}u < +\infty$$

and

$$F = \mathbb{E}(F) + \int_0^T f_u\,\mathrm{d}X_u. \tag{13}$$

For $k > 1$, just observe that for every square integrable smooth functional F (see e.g. [2]) with representation (13), one has for f_u (as an element of $L^2(\mathcal{F}_u)$) a representation of the type

$$f_u = \mathbb{E}(f_u \mid X_u) + \int_0^u \phi_{u,s}(X_u; X_h, h \leqslant s)\,\mathrm{d}X_s^{(u)}$$
$$:= \mathbb{E}(f_u \mid X_u) + \int_0^u f_s^{(u)}\,\mathrm{d}X_s^{(u)},$$

where the application

$$(u,x) \mapsto \mathbb{E}(f_u \mid X_u = x)$$

is measurable, whereas for $f_s^{(u)}$ holds

$$f_s^{(u)} = \mathbb{E}\big(f_s^{(u)} \mid X_u, X_s\big) + \int_0^s \phi_{u,s,r}(X_u, X_s; X_h, h \leqslant r)\,\mathrm{d}X_r^{(s)}, \tag{14}$$

where

$$(u,s;x,y) \mapsto \mathbb{E}\big(f_s^{(u)} \mid X_u = x, X_s = y\big)$$

defines a measurable application, and so on (to prove (14) for a generic element of $L^2(\mathcal{F}_s^{(u)})$, consider functions of the form

$$g(X_u)h(X_s)F\big(X_w^{(s)}, w \leqslant s\big),$$

where F is smooth, which are total in $L^2(\mathcal{F}_s^{(u)})$). The general result is achieved by a density argument.

Incidentally, note that the measurable application introduced in formula (11) is uniquely determined, for every k, outside some $\mathrm{d}u_1 \cdots \mathrm{d}u_k \otimes \mathrm{d}\mathbb{P}$-null set: we will call such an application the kth *Itô integrand* of F.

We now present two results which are related respectively to Proposition 3.1 and Proposition 3.3 of Föllmer, Wu and Yor [1].

PROPOSITION 7. – *Under* $(\mathbf{H}_n)$, *for a zero mean r.v.* F, $F \in \overline{\Pi}_n^{\perp}$ *if, and only if, for every* $i = 1, \ldots, n$,

$$\mathbb{E}\big(\phi_{u_1,\ldots,u_i}^{(F,i)} \mid X_{u_1}, \ldots, X_{u_i}\big) = 0, \quad \textit{a.e.}\text{-Leb}_{\Delta^i}(\mathrm{d}u_1, \ldots, \mathrm{d}u_i), \tag{15}$$

where $\phi_{u_1,\ldots,u_i}^{(F,i)}$ *is the* i*th Itô integrand of* F.

Proof. – Clearly if (15) is valid then, for every $k \leqslant n+1$,

$$F = \int_0^T \int_0^{u_1} \cdots \int_0^{u_{k-1}} \phi_{u_1,\ldots,u_k}^{(F,k)}\,\mathrm{d}X_{u_k}^{(u_{k-1})}\,\mathrm{d}X_{u_{k-1}}^{(u_{k-2})} \cdots \mathrm{d}X_{u_1}$$

which implies $F \in \overline{\Pi}_{k-1}^{\perp}$.

On the other hand, suppose $F \in \overline{\Pi}_k^{\perp}$ (for $k \leqslant n$), and write, due to $(\mathbf{H}_n)$,

$$F = \int_0^T \int_0^{u_1} \cdots \int_0^{u_{k-1}} \phi_{u_1,\ldots,u_k}^{(F,k)} \, \mathrm{d}X_{u_k}^{(u_{k-1})} \, \mathrm{d}X_{u_{k-1}}^{(u_{k-2})} \cdots \mathrm{d}X_{u_1}.$$

Thus,

$$\int_0^T \cdots \int_0^{u_{k-1}} \mathbb{E}\big[\mathbb{E}\big(\phi_{u_1,\ldots,u_k}^{(F,k)} \mid X_{u_1},\ldots,X_{u_k}\big) h(u_1,X_{u_1};\ldots;u_k,X_{u_k})\big] \, \mathrm{d}u_k \cdots \mathrm{d}u_1 = 0$$

for every $h \in L^2(\mu_k)$. And the result is proved, by choosing functions h of the kind

$$1_{A^{(k)}}(u_1,\ldots,u_k) \exp(\mathrm{i}\lambda_1 X_{u_1}) \cdots \exp(\mathrm{i}\lambda_k X_{u_k})$$

with $A^{(k)}$ a generic subset of Δ^k. $\square$

The second announced result is the following

PROPOSITION 8. – *Under* $(\mathbf{H}_n)$, *consider an integer* $m > 0$. *Then,* $F \in \overline{\Pi}_{n+m}^{\perp}$ *if, and only if, for every* $k \leqslant n$,

$$\mathbb{E}\big(\phi_{u_1,\ldots,u_k}^{(F,k)} \mid X_{u_1},\ldots,X_{u_k}; X_{t_1},\ldots,X_{t_{n+m-k}}\big) = 0$$

a.e. $\mathrm{Leb}_{\Delta^k}(\mathrm{d}u_1,\ldots,\mathrm{d}u_k)$, *for every* $t_1 < \cdots < t_{n+m-k} < u_k$.

Proof. – The sufficiency of the above condition can be proved by simply mimicking the first part of the proof of Proposition 3.3 in Föllmer, Wu and Yor [1]. One can deduce necessity by observing that, under $(\mathbf{H}_n)$, a r.v. of the kind

$$H = f_1(X_{t_1}) \cdots f_{n+m-k}(X_{t_{n+m-k}}) Y$$

belongs to $\overline{\Pi}_{n+m}$ for every Y in K_k and for every $(n+m-k)$-ple of bounded functions. As a matter of fact, since $K_k \subset \overline{\Pi}_k$, Y is the L^2-limit of linear combinations of r.v.'s of the form

$$g_1(X_{s_1}) \cdots g_k(X_{s_k})$$

and therefore – due to the boundedness of the f_i's – H is the L^2-limit of linear combinations of r.v.'s of the form

$$f_1(X_{t_1}) \cdots f_{n+m-k}(X_{t_{n+m-k}}) g_1(X_{s_1}) \cdots g_k(X_{s_k}).$$

One can then choose, due to $(\mathbf{H}_n)$,

$$Y = \int_0^T \int_0^{u_1} \cdots \int_0^{u_{k-1}} h(u_1,X_{u_1};\ldots;u_k,X_{u_k}) \, \mathrm{d}X_{u_k}^{(u_{k-1})} \, \mathrm{d}X_{u_{k-1}}^{(u_{k-2})} \cdots \mathrm{d}X_{u_1}$$

with

$$h(u_1, X_{u_1}; \ldots; u_k, X_{u_k}) = 1_{A^{(k)}}(u_1, \ldots, u_k) \exp(\mathrm{i}\lambda_1 X_{u_1}) \cdots \exp(\mathrm{i}\lambda_k X_{u_k}),$$

where now $A^{(k)}$ is taken to be an arbitrary subset of Δ^k such that

$$\forall (u_1, \ldots, u_k) \in A^{(k)}\colon\ u_k > t_{n+m-k}.$$

If one sets finally

$$f_j(x) = \exp(\mathrm{i}\gamma_j x)$$

the result is easily achieved, once F is written in the form

$$F = \int_0^T \int_0^{u_1} \cdots \int_0^{u_{k-1}} \phi^{(F,k)}_{u_1,\ldots,u_k} \, \mathrm{d}X^{(u_{k-1})}_{u_k} \, \mathrm{d}X^{(u_{k-2})}_{u_{k-1}} \cdots \mathrm{d}X_{u_1}$$

thanks to $(\mathbf{H}_n)$ and Proposition 7 □

From Proposition 7 and Proposition 8 we deduce a useful

LEMMA 9. – *Under* $(\mathbf{H}_n)$, *let*

$$\left\{Y_{u_1,\ldots,u_n}, (u_1, \ldots, u_n) \in \Delta^n\right\}$$

be such that

$$\int_0^T \cdots \int_0^{u_{n-1}} \mathbb{E}\left(Y^2_{u_1,\ldots,u_n}\right) \mathrm{d}u_1 \cdots \mathrm{d}u_n < +\infty$$

and

$$\mathbb{E}(Y_{u_1,\ldots,u_n} \mid X_{u_1}, \ldots, X_{u_n}) = 0$$

for every $(u_1, \ldots, u_n) \in \Delta^n$. *Then*

$$H^{(s)} := \int_0^T \int_0^{u_1} \cdots \int_0^{u_{n-1}} \mathbb{E}(Y_{u_1,\ldots,u_n} \mid X_{u_1}, \ldots, X_{u_n}; X_s) 1_{(u_n > t)} \, \mathrm{d}X^{(u_{n-1})}_{u_n} \, \mathrm{d}X^{(u_{n-2})}_{u_{n-1}} \cdots \mathrm{d}X_{u_1}$$

belongs to K_{n+1} *for every* (s, t) *such that* $s < t < T$.

Proof. – From Proposition 7 and $\mathbb{E}(Y_{u_1,\ldots,u_n} \mid X_{u_1,\ldots,u_n}) = 0$, one gets

$$H^{(s)} \in \bigoplus_{m=n+1}^{\infty} K_m.$$

More to the point, if one considers $F \in K_m$, for $m > n + 1$, one has

$$\mathbb{E}\big(H^{(s)} F\big) = \int_0^T \cdots \int_0^{u_{n-1}} \mathbb{E}\big[\phi^{(F,n)}_{u_1,\ldots,u_n} \mathbb{E}(Y_{u_1,\ldots,u_n} \mid X_{u_1}, \ldots, X_{u_n}; X_s)\big] 1_{(u_n > t)} \, \mathrm{d}u_n \cdots \mathrm{d}u_1 = 0$$

as, according to Proposition 8,

$$\mathbb{E}\big(\phi^{(F,n)}_{u_1,\dots,u_n} \mid X_{u_1},\dots,X_{u_n}; X_s\big) = 0$$

a.s.-$\mathrm{Leb}_{\Delta^n}(\mathrm{d}u_1\cdots\mathrm{d}u_n)$, for every $s < u_n$. Therefore, $H \in K_{n+1}$. □

We are now in a position to prove Theorem 1 in the general case: to do this, we will show that, if $(\mathbf{H}_n)$ is verified, then $(\mathbf{H}_{n+1})$ is necessarily true.

Before doing this, we observe that, throughout the sequel, we will use the following fact, which stems straightforwardly from the Markov property of X: consider the instants

$$u_1 > \cdots > u_n > t \geqslant s,$$

then, for every bounded f,

$$\mathbb{E}\big(f(X_t) \mid X_{u_1},\dots,X_{u_n},\mathcal{F}_s\big) = \mathbb{E}\big(f(X_t) \mid X_{u_n},\mathcal{F}_s\big)$$

so that, for instance, under the law $\mathbb{P}(\cdot \mid X_{u_1} = x_1,\dots,X_{u_n} = x_n)$ the process $(X_t, t \leqslant u_n)$ is still a Brownian bridge of length u_n, from 0 to x_n.

Eventually, let $(\mathbf{H}_n)$ be verified, and consider a r.v. of the form

$$H = f_1(X_{t_1})\cdots f_k(X_{t_k})\cdots f_{n+1}(X_{t_{n+1}}),$$

where (due again to a density argument) every f_k is supposed to be such that

$$f_k(X_{t_k}) = \exp(\mathrm{i}\lambda_k X_{t_k})$$

and $t_1 < \cdots < t_{n+1}$. We claim that, given $(\mathbf{H}_n)$, the projection of H on K_{n+1} must be

$$\int_0^T\int_0^{u_1}\cdots\int_0^{u_n} \phi^{(H,n+1)}_{u_1,\dots,u_{n+1}}(X_{u_1},\dots,X_{u_n},X_{u_{n+1}})\,\mathrm{d}X^{(u_n)}_{u_{n+1}}\,\mathrm{d}X^{(u_{n-1})}_{u_n}\cdots\mathrm{d}X_{u_1}, \tag{16}$$

where

$$\begin{aligned}
&\phi^{(H,n+1)}_{u_1,\dots,u_{n+1}}(X_{u_1},\dots,X_{u_n},X_{u_{n+1}})\\
&\quad = 1_{(t_n,t_{n+1})}(u_1)\,P_{t_{n+1}-u_1} f'_{n+1}(X_{u_1})\\
&\qquad \times \prod_{k=2}^{n+1} \frac{\partial}{\partial z} h^{(u_{k-1},X_{u_{k-1}})}_{f_{(n+2)-k},t_{(n+2)-k}}(u_k,z)\bigg|_{z=X_{u_k}} 1_{(t_{(n+1)-k},t_{(n+2)-k})}(u_k)
\end{aligned}$$

with

$$h^{(u,x)}_{f_k,t_k}(s,z) := \exp\left[\mathrm{i}\lambda_k \frac{(u-t_k)z + (t_k-s)x}{u-s} - \frac{1}{2}\lambda_k^2 \frac{(u-t_k)(t_k-s)}{u-s}\right]$$

and $t_0 = 0$.

To see this, we introduce the following convention: for two r.v.'s C and B we write

$$C \overset{\mathrm{mod}(n)}{=} B$$

whenever $B \in \overline{\Pi}_{n+1}$ and $C - B \in \overline{\Pi}_n$. With such a notation, one has, for H defined as above, the $n+1$ equivalence relations

$$\begin{aligned}
H &\overset{\text{mod}(n)}{=} \int_{t_n}^{t_{n+1}} P_{t_{n+1}-u_1} f'_{n+1}(X_{u_1})\,\mathrm{d}X_{u_1} \times f_n(X_{t_n}) \cdots f_1(X_{t_1}) \\
&\overset{\text{mod}(n)}{=} \int_{t_n}^{t_{n+1}} P_{t_{n+1}-u_1} f'_{n+1}(X_{u_1}) \\
&\qquad \times \int_{t_{n-1}}^{t_n} \frac{\partial}{\partial z} h^{(u_1, X_{u_1})}_{f_n, t_n}(u_2, z)\bigg|_{z = X_{u_2}} \mathrm{d}X^{(u_1)}_{u_2}\,\mathrm{d}X_{u_1} \times f_{n-1}(X_{t_{n-1}}) \cdots f_1(X_{t_1}) \\
&\cdots \\
&\overset{\text{mod}(n)}{=} \int_0^T \int_0^{u_1} \cdots \int_0^{u_n} \phi^{(H, n+1)}_{u_1, \ldots, u_{n+1}}(X_{u_1}, \ldots, X_{u_n}, X_{u_{n+1}})\,\mathrm{d}X^{(u_n)}_{u_{n+1}}\,\mathrm{d}X^{(u_{n-1})}_{u_n} \cdots \mathrm{d}X_{u_1}
\end{aligned}$$

the simplification being justified by the following procedure.

– For the first step, we write

$$f_{n+1}(X_{t_{n+1}}) = P_{t_{n+1}-t_n} f_{n+1}(X_{t_n}) + \int_{t_n}^{t_{n+1}} P_{t_{n+1}-u_1} f'_{n+1}(X_{u_1})\,\mathrm{d}X_{u_1}$$

  and we eliminate the r.v.

$$P_{t_{n+1}-t_n} f_{n+1}(X_{t_n}) f_n(X_{t_n}) \cdots f_1(X_{t_1}) \in \overline{\Pi}_n.$$

– For the kth step ($k = 2, \ldots, n$) we write

$$\begin{aligned}
f_{(n+2)-k}(X_{t_{(n+2)-k}}) &= \mathbb{E}\big(f_{(n+2)-k}(X_{t_{(n+2)-k}}) \mid X_{u_{k-1}}, X_{t_{(n+2)-k}}\big) \\
&= \mathbb{E}\big(f_{(n+2)-k}(X_{t_{(n+2)-k}}) \mid X_{u_{k-1}}, X_{t_{(n+1)-k}}\big) \\
&\quad + \int_{t_{(n+1)-k}}^{t_{(n+2)-k}} \frac{\partial}{\partial z} h^{(u_{k-1}, X_{u_{k-1}})}_{f_{(n+2)-k}, t_{(n+2)-k}}(u_k, z)\bigg|_{z = X_{u_k}} \mathrm{d}X^{(u_{k-1})}_{u_k}
\end{aligned}$$

  thanks to Proposition 6 and Itô's formula since, under $\mathbb{P}(\cdot \mid X_{u_{k-1}} = x)$, X is a Brownian bridge of length u_{k-1}, from 0 to x. Then, we elide a r.v. of the form

$$\begin{aligned}
G &= f_1(X_{t_1}) \cdots f_{(n+1)-k}(X_{t_{(n+1)-k}}) \\
&\quad \times \int_{t_n}^{t_{n+1}} \cdots \int_{t_{(n+2)-k}}^{t_{(n+3)-k}} \Phi_{u_1, \ldots, u_{k-1}}(X_{u_1}, \ldots, X_{u_{k-1}}; X_{t_{(n+1)-k}})\,\mathrm{d}X^{(u_{k-2})}_{u_{k-1}} \cdots \mathrm{d}X_{u_1}
\end{aligned}$$

  since, due to Proposition 8, we have that G is orthogonal to $\overline{\Pi}_{n+1} \cap \overline{\Pi}_n^{\perp} = K_{n+1}$, and therefore is in $\overline{\Pi}_n$.

– To conclude, we write

$$f_1(X_{t_1}) = \mathbb{E}(f_1(X_{t_1}) \mid X_{u_n}) + \int_0^{t_1} \frac{\partial}{\partial z} h_{f_1,t_1}^{(u_n,X_{u_n})}(u_{n+1}, z)\Big|_{z=X_{u_{n+1}}} \mathrm{d}X_{u_{n+1}}^{(u_n)}$$

and we eliminate a r.v. of the form

$$\int_{t_n}^{t_{n+1}} \cdots \int_{t_1}^{t_2} \Phi_{u_1,\dots,u_n}(X_{u_1}, \dots, X_{u_n}) \,\mathrm{d}X_{u_n}^{(u_{n-1})} \cdots \mathrm{d}X_{u_1}$$

since, thanks to $(\mathbf{H}_n)$, it is an element of $K_n \subset \overline{\Pi}_n$.

The representation of Theorem 1 is therefore true for the orthogonal projection on K_{n+1} of every linear combination of r.v.'s of the kind

$$\exp(\mathrm{i}\lambda_1 X_{t_1}) \cdots \exp(\mathrm{i}\lambda_k X_{t_k}) \cdots \exp(\mathrm{i}\lambda_{n+1} X_{t_{n+1}}),$$

and the extension to K_{n+1} is achieved by observing that random variables of the type

$$\int_0^T \int_0^{u_1} \cdots \int_0^{u_n} h(u_1, X_{u_1}; \dots; u_{n+1}, X_{u_{n+1}}) \,\mathrm{d}X_{u_{n+1}}^{(u_n)} \,\mathrm{d}X_{u_n}^{(u_{n-1})} \cdots \mathrm{d}X_{u_1} \tag{17}$$

form an Hilbert subspace of $L^2(\mathbb{P})$.

To prove that representation (17) is also sufficient for a random variable F to be in K_{n+1}, one can consider a r.v. F with such a representation and orthogonal to K_{n+1}. In particular, F will be orthogonal to every r.v. H with the form (16) and such that

$$\begin{aligned}\phi_{u_1,\dots,u_{n+1}}^{(H,n+1)}(X_{u_1}, \dots, X_{u_n}, X_{u_{n+1}}) = \prod_{k=1}^{n} \exp(\mathrm{i}\lambda_k X_{u_k}) 1_{(t_{(n+1)-k}, t_{(n+2)-k}-\varepsilon_{(n+2)-k})}(u_k) \\ \times \frac{\partial}{\partial z} h_{f_1,t_1}^{(u_n,X_{u_n})}(u_{n+1}, z)\Big|_{z=X_{u_{n+1}}} 1_{(0,t_1-\varepsilon_1)}(u_{n+1}),\end{aligned} \tag{18}$$

where $t_{n+1} > t_n > \cdots > t_1 > t_0 = 0$, $\varepsilon_k \in (0, t_k - t_{k-1})$ for every k and

$$f_1(X_{t_1}) = \exp(\mathrm{i}\gamma X_{t_1}). \tag{19}$$

Such a variable H belongs to K_{n+1} due to Lemma 9: as a matter of fact, Proposition 6 implies that H can be represented in the following way

$$\begin{aligned}H = \int_{t_n}^{t_{n+1}-\varepsilon_{n+1}} \int_{t_{n-1}}^{t_n-\varepsilon_n} \cdots \int_{t_1}^{t_2-\varepsilon_2} \Phi(X_{u_1}; \dots; X_{u_n}) \\ \times \mathbb{E}(Y_{u_1,\dots,u_n} \mid X_{u_1}, \dots, X_{u_n}; X_{t_1-\varepsilon_1}) \,\mathrm{d}X_{u_n}^{(u_{n-1})} \,\mathrm{d}X_{u_{n-1}}^{(u_{n-2})} \cdots \mathrm{d}X_{u_1},\end{aligned}$$

where the $Y_{u_1,\dots,u_n}$ are such that

$$
\begin{aligned}
Y_{u_1,\dots,u_n} &= f_1(X_{t_1}) - \mathbb{E}\big(f_1(X_{t_1}) \mid X_{u_1},\dots,X_{u_n}\big) \\
&= f_1(X_{t_1}) - \mathbb{E}\big(f_1(X_{t_1}) \mid X_{u_n}\big)
\end{aligned}
$$

and therefore

$$\mathbb{E}(Y_{u_1,\dots,u_n} \mid X_{u_1},\dots,X_{u_n}) = 0.$$

Since t_{n+1} and ε_k ($k = 1,\dots,n+1$) are arbitrary, orthogonality implies, a.e. $\mathrm{Leb}_{\Delta^{n+1}}$,

$$\mathbb{E}\left[h(u_1, X_{u_1};\dots;u_{n+1}, X_{u_{n+1}})\left(\prod_{k=1}^{n} \mathrm{e}^{\mathrm{i}\lambda_k X_{u_k}}\right)\mathrm{e}^{\mathrm{i}\gamma\frac{(u_n-t_1)X_{u_{n+1}}+(t_1-u_{n+1})X_{u_n}}{u_n-u_{n+1}}}\right] = 0$$

on $[t_n, T] \times [t_{n-1}, t_n] \times \cdots \times [0, t_1]$ for every $(\lambda_1,\dots,\lambda_n,\gamma)$. Set finally, for fixed $u_1,\dots,u_{n+1}$

$$\gamma' = \gamma\frac{u_n - u_{n+1}}{u_n - t_1}, \qquad \lambda_n' = \lambda_n - \gamma\frac{t_1 - u_{n+1}}{u_n - t_1}$$

to have that, for every $(\lambda_1,\dots,\lambda_n,\gamma)$

$$\mathbb{E}\left[h(u_1, X_{u_1};\dots;u_{n+1}, X_{u_{n+1}})\left(\prod_{k=1}^{n} \mathrm{e}^{\mathrm{i}\lambda_k X_{u_k}}\right)\mathrm{e}^{\mathrm{i}\gamma X_{u_{n+1}}}\right] = 0$$

which implies

$$h\big(u_1, X_{u_1}(\omega);\dots;u_{n+1}, X_{u_{n+1}}(\omega)\big) = 0 \quad \text{a.e.-}\mathrm{Leb}_{\Delta^{n+1}} \otimes \mathrm{d}\mathbb{P}$$

on $[t_n, T] \times \cdots \times [0, t_1]$, and the result is completely proved, as $t_1,\dots,t_n$ have been arbitrarily chosen.

To conclude the section, we can drop the recursive structure from the results presented at the beginning of this paragraph, so obtaining – as a combination of Proposition 7 and Proposition 8 – an actual generalization of Proposition 3.1, 3.2 and 3.3 of Föllmer, Wu and Yor [1].

PROPOSITION 10. – *For a r.v. F with $\mathbb{E}(F) = 0$, the following conditions are equivalent*:

(1) $F \in \overline{\Pi}_n^{\perp}$.

(2) *For every $i = 1,\dots,n$*

$$\mathbb{E}\big(\phi^{(F,i)}_{u_1,\dots,u_i} \mid X_{u_1},\dots,X_{u_i}\big) = 0, \quad \textit{a.e.-}\mathrm{Leb}_{\Delta^i}(\mathrm{d}u_1,\dots,\mathrm{d}u_i),$$

where $\phi^{(F,i)}_{u_1,\dots,u_i}$ is the ith Itô integrand of F.

(3) *For every $k < n$*

$$\mathbb{E}\big(\phi^{(F,k)}_{u_1,\dots,u_k} \mid X_{u_1},\dots,X_{u_k}; X_{t_1},\dots,X_{t_{n-k}}\big) = 0$$

a.e. $\mathrm{Leb}_{\Delta^k}(\mathrm{d}u_1,\dots,\mathrm{d}u_k)$, *for every $t_1 < \cdots < t_{n-k} < u_k$.*

6. Conclusion

It is clear that our results are intimately related with many subjects of current study.

Consider indeed the theory of weak Brownian motions, as exposed in Föllmer, Wu and Yor [1]. We define a weak Brownian motion (WBM) of order n to be a stochastic process Y whose marginal laws up to the nth order coincide with the marginals of BM, though Y is not a BM. Of specific interest is therefore the study of those WBM laws which are absolutely continuous with respect to $\mathbb{P}$. In particular, one can show that the proof of the existence – along with the subsequent characterization – of such laws relies heavily on the characterization of those $\Psi \in L^2(\mathbb{P})$ such that, for a fixed n,

$$\mathbb{E}(\Psi \mid X_{t_1}, \ldots, X_{t_n}) = 0, \quad \forall (t_1, \ldots, t_n) \in [0, T]^n$$

i.e. of the elements of $\overline{\Pi}_n^{\perp}$. Theorem 1 then furnishes a complete description (in terms of multiple time-space integrals) of such functionals, and seems to be of a certain interest to proceed with the analysis started in the above quoted reference.

On the other hand, as already pointed out, the relation between the CTSRP and the standard *chaotic representation property* (CRP) needs to be clarified, and this will be the object of a separate paper. For now, we stress the fact that such a relation is far from being straightforward, as shown in the following example. Consider indeed a zero mean r.v. $Y \in L^2(\mathbb{P})$ which is also an element of the second Wiener chaos (noted C_2), i.e. there exists a measurable function f, defined on Δ^2 and such that

$$\int_0^T \int_0^u f^2(u, s)\,\mathrm{d}s\,\mathrm{d}u < +\infty$$

and

$$Y = \int_0^T \int_0^u f(u, s)\,\mathrm{d}X_s\,\mathrm{d}X_u.$$

Then, from simple Hilbert space arguments, there exists a pair of measurable functions f_1 and f_2, defined respectively on $[0, T]$ and Δ^2, such that

$$\int_0^T f_1^2(u) u\,\mathrm{d}u < +\infty \qquad \int_0^T \int_0^u f_2^2(u, s)\,\mathrm{d}s\,\mathrm{d}u < +\infty,$$

and

$$\begin{aligned} Y &= \int_0^T \int_0^u f_1(u)\,\mathrm{d}X_s\,\mathrm{d}X_u + \int_0^T \int_0^u f_2(u, s)\,\mathrm{d}X_s^{(u)}\,\mathrm{d}X_u \\ &= \int_0^T f_1(u) X_u\,\mathrm{d}X_u + \int_0^T \int_0^u f_2(u, s)\,\mathrm{d}X_s^{(u)}\,\mathrm{d}X_u \\ &=: Y_1 + Y_2. \end{aligned} \tag{20}$$

It is evident that (20) corresponds to the orthogonal decomposition of Y in terms of K_1 and K_2, i.e. the first and second time-space Brownian chaos. As a consequence, we can conclude that (i) r.v.'s of the form of Y_2 *do not* exhaust C_2 and (ii) the orthogonal of C_1 (the first Wiener chaos) in K_1 *strictly contains* the projection of C_2 on K_1, since, e.g., for any function h bounded and measurable, and for any odd integer k

$$\int_0^T h(u) X_u^k \,\mathrm{d}X_u \in C_1^{\perp} \cap K_1.$$

It is eventually natural to ask whether the CTSRP is specific of BM, or is shared by other processes. In particular, we think about the first Azéma martingale and the compensated Poisson process, which have been shown to possess the CRP. More generally Föllmer, Wu and Yor [1] have shown that the non totality of the Π_k's (see the introduction) is valid for a wide class of Markov processes, and it is therefore arguable that such a class enjoys some analogue of the CTSRP for BM. Of course, the very difficulty of proving the above claim relies in the construction of multiple time-space Wiener integrals, for which a pervasive use of the theory of (initial) *enlargements of filtrations* (as developed by Jeulin et al.) must be performed.

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REFERENCES

[1] Föllmer H., Wu C.-T., Yor M., On weak Brownian motions of arbitrary order, Annales de l'Institut H. Poincaré 36 (4) (2000) 447–487.
[2] Nualart D., The Malliavin Calculus and Related Topics, Springer, Berlin, 1995.
[3] Stroock D.W., Homogeneous chaos revisited, in: Séminaire de Probabilités XXI, Lecture Notes in Mathematics, Vol. 1247, Springer, 1987, pp. 1–8.
[4] Wiener N., The homogeneous chaos, Amer. J. Math. 60 (1938) 879–936.
[5] Yor M., Some Aspects of Brownian Motion. Part I: Some Special Functionals, ETH Zürich Lectures in Mathematics, Birkhäuser, Basel, 1992.