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Asymptotics for an Arcsin type result

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ABSTRACT. — Let A_t be the amount of time that a Brownian motion spends above 0 before time t. For fixed t the ratio A_t/t has distribution independent of t; viewed as a function of time A_t/t can become arbitrarily small. In this paper we consider the effect of modifying the denominator. In particular, if f is monotonic then $\lim \inf A_t/tf(t)=0$ or ∞ according as $\int^{\infty} \sqrt{f(t)}\,(dt/t)$ diverges or converges.

The proof considers A_t at the ends of "long" negative excursions and involves showing the existence of infinitely many such excursions.

Key words : Brownian sample path, Arcsin law, Excursion theory.

RÉSUMÉ. — Soit A_t le temps passé par le mouvement brownien au-dessus de zéro avant le temps t. Pour t donné, le rapport A_t/t a une distribution indépendante de t; et $\lim \inf A_t/t=0$. Dans cet article, nous nous intéressons aux conséquences d'une modification du dénominateur. Nous montrons que, si f est une fonction décroissante, alors $\liminf_{t \uparrow \infty} A_t/tf(t)=0$ si $\int^{\infty} \sqrt{f(t)}\,(dt/t)=\infty$ et $\lim_{t \uparrow \infty} A_t/tf(t)=\infty$ si $\int^{\infty} \sqrt{f(t)}\,(dt/t)<\infty$.

La démonstration s'intéresse à A_t à la suite d'excursions négatives prolongées et met en relief l'existence d'une infinité d'excursions.

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1. MAIN RESULTS

The proportion of time that a Brownian sample path spends above zero before a fixed time t is independent of t and is characterised by the arcsin law. Many other functionals of sample paths inherit time-scaling properties from Brownian motion. In this paper we consider the asymptotic behaviour of these functionals.

Define

$$A_t = \int_0^t I_{\{B_s > 0\}} ds. \tag{1}$$

Then A_t/t has the arcsin distribution:

$$\mathbb{P}\left(\frac{A_t}{t} \leqq u\right) = \frac{2}{\pi} \sin^{-1}(\sqrt{u}). \tag{2}$$

We consider

$$\liminf_t \frac{A_t}{t f(t)} \tag{3}$$

for suitable functions f. For f the unit function $f(t) \equiv 1$ we can use (2) and the Hewitt-Savage 0-1 Law to conclude

$$\liminf_t \frac{A_t}{t f(t)} = 0 \quad \text{a. s.} \tag{4}$$

We consider when (4) is true for other functions f.

Suppose that C_t is an adapted functional of a Brownian sample path and that C_t has the properties

C 1 $0 < C_t \leqq t$ and C_t increasing;

C 2 C_t constant on negative Brownian excursions;

C 3 C_t/t has distribution F independent of t, and

$$\lim_{u \downarrow 0} u^{-1/2} F(u) = c > 0;$$

We prove our results in terms of the functional C_t; notice that A_t satisfies these conditions, as does J_t where

$$J_t = \sup_{s \leqq t} \{ s : B_s = 0 \}.$$

THEOREM 1. — *Suppose f is decreasing. Then*

(i)

$$\int^{\infty} \frac{dt}{t}\sqrt{f(t)} < \infty \quad \Rightarrow \quad \liminf_{t \uparrow \infty} \frac{C_t}{tf(t)} = \infty \quad \text{a.s.} \tag{5}$$

(ii)

$$\int^{\infty} \frac{dt}{t}\sqrt{f(t)} = \infty \quad \Rightarrow \quad \liminf_{t \uparrow \infty} \frac{C_t}{tf(t)} = 0 \quad \text{a.s.} \tag{6}$$

Since we obtain an integral test for f some monotonicity condition is essential. Also we only need the hypothesis to hold in a neighbourhood of infinity, since we can change the definition of f on some compact interval without altering either the statements or conclusions of Theorem 1. The analogous theorem for small times is as follows:

THEOREM 2. — *Suppose g is increasing. Then*

(i)

$$\int_{0+} \frac{ds}{s}\sqrt{g(s)} < \infty \quad \Rightarrow \quad \liminf_{s \downarrow 0} \frac{C_s}{sg(s)} = \infty \quad \text{a.s.} \tag{7}$$

(ii)

$$\int_{0+} \frac{ds}{s}\sqrt{g(s)} = \infty \quad \Rightarrow \quad \liminf_{s \downarrow 0} \frac{C_s}{sg(s)} = 0 \quad \text{a.s.} \tag{8}$$

A higher dimensional version of this question has been investigated by Mountford [3] and Meyre [2]. Mountford considered the proportion of time that planar Brownian motion spends in a wedge, normalised by the functions $f(t) = (\log t)^{-p}$. By considering the wedge of angle π we can specialise his result to one dimension:

$$\liminf_{t \uparrow \infty} (\log t)^p \frac{A_t}{t} = \begin{cases} 0 & p < 2 \\ \infty & p > 2 \end{cases} \tag{9}$$

Meyre considered the d-dimensional case, and the amount of time spent in a closed cone with vertex O by a Brownian motion. Again considering functions of the form $f(t) = (\log t)^{-p}$, he extended the results of Mountford and showed in particular that for $p = 2$

$$\liminf_{t \uparrow \infty} (\log t)^2 \frac{A_t}{t} = 0 \tag{10}$$

Both authors derived their results by finding an increasing unbounded sequence of times at which the Brownian motion was relatively far from the origin, and then showing that it would sometimes take a "long" time to leave the relevant wedge or cone. In this paper we use excursion arguments which are specific to one-dimension to prove a more sensitive result.

The remainder of the paper is structured as follows. In section 2 we use Borel-Cantelli type arguments to prove

$$\int^{\infty} \frac{dt}{t}\sqrt{f(t)} < \infty \quad \Rightarrow \quad \liminf_{t \uparrow \infty} \frac{C_t}{tf(t)} = \infty. \tag{11}$$

In section 3 we complete the proof of Theorem 1 by considering the value of C_t at the ends of "long" excursions from zero. In Section 4 we deduce Theorem 2 by considering the map $B_t \to t B_{1/t}$.

I am pleased to acknowledge the assistance with the excursion arguments provided by Professor Chris Rogers.

2. PROOF OF THEOREM 1 (i)

For fixed $r > 1$, K

$$\mathbb{P}\left(\frac{C_{r^n}}{r^n f(r^n)} < K\right) = F(K f(r^n)) \tag{12}$$

Then using the fact that f is decreasing we have

$$\sum_n \mathbb{P}\left(\frac{C_{r^n}}{r^n f(r^n)} < K\right) < \infty \quad \text{if and only if} \quad \int^{\infty} \frac{dy}{y} F(K f(y)) < \infty$$

and then by the First Borel-Cantelli Lemma

$$\int^{\infty} \frac{dy}{y} F(K f(y)) < \infty \quad \Rightarrow \quad \mathbb{P}\left(\frac{C_{r^n}}{r^n f(r^n)} > K, \text{ eventually}\right) = 1.$$

For $r^n \leqq t < r^{n+1}$

$$\frac{C_t}{tf(t)} > \frac{C_{r^n}}{r^{n+1} f(r^n)}$$

and $C_t/tf(t) > K/r$ eventually, almost surely.

3. PROOF OF THEOREM 1 (ii)

In order to prove Theorem 1 (ii) we consider the excursion process of B_t from 0. The principle underlying the proof is that low values of $C_t/tf(t)$ will occur near the ends of long negative excursions. We define "long" excursions in an f-dependent way such that $C_t/tf(t)$ is less than ε near the end of such excursions; the result follows if we can prove that, for each ε, there are an infinite number of long excursions.

A feature of the proof is that it is not necessary to consider combinations of two or more large negative excursions following soon after one another.

By hypothesis f is decreasing and without loss of generality we may assume that $f(t) \leqq 1$ and $f(t) \downarrow 0$; otherwise replace f by $f(t) \wedge (\log(e+t))^{-1}$.

First we establish some suitable notation: let L_t be the local time (of B) at 0 by time t, and let $\gamma_l \equiv \inf\{t: L_t > l\}$ so that γ_l is a subordinator. Classify an excursion at local time l as a *long* excursion if its lifetime ξ satisfies

$$\xi f(\xi) > \frac{l^2}{\varepsilon}.$$

Classify it as a *long-early* excursion if also $\gamma_l \leqq l^2$. If $\tau = \gamma_{l-}$ is the time at which a long-early excursion begins then,

$$\frac{C_\xi}{\xi f(\xi)} \leqq \frac{C_{\tau+\xi-}}{\xi f(\xi)} = \frac{C_\tau}{\xi f(\xi)} < \varepsilon \frac{C_\tau}{l^2} \leqq \varepsilon \frac{C_\tau}{\tau} \leqq \varepsilon. \tag{13}$$

It remains to show that there is an increasing unbounded sequence of negative long-early excursions; Theorem 1 (ii) then follows immediately.

Define

$$\begin{aligned} \phi(l) &= \int_{\{\xi : \xi f(\xi) > l^2/\varepsilon\}} \frac{d\xi}{\sqrt{2\pi}\,\xi^{3/2}} \\ &= \text{rate of long excursions at local time } l. \end{aligned} \tag{14}$$

The integrand in (14) follows from the fact that the rate of excursions whose lifetime exceeds ξ is $(2/\pi\xi)^{1/2}$ (*see* for example [1], p. 129, but beware normalisations of the local time which introduce extra factors of 2). Note that since $f(\xi) \leqq 1$

$$\begin{aligned} \phi(l) &\leqq \int_{\{\xi : \xi > l^2/\varepsilon\}} \frac{d\xi}{\sqrt{2\pi}\,\xi^{3/2}} \\ &= \left(\frac{2\varepsilon}{\pi}\right)^{1/2} \frac{1}{l}. \end{aligned} \tag{15}$$

For $l > 1$ let N_l be the number of negative long-early excursions between local times 1 and l. (We start counting at local time 1 to avoid the potential complication of an infinite number of long excursions occuring immediately at time 0.) Our proof that

$$N_\infty = \infty \text{ a.s. if and only if } \int^\infty \frac{dt}{t}\sqrt{f(t)} = \infty$$

is completed in two stages.

PROPOSITION 3.1:

$$N_\infty = \infty \text{ a.s.} \Leftrightarrow \int_1^\infty I_{\{\gamma_u \leqq u^2\}} \phi(u)\, du = \infty \quad \text{a.s.}$$

Proof. – Let

$$V_l = \frac{1}{2} \int_1^l I_{\{\gamma_u \leqq u^2\}} \phi(u)\, du$$

and define $M_l = N_l - V_l$. The rate of negative long-early excursions at local time u is a multiple $\frac{1}{2} I_{\{\gamma_u \leqq u^2\}}$ of the rate of long excursions given in (14). Thus V is the compensator for N and M is a martingale. We need to prove that $N_\infty = \infty$ almost surely if and only if $V_\infty = \infty$ almost surely.

M is a martingale with bounded jumps. M stopped when it first falls to $-k$ is a martingale bounded below; thus it converges almost surely to a finite limit. If $N_\infty = \infty$ then the stopped martingale can only converge by reaching $-k$; since k is arbitrary it follows that lim inf $M_l = -\infty$; thus $V_\infty = \infty$.

Conversely if $V_\infty = \infty$ then a similar argument applied to M stopped on first rising above k shows that lim sup $M_l = \infty$ almost surely; hence $N_\infty = \infty$. □

PROPOSITION 3.2:

$$\int^\infty I_{\{\gamma_u \leqq u^2\}} \phi(u)\, du = \infty \text{ a.s.} \Leftrightarrow \int^\infty \frac{dt}{t}\sqrt{f(t)} = \infty \tag{16}$$

Proof. – If $V_\infty = \infty$ then certainly

$$\int^\infty \phi(u)\, du = \infty$$

which can be shown by a few lines of calculus to be equivalent to the right hand side of (16).

Conversely let $K_l = I_{\{\gamma_l \leqq l^2\}}$ and $\mu_l = \mathbb{E}(V_l)$. We prove that our integral condition on f is equivalent to $\mu_l \uparrow \infty$ and hence that ($V_\infty = \infty$, a.s.) as required.

Note that $\mathbb{P}\{\gamma_s \leqq \alpha\} = \mathbb{P}\{L_\alpha \geqq s\} = \mathbb{P}\{S_\alpha \geqq s\} = \mathbb{P}\{|N| \geqq s/\sqrt{\alpha}\}$ where L and S are the local time and maximum processes of a Brownian motion and N is a standard normal random variable. In particular define $c=\mathbb{E}(K_s)\equiv\mathbb{P}\{\gamma_s\leqq s^2\}$, independent of s; then $\mu_t=\frac{1}{2}c\int_1^t\phi(u)\,du$.

For $s\leqq t$

$$\mathbb{P}\{\gamma_t\leqq t^2\,|\,\gamma_s\leqq s^2\}=\int_0^{s^2}\mathbb{P}(\gamma_s\in du)\frac{\mathbb{P}\{\gamma_{t-s}\leqq t^2-u\}}{\mathbb{P}\{\gamma_s\leqq s^2\}}$$

and

$$\begin{aligned}0&\leqq\mathbb{P}\{\gamma_t\leqq t^2\,|\,\gamma_s\leqq s^2\}-\mathbb{P}\{\gamma_t\leqq t^2\}\\&=\int_0^{s^2}\frac{\mathbb{P}\{\gamma_s\in du\}}{\mathbb{P}\{\gamma_s\leqq s^2\}}\mathbb{P}\left\{|N|\in\left(\frac{t-s}{\sqrt{t^2-u}},1\right)\right\}\\&\leqq k\frac{s}{t}\end{aligned}$$

for some constant k. This estimate can be used as follows:

$$\begin{aligned}\operatorname{Var}(V_t)&=\frac{1}{2}\int_1^t ds\,\phi(s)\int_1^s du\,\phi(u)\,\mathbb{E}(K_sK_u-c^2)\\&\leqq\frac{kc}{2}\int_1^t ds\,\phi(s)\left(\frac{1}{s}\int_1^s u\,\phi(u)\,\mathrm{du}\right)\\&\leqq D\,\mathbb{E}(V_t)\end{aligned}$$

for some constant D, this last line following from (15). Since $\mu_t\uparrow\infty$ we have

$$\frac{V_t}{\mathbb{E}(V_t)}\xrightarrow{L^2}1.$$

For any large constant Λ we can choose s sufficiently large so that $\mu_s>2\Lambda$, whence using Tchebyshev

$$\begin{aligned}&\mathbb{P}(V_\infty<\Lambda)\leqq\mathbb{P}(V_\infty<\mu_s/2)\leqq\mathbb{P}(V_s<\mu_s/2)\\&\qquad\leqq\mathbb{P}\left(\frac{|V_s-\mu_s|}{\mu_s}>\frac{1}{2}\right)\leqq\frac{4\operatorname{Var}(V_s)}{\mathbb{E}(V_s)^2}\to 0.\end{aligned}$$

Thus ($V_\infty=\infty$, a.s.) as required. □

4. ASYMPTOTIC RESULTS FOR SMALL TIMES

In this section we deduce Theorem 2. Suppose $g:(0,1]\to\mathbb{R}^+$ is increasing.

PROPOSITION 4.1:

$$\int_{0+} \frac{ds}{s} \sqrt{g(s)} < \infty \quad \Rightarrow \quad \liminf_{s \downarrow 0} \frac{C_s}{s g(s)} = \infty \quad \text{a.s.} \tag{17}$$

Proof. – By considering $C_s/sg(rs)$ down the geometric sequence r^{-n} and mirroring the proof of § 2 it is possible to obtain

$$\int_{0+} \frac{ds}{s} F(K g(s)) < \infty \quad \Rightarrow \quad \mathbb{P}\left(\frac{C_{r^{-n}}}{r^{-n} g(r^{-(n-1)})} > K, \text{ eventually} \right) = 1.$$

Using $g(s)$ increasing we have

$$\frac{C_s}{sg(s)} \geqq \frac{C_{r^{-n}}}{r^{-(n-1)} g(r^{-(n-1)})} \qquad r^{-n} \leqq s < r^{-(n-1)}$$

and $C_s/sg(s) > K/r$ eventually. □

PROPOSITION 4.2:

$$\int_{0+} \frac{ds}{s} \sqrt{g(s)} = \infty \quad \Rightarrow \quad \liminf_{s \downarrow 0} \frac{C_s}{sg(s)} = 0 \quad \text{a.s.} \tag{18}$$

Proof. – For the same reasons as in § 3 we may assume that $g(s) \leq 1$ and $g(0+) = 0$. Define $f: [1, \infty) \to [0, 1]$ by $f(t) = g(s)$ where $t = 1/s$. Then f is decreasing. Also

$$\int_{0+} \frac{ds}{s} \sqrt{g(s)} = \infty \quad \Leftrightarrow \quad \int^{\infty} \frac{dt}{t} \sqrt{f(t)} = \infty.$$

By the results of § 3 there are an infinite number of negative excursions (t_0, t_1) of lifetime $\xi = t_1 - t_0$ which satisfy both $\xi f(\xi) > t_0/\varepsilon$ and $\xi > t_0$. This second condition is not restrictive for large times since $f(t) \downarrow 0$.

If B_t is a Brownian motion then so is $W_s \equiv s B_t$. Hence if (t_0, t_1) is a Brownian excursion for B then (s_1, s_0) is a W-excursion, where $s_i = t_i^{-1}$, $i = 0, 1$. Then if C_s is defined relative to the Brownian motion W_s and if (t_0, t_1) is a long excursion for $B_t \equiv W_s/s$

$$\frac{C_{s_0}}{s_0 g(s_0)} = \frac{C_{s_1}}{s_1} \frac{s_1}{s_0 g(s_0)}$$

But $C_{s_1} \leqq s_1$ and since $\xi > t_0$ and f is decreasing

$$\frac{s_1}{s_0 g(s_0)} = \frac{t_0}{t_1 f(t_0)} < \frac{t_0}{\xi f(\xi)} < \varepsilon \quad \square$$

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