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Asymptotic expansion of stochastic oscillatory integrals with rotation invariance

by

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ABSTRACT. – Asymptotics of oscillatory integrals on the classical 2-dimensional Wiener space, whose phase functional is the stochastic line integral of a 1-form, is considered. Under the assumption that the exterior derivative of the 1-form is rotation invariant, an asymptotic expansion is obtained. This result is extended to the process associated to a general rotation invariant metric. © Elsevier, Paris

Key words: Wiener integrals, Asymptotic expansions, Oscillatory integrals, Diffusion processes

RÉSUMÉ. – Nous étudions le comportement asymptotique d'intégrales oscillantes sur l'espace de Wiener classique en dimension 2, lorsque la phase est l'intégrale stochastique d'une 1-forme. Sous l'hypothèse que la dérivée extérieure de la 1-forme est invariante par rotation, nous obtenons ce développement asymptotique. Nous étendons ce résultat à la diffusion associée à une métrique invariante par rotation. © Elsevier, Paris

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1. INTRODUCTION

Let (W, P) be the 2-dimensional Wiener space: $W = \{w \in C([0, \infty) \to \mathbb{R}^2)\colon w(0) = 0\}$ and P is the Wiener measure on W. Let b be a C^1 differential 1-form on $\mathbb{R}^2$ satisfying the following rotation invariance condition:

$$db = f(|x|)\,dx^1 \wedge dx^2. \tag{1.1}$$

In this paper we investigate the asymptotic behavior of

$$I(\xi; a) = E\left[\exp\left(\mathrm{i}\xi \int_0^1 b\big(w(t)\big) \circ dw(t)\right) a\big(w(\cdot)\big) \middle| w(1) = 0\right] \tag{1.2}$$

and

$$\mathcal{I}(\xi; a) = E\left[\exp\left(\mathrm{i}\xi \int_0^1 b\big(w(t)\big) \circ dw(t)\right) a\big(w(\cdot)\big)\right] \tag{1.3}$$

as $\xi \to \infty$, where $\mathrm{i} = \sqrt{-1}$ and $\circ dw(t)$ is the Stratonovitch stochastic integral. Under some conditions on the function f in (1.1) and the amplitude functional a, we give an asymptotic expansion as

$$I(\xi; a) \sim \exp\big(-\xi f(0)/2 - \sqrt{\xi}\mu_0^1 - \mu_0^2\big)\xi\left\{f(0)a(0) + \sum_{n=1}^{\infty} \xi^{-n/2} I_n\right\} \tag{1.4}$$

and a similar expansion for $\mathcal{I}(\xi; a)$. Moreover we extend these results to the case, where the Wiener process w is replaced by the process associated to a general rotation invariant metric and the 1-form b may degenerate finitely at the origin.

The integrals (1.2) and (1.3) are called stochastic oscillatory integrals. These integrals appear, for example, in the study of the Schrödinger operator with a magnetic field and the 2-form db in (1.1) corresponds to the magnetic field (cf. [21]). For these integrals the asymptotic behavior is estimated from above in more general setting [3,4,12–14,29]. For the exact leading term, Ikeda–Manabe studied the following two cases: one is the case where the phase functional is a quadratic functional, and the other is the above rotation invariant case [9]. For the quadratic case, we have a general exact formula and many related results are obtained. For this aspect, see [7,9,14,23,24,26,27] and references therein. However, as opposed to finite-dimensional oscillatory integrals, it is difficult to obtain the asymptotic behavior for more general phase functionals from these results. On the other hand, the rotation invariant case is the simplest nonquadratic case for which the asymptotic behavior can be investigated in detail. In this paper we will extend the results in [9] on the rotation invariant case. As for the asymptotic expansion, Ben Arous gave for a function corresponding to (1.3), where $w(s)$ is replaced by $w(s)/\sqrt{\xi}$ in more general setting without rotation invariance conditions [1].

Under the rotation invariance condition (1.1), by taking the expectation in the spherical direction, our problem is reduced to that on Laplace type asymptotic behavior. For example, $I(\xi; 1)$ is written as

$$I(\xi; 1) = E\left[\exp\left(-\frac{\xi^2}{2}\int_0^1 F\left(|w(t)|\right)^2 dt\right)\middle| w(1) = 0\right], \tag{1.5}$$

where

$$F(r) = \int_0^1 f(ur)ur\,du.$$

For more general cases, see Lemmas 2.1 and 3.1 below. By the Feynman–Kac formula the right hand side is related to the integral kernel of the heat semigroup of the Schrödinger operator $-\Delta + \xi^2 F^2$. Then the problem on the asymptotic behavior corresponds to a problem on the semiclassical approximation. We now use the technique of Simon [22] for the asymptotic expansion of the eigenvalues and eigenfunctions in the semiclassical limit. For this Laplace type asymptotic behavior, more general results are obtained for the case where $w(s)$ is replaced by $w(s)/\xi$. For these results, see [2] and references therein.

The aim of Ikeda–Manabe [9] was to show an infinite dimensional analogue of the principle of the stationary phase, in the setting of Wiener functional integrals. In fact, we can formally regard our results as examples of this principle. For this aspect, see Remark 3.2(i) below.

For a general rotation invariant metric

$$dr \otimes dr + g(r)^2\, d\theta \otimes d\theta$$

in terms of the polar coordinate (r, θ), $r \geqslant 0$, $\theta \in S^1$, we reduce the problem to that on the radial process and use techniques of the theory of the 1-dimensional diffusion processes and ordinary differential equations.

Since the asymptotic behavior of our oscillatory integrals are determined by the behavior of the metric $g(r)$ and the 1-form b at the boundary $r = 0$, we need precise analysis at this boundary. Under the condition that $\lim_{r\to 0} g(r)/r^{\alpha} = 1$ for some $\alpha \in \mathbb{R}$, $r = 0$ is an entrance boundary point for the radial process if $\alpha \geqslant 1$, a regular boundary point if $\alpha \in (-1, 1)$, and an exit boundary point if $\alpha \leqslant -1$, in the sense of Feller [5]. Accordingly we pose the Neumann boundary condition at $r = 0$ for $\alpha \geqslant -1$ and the Dirichlet boundary condition for $\alpha \leqslant 1$. If we moreover assume that $g(r)/r^{\alpha} = 1 + o(r^2)$ as $r \downarrow 0$, then the generator is regarded as a simple perturbation of that for the case $g(r) = r^{\alpha}$. Then we use the explicit representation of the transition density of the Bessel process with index $\alpha + 1$ for $\alpha \geqslant -1$ and the corresponding representation for the Dirichlet case with $\alpha \leqslant 1$ (see Lemma 5.1 below). When this condition is not satisfied, the existence of the function such as $I(\xi; a)$ in (1.3) is not trivial. However, by a work of Hille [6] (see also Matsumoto [15]) on the continuity of the transition density at the boundary, we obtain a sharp result for the case of the Neumann condition with $-1 < \alpha < 3$, which corresponds to the limit circle case (see Theorem 6 below).

The organization of this paper is as follows: In Section 2, we treat a fundamental case for the function $I(\xi; a)$ in (1.2) with a simple amplitude functional a. In Section 3, we consider more general amplitude functionals a for both functions $I(\xi; a)$ and $\mathcal{I}(\xi; a)$. In this section, we also give a fundamental remark for our problem (see Remark 3.2 below). In Sections 4 and 5, we consider the case of a general rotation invariant metric. Section 4 is devoted to the case of Neumann condition and Section 5 is devoted to the case of Dirichlet condition.

2. A FUNDAMENTAL CASE

On $\mathbb{R}^2$ with the standard metric, we take a C^1 differential 1-form b satisfying the following condition:

(A.1)

(i) $db = f(r)\,dx^1 \wedge dx^2,\ r = |x|$;

(ii) $f(0) > 0$;

(iii) $f(r) \geqslant 0$;

(iv) $\underline{\lim}_{r\to\infty} r^{1-\varepsilon} f(r) > 0$ for some $\varepsilon > 0$;

(v) f is smooth on the open interval $(0, \infty)$ and each derivative of f is dominated by a polynomial.

We take a functional a on the Wiener space W. For this functional, we introduce the following conditions:

(A.2) $a(w) = A(r(t_1), r(t_2), \ldots, r(t_K))$, where $r(\cdot) = |w(\cdot)|$, $0 < t_1 < t_2 < \cdots < t_K < 1$ and A is a smooth function on $[0, \infty)^K$ whose derivatives are dominated by polynomials.

Let $\mu_0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$ be the eigenvalues of the harmonic oscillator

$$H = \frac{-\Delta}{2} + \frac{1}{2}\left(\frac{f(0)r}{2}\right)^2$$

on $L^2(\mathbb{R}^2)$, $\varphi_0, \varphi_1, \varphi_2, \ldots$ be the corresponding normalized eigenfunctions, and $P_\perp$ be the orthogonal projection to the orthogonal complement to the ground states, where $\Delta = \sum_{j=1}^{2} \partial^2/\partial(x^j)^2$. In this case, $\mu_0 = f(0)/2$ and φ_0 can be taken as

$$\varphi_0(x) = \sqrt{\frac{f(0)}{2\pi}} \exp\left(-\frac{f(0)}{4} r^2\right).$$

We denote the norm of $L^2(\mathbb{R}^2)$ by $\|\cdot\|$ and the inner product by $(\cdot, \cdot)$.

For the function $I(\xi; a)$, $\xi \in \mathbb{R}$, defined in (1.2), we prove the following in this section:

THEOREM 1. – *Under the conditions* (A.1)–(A.2), *we have the following asymptotic expansion as* $\xi \to \infty$:

$$I(\xi; a) \sim \exp(-\xi\mu_0 - \sqrt{\xi}\mu_0^1 - \mu_0^2)\xi \sum_{n=0}^{\infty} \xi^{-n/2} I_n,$$

in the sense that

$$I(\xi; a) \exp\left(\xi\mu_0 + \sqrt{\xi}\mu_0^1 + \mu_0^2\right)\frac{1}{\xi} - \sum_{n=0}^{N-1} \xi^{-n/2} I_n = \mathrm{O}(\xi^{-N/2})$$

for any $N \geqslant 1$, *where*

$$\mu_0 = \frac{f(0)}{2},$$

$$\mu_0^1 = \frac{f'(0)}{2}\sqrt{\frac{\pi}{2f(0)}},$$

$$\mu_0^2 = \frac{f''(0)}{2f(0)} + \left(\frac{2f'(0)}{3f(0)}\right)^2 - \left(\frac{f(0)f'(0)}{6}\|(H - \mu_0)^{-1/2} P_\perp r^3 \varphi_0\|\right)^2,$$

$$I_0 = f(0)a(0),$$

and I_n, $n \in \mathbb{N}$, *are finite linear combinations of* $\{(\partial^\alpha A)(0)\colon \alpha \in \mathbb{Z}_+^K\}$ *whose coefficients are polynomials of*

$$\{f^{(m)}(0), (\varphi_0, r^{\ell(0)}(\mu_0 - H)^{-1} P_\perp \cdots r^{\ell(m-1)}(\mu_0 - H)^{-1} P_\perp r^{\ell(m)} \varphi_0)\colon m, \ell(0), \ell(1), \dots, \ell(m) \in \mathbb{Z}_+\}.$$

We first show the following:

LEMMA 2.1. –

$$I(\xi; a) = E\left[\exp\left(-\frac{1}{2}\int_0^\xi F(r(t), \xi)^2\, dt\right) A\left(\frac{r(\xi\cdot)}{\sqrt{\xi}}\right) \middle| w(\xi) = 0\right] \tag{2.1}$$

where

$$F(r, \xi) = \int_0^1 f\left(\frac{ur}{\sqrt{\xi}}\right) ur\, du.$$

Proof. – By the stochastic version of Stokes' theorem (cf. [8,25]), we have

$$\int_0^1 b(w(t)) \circ dw(t) = \int_0^1 f_1(r(t)) \circ dS(t) + \int_0^1 b(uw(1)) \cdot w(1)\, du, \tag{2.2}$$

where

$$f_1(r) = \int_0^1 f(ur) 2u\, du$$

and $S(t)$ is Lévy's stochastic area defined by

$$S(t) = \frac{1}{2}\int_0^t \{w^1(s)\, dw^2(s) - w^2(s)\, dw^1(s)\}.$$

As is explained in Section VI-6 of [10], the stochastic area is represented as

$$S(t) = B\left(\frac{1}{4}\int_0^t r(s)^2\, ds\right), \tag{2.3}$$

where $B(\cdot)$ is a 1-dimensional Brownian motion independent of the process $\{r(\cdot)\}$. By taking the expectation in the Brownian motion $B(\cdot)$, we have

$$I(\xi;a)=E\left[\exp\left(-\frac{\xi^2}{2}\int_0^1 F(r(t),1)^2\,dt\right)A(r(\cdot))\,\middle|\,w(1)=0\right]. \tag{2.4}$$

By using the scaling property of the Brownian motion

$$\{w(\cdot)\}\overset{\mathcal{L}}{\sim}\left\{\frac{w(\xi\cdot)}{\sqrt{\xi}}\right\},$$

we obtain (2.1). □

We introduce an operator

$$H(\xi)=\frac{1}{2}(-\Delta+F(r,\xi)^2),$$

where $F(r,\xi)$ is the function defined in Lemma 2.1. This operator is essentially self adjoint on $C_0^\infty(\mathbb{R}^2)$ by Theorem X.28 in [18]. We denote the self adjoint extension by the same symbol. The integral kernel $e^{-tH(\xi)}(x,x')$, $t>0$, $x,x'\in\mathbb{R}^2$ of the heat semigroup $e^{-tH(\xi)}$, which we call the heat kernel of $H(\xi)$ in the following, has the following representation:

$$e^{-tH(\xi)}(x,x')=E\left[\exp\left(-\frac{1}{2}\int_0^t F(r(s),\xi)^2\,ds\right)\,\middle|\,x+w(t)=x'\right]\times\frac{\exp(-|x-x'|^2/(2t))}{2\pi t}$$

(cf. [21] Theorem 6.2). Thus (2.4) is rewritten as follows:

$$\begin{aligned}I(\xi;a)=2\pi\xi\int_{\mathbb{R}^{2K}} & e^{-\xi t_1 H(\xi)}(0,x_1)e^{-\xi(t_2-t_1)H(\xi)}(x_1,x_2)\\ &\times\cdots\times e^{-\xi(t_K-t_{K-1})H(\xi)}(x_{K-1},x_K)e^{-\xi(1-t_K)H(\xi)}(x_K,0)\\ &\times A\left(\frac{r_1}{\sqrt{\xi}},\frac{r_2}{\sqrt{\xi}},\dots,\frac{r_K}{\sqrt{\xi}}\right)dx_1\,dx_2\cdots dx_K.\end{aligned}$$

On the other hand, by the condition (A.1) (iv), the operator $H(\xi)$ has purely discrete spectrum (cf. [19] Theorem XIII.67). We denote the eigenvalues by $\mu_0(\xi) \leqslant \mu_1(\xi) \leqslant \mu_2(\xi) \leqslant \cdots$ and the corresponding normalized eigenfunctions by $\varphi_0(x,\xi), \varphi_1(x,\xi), \varphi_2(x,\xi), \ldots$. Since the heat semigroup $e^{-tH(\xi)}$ generated by $H(\xi)$ is positivity improving, we see that $\mu_0(\xi) < \mu_1(\xi)$ and we can take $\varphi_0(x,\xi)$ as a positive function (see [19] Theorem XIII.44). By Mercer's expansion theorem, the heat kernel $e^{-tH(\xi)}(x,y)$ is represented as the following convergent series:

$$e^{-tH(\xi)}(x,y) = \sum_{n=0}^{\infty} e^{-t\mu_n(\xi)}\varphi_n(x,\xi)\varphi_n(y,\xi). \tag{2.5}$$

For the asymptotic behavior as $\xi \to \infty$, we follow the argument of Simon [22]. We first show the following:

PROPOSITION 2.1. – *For each $n \geqslant 0$, $\lim_{\xi\to\infty}\mu_n(\xi) = \mu_n$.*

Proof. – For the upper bound, we use the Rayleigh–Ritz principle:

$$\mu_n(\xi) \leqslant \text{the largest eigenvalue of } \big((\varphi_j, H(\xi)\varphi_k)\big)_{1\leqslant j,k\leqslant n}.$$

Since each φ_j is a polynomial times a Gaussian function, we easily see that

$$\big(\varphi_j, H(\xi)\varphi_k\big) = \mu_j\delta_{jk} + O\left(\frac{1}{\sqrt{\xi}}\right).$$

Therefore we have

$$\varlimsup_{\xi\to\infty} \mu_n(\xi) \leqslant \mu_n.$$

We next show the lower bound. We take smooth functions $\{h,k\}$ on $\mathbb{R}^2$ so that $h^2 + k^2 \equiv 1$, $\operatorname{supp} h \subset \{|x| < 2\}$ and $\operatorname{supp} k \subset \{|x| > 1\}$. For any $\xi > 0$, we set $h_\xi(x) = h(x/\xi^{1/10})$ and $k_\xi(x) = k(x/\xi^{1/10})$. By the Ismagilov–Morgan–Sigal–Simon localization ([22] Lemma 3.1), we have

$$\big(\varphi, H(\xi)\varphi\big) \geqslant \big(h_\xi\varphi, H(\xi)h_\xi\varphi\big) + \big(k_\xi\varphi, H(\xi)k_\xi\varphi\big) - \frac{C_1}{\xi^{1/5}}\|\varphi\|^2$$

for some $C_1 > 0$ and any $\varphi \in C_0^\infty(\mathbb{R}^2)$. For the first term, we easily see that

$$\big(h_\xi\varphi, H(\xi)h_\xi\varphi\big) \geqslant (h_\xi\varphi, Hh_\xi\varphi) - \frac{C_2}{\xi^{1/5}}\|\varphi\|^2.$$

For the second term, we estimate as follows:

$$\left(k_\xi\varphi, H(\xi)k_\xi\varphi\right) \geqslant \frac{1}{2}\left\|F(r,\xi)k_\xi\varphi\right\|^2 \geqslant \frac{1}{2}\Big(\inf_{r>\xi^{1/10}} F(r,\xi)\Big)^2 \|k_\xi\varphi\|^2.$$

By the conditions (A.1) (ii)–(iv), we can estimate as

$$\inf_{r>\xi^{1/10}} F(r,\xi) \geqslant C_3\xi^{1/10}$$

for some $C_3 > 0$.

We now take $\mu \in (\mu_n, \mu_{n+1})$ and set $G = h_\xi(H-\mu)P[\mu]h_\xi$, where $P[\mu]$ is the orthogonal projection onto the eigenspace of H corresponding to the eigenvalues less than μ and h_ξ is regarded as a multiplication operator. Then, by the above estimates, we have

$$\left(\varphi, H(\xi)\varphi\right) \geqslant \mu\|\varphi\|^2 + (\varphi, G\varphi) - \frac{C_4}{\xi^{1/5}}\|\varphi\|^2$$

for large enough ξ. Since the rank of G is at most n, we have

$$\mu_{n+1}(\xi) \geqslant \mu - \frac{C_4}{\xi^{1/5}}$$

by the min-max principle. From this we obtain

$$\varliminf_{\xi\to\infty} \mu_n(\xi) \geqslant \mu_n. \tag{2.6}$$

□

Moreover, by the same procedure as in Simon [22], we have the following:

PROPOSITION 2.2. – (i) *As $\xi \to \infty$, we have*

$$\mu_0(\xi) \sim \mu_0 + \sum_{n=1}^{\infty} \frac{\mu_0^n}{\sqrt{\xi}^n},$$

where μ_0^1, μ_0^2 are those of Theorem 1.

(ii) *As $\xi \to \infty$, we have*

$$\varphi_0(x,\xi) \sim \varphi_0(x) + \sum_{n=1}^{\infty} \frac{\varphi_0^n(x)}{\sqrt{\xi}^n},$$

where φ_0^n, $n \in \mathbb{N}$, are continuous functions such that $r^m \varphi_0^n \in L^2$ for any $m \geqslant 0$, in the following sense:

$$\sup_{x \in K} \left| \varphi_0(x, \xi) - \varphi_0(x) - \sum_{n=1}^{N-1} \frac{\varphi_0^n(x)}{\sqrt{\xi}^n} \right| = \mathrm{O}(\xi^{-N/2}) \tag{2.7}$$

for any compact set K in $\mathbb{R}^2$, and

$$\left\| r^m \left\{ \varphi_0(x, \xi) - \varphi_0(x) - \sum_{n=1}^{N-1} \frac{\varphi_0^n(x)}{\sqrt{\xi}^n} \right\} \right\| = \mathrm{O}(\xi^{-N/2}) \tag{2.8}$$

for any $m \geqslant 0$.

Proof. – The proof of (i) and (2.8) with $m = 0$ is identical to that of Simon [22] Theorem 4.1. The rest is also proven by modifying slightly the proof of Simon [22] Theorem 4.1. In fact, if we write as

$$\varphi_0(x, \xi) = \varphi_0(x) + \sum_{n=1}^{N-1} \frac{\varphi_0^n(x)}{\sqrt{\xi}^n} + \frac{\varphi_0^{[N]}(x, \xi)}{\sqrt{\xi}^N},$$

then φ_0^n are linear combinations of

$$\int_{|z-\mu_0|=\varepsilon} \frac{1}{(z-\mu_0)} (z-H)^{-1} r^{k(1)} (z-H)^{-1} r^{k(2)} \cdots (z-H)^{-1} r^{k(m)} \varphi_0 \, dz,$$

$m, k(1), k(2), \ldots, k(m) \in \mathbb{Z}_+$ and $\varphi_0^{[N]}$ is also some polynomial of the functions dominated by functions of this type and

$$\int_{|z-\mu_0|=\varepsilon} \frac{1}{(z-\mu_0)} \{(z-H)^{-1} V(r, \xi)\}^N \left(z - H(\xi)\right)^{-1} \varphi_0 \, dz,$$

where $V(r, \xi) = F(r, \xi)^2 - (f(0)r/2)^2$ and $\varepsilon > 0$ is taken small enough. Then to prove (2.8), as in the proof of Theorem 4.1 of [22], it is enough to show the subsequent Lemma 2.2. To prove (2.7), by the Sobolev lemma, it is enough to show that

$$\left\| (1-\Delta)\zeta \left\{ \varphi_0(x, \xi) - \varphi_0(x) - \sum_{n=1}^{N-1} \frac{\varphi_0^n(x)}{\sqrt{\xi}^n} \right\} \right\| = \mathrm{O}(\xi^{-N/2})$$

for any $\zeta \in C_0^\infty(\mathbb{R}^2)$. This is proven by the subsequent Lemmas 2.2 and 2.3. □

LEMMA 2.2 (Lemma 4.2 of [22]). – *For each fixed $m \in \mathbb{Z}_+$, we have*

$$\sup_{|z-\mu_0|=\varepsilon} \left\|(1+r^2)^{m/2}(z-H)^{-1}(1+r^2)^{-m/2}\right\|_{op} < \infty$$

and

$$\sup_{|z-\mu_0|=\varepsilon} \left\|(1+r^2)^{m/2}\left(z-H(\xi)\right)^{-1}(1+r^2)^{-m/2}\right\|_{op} < \infty,$$

where $\|\cdot\|_{op}$ is the operator norm and $\varepsilon > 0$ is taken small enough.

LEMMA 2.3. – *For any $\zeta \in C_0^\infty(\mathbb{R}^2)$, we have*

$$\sup_{|z-\mu_0|=\varepsilon} \left\|(1-\Delta)\zeta(z-H)^{-1}\right\|_{op} < \infty$$

and

$$\sup_{|z-\mu_0|=\varepsilon} \left\|(1-\Delta)\zeta\left(z-H(\xi)\right)^{-1}\right\|_{op} < \infty.$$

Proof of Theorem 1. – We decompose as

$$I(\xi;a) = I_0(\xi;a) + I_R(\xi;a),$$

where

$$I_0(\xi;a) = 2\pi\xi e^{-\xi\mu_0(\xi)}\varphi_0(0,\xi)^2 \int_{\mathbb{R}^{2K}} \varphi_0(x_1,\xi)^2\varphi_0(x_2,\xi)^2\cdots\varphi_0(x_K,\xi)^2$$
$$\times A\left(\frac{r_1}{\sqrt{\xi}},\frac{r_2}{\sqrt{\xi}},\ldots,\frac{r_K}{\sqrt{\xi}}\right)dx_1\,dx_2\cdots dx_K$$

and $I_R(\xi;a) = I(\xi;a) - I_0(\xi;a)$. $I_0(\xi;a)$ is expanded by Proposition 2.2 and $I_R(\xi;a)$ is negligible by the subsequent Lemma 2.4. □

LEMMA 2.4. –

$$\overline{\lim_{\xi\to\infty}} \frac{1}{\xi}\log|I_R(\xi;a)| < -\mu_0.$$

Proof. – Since A is of polynomial growth, it is enough to show that

$$\overline{\lim_{\xi\to\infty}} \frac{1}{\xi}\log|I_{<m>}(\xi;\alpha_1,\alpha_2,\ldots,\alpha_K)| < -\mu_0, \tag{2.9}$$

for any $m \in \{0, 1, \ldots, K\}, \alpha_1, \alpha_2, \ldots, \alpha_K \in \mathbb{Z}_+$, where

$$\begin{aligned}
&I_{\langle m\rangle}(\xi; \alpha_1, \alpha_2, \ldots, \alpha_K) \\
&\quad = \mathrm{e}^{-\xi t_m \mu_0(\xi)} \varphi_0(0, \xi) \int\limits_{\mathbb{R}^{2K}} \varphi_0(x_1, \xi)^2 \varphi_0(x_2, \xi)^2 \cdots \varphi_0(x_{m-1}, \xi)^2 \varphi_0(x_m, \xi) \\
&\qquad \times \big(\mathrm{e}^{-\xi(t_{m+1}-t_m)H(\xi)}(x_m, x_{m+1}) \\
&\qquad\quad - \mathrm{e}^{-\xi(t_{m+1}-t_m)\mu_0(\xi)} \varphi_0(x_m, \xi) \varphi_0(x_{m+1}, \xi)\big) \\
&\qquad \times \mathrm{e}^{-\xi(t_{m+2}-t_{m+1})H(\xi)}(x_{m+1}, x_{m+2}) \cdots \mathrm{e}^{-\xi(1-t_K)H(\xi)}(x_K, 0) \\
&\qquad \times r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_K^{\alpha_K} \, dx_1 \, dx_2 \cdots dx_K. \qquad (2.10)
\end{aligned}$$

We take $\delta > 0$ small enough. By using (2.5) and Schwarz's inequality we estimate as follows:

$$\begin{aligned}
&\mathrm{e}^{-\xi t H(\xi)}(x, y) \\
&\quad \leqslant \sqrt{\sum_{n=0}^{\infty} \mathrm{e}^{-\xi t \mu_0(\xi)} \varphi_0(x, \xi)^2} \sqrt{\sum_{n=0}^{\infty} \mathrm{e}^{-\xi t \mu_0(\xi)} \varphi_0(y, \xi)^2} \\
&\quad \leqslant \mathrm{e}^{-(\xi t - \delta)\mu_0(\xi)} \sqrt{\mathrm{e}^{-\delta H(\xi)}(x, x)} \sqrt{\mathrm{e}^{-\delta H(\xi)}(y, y)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathrm{e}^{-\xi t H(\xi)}(x, y) - \mathrm{e}^{-\xi t \mu_0(\xi)} \varphi_0(x, \xi) \varphi_0(y, \xi)\right| \\
&\quad \leqslant \sqrt{\sum_{n=1}^{\infty} \mathrm{e}^{-\xi t \mu_0(\xi)} \varphi_0(x, \xi)^2} \sqrt{\sum_{n=1}^{\infty} \mathrm{e}^{-\xi t \mu_0(\xi)} \varphi_0(y, \xi)^2} \\
&\quad \leqslant \mathrm{e}^{-(\xi t - \delta)\mu_1(\xi)} \sqrt{\mathrm{e}^{-\delta H(\xi)}(x, x)} \sqrt{\mathrm{e}^{-\delta H(\xi)}(y, y)}.
\end{aligned}$$

Moreover we use the following: for each $k \in \mathbb{N}$,

$$\sup_{\xi} \left\| \varphi_0(x, \xi) r^k \right\| < \infty \qquad (2.11)$$

and there are $C > 0$ and $N \in \mathbb{N}$ such that

$$\int\limits_{\mathbb{R}^2} \mathrm{e}^{-\delta H(\xi)}(x, x) r^k \, dx \leqslant C\big(1 + \xi^N\big). \qquad (2.12)$$

(2.11) is deduced from Proposition 2.2 and (2.12) can be shown as follows: by using the Feynman–Kac formula, Jensen's inequality, the translation invariance of the Lebesgue measure and the assumption (A.1) (iv), we estimate as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathrm{e}^{-\delta H(\xi)}(x,x) r^k\, dx \\
&\leqslant \int_{\mathbb{R}^2} dx \frac{r^k}{2\pi\delta^2} E\left[\int_0^\delta dt \exp\left(-\frac{\delta}{2} F\left(|x+w(t)|,\xi\right)^2\right)\middle| w(\delta)=0\right] \\
&= E\left[\int_0^\delta dt \int_{\mathbb{R}^2} dx \frac{|x-w(t)|^k}{2\pi\delta^2} \exp\left(-\frac{\delta}{2} F(r,\xi)^2\right)\middle| w(\delta)=0\right] \\
&\leqslant C_1\left\{1+\int_0^\infty dr\cdot r^{k+1}\exp\left(-\frac{\delta}{2}F(r,\xi)^2\right)\right\} \\
&\leqslant C_2(1+\xi^N)
\end{aligned}$$

for some $N>0$.

Now we can show (2.9) by using Schwarz's inequality and these estimates in the right hand side of (2.10). □

3. OTHER AMPLITUDES

In this section we consider more general amplitude functionals. We first give a fundamental theorem for the function $\mathcal{I}(\xi;a)$, $\xi\in\mathbb{R}$, defined in (1.3). For this we introduce the following conditions:

(A.3) $b(0)=0$, b is smooth, and each derivative of b is dominated by a polynomial.

(A.4) $a(w)=A(r(t_1),r(t_2),\ldots,r(t_K))C(w(1))$, where $A(r(t_1),r(t_2),\ldots,r(t_K))$ is same as in (A.2) and C is a smooth function on $\mathbb{R}^2$ whose derivatives are dominated by polynomials.

Then we have the following:

THEOREM 2. – *Under the conditions* (A.1), (A.3) *and* (A.4), *we have the following asymptotic expansion as* $\xi\to\infty$:

$$\mathcal{I}(\xi;a)\sim\exp\left(-\xi\mu_0-\mu_0^2\right)\sum_{n=0}^{\infty}\xi^{-n}\mathcal{I}_n,$$

in the same sense as in Theorem 1, *where* μ_0, μ_0^2 *are same as in Theorem* 1, $\mathcal{I}_0=a(0)/\sqrt{\det\{E_2-\mathrm{i}(\nabla b(0)+{}^t\nabla b(0))/f(0)\}}$, E_2 *is the*

2×2 *identity matrix and* $\mathcal{I}_n$, $n \in \mathbb{N}$, *are finite linear combinations of* $\{(\partial^\alpha A)(0) \times (\partial^\beta C)(0)\colon\ \alpha \in \mathbb{Z}_+^K, \beta \in \mathbb{Z}_+^2\}$ *whose coefficients are polynomials of*

$$\{f^{(2m)}(0), (\varphi_0, r^{\ell(0)}(\mu_0 - H)^{-1} P_\perp \cdots r^{\ell(m-1)}(\mu_0 - H)^{-1} P_\perp r^{\ell(m)} \varphi_0)\colon$$
$$m, \ell(0), \ell(1), \ldots, \ell(m) \in \mathbb{Z}_+\}.$$

Remark 3.1. – Under the condition (A.3), $f'(0) = \mu_0^1 = 0$.

To consider more general amplitude functionals, we introduce the following conditions:

(A.5) $\underline{\lim}_{r\to\infty} f(r) > 0$.

(A.6) $a = A(r(\cdot)) \in L^1(P(\cdot|w(1) = 0))$, $\lim_{\|r(\cdot)\|_2 \to 0} A(r(\cdot)) = A(0)$ and $|A(0)| < \infty$, where $\|\cdot\|_2$ is the norm on the L^2 space on the interval $[0, 1]$.

(A.7) $a = A(r(\cdot))C(w(1))$ where $A \in L^1(P)$, $|A(0)| < \infty$,

$$\lim_{\|r(\cdot)\|_2\to 0} A(r(\cdot)) = A(0),$$

C is C^1 and C, ∇C are bounded.

Then we have the following:

THEOREM 3. – *Under the conditions* (A.1), (A.5) *and* (A.6), *we have*

$$I(\xi; a) = \exp\left(-\xi\mu_0 - \sqrt{\xi}\mu_0^1 - \mu_0^2\right)\xi\left(I_0 + o(1)\right),$$

as $\xi \to \infty$, *where* μ_0, μ_0^1, μ_0^2 *and* I_0 *are given in Theorem* 1.

THEOREM 4. – *Under the conditions* (A.1), (A.3), (A.5) *and* (A.7), *we have*

$$\mathcal{I}(\xi; a) = \exp\left(-\xi\mu_0 - \mu_0^2\right)\left(\mathcal{I}_0 + o(1)\right),$$

as $\xi \to \infty$, *where* μ_0, μ_0^1, μ_0^2, $\mathcal{I}_0$ *are given in Theorem* 2.

Moreover, we introduce the following condition:

(A.8) $a = A(r(\cdot))$, $A \geqslant 0$, $\sup A < \infty$ and

$$\inf\{A(r(\cdot))\colon \sup_{[0,1]} r(t) \leqslant R\} > 0 \quad \text{for some } R > 0.$$

Then, for the function $I(\xi; a)$, we have the following:

THEOREM 5. – *Under the conditions* (A.1) *and* (A.8), *we have*

$$\lim_{\xi\to\infty} \frac{1}{\xi} \log I(\xi; a) = -\frac{f(0)}{2}.$$

We here give general remarks:

Remark 3.2. – (i) As is explained in Ikeda–Manabe [9], our results are regarded formally as an infinite-dimensional example of the principle of the stationary phase. We here discuss this aspect. We assume $b(0)=0$. On the Cameron–Martin subspace H of W, we consider a functional $\beta(h)$, $h\in H$, defined by

$$\beta(h)=\int_0^1 b\big(h(s)\big)\cdot dh(s).$$

This functional is sometimes called the skeleton of our phase functional. For $k\in H$ such that $k(1)=0$, the derivative in the direction k is

$$D\beta(h)[k]=\int_0^1\int_0^t f\big(h(s)\big)\big\{dh^2(s)\,dk^1(t)-dh^1(s)\,dk^2(t)\big\}.$$

Therefore, $h\equiv 0$ is the only stationary point. The Hessian at $h\equiv 0$ is

$$\begin{aligned}(D^2\beta)(0)[h,k]&=\langle\nabla b(0),h(1)\otimes k(1)\rangle\\&\quad+f(0)\int_0^1\big\{h^1(t)\,dk^2(t)-h^2(t)\,dk^1(t)\big\}\end{aligned}$$

for $h,k\in H$. Another expression of this is

$$(D^2\beta)(0)[h,k]=(Bh,k)_H,$$

where B is a linear map on H determined by

$$\frac{d}{dt}(Bh)(t)=\nabla\big\{h(1)\cdot b\big\}(0)+f(0)\begin{pmatrix}0&-1\\1&0\end{pmatrix}h(t).$$

This operator B is one to one. Thus $h\equiv 0$ satisfies the condition of the nondegenerate stationary point except that the operator B may not be onto (cf. [17]). On the other hand, our results state that the asymptotic behavior of the oscillatory integrals $I(\xi;a)$ and $\mathcal{I}(\xi;a)$ as $\xi\to\infty$ are determined mainly by the behavior of the functionals $\beta(h)$ and a at the point $h\equiv 0$ on H. More precisely, if we assume b is smooth at 0 and put $\beta_2(w)=f(0)S(1)$ and $I_2(\xi)=E[\exp(\mathrm{i}\xi\beta_2(w))\mid w(1)=0]$, then we have

$$\lim_{\xi\to\infty} I(\xi,1)/I_2(\xi) = \exp(-\mu_0^2).$$

The right hand side becomes 1 if we replace $I_2(\xi)$ by

$$I_4(\xi) = E\left[\exp(\mathrm{i}\xi\beta_4(w))\,|\,w(1)=0\right],$$

where

$$\beta_4(w) = \beta_2(w) + \frac{f''(0)}{8}\int_0^1\int_0^t |w(s)|^2\left(dw^2(s)\,dw^1(t) - dw^1(s)\,dw^2(t)\right).$$

The skeleton of $\beta_2(w)$ and $\beta_4(w)$ are the 1st and 2nd approximating functionals appearing in the Taylor expansion of $\beta(h)$ around the critical point $h=0$, respectively.

(ii) In Ikeda–Manabe [9], Theorem 3 is obtained in the case that $f(r) = \sqrt{1+r^\beta}$ for some $\beta > 0$ and a is represented as

$$a(w) = A\left(\int_0^1 r(s)^2 h_1(s)\,ds, \int_0^1 r(s)^2 h_2(s)\,ds, \ldots, \int_0^1 r(s)^2 h_m(s)\,ds\right),$$

where A is a bounded uniformly continuous function and $h_1, h_2, \ldots, h_m$ are smooth functions on the interval [0.1].

(iii) The estimate in [28] asserts that

$$\overline{\lim_{\xi\to\infty}}\frac{1}{\xi}\log|I(\xi,1)| \leqslant -\inf_{r\geqslant 0}\frac{f(r)}{2} \tag{3.1}$$

for the present situation. Since $\inf_{r\geqslant 0} f(r)$ may be less than $f(0)$, the result in this paper is an example that (3.1) is not best possible.

Theorem 2 is proven as in the last section by using the following:

LEMMA 3.1. –

$$\mathcal{I}(\xi;a) = E\left[\exp\left(-\frac{1}{2}\int_0^\xi F(r(t),\xi)^2\,dt\right.\right.$$
$$\left.+\mathrm{i}\sqrt{\xi}\int_0^1 b\left(\frac{uw(\xi)}{\sqrt{\xi}}\right)\cdot w(\xi)\,du\right)$$

$$\times A\left(\frac{r(\xi\cdot)}{\sqrt{\xi}}\right)C\left(\frac{w(\xi)}{\sqrt{\xi}}\right)\Bigg]. \tag{3.2}$$

The following proof is essentially that of Theorem 3.1 of [9]:

Proof of Theorem 3. – Taking $K > 0$ large, we decompose the right hand side of (2.4) as

$$I(\xi; a) = I(\xi; 1)a(0) + I_1(\xi; a) + I_2(\xi; a),$$

where

$$I_1(\xi; a) = E\left[\exp\left(-\frac{\xi^2}{2}\int_0^1 F\big(r(t), 1\big)^2 dt\right)\right.$$
$$\left.\times \big\{A\big(r(\cdot)\big) - A(0)\big\}\colon \|r(\cdot)\|_2 > \sqrt{\frac{K}{\xi}}\,\right| w(1) = 0\Bigg]$$

and

$$I_2(\xi; a) = E\left[\exp\left(-\frac{\xi^2}{2}\int_0^1 F\big(r(t), 1\big)^2 dt\right)\right.$$
$$\left.\times \big\{A\big(r(\cdot)\big) - A(0)\big\}\colon \|r(\cdot)\|_2 \leqslant \sqrt{\frac{K}{\xi}}\,\right| w(1) = 0\Bigg].$$

By the conditions (A.1) (ii), (iii) and (A.5), there exists $C_1 > 0$ such that

$$F(r, 1) \geqslant C_1 r.$$

Then, by Hölder's inequality, we have

$$|I_1(\xi; a)| \leqslant C_2 \exp(-C_3 \xi K).$$

This is negligible if K is taken large enough.

For any $\varepsilon > 0$, by the condition (A.6), there exists $\xi_\varepsilon > 0$ such that $|A(r(\cdot)) - A(0)| < \varepsilon$ if $\|r(\cdot)\|_2 \leqslant \sqrt{K/\xi_\varepsilon}$. Therefore we have

$$|I_2(\xi; a)| \leqslant \varepsilon |I(\xi; 1)| \quad \text{for any } \xi \geqslant \xi_\varepsilon.$$

By all these, we can complete the proof. □

The proof of Theorem 4 is almost identical with that of Theorem 3.

Proof of Theorem 5. – For large enough $\xi > 0$, we have

$$I(\xi;a) \geqslant CE\left[\exp\left(-\frac{\xi^2}{2}\int_0^1 F\big(r(t),1\big)^2 dt\right):\ \sup_{[0,1]} r(t) < \frac{1}{\xi}\,\middle|\, w(1)=0\right]$$

$$= CE\left[\exp\left(-\frac{1}{2}\int_0^\xi F\big(r(t),\xi\big)^2 dt\right):\ \sup_{[0,\xi]} r(t) < 1\,\middle|\, w(\xi)=0\right],$$

where $C = \inf\{A(r(\cdot)):\ \sup_{[0,1]} r(t) < R\}$. From this we have the lower bound. □

Remark 3.3. – (i) In this proof of Theorem 5, it is essential that the integrand of $I(\xi;a)$ is real nonnegative. Such an expression is not obtained for $\mathcal{I}(\xi;a)$ (cf. Lemma 3.1).

(ii) By using Hörder's inequality, the condition (A.8) is weakend as follows:

$$0 \leqslant a = A\big(r(\cdot)\big) \in \bigcap_{p\geqslant 1} L^p\big(P(\cdot \mid w(1)=0)\big),$$

$$a^{-1}\chi_R \in \bigcap_{p\geqslant 1} L^p\big(P(\cdot \mid w(1)=0)\big)$$

for some $R > 0$, where χ_R is the indicator function of the set $\{w:\ \sup_{[0,1]} r(t) < R\}$ on W.

4. A GENERAL ROTATION INVARIANT METRIC

In this section we extend our results in last sections to the case where the Wiener process w is replaced by a process associated to a general rotation invariant metric and the 1-form b may degenerate finitely at the origin. For the formulation, we refer to a work of Sheu [20]. Using the polar coordinate (r,θ), we assume that a Riemannian metric g on $\mathbb{R}^2$ is given by

$$g = dr \otimes dr + g(r)^2\, d\theta \otimes d\theta,$$

where $g(r)$ is a positive smooth function on the interval $(0,\infty)$ satisfying the following:

(g-i) As $r \to 0$,

$$g(r) \sim r^\alpha \sum_{n\geqslant 0} g_n r^{\alpha(n)}$$

where $\alpha \in \mathbb{R}$, $g_0 = 1$ and $0 = \alpha(0) < \alpha(1) < \alpha(2) < \cdots \nearrow \infty$;

(g-ii) $\inf_{r>1} g(r)/r^{\alpha'} > 0$ for some $\alpha' \in \mathbb{R}$;

(g-iii) $\sup_{r>1} |g'(r)|/r^{\alpha''} < \infty$ for some $\alpha'' < \alpha'$ and $\sup_{r>1} |g''(r)|/r^{\alpha'} < \infty$;

(g-iv) $-1 < \alpha < 3$ and $\alpha(1) > (1 - |\alpha|)/2$.

The corresponding Laplace–Beltrami operator is represented as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{g'}{g}(r)\frac{\partial}{\partial r} + \frac{1}{g(r)^2}\frac{\partial^2}{\partial \theta^2}$$

on $\{r > 0\}$. Its radial part is a Sturm–Liouville operator

$$\Delta_r = \frac{d^2}{dr^2} + \frac{g'}{g}(r)\frac{d}{dr} = \frac{d}{g(r)\,dr} g(r)\frac{d}{dr}$$

on $(0, \infty)$. The boundary 0 is entrance if $\alpha \geqslant 1$ and regular if $-1 < \alpha < 1$ in the sense of Feller [5]. The boundary ∞ is always natural. Let $r(t, r)$, $t \geqslant 0$, $r \geqslant 0$ on $[0, \infty)$ be the diffusion process generated by $\Delta_r/2$ with the domain

$$\mathcal{D} = \{\varphi(r) = \Phi(r^2) \colon \Phi \in C_0^\infty([0, \infty))\}. \tag{4.1}$$

Then we can construct a diffusion process $X(t, x)$, $t \geqslant 0$, $x \neq 0$, generated by $\Delta/2$ as the skew product of the process $r(t, r)$ and an independent spherical Brownian motion $BM(S^1)$ run with the clock $\int_0^t g^{-2}(r(s, r))\,ds$ (cf. [11]).

We consider a differential 1-form b given by

$$b = k(r)\,d\theta,$$

where $k(r)$ is a positive smooth function on $(0, \infty)$. The fibre norm of this form with respect to the above metric is $F(r) := k(r)/g(r)$. We assume that this function F satisfies the following:

(b-i) As $r \to 0$,

$$F(r) \sim r^\rho \sum_{n \geqslant 0} f_n r^{\rho(n)}$$

where $\rho > 0$, $f_0 > 0$, $0 = \rho(0) < \rho(1) < \rho(2) < \cdots \nearrow \infty$;

(b-ii) There are $0 < \rho' < \rho''$ and $c', c'' > 0$ such that

$$c' r^{\rho'} \leqslant F(r) \leqslant c'' r^{\rho''}$$

for any $r \geqslant 1$.

We consider the asymptotic behavior, as $\xi \to \infty$, of a function defined by

$$I(\xi) := \lim_{R' \to 0} \lim_{R \to 0} \int_{B(R)} E\left[\exp\left(\mathrm{i}\xi \int_0^1 b(X(t,x)) \circ dX(t,x)\right) : X(1,x) \in B(R')\right] d\,\mathrm{vol}(x) \times \left\{ \int_{B(R)} P(X(1,x) \in B(R'))\, d\,\mathrm{vol}(x) \right\}^{-1}, \tag{4.2}$$

where vol is the volume with respect to the above Riemannian metric and $B(R) := \{x \in \mathbb{R}^2 \colon |x| < R\}$ for $R > 0$. By taking the expectation in $BM(S^1)$, we can rewrite this as

$$I(\xi) = \lim_{R' \to 0} E\left[\exp\left(-\frac{\xi^2}{2} \int_0^1 F(r(t))^2\, dt\right) \middle| r(1) < R'\right], \tag{4.3}$$

where $r(t) := r(t, 0)$.

To state the theorem, we introduce an operator

$$H = \frac{1}{2}(-\Delta_\alpha + F_0(r)^2)$$

with the domain $\mathcal{D}$ defined in (4.1), where

$$\Delta_\alpha = \frac{d^2}{dr^2} + \frac{\alpha}{r}\frac{d}{dr}$$

and $F_0(r) := f_0 r^\rho$. By Lemma 4.1 below, this operator is essentially self adjoint as an operator on $L^2((0,\infty), m^\alpha)$, where $m^\alpha(dr) := r^\alpha\, dr$. We denote the self adjoint extension by the same symbol. By Lemma 4.3 below, this operator has purely discrete spectrum. We denote the eigenvalues by $\mu_0 < \mu_1 < \mu_2 < \cdots$, the corresponding normalized eigenfunctions by $\varphi_0, \varphi_1, \varphi_2, \ldots$, and the orthogonal projection to the orthogonal complement to the ground states by $P_\perp$. We easily see that $\mu_0 > 0$ and that φ_0 can be taken as a positive continuous function on the closed interval $[0, \infty)$. Let $\mathrm{e}^{t\Delta_r/2}(r, r')$, $t > 0$, $r \in [0, \infty)$, $r' \in (0, \infty)$, be the transition density of the process $r(t, r)$ with respect to the speed measure

$m(dr) := g(r)\,dr$. By the proof of Lemma 4.7 below, we see that

$$\mathrm{e}^{t\Delta_r/2}(0,0) = \lim_{R\to 0} \int_0^R \mathrm{e}^{t\Delta_r/2}(0,r)m(dr)/m((0,R))$$

exists. This value is positive (see, e.g., McKean [16]). Let

$$\mathcal{A} = \{A = (k_1, k_2, \ldots, k_m; \ell_1, \ell_2, \ldots, \ell_n)\colon m, k_1, k_2, \ldots, k_m, \\ \ell_1, \ell_2, \ldots, \ell_n \in \mathbb{N}, n \in \mathbb{Z}_+\}.$$

For $A = (k_1, k_2, \ldots, k_m; \ell_1, \ell_2, \ldots, \ell_n) \in \mathcal{A}$, we set

$$\iota(A) = \rho(k_1) + \rho(k_2) + \cdots + \rho(k_m) + \alpha(\ell_1) + \alpha(\ell_2) + \cdots + \alpha(\ell_n).$$

Let

$$\mathcal{B} = \{B = (A_0, A_1, \ldots, A_m)\colon m \in \mathbb{Z}_+, A_0, A_1, \ldots, \\ A_m \in \mathcal{A}, \iota(A_1), \iota(A_2), \ldots, \iota(A_m) > 2\}.$$

For $B = (A_0, A_1, \ldots, A_m) \in \mathcal{B}$, we set

$$\kappa(B) = \iota(A_0) + \iota(A_1) + \cdots + \iota(A_m) - 2m.$$

The main theorem is the following:

THEOREM 6. – *Under the conditions* (g-i)–(g-iv) *and* (b-i)–(b-ii), *we have the following asymptotic expansion as* $\xi \to \infty$:

$$I(\xi) \sim \exp\left\{ -\xi^{2\gamma}\mu_0 - \sum_{A\in\mathcal{A}:\ \iota(A)\leqslant 2} \xi^{\gamma(2-\iota(A))}\mu_0^A \right\}\xi^{(\alpha+1)\gamma} \\ \times \left\{ I_0 + \sum_{B\in\mathcal{B}} I_B \xi^{-\gamma\kappa(B)} \right\}, \tag{4.4}$$

where $\gamma = 1/(\rho+1)$, $\{\mu_0^A\colon A \in \mathcal{A}\}$ *are polynomials of*

$\{g_n, f_n, (r^{k(0)}(r^{-\varepsilon(0)}d/dr)^{\ell(0)}\varphi_0, (\mu_0 - H)^{-1}P_\perp r^{k(1)}(r^{-\varepsilon(1)}d/dr)^{\ell(1)} \cdots$
$(\mu_0 - H)^{-1}P_\perp r^{k(n)}(r^{-\varepsilon(n)}d/dr)^{\ell(n)}\varphi_0)\colon n \in \mathbb{Z}_+, k(0), k(1), \ldots, k(n)$
$\geqslant 0, 0 \leqslant \varepsilon(0), \varepsilon(1), \ldots, \varepsilon(n) \leqslant 1 - \alpha(1), \ell(0), \ell(1), \ldots, \ell(n) \in \{0, 1\}\}$,

$$I_0 = \frac{\varphi_0(0)^2}{\mathrm{e}^{\Delta_r/2}(0,0)},$$

and $\{I_B\colon\ B\in\mathcal{B}\}$ *are polynomials of*

$$\{g_n, f_n, \big(r^{k(0)}\big(r^{-\varepsilon(0)}d/dr\big)^{\ell(0)}\varphi_0, (\mu_0-H)^{-1}P_\perp r^{k(1)}\big(r^{-\varepsilon(1)}d/dr\big)^{\ell(1)}\cdots$$
$$(\mu_0-H)^{-1}P_\perp r^{k(n)}\big(r^{-\varepsilon(n)}d/dr\big)^{\ell(n)}\varphi_0\big),$$
$$\big((\mu_0-H)^{-1}P_\perp r^{k(0)}\big(r^{-\varepsilon(0)}d/dr\big)^{\ell(0)}\cdots$$
$$(\mu_0-H)^{-1}P_\perp r^{k(n)}\big(r^{-\varepsilon(n)}d/dr\big)^{\ell(n)}\varphi_0\big)(0)\colon\ n\in\mathbb{Z}_+, k(0),k(1),\ldots,k(n)$$
$$\geqslant 0, 0\leqslant \varepsilon(0),\varepsilon(1),\ldots,\varepsilon(n)\leqslant 1-\alpha(1), \ell(0),\ell(1),\ldots,\ell(n)\in\{0,1\}\}.$$

The condition (g-iv) $\alpha<3$ ensures that the boundary 0 is of limit circle type for the operator Δ_r (see, e.g., [18] Appendix to X.1). When this condition is not satisfied, we assume the following:

(g-v) $\alpha>-1$ and $\alpha(1)>2$.

Then we have the following:

THEOREM 7. – *Under the conditions* (g-i)–(g-iii), (g-v) *and* (b-i)–(b-ii), *we have the following asymptotic expansion as* $\xi\to\infty$:

$$I(\xi)\sim\exp\left\{-\xi^{2\gamma}\mu_0-\sum_{A\in\mathcal{A}\colon\ \iota(A)\leqslant 2}\xi^{\gamma(2-\iota(A))}\mu_0^A\right\}\xi^{(\alpha+1)\gamma}\big(I_0+\mathrm{o}(1)\big),\tag{4.5}$$

where γ, μ_0^A *and* I_0 *are those in Theorem 6.*

Remark 4.1. – (i) Under the condition $\alpha(1)>2$, $\{\mu_0^A\colon\ A\in\mathcal{A}, \iota(A)\leqslant 2\}$ are polynomials of only

$$\{f_n, \big(r^{k(0)}\varphi_0, (\mu_0-H)^{-1}P_\perp r^{k(1)}\cdots(\mu_0-H)^{-1}P_\perp r^{k(n)}\varphi_0\big)\colon\ n\in\mathbb{Z}_+,$$
$$k(0),k(1),\ldots,k(n)>0\}.$$

(ii) We can treat also general amplitudes as in Sections 2 and 3. However, since the discussion is almost identical, we omit it.

In the following we always assume only (g-i)–(g-iii), $\alpha>-1$ and (b-i)–(b-ii), unless otherwise stated.

To analyze the function $I(\xi)$, we first represent this as

$$I(\xi)=\mathrm{e}^{-\mathcal{H}(\xi)}(0,0)/\mathrm{e}^{\Delta_r/2}(0,0),\tag{4.6}$$

where $\mathrm{e}^{-t\mathcal{H}(\xi)}(r,r')$ and $\mathrm{e}^{t\Delta_r/2}(r,r')$, $t>0$, $r,r'>0$ are the heat kernels of the operators

$$\mathcal{H}(\xi)=\frac{1}{2}\big(-\Delta_r+\xi^2F(r)^2\big)$$

and $-\Delta_r/2$, respectively, with respect to the measure $m(dr)$. To define these operators, we use the following lemma, which is proven by the same technique of Sheu [20]:

LEMMA 4.1. – *The operators $\mathcal{H}(\xi)$ and Δ_r are essentially self adjoint on the domain $\mathcal{D}$ defined in* (4.1) *as operators on* $L^2((0,\infty),m)$.

Moreover, by this lemma, we can represent the heat kernel as follows:

$$\begin{aligned} &\mathrm{e}^{-t\mathcal{H}(\xi)}(r,r') \\ &\quad = E\left[\exp\left(-\frac{\xi^2}{2}\int_0^t F\big(r(s,r)\big)^2\,ds\right)\middle| r(t,r)=r'\right]\mathrm{e}^{t\Delta_r/2}(r,r') \end{aligned} \tag{4.7}$$

(see, e.g., [21] Theorem 1.1). By a general theory (see, e.g., [11] Section 4.11), the heat kernel $\mathrm{e}^{-t\mathcal{H}(\xi)}(r,r')$ is a positive smooth function of $(r,r')\in(0,\infty)^2$. For the representation (4.6), we use the following:

LEMMA 4.2. – *$\mathrm{e}^{-t\mathcal{H}(\xi)}(r,r')$ is a positive continuous function of $(r,r')\in[0,\infty)^2$.*

Proof. – When $\alpha<1$, this fact is well known since the boundary 0 is regular (see, e.g., McKean [16]). When $\alpha\geqslant 1$, we use a result of Hille [6] as follows (see also Matsumoto [15]). We assume $\xi=1$ and write $\mathcal{H}:=\mathcal{H}(1)$. We use a canonical scale $x=T(r)$, where

$$T(r)=\int_1^r \frac{ds}{g(s)}. \tag{4.8}$$

Then the operator $\mathcal{H}$ is represented as

$$\mathcal{H}=\frac{1}{2}\left(-\frac{d}{dM(x)}\frac{d}{dx}+F\big(T^{-1}(x)\big)^2\right),$$

where $dM(x)=g(T^{-1}(x))^2\,dx$. Since the boundary 0 is entrance, we have $\lim_{r\downarrow 0}T(r)=-\infty$ and

$$\int_{-\infty}^0 (-x)\,dM(x)=\int_0^1\int_r^1 \frac{ds}{g(s)}g(r)\,dr<\infty.$$

Moreover, if we set

$$\rho(x) = \sup_{x \geqslant y} \left\{ -y \int_{-\infty}^{y} dM(z) \right\},$$

then we have

$$\int_{-\infty}^{0} \frac{\rho(x)}{-x} dx = \int_{0}^{1} \left(g(r) \int_{r}^{1} \frac{ds}{g(s)} \right)^{-1} \left(\sup_{r \geqslant s} \int_{0}^{1} \frac{dt}{g(t)} \int_{0}^{s} g(t)\, dt \right) dr < \infty.$$

Therefore, by the same method of the proof of Theorem 4.1 of Matsumoto [15] (cf. Hille [6]), we have

$$\varlimsup_{n \to \infty} \frac{1}{m_n} \log \sup_{x \leqslant x_0} |\Phi_n(x)| \leqslant 0 \tag{4.9}$$

for any $x_0 \in \mathbb{R}$, where $m_1 < m_2 < \cdots$ are the eigenvalues of $\mathcal{H}$ and $\Phi_1, \Phi_2, \ldots$ are the corresponding normalized eigenfunctions. The discreteness of the spectrum of $\mathcal{H}$ is ensured by the subsequent Lemma 4.3. Moreover by the same lemma and Mercer's expansion theorem, we see that $e^{-t\mathcal{H}}(r, r')$ is a bounded continuous function of $(r, r') \in [0, \infty)^2$. Since the boundary 0 is entrance, we have

$$\lim_{x \to -\infty} \Phi_n(x) \neq 0.$$

Thus we have

$$e^{-t\mathcal{H}}(r, r') > 0$$

for any $(r, r') \in [0, \infty)^2$. □

LEMMA 4.3. – *The heat semigroup generated by $\mathcal{H}(\xi)$ consists of trace class operators.*

Proof. – As in the last proof, we consider only $\mathcal{H} := \mathcal{H}(1)$. We set

$$\mu_n(\mathcal{H}) := \sup_{\varphi_1, \varphi_2, \ldots, \varphi_{n-1} \in L^2(g dr)} \inf \left\{ \frac{1}{2} (\|\psi'\|_g^2 + \|F\psi\|_g^2) : \psi \in \mathcal{D}, \right.$$
$$\left. \|\psi\|_g = 1, (\psi, \varphi_j)_g = 0 \text{ for } j = 1, 2, \ldots, n-1 \right\},$$

where $\|\cdot\|_g$ and $(\cdot,\cdot)_g$ are the norm and the inner product of $L^2(g\,dr)$, respectively. By the condition (g-i), we can take positive smooth functions $g_1(r)$ and $g_2(r)$ on the interval $(0,\infty)$ such that

$$\begin{cases} g_1(r) \leqslant g(r) \leqslant g_2(r), & \text{on } (0,\infty), \\ g_1(r) = C_1 r^\alpha,\, g_2(r) = C_2 r^\alpha, & \text{on } (0,1), \\ g_1(r) = g(r) = g_2(r), & \text{on } (2,\infty), \end{cases}$$

for some $0 < C_1 < C_2$. Then we have

$$\mu_n(\mathcal{H}) \geqslant \mu_n(\widehat{\mathcal{H}}),$$

where

$$\mu_n(\widehat{\mathcal{H}}) := \sup_{\varphi_1,\varphi_2,\ldots,\varphi_{n-1}\in L^2(g_1 dr)} \inf\left\{\frac{1}{2}\left(\|(J\psi)'\|_{g_1}^2 + \|FJ\psi\|_{g_1}^2\right):\ \psi \in \mathcal{D},\right.$$

$$\left.\|\psi\|_{g_1} = 1,\, (\psi,\varphi_j)_{g_1} = 0 \text{ for } j = 1,2,\ldots,n-1\right\},$$

and $J(r) := \sqrt{g_1(r)/g_2(r)}$. By the min-max principle, it is enough to show that the self adjoint operator

$$\widehat{\mathcal{H}} = \frac{1}{2}\left(-\frac{J}{g_1}\frac{d}{dr}g_1\frac{d}{dr}J + (FJ)^2\right)$$

on $L^2(g_1\,dr)$ generates a heat semigroup consisting of trace class operators. We use a scale $\sigma = S(r)$ defined by

$$S(r) = \int_0^r \frac{ds}{J(s)}.$$

Then $\widehat{\mathcal{H}}$ is regarded as a self adjoint operator

$$\widehat{\mathcal{H}} = \frac{1}{2}\left(-\frac{1}{\mathcal{G}(\sigma)}\frac{d}{d\sigma}\mathcal{G}(\sigma)\frac{d}{d\sigma} + \mathcal{V}(\sigma)\right),$$

on $L^2((0,\infty),\mathcal{G}(\sigma)d\sigma)$, where

$$\mathcal{G}(\sigma) := (g_1 J)\big(S^{-1}(\sigma)\big)$$

and

$$\mathcal{V}(\sigma) := \big\{(FJ)^2 - JJ'g_1'/g_1 - JJ''\big\}\big(S^{-1}(\sigma)\big).$$

By the unitary operator $\sqrt{\mathcal{G}(\sigma)/\sigma^\alpha}$ from $L^2(\mathcal{G}(\sigma)\,d\sigma)$ to $L^2(\sigma^\alpha\,d\sigma)$, the operator $\widehat{\mathcal{H}}$ is transformed to a self adjoint operator

$$\widetilde{\mathcal{H}} := \frac{1}{2}\big(- \Delta_\alpha - \widetilde{\mathcal{G}}(\sigma) + \mathcal{V}(\sigma)\big)$$

on $L^2(\sigma^\alpha\,d\sigma)$, where

$$\widetilde{\mathcal{G}}(\sigma) = \frac{\alpha(\alpha-2)}{4\sigma^2} + \frac{\mathcal{G}'^2 - 2\mathcal{G}\mathcal{G}''}{4\mathcal{G}^2}.$$

By the conditions (g-ii) and (g-iii), $\widetilde{\mathcal{G}}(\sigma)$ is a bounded continuous function on $[0,\infty)$. Therefore the heat kernel of the operator $\widetilde{\mathcal{H}}$ has the following representation:

$$\begin{aligned} &e^{-t\widetilde{\mathcal{H}}}(\sigma,\sigma') \\ &\quad = E\left[\exp\left(\frac{1}{2}\int_0^t (\widetilde{\mathcal{G}} - \mathcal{V})\big(r_\alpha(s,\sigma)\big)\,ds\right)\middle| r_\alpha(t,\sigma) = \sigma'\right] e^{t\Delta_\alpha/2}(\sigma,\sigma') \end{aligned}$$

for $(\sigma,\sigma') \in [0,\infty)^2$, where $r_\alpha(t,\sigma)$, $t \geqslant 0$, $\sigma \geqslant 0$, is the Bessel diffusion process with index $\alpha+1$ and $e^{t\Delta_\alpha/2}(\sigma,\sigma')$ is its transition density function:

$$e^{t\Delta_\alpha/2}(\sigma,\sigma') = \frac{\exp(-(\sigma^2+\sigma'^2)/(2t))}{2^{(\alpha-1)/2}t^{(\alpha+1)/2}} \sum_{n=0}^{\infty} \frac{(\sigma\sigma'/(2t))^{2n}}{n!\Gamma(n+(\alpha+1)/2)} \tag{4.10}$$

(cf., e.g., [10] IV-(8.20) and [21] Theorem 6.1). By this explicit representation, we have

$$\sup_{\sigma,\sigma'} e^{t\Delta_\alpha/2}(\sigma,\sigma')/\big(1+(\sigma\sigma')^2\big) < \infty \tag{4.11}$$

for any $t > 0$. Moreover if we take large $R > 0$, then we have

$$e^{-t\widetilde{\mathcal{H}}}(\sigma,\sigma) < C_3 \exp\big(- C_4 \sigma^{2(1\wedge\rho')}\big) \tag{4.12}$$

for any $\sigma \geqslant R$, where C_3 and C_4 are positive constants depending only on R. In fact, if we divide as

$$e^{-\widetilde{\mathcal{H}}}(\sigma,\sigma) = E\left[\exp\left(\frac{1}{2}\int_0^1 (\widetilde{\mathcal{G}} - \mathcal{V})(r_\alpha(t,\sigma))\,dt\right):\right.$$

$$\left. \inf_{0\leqslant t\leqslant 1} r_\alpha(t,\sigma) \leqslant \frac{\sigma}{2} \right| r_\alpha(1,\sigma)=\sigma \Bigg] e^{\Delta_\alpha/2}(\sigma,\sigma)$$

$$+E\Bigg[\exp\left(\frac{1}{2}\int_0^1 (\widetilde{\mathcal{G}}-\mathcal{V})(r_\alpha(t,\sigma))\,dt\right):$$

$$\left. \inf_{0\leqslant t\leqslant 1} r_\alpha(t,\sigma) > \frac{\sigma}{2} \right| r_\alpha(1,\sigma)=\sigma \Bigg] e^{\Delta_\alpha/2}(\sigma,\sigma),$$

then the first term is estimated by the Chapman–Kolmogorov equality, (4.11) and an estimate of stopping times (cf. [10] Lemma V-10.5) as follows:

$$E\left[\exp\left(\frac{1}{2}\int_0^1 (\widetilde{\mathcal{G}}-\mathcal{V})(r_\alpha(t,\sigma))\,dt\right): \left. \inf_{0\leqslant t\leqslant 1} r_\alpha(t,\sigma) \leqslant \frac{\sigma}{2} \right| r_\alpha(1,\sigma)=\sigma \right]$$

$$\times e^{\Delta_\alpha/2}(\sigma,\sigma)$$

$$\leqslant C_5 P\left(\left. \inf_{0\leqslant t\leqslant 1} r_\alpha(t,\sigma) \leqslant \frac{\sigma}{2} \right| r_\alpha(1,\sigma)=\sigma \right) e^{\Delta_\alpha/2}(\sigma,\sigma)$$

$$=2C_5\int_0^\infty P\left(\left. \inf_{0\leqslant t\leqslant 1/2} r_\alpha(t,\sigma) \leqslant \frac{\sigma}{2} \right| r_\alpha\left(\frac{1}{2},\sigma\right)=\sigma' \right)$$

$$\times e^{\Delta_\alpha/4}(\sigma,\sigma')^2 \sigma'^\alpha \,d\sigma'$$

$$\leqslant C_6 E\left[\left(1+r_\alpha\left(\frac{1}{2},\sigma\right)^2\sigma^2\right): \inf_{0\leqslant t\leqslant 1/2} r_\alpha(t,\sigma) \leqslant \frac{\sigma}{2} \right]$$

$$\leqslant C_6 E\left[\left(1+r_\alpha\left(\frac{1}{2},\sigma\right)^2\sigma^2\right)^2\right]^{1/2} P\left(\inf_{0\leqslant t\leqslant 1/2} r_\alpha(t,\sigma) \leqslant \frac{\sigma}{2} \right)^{1/2}$$

$$\leqslant C_7 \exp\left(-C_8\sigma^2\right),$$

where C_7 and C_8 are positive constants depending only on R. The second term is estimated as

$$E\left[\exp\left(\frac{1}{2}\int_0^1 (\widetilde{\mathcal{G}}-\mathcal{V})(r_\alpha(t,\sigma))\,dt\right): \left. \inf_{0\leqslant t\leqslant 1} r_\alpha(t,\sigma) > \frac{\sigma}{2} \right| r_\alpha(t,\sigma)=\sigma \right]$$

$$\times e^{t\Delta_\alpha/2}(\sigma,\sigma)$$

$$\leqslant C_9 \exp\left(-C_{10}\sigma^{2\rho'}\right)$$

by the condition (b-ii). From (4.11) and (4.12), we obtain

$$\int_0^\infty \mathrm{e}^{-\widetilde{\mathcal{H}}}(\sigma,\sigma)\sigma^\alpha\, d\sigma < \infty,$$

which completes the proof. □

As in Section 2, we rewrite (4.6) as

$$I(\xi) = \xi^{(\alpha+1)\gamma}\mathrm{e}^{-\xi^{2\gamma}H(\xi)}(0,0)\Big/\mathrm{e}^{\Delta_r/2}(0,0), \tag{4.13}$$

where $\mathrm{e}^{-tH(\xi)}(r,r'), t>0, r, r' \geqslant 0$ is the heat kernel of the self adjoint operator

$$H(\xi) = \frac{1}{2}\big(-\Delta_\xi + F(r,\xi)^2\big)$$

on $L^2(m_\xi)$ with respect to the speed measure $m_\xi(dr) = g_\xi(r)\,dr$,

$$\Delta_\xi = \frac{d^2}{dr^2} + \frac{g_\xi'}{g_\xi}(r)\frac{d}{dr},$$

$$F(r,\xi) = \xi^{1-\gamma}\frac{k}{g}\left(\frac{r}{\xi^\gamma}\right)$$

and $g_\xi(r) = \xi^{\alpha\gamma}g(r/\xi^\gamma)$. (4.13) is obtained from the scaling property

$$\{r(\cdot)\} \overset{\mathcal{L}}{\sim} \left\{\frac{r_\xi(\xi^{2\gamma}\cdot)}{\xi^\gamma}\right\}, \tag{4.14}$$

where $r_\xi(\cdot)$ is the diffusion process generated by $\Delta_\xi/2$. Moreover we rewrite (4.13) as

$$I(\xi) = \xi^{(\alpha+1)\gamma}\sum_{n=0}^\infty \mathrm{e}^{-\xi^{2\gamma}\mu_n(\xi)}\varphi_n(0,\xi)^2/\mathrm{e}^{\Delta_r/2}(0,0),$$

where $\mu_0(\xi) < \mu_1(\xi) < \mu_2(\xi) < \cdots$ are the eigenvalues of $H(\xi)$ and $\varphi_0(r,\xi), \varphi_1(r,\xi), \varphi_2(r,\xi), \ldots$ are the corresponding normalized eigenfunctions.

The aymptotic behavior of $\mu_n(\xi)$ is as follows:

PROPOSITION 4.1. – $\lim_{\xi\to\infty}\mu_n(\xi) = \mu_n$ *for each* $n \geqslant 0$.

PROPOSITION 4.2. – *We have the following asymptotic expansion as* $\xi \to \infty$:

$$\mu_0(\xi) \sim \mu_0 + \sum_{A \in \mathcal{A}} \frac{\mu_0^A}{\xi^{\gamma \iota(A)}},$$

where $\mathcal{A}$, μ_0^A, γ *and* $\iota(A)$ *are those in Theorem* 6.

To prove these propositions we use the following:

LEMMA 4.4. – *For each* φ_n, *we have the following*:
(i) *For some* $c_1, c_2, c_3 > 0$ *depending only on* n, *we have*

$$|\varphi_n(r)| \leqslant c_1 \exp\left(-c_2 r^{c_3}\right). \tag{4.15}$$

(ii) $\lim_{r \downarrow 0} r^{\alpha \wedge 0} \varphi_n'(r) = 0$.
(iii) *For any* $N = 0, 1, 2, \ldots$, *we have*

$$\int_0^\infty |\varphi_n'|^2 r^N m^\alpha(dr) < \infty. \tag{4.16}$$

(iv) $\varphi_n \in \mathrm{Dom}(H(\xi))$.

LEMMA 4.5. – *For each fixed* $m \in \mathbb{Z}_+$ *and* $a \in [0, 1 \wedge \{(|\alpha| + 1)/2\})$, *we have*

$$\sup_{|z-\mu_0|=\varepsilon} \left\|(1+r^2)^{m/2}(z-H)^{-1}(1+r^2)^{-m/2}\right\|_{op} < \infty \tag{4.17}$$

and

$$\sup_{|z-\mu_0|=\varepsilon} \left\|(1+r^2)^{m/2} \frac{1}{r^a} \frac{d}{dr}(z-H)^{-1}(1+r^2)^{-m/2}\right\|_{op} < \infty, \tag{4.18}$$

where $\|\cdot\|_{op}$ *is the operator norm on* $L^2(m^\alpha)$.

Proof of Lemma 4.4. – (i) By estimating the probabilistic representation

$$\begin{aligned} & e^{-tH}(r, r') \\ & = E\left[\exp\left(-\frac{1}{2}\int_0^t F_0\big(r_\alpha(s,r)\big)^2 ds\right) \middle| r_\alpha(t,r) = r'\right] e^{t\Delta_\alpha/2}(r, r') \end{aligned} \tag{4.19}$$

as in the proof of (4.12), we obtain (4.15) with $c_3 = 2(1 \wedge \rho')$.

(ii) Let ψ be a smooth function on $(0,\infty)$ such that $\psi(r)=r^{-\delta}$ on $(0, R_1]$ and $\psi(r)\equiv 0$ on $[R_2,\infty)$ for some small $\delta>0$ and some $0<R_1<R_2$. For $\varepsilon\in(0,R_1]$, we have

$$\mu_n\int_\varepsilon^\infty \varphi_n\psi m^\alpha(dr)=\int_\varepsilon^\infty (H\varphi_n)\psi m^\alpha(dr)$$
$$=\frac{1}{2}\int_\varepsilon^\infty\{\varphi_n'\psi'+F_0(r)^2\varphi_n\psi\}m^\alpha(dr)+\frac{1}{2}\varepsilon^{\alpha-\delta}\varphi_n'(\varepsilon).$$

Since the left hand side and the first term of the right hand side have limits as $\varepsilon\downarrow 0$, the limit $\lim_{r\downarrow 0} r^{\alpha-\delta}\varphi_n'(r)$ should exist. Thus we have $\lim_{r\downarrow 0} r^\alpha\varphi_n'(r)=0$. On the other hand, we have

$$\left|(r^\alpha\varphi_n')'\right|\leqslant C_2r^\alpha\quad\text{for } 0<r\leqslant C_3$$

by the characteristic equation and (i) of this lemma. Thus we have

$$|r^\alpha\varphi_n'|\leqslant\int_0^r\left|(r^\alpha\varphi_n')'\right|dr\leqslant C_4r^{\alpha+1},$$

from which we obtain $\lim_{r\downarrow 0}\varphi_n'(r)=0$.

(iii) (4.16) with $N=0$ follows from $\varphi_n\in\mathrm{Dom}(H)$. For $N\geqslant 1$, we use the integration by parts and the characteristic equation for φ_n as follows:

$$\int|\varphi_n'|^2r^{\alpha+N}dr=-\int\varphi_n(\Delta_\alpha\varphi_n)r^{\alpha+N}dr-N\int\varphi_n\varphi_n'r^{\alpha+N-1}dr.$$

The first term equals to

$$\int(2\mu_n-F_0(r)^2)\varphi_n^2r^{\alpha+N}dr,$$

which is finite by (4.15). The second term is dominated by

$$\sqrt{\int\varphi_n^2r^{\alpha+2N-2}\,dr}\sqrt{\int|\varphi_n'|^2r^\alpha\,dr},$$

which is finite by (4.15) and (4.16) with $N=0$.

(iv) We define a domain by

$$\widetilde{\mathcal{D}} := \left\{ \varphi \in C^2((0,\infty))\colon \ \varphi \text{ satisfies (i)–(iii) of this lemma, and} \right.$$

$$\left. \int_0^\infty |\Delta_\alpha \varphi|^2 r^N m^\alpha(dr) < \infty \text{ for any } N = 0, 1, 2, \ldots \right\}.$$

We easily see that $\mathcal{D} \subset \widetilde{\mathcal{D}}$ and that $H(\xi)$ is defined as a symmetric operator on $\widetilde{\mathcal{D}}$ by the condition (g-iv). Therefore, by Lemma 4.1, we have $\widetilde{\mathcal{D}} \subset \mathrm{Dom}(H(\xi))$. On the other hand, we have $\varphi_n \in \widetilde{\mathcal{D}}$ by (i)–(iii) of this lemma. Thus we have $\varphi_n \in \mathrm{Dom}(H(\xi))$. □

Lemma 4.5 is proven by the technique in McKean [16].

The asymptotic behavior of $\varphi_0(0,\xi)$ is as follows:

PROPOSITION 4.3. – *Under the conditions of Theorem* 6, *we have the following asymptotic expansion as* $\xi \to \infty$:

$$\varphi_0(0,\xi) \sim \varphi_0(0) + \sum_{A \in \mathcal{A}} \frac{\varphi_0^A(0)}{\xi^{\gamma \iota(A)}},$$

where $\mathcal{A}$, γ, $\iota(A)$ *are those in Theorem* 6 *and* $\{\varphi_0^A(0)\colon A \in \mathcal{A}\}$ *are polynomials of*

$$\begin{aligned}
&\{g_n, f_n, (r^{k(0)}(r^{-\varepsilon(0)}d/dr)^{\ell(0)}\varphi_0, (\mu_0 - H)^{-1}P_\perp r^{k(1)}(r^{-\varepsilon(1)}d/dr)^{\ell(1)} \cdots \\
&\quad (\mu_0 - H)^{-1}P_\perp r^{k(n)}(r^{-\varepsilon(n)}d/dr)^{\ell(n)}\varphi_0), \\
&\quad ((\mu_0 - H)^{-1}P_\perp r^{k(0)}(r^{-\varepsilon(0)}d/dr)^{\ell(0)} \cdots \\
&\quad (\mu_0 - H)^{-1}P_\perp r^{k(n)}(r^{-\varepsilon(n)}d/dr)^{\ell(n)}\varphi_0)(0)\colon n \in \mathbb{Z}_+, k(0), k(1), \ldots, k(n) \\
&\quad \geqslant 0, 0 \leqslant \varepsilon(0), \varepsilon(1), \ldots, \varepsilon(n) \leqslant 1 - \alpha(1), \ell(0), \ell(1), \ldots, \ell(n) \in \{0,1\}\}.
\end{aligned}$$

To prove this proposition, we use the following:

LEMMA 4.6. – *Under the conditions of Theorem* 6, *we have*

$$\sup_{|z-\mu_0|=\varepsilon} \sup_{\|\varphi\|=1} \left|\{(z-H)^{-1}\varphi\}(0)\right| < \infty \tag{4.20}$$

and

$$\sup_{\xi \geqslant 1} \sup_{|z-\mu_0|=\varepsilon} \sup_{\|\varphi\|=1} \left|\{(z - H(\xi))^{-1}(1+r^2)^{-m/2}\varphi\}(0)\right| < \infty \tag{4.21}$$

for large enough $m \geqslant 0$.

The proof of this lemma is reduced to the following:

LEMMA 4.7. – *Under the conditions* (g-i)–(g-iii) *and* (b-i)–(b-ii), *we have*

$$\sup_{\xi\geqslant 1}\sup_{0<\delta\leqslant 1} \mathrm{e}^{-H(\xi,\delta)}(0,0)<\infty, \tag{4.22}$$

where

$$H(\xi,\delta)=\frac{1}{2}\left(-\Delta_\xi+\delta F(r,\xi)^2\right)$$

is a self adjoint operator on $L^2(m_\xi)$.

Proof. – We note that

$$\begin{aligned}
&\mathrm{e}^{-H(\xi,\delta)}(0,0)\\
&=\lim_{\varepsilon\to 0} E\left[\exp\left(-\frac{\delta}{2}\int_0^1 F\big(r_\xi(t),\xi\big)^2\,dt\right):\ r_\xi(1)<\varepsilon\right]\Big/ m_\xi\big((0,\varepsilon)\big)
\end{aligned}$$

and that $R_\xi(t):=r_\xi(t)^2$, $t\geqslant 1$ is regarded as the pathwise unique solution of the stochastic integral equation

$$R_\xi(t)=\int_0^t 2\sqrt{R_\xi(s)}\,dw(s)+\int_0^t G_\xi\big(R_\xi(s)\big)\,ds,$$

where $w(t)$ is the 1-dimensional standard Brownian motion with $w(0)=0$ and

$$G_\xi(R)=\sqrt{R}\frac{g_\xi'}{g_\xi}(\sqrt{R})+1$$

(cf. [10] Examples IV-8.2 and 3). By the conditions (g-i)–(g-iii), we have

$$G_\xi(R)\geqslant G_0(R)$$

for any $\xi\geqslant 1$, where

$$G_0(R)=\sqrt{R}\frac{g_0'}{g_0}(\sqrt{R})+1,$$

$g_0(r)$ is a smooth function on $(0,\infty)$ such that

$$g_0(r)=\begin{cases} r^\alpha\exp(C_1 r^{\alpha(1)}), & \text{for } 0\leqslant r\leqslant 1,\\ C_2 r^{-2}\exp(-r^3), & \text{for } r\geqslant C_3,\end{cases}$$

and C_1, C_2, C_3 are some positive constants. Let $R_0(t,R)$, $t \geqslant 0$, $R \geqslant 0$, be the solution of

$$R_0(t,R) = R + \int_0^t 2\sqrt{R_0(s,R)}\,dw(s) + \int_0^t G_0\big(R_0(s,R)\big)\,ds$$

and let $r_0(t,r) := \sqrt{R_0(t,r^2)}$ for $r \geqslant 0$ and $r_0(t) := r_0(t,0)$. Then, by the comparison theorem (cf. Ikeda–Watanabe [10] Theorem VI-1.1), we have

$$r_\xi(t) \geqslant r_0(t)$$

and

$$E\left[\exp\left(-\frac{\delta}{2}\int_0^1 F\big(r_\xi(t),\xi\big)^2\,dt\right):\ r_\xi(1) < \varepsilon\right] \leqslant P\big(r_0(1) < \varepsilon\big).$$

The generator of the process $\{r_0(t,r)\}$ is $\Delta_0/2$, where

$$\Delta_0 = \frac{d}{g_0(r)\,dr} g_0(r) \frac{d}{dr}.$$

For this operator, the ∞ is entrance and the boundary 0 is same as for the operator Δ_r or Δ_α. Thus the resolvent $(\mu - \Delta_0)^{-1}$ for each $\mu > 0$ is a trace class operator (see, e.g., McKean [16]). Therefore, as in Lemma 4.2, we can show the existence of

$$\mathrm{e}^{\Delta_0/2}(0,0) = \lim_{\varepsilon\to 0} \frac{P(r_0(1) < \varepsilon)}{m_0((0,\varepsilon))},$$

where $m_0(dr) = g_0(r)\,dr$ (cf. Hille [6], Matsumoto [15]). On the other hand, by the condition (g-i), we easily see that

$$\sup_{\xi\geqslant 1} \overline{\lim_{\varepsilon\to 0}} \frac{m_0((0,\varepsilon))}{m_\xi((0,\varepsilon))} < \infty.$$

Therefore we obtain (4.22). □

Now Theorem 6 is proven similarly as in Section 2 by using Propositions 4.1–4.3.

For the proof of Theorem 7, we use the following:

LEMMA 4.8. – *Under the conditions of Theorem 7, we have*

$$\lim_{\xi\to\infty} \varphi_0(0,\xi)^2 = \varphi_0(0)^2.$$

Proof. – By the unitary operator $\sqrt{g_\xi/r^\alpha}$ from $L^2(m_\xi)$ to $L^2(m^\alpha)$, the operator $H(\xi)$ is transformed to a self adjoint operator

$$\widetilde{H}(\xi) = \frac{1}{2}(-\Delta_\alpha - \widetilde{g_\xi}(r) + F(r,\xi)^2)$$

on $L^2(m^\alpha)$, where

$$\widetilde{g_\xi}(r) = \frac{\alpha(\alpha-2)}{4r^2} + \frac{g_\xi'^2 - 2g_\xi g_\xi''}{4g_\xi^2}.$$

Under the conditions of Theorem 7, we can show that $\widetilde{g_\xi}$ is a bounded continuous function on the closed interval $[0,\infty)$. Therefore the heat kernel has the following representation:

$$e^{-tH(\xi)}(r,r') = \sqrt{\frac{r^\alpha}{g_\xi(r)}} E\left[\exp\left(\frac{1}{2}\int_0^t \{\widetilde{g_\xi}(r_\alpha(s,r)) - F(r_\alpha(s,r),\xi)^2\}\,ds\right)\middle| r_\alpha(t,r) = r'\right] e^{t\Delta_\alpha/2}(r,r')\sqrt{\frac{r'^\alpha}{g_\xi(r')}}$$

for $(r,r') \in [0,\infty)^2$. By the Lebesgue convergence theorem, we have

$$\lim_{\xi\to\infty} e^{-tH(\xi)}(0,0) = e^{-tH}(0,0).$$

By regarding

$$e^{-tH(\xi)}(0,0) = \sum_{n=0}^\infty e^{-t\mu_n(\xi)}\varphi_n(0,\xi)^2$$

as a Laplace transform, we have

$$\sum_{n=0}^\infty \varphi_n(0,\xi)^2\delta_{\mu_n(\xi)}(\lambda)\,d\lambda \to \sum_{n=0}^\infty \varphi_n(0)^2\delta_{\mu_n}(\lambda)\,d\lambda \tag{4.23}$$

vaguely, as $\xi\to\infty$, from which we can obtain the result. □

5. AN ABSORBING CASE

In this section, we assume the conditions (g-i)–(g-iii) and
(g-vi) $\alpha < 1$ and $\alpha(1) > 2$.

In this case, the boundary 0 is exit if $\alpha \leqslant -1$ and regular if $-1 < \alpha < 1$ for the operator Δ_r. Thus, to obtain a radial process on $(0, \infty)$, we should take the boundary 0 as its absorbing barrier especially if $\alpha \leqslant -1$. We assume this absorbing condition also for $-1 < \alpha < 1$. We denote the corresponding process by $r^0(t, r)$, $t \geqslant 0$, $r > 0$. As in the last section, we construct a diffusion process $X^0(t, x)$, $t \geqslant 0$, $x \neq 0$, as the skew product of the process $r^0(t, r)$ and an independent spherical Brownian motion $BM(S^1)$ run with the clock $\int_0^t g^{-2}(r^0(s, r))\, ds$ (cf. [11]). Then $X^0(t, x)$ is a diffusion process generated by the half of the Laplace–Beltrami operator $\Delta/2$ with the point 0 as its terminal point. Let b be a differential 1-form satisfying the conditions (b-i)–(b-ii) in the last section.

As in the last section, we consider the asymptotic behavior of a function defined by

$$I^0(\xi) := \lim_{(R,R')\to(0,0)} \int_{B(R)} E\left[\exp\left(\mathrm{i}\xi \int_0^1 b\big(X^0(t,x)\big) \circ dX^0(t,x)\right) : X^0(1,x) \in B(R')\right] d\,\mathrm{vol}(x)$$
$$\times \left\{ \int_{B(R)} P\big(X^0(1,x) \in B(R')\big)\, d\,\mathrm{vol}(x) \right\}^{-1}. \tag{5.1}$$

By taking the expectation in $BM(S^1)$, we can rewrite this as

$$I^0(\xi) = \lim_{(R,R')\to(0,0)} \int_0^R E\left[\exp\left(-\frac{\xi^2}{2}\int_0^1 F\big(r^0(t,r)\big)^2 dt\right) : r^0(1,r) < R'\right] m(dr) \left\{ \int_0^R P\big(r^0(1,r) < R'\big) m(dr) \right\}^{-1}. \tag{5.2}$$

We define a self adjoint operator

$$H^0 = \frac{1}{2}\big(-\Delta_\alpha + F_0(r)^2\big)$$

on $L^2((0,\infty), m^\alpha)$ as the Friedrichs extension of the corresponding operator on $C_0^\infty((0,\infty))$. This operator has purely discrete spectrum. We denote the eigenvalues by $\mu_0^0 < \mu_1^0 < \mu_2^0 < \cdots$, the corresponding normalized eigenfunctions by $\varphi_0^0, \varphi_1^0, \varphi_2^0, \ldots$, and the orthogonal projection to the orthogonal complement to the ground states by $P_\perp^0$. We easily see that $\mu_0^0 > 0$ and the nonzero existence of the derivative

$$\lim_{r\to 0} r^\alpha \varphi_0^{0\prime}(r) =: \varphi_{0,+}^0(0)$$

with respect to the canonical scale (see, e.g., McKean [16]). Let $e^{t\Delta_r^0/2}(r,r')$, $t, r, r' > 0$, be the transition density of the process $r^0(t,r)$ with respect to the speed measure $m(dr) := g(r)\,dr$. The nonzero existence of

$$\lim_{(r,r')\to(0,0)} g(r)\frac{\partial}{\partial r} g(r')\frac{\partial}{\partial r'} e^{\Delta_r^0/2}(r,r') =: e_{++}^{\Delta_r^0/2}(0,0)$$

is proven by using Lemma 5.2 below. Let $\mathcal{A}$, $\iota(\mathcal{A})$, $\mathcal{B}$ and $\mathcal{K}(\mathcal{B})$ be as in the last section.

The main theorem in this section is the following:

THEOREM 8. – *We assume the conditions* (g-i)–(g-iii), (g-vi) *and* (b-i)–(b-ii).

(i) *When $\alpha > -1$, we have the following asymptotic expansion as $\xi \to \infty$:*

$$I^0(\xi) \sim \exp\left\{-\xi^{2\gamma}\mu_0^0 - \sum_{A\in\mathcal{A}:\ \iota(A)\leqslant 2} \xi^{\gamma(2-\iota(A))}\mu_0^{0,A}\right\}\xi^{(3-\alpha)\gamma} \times\left\{I_0^0 + \sum_{B\in\mathcal{B}} I_B^0 \xi^{-\gamma\kappa(B)}\right\}, \tag{5.3}$$

where $\gamma = 1/(\rho+1)$, $\{\mu_0^{0,A}\colon A\in\mathcal{A}\}$ are polynomials of

$$\{g_n, f_n, (r^{k(0)}\varphi_0^0, (\mu_0^0 - H^0)^{-1} P_\perp^0 r^{k(1)} \cdots (\mu_0^0 - H^0)^{-1} P_\perp^0 r^{k(n)}\varphi_0^0)\colon n\in\mathbb{Z}_+, k(0), k(1), \ldots, k(n) \geqslant 0\},$$

$$I_0^0 = \frac{\varphi_{0,+}^0(0)^2}{e_{++}^{\Delta_r^0/2}(0,0)},$$

and $\{I_B^0\colon B\in\mathcal{B}\}$ are polynomials of

$$\{g_n, f_n, (r^{k(0)}\varphi_0^0, (\mu_0^0 - H^0)^{-1} P_\perp^0 r^{k(1)} \cdots (\mu_0^0 - H^0)^{-1} P_\perp^0 r^{k(n)} \varphi_0^0),$$
$$\lim_{r\to 0} r^{\alpha-1}((\mu_0^0 - H^0)^{-1} P_\perp^0 r^{k(0)} \cdots (\mu_0^0 - H^0)^{-1} P_\perp^0 r^{k(n)} \varphi_0^0)(r):$$
$$n \in \mathbb{Z}_+, k(0), k(1), \ldots, k(n) \geqslant 0\}.$$

(ii) *For any $\alpha < 1$, we have the following asymptotic expansion as $\xi \to \infty$:*

$$I^0(\xi) \sim \exp\left\{-\xi^{2\gamma}\mu_0^0 - \sum_{A\in\mathcal{A}:\, \iota(A)\leqslant 2} \xi^{\gamma(2-\iota(A))}\mu_0^{0,A}\right\}\xi^{(3-\alpha)\gamma}\{I_0^0 + \mathrm{o}(1)\}. \tag{5.4}$$

As in Theorem 7, $\{\mu_0^{0,A}\colon A \in \mathcal{A}, \iota(A) \leqslant 2\}$ are independent of $\{g_n\colon n \geqslant 1\}$.

The fundamental tool is the following representation of the transition density $\mathrm{e}^{t\Delta_\alpha^0/2}(r,r')$, t, r, $r' > 0$, with respect to the speed measure $m^\alpha(dr)$, of the process $r_\alpha^0(t,r)$ generated by the operator $\Delta_\alpha/2$ with the absorbing condition at the boundary 0:

LEMMA 5.1. –

$$\mathrm{e}^{t\Delta_\alpha^0/2}(r,r') = \exp\left(-\frac{r^2+r'^2}{2t}\right)\frac{1}{t}\left(\frac{rr'}{\sqrt{2t}}\right)^{1-\alpha}\sum_{n=0}^{\infty}\frac{(rr'/(2t))^{2n}}{n!\Gamma(n+(3-\alpha)/2)}. \tag{5.5}$$

We can prove this samely as in Example IV-8.3 of [10].

Remark 5.1. – In terms of the modified Bessel function

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{n!\Gamma(\nu+n+1)},$$

(5.5) is rewritten as

$$\mathrm{e}^{t\Delta_\alpha^0/2}(r,r') = \exp\left(-\frac{r^2+r'^2}{2t}\right)\frac{(rr')^{(1-\alpha)/2}}{t} I_{(1-\alpha)/2}\left(\frac{rr'}{t}\right).$$

This expression is similar to that for the usual Bessel process with index $\alpha+1$ for $\alpha > -1$:

$$\mathrm{e}^{t\Delta_\alpha/2}(r,r') = \exp\left(-\frac{r^2+r'^2}{2t}\right)\frac{(rr')^{(1-\alpha)/2}}{t} I_{(\alpha-1)/2}\left(\frac{rr'}{t}\right).$$

By this lemma and the subsequent Lemma 5.2, we can represent as

$$I^0(\xi) = \mathrm{e}_{\alpha\alpha}^{-\mathcal{H}^0(\xi)}(0,0)/\mathrm{e}_{\alpha\alpha}^{\Delta_r^0/2}(0,0), \tag{5.6}$$

where

$$\mathrm{e}_{\alpha\alpha}^{-t\mathcal{H}^0(\xi)}(0,0) = \lim_{(r,r')\to(0,0)} r^{\alpha-1}r'^{\alpha-1}\mathrm{e}^{-t\mathcal{H}^0(\xi)}(r,r'),$$

$$\mathrm{e}_{\alpha\alpha}^{t\Delta_r^0/2}(0,0) = \lim_{(r,r')\to(0,0)} r^{\alpha-1}r'^{\alpha-1}\mathrm{e}^{t\Delta_r^0/2}(r,r')$$

and

$$\mathcal{H}^0(\xi) = \frac{1}{2}(-\Delta_r + \xi^2 F(r)^2)$$

on $L^2((0,\infty),m)$ is the Friedrichs extension of the corresponding operator on $C_0^\infty((0,\infty))$.

LEMMA 5.2. – *The heat kernel* $\mathrm{e}^{-t\mathcal{H}^0(\xi)}(r,r')$ *and the transition density* $\mathrm{e}^{t\Delta_r^0/2}(r,r')$ *have the representations*

$$\mathrm{e}^{-t\mathcal{H}^0(\xi)}(r,r') = \sqrt{\frac{r^\alpha}{g(r)}}E\left[\exp\left(\frac{1}{2}\int_0^t (\tilde{g} - \xi^2F^2)(r_\alpha^0(s,r))\,ds\right)\middle| r_\alpha^0(t,r) = r'\right]\mathrm{e}^{t\Delta_\alpha^0/2}(r,r')\sqrt{\frac{r'^\alpha}{g(r')}} \tag{5.7}$$

and

$$\mathrm{e}^{t\Delta_r^0/2}(r,r') = \sqrt{\frac{r^\alpha}{g(r)}}E\left[\exp\left(\frac{1}{2}\int_0^t \tilde{g}(r_\alpha^0(s,r))\,ds\right)\middle| r_\alpha^0(t,r) = r'\right] \times \mathrm{e}^{t\Delta_\alpha^0/2}(r,r')\sqrt{\frac{r'^\alpha}{g(r')}}, \tag{5.8}$$

where

$$\tilde{g}(r) = \frac{\alpha(\alpha-2)}{4r^2} + \frac{g'^2 - 2gg''}{4g^2}.$$

As in the last section, we use the scaling property to rewrite (5.6) as

$$I^0(\xi) = \xi^{(3-\alpha)\gamma}\mathrm{e}_{\alpha\alpha}^{-\xi^{2\gamma}H^0(\xi)}(0,0)/\mathrm{e}_{\alpha\alpha}^{\Delta_r^0/2}(0,0),$$

where

$$H^0(\xi) = \frac{1}{2}\big(- \Delta_\xi + F(r,\xi)^2\big).$$

Moreover we rewrite this as

$$I^0(\xi) = \xi^{(3-\alpha)\gamma} \sum_{n=0}^{\infty} \mathrm{e}^{-\xi^{2\gamma}\mu_n^0(\xi)} \varphi_{n,\alpha}^0(0,\xi)^2 / \mathrm{e}^{\Delta_r^0/2}(0,0),$$

where

$$\varphi_{n,\alpha}^0(0,\xi) = \lim_{r\to 0} r^{\alpha-1}\varphi_n^0(r,\xi),$$

$\mu_0^0(\xi) < \mu_1^0(\xi) < \mu_2^0(\xi) < \cdots$ are the eigenvalues of $H^0(\xi)$ and $\varphi_0^0(r,\xi)$, $\varphi_1^0(r,\xi), \varphi_2^0(r,\xi), \ldots$ are the corresponding normalized eigenfunctions.

As in the last section, we have the following:

PROPOSITION 5.1. – (i) $\lim_{\xi\to\infty} \mu_n^0(\xi) = \mu_n^0$ *for each* $n \geqslant 0$.
(ii) *We have the following asymptotic expansion as* $\xi \to \infty$:

$$\mu_0^0(\xi) \sim \mu_0^0 + \sum_{A\in\mathcal{A}} \frac{\mu_0^{0,A}}{\xi^{\gamma\iota(A)}},$$

where $\mathcal{A}$, μ_0^A, γ *and* $\iota(A)$ *are those in Theorem* 8.

PROPOSITION 5.2. – *Under the conditions of Theorem* 8(i), *we have the following asymptotic expansion as* $\xi \to \infty$:

$$\varphi_{0,\alpha}^0(0,\xi) \sim \varphi_{0,\alpha}^0(0) + \sum_{A\in\mathcal{A}} \frac{\varphi_0^{0,A}(0)}{\xi^{\gamma\iota(A)}},$$

where $\mathcal{A}$, γ, $\iota(A)$ *are those in Theorem* 8,

$$\varphi_{0,\alpha}^0(0) = \lim_{r\to 0} r^{\alpha-1}\varphi_0^0(r)$$

and $\{\varphi_0^{0,A}(0)\colon\ A \in \mathcal{A}\}$ *are polynomials of*

$$\begin{aligned}&\big\{g_n, f_n, \big(r^{k(0)}\varphi_0^0, \big(\mu_0^0 - H^0\big)^{-1} P_\perp^0 r^{k(1)} \cdots \big(\mu_0^0 - H^0\big)^{-1} P_\perp^0 r^{k(n)}\varphi_0^0\big),\\ &\lim_{r\to 0} r^{\alpha-1}\big(\big(\mu_0^0 - H^0\big)^{-1} P_\perp^0 r^{k(0)} \cdots \big(\mu_0^0 - H^0\big)^{-1} P_\perp^0 r^{k(n)}\varphi_0^0\big)(r)\colon\\ &n \in \mathbb{Z}_+,\ k(0), k(1), \ldots, k(n) \geqslant 0\big\}.\end{aligned}$$

By these propositions, Theorem 8 is proven similarly as in the last section, since $\varphi^0_{0,\alpha}(0) = \varphi^0_{0,+}(0)/(1-\alpha)$ and

$$e^{t\Delta^0_r/2}_{\alpha\alpha}(0,0) = \frac{1}{(1-\alpha)^2} e^{t\Delta^0_r/2}_{++}(0,0).$$

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