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On weak convergence of sequences of continuous local martingales

by

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SUMMARY. — Let $\{M^n(t)\}_{n=1,2,\dots}$ be a sequence of Hilbert space valued continuous local martingales. We give a necessary and sufficient condition for which $\{M^n\}$ is tight in C in terms of $\{\langle M^n \rangle\}$. Using this result we show the preservation of the local martingale property under the weak convergence in C of $\{M^n\}$.

Mots-clés : Continuous local martingale. Weak convergence.

RÉSUMÉ. — Soit $\{M^n(t)\}_{n=1,2,\dots}$ une suite de martingales locales continues à valeurs dans un espace de Hilbert. Nous donnons une condition nécessaire et suffisante pour que $\{M^n\}$ soit tendue dans C à l'aide d'une condition de tension sur $\{\langle M^n \rangle\}$. A l'aide de ce résultat, nous montrons que la propriété de martingale faible locale est conservée par la convergence faible dans C .

1. INTRODUCTION

Let H be a real separable Hilbert space with inner product (\cdot, \cdot) and $\{e_j\}_{j \in J}$ be an orthonormal basis of H . Consider a sequence $\{M^n(t)\}_{n=1,2,\dots}$

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of H -valued continuous local martingales starting at 0, where the time interval I is $[0, T]$ ($0 < T < +\infty$) or $[0, +\infty)$. Denote by $\text{dia} \langle M^n \rangle (t)$ the vector of all diagonal components of $(\langle (M^n, e_k), (M^n, e_l) \rangle (t))_{k, l \in J}$, where $\langle (M^n, e_k), (M^n, e_l) \rangle$ is the quadratic variational process of the martingales (M^n, e_k) and (M^n, e_l) (cf. [3] [6] [9]). We then regard $\text{dia} \langle M^n \rangle$ as an l^1 -valued continuous process in case that H is infinite-dimensional.

Rebolledo [11] proved, using Lenglart's inequality [8], that in case of $H = \mathbb{R}$ the tightness in C of $\{M^n(t)\}_{n=1,2,\dots}$ and the tightness in C of $\{\text{dia} \langle M^n \rangle (t)\}_{n=1,2,\dots}$ ($= \{\langle M^n \rangle (t)\}_{n=1,2,\dots}$) are equivalent to each other, where C is the space of continuous sample functions. A main purpose of this article is to show the above equivalence holds even if H is infinite-dimensional (Theorem A in Section 2).

By applying Theorem A we show that any accumulation point of $\{M^n(t)\}_{n=1,2,\dots}$ under the weak convergence in C is also a continuous local martingale (Theorem B in Section 3). Moreover the continuity of the mapping $\text{dia} \langle \cdot \rangle$ under the weak convergence in C is stated in Theorem C in Section 3.

In Appendix we give an elementary proof of Theorem A in case of $H = \mathbb{R}$ which is best adapted to the situation where all the processes considered are continuous and which is based on change of time.

2. TIGHTNESS OF $\{M^n\}$ AND $\{\text{dia} \langle M^n \rangle\}$

For a real separable Banach space E , set

$$C_0(I, E) = \{w; w: I \rightarrow E, w \text{ is continuous and } w(0) = 0\}$$

and we define the usual metric on $C_0(I, E)$ (see [5]); that is, $\{w_n\}$ converges to w if and only if $\{w_n\}$ converges uniformly to w on each compact interval in I . Let $\mathcal{B}(C_0(I, E))$ be the topological σ -algebra of $C_0(I, E)$. We denote the space of all probability measures on $(C_0(I, E), \mathcal{B}(C_0(I, E)))$ by $\mathcal{P}(C_0(I, E))$ and endow $\mathcal{P}(C_0(I, E))$ with the weak convergence topology. Consider the following subspace of $\mathcal{P}(C_0(I, E))$:

$$\mathcal{P}_{lm}(C_0(I, E)) = \{P \in \mathcal{P}(C_0(I, E)); \langle w(t), f \rangle$$

is continuous local martingale for every $f \in E'\}$,

where E' is the dual space of E .

For an orthonormal basis $\{e_j\}_{j \in J}$ of H , define the mapping $\phi: C_0(I, H) \rightarrow C_0(I, l^2(J))$ by

$$(2.1) \quad \phi(w)(t) = ((w(t), e_j))_{j \in J} \quad \text{for } w \in C_0(I, H), t \in I,$$

where for $p = 1, 2$

$$l^p(J) = \begin{cases} l^p & \text{if } \dim H = +\infty \\ \mathbb{R}^d & \text{if } \dim H = d < +\infty. \end{cases}$$

Then it is obvious that

$$(2.2) \quad \phi: C_0(I, H) \rightarrow C_0(I, l^2(J)) \quad \text{is a homeomorphism.}$$

According to the mapping ϕ , we introduce the mapping Φ :

$$\mathcal{P}(C_0(I, H)) \rightarrow \mathcal{P}(C_0(I, l^2(J)))$$

by

$$(2.3) \quad \Phi(P) = P \circ \phi^{-1} \quad \text{for } P \in \mathcal{P}(C_0(I, H)).$$

We have then by (2.2)

$$(2.4) \quad \Phi: \mathcal{P}(C_0(I, H)) \rightarrow \mathcal{P}(C_0(I, l^2(J))) \quad \text{is a homeomorphism}$$

and

$$(2.5) \quad \Phi: \mathcal{P}_{lm}(C_0(I, H)) \rightarrow \mathcal{P}_{lm}(C_0(I, l^2(J))) \quad \text{is a homeomorphism.}$$

We denote by $\hat{w}(t) = (\hat{w}_j(t))_{j \in J}$ the sample path of $C_0(I, l^2(J))$. Since the process $(\text{dia} \langle \hat{w} \rangle, \hat{P}) = ((\langle \hat{w}_j \rangle)_{j \in J}, \hat{P})$ is a $l^1(J)$ -valued continuous process for $\hat{P} \in \mathcal{P}_{lm}(C_0(I, l^2(J)))$, we can define the mapping $\text{dia} \langle \cdot \rangle$:

$$\mathcal{P}_{lm}(C_0(I, l^2(J))) \rightarrow \mathcal{P}(C_0(I, l^1(J)))$$

in the following manner; for $\hat{P} \in \mathcal{P}_{lm}(C_0(I, l^2(J)))$

$$(2.6) \quad \text{dia} \langle \hat{P} \rangle = \text{the probability measure on } C_0(I, l^1(J)) \text{ induced by } (\text{dia} \langle \hat{w} \rangle, \hat{P}).$$

We get the following theorem about the tightness of a family of H -valued continuous local martingales.

THEOREM A. — Let $P_\alpha \in \mathcal{P}_{lm}(C_0(I, H))$ ($\alpha \in A$). Then the following three conditions are equivalent to each other.

- i) $\{P_\alpha; \alpha \in A\}$ is tight in $C_0(I, H)$.
- ii) $\{\Phi(P_\alpha); \alpha \in A\}$ is tight in $C_0(I, l^2(J))$.
- iii) $\{\text{dia} \langle \Phi(P_\alpha) \rangle; \alpha \in A\}$ is tight in $C_0(J, l^1(J))$.

As stated in the introduction Rebolledo [11] proved the above theorem using Lengart's inequality in case of $H = \mathbb{R}$. So we will prove the above theorem in case of $\dim H = +\infty$. For this purpose we estimate the tail behaviors of $\{\Phi(P_\alpha); \alpha \in A\}$ and $\{\text{dia} \langle \Phi(P_\alpha); \alpha \in A \rangle\}$ uniformly in $\alpha \in A$.

Let K be a subset of l^p ($p = 1, 2$). It is well known ([4]) that K is relatively compact if and only if

$$(2.7) \quad \begin{cases} \sup_{c \in K} \|c\|_{l^p} < +\infty \\ \lim_{n \rightarrow \infty} \sup_{c \in K} \sum_{m=n}^{\infty} |c_m|^p = 0, \quad \text{where } c = (c_1, c_2, \dots). \end{cases}$$

Let $I = [0, T]$ and $Q_\alpha \in \mathcal{P}(C_0(I, l^p))$ ($\alpha \in A$). Noting (2.7), it is easy to see that $\{Q_\alpha; \alpha \in A\}$ is tight in $C_0(I, l^p)$ if and only if for any $\varepsilon > 0$ and $i = 1, 2, \dots$ there exist a compact set $K \subset l^p$ and $\delta(i) > 0$ such that

$$(2.8) \quad \sup_{\alpha \in A} Q_\alpha \left(\sup_{\substack{0 \leq s, t \leq T \\ |s-t| \leq \delta(i)}} |w_i(t) - w_i(s)| < \varepsilon \right) > 1 - \varepsilon, \quad i = 1, 2, \dots$$

and

$$(2.9) \quad \sup_{\alpha \in A} Q_\alpha(w(t) \in K \quad \text{for any } t \in I) > 1 - \varepsilon,$$

where $w = (w_1, w_2, \dots) \in C_0(I, l^p)$.

From now on we denote by $\hat{w} = (\hat{w}_1, \hat{w}_2, \dots)$ the sample path of $C_0(I, l^1)$. We prepare a lemma for the proof of Theorem A.

LEMMA 2.1. — Let $I = [0, T]$ and $\hat{P}_\alpha \in \mathcal{P}_m(C_0(I, l^2))$ ($\alpha \in A$). Then the following two properties are equivalent to each other.

i) For any $\varepsilon > 0$ there exists a compact set $K \subset l^2$ such that

$$(2.10) \quad \hat{P}_\alpha(\hat{w}(t) \in K \quad \text{for any } t \in I) \geq 1 - \varepsilon \quad \text{for } \alpha \in A.$$

ii) For any $\varepsilon > 0$ there exists a compact set $\hat{K} \subset l^1$ such that

$$(2.11) \quad \text{dia} \langle \hat{P}_\alpha \rangle(\hat{w}(t) \in \hat{K} \quad \text{for any } t \in I) \geq 1 - \varepsilon \quad \text{for } \alpha \in A.$$

Proof. — We show that i) implies ii). Assume that (2.10) holds. Set

$$\gamma = \begin{cases} \inf \{ t \in I; \hat{w}(t) \notin K \} \\ T \quad \text{if } \{ \} = \phi. \end{cases}$$

Since it holds that

$$\begin{aligned} \sup_{\alpha \in A} E^{\hat{P}_\alpha} [\|\text{dia} \langle \hat{w} \rangle(T \wedge \gamma)\|_{l^1}] &= \sup_{\alpha \in A} E^{\hat{P}_\alpha} [\|\hat{w}(T \wedge \gamma)\|_{l^2}^2] \\ &\leq \sup_{c \in K} \|c\|_{l^2}^2 = \beta < +\infty, \end{aligned}$$

we get by Chebyshev's inequality

$$(2.12) \quad \sup_{\alpha \in A} \mathring{P}_\alpha(\|\text{dia} \langle \mathring{w} \rangle (T \wedge \gamma)\|_{l^1} > G) \leq \frac{\beta}{G} \quad \text{for } G > 0.$$

On the other hand it is easy to see that

$$(2.13) \quad \lim_{n \rightarrow \infty} \sup_{\alpha \in A} E^{\mathring{P}_\alpha} \left[\sum_{m=n}^{\infty} \langle \mathring{w}_m \rangle (T \wedge \gamma) \right] \leq \lim_{n \rightarrow \infty} \sup_{c \in K} \sum_{m=n}^{\infty} c_m^2 = 0.$$

Let $\{\varepsilon_r\}_{r=1,2,\dots}$ be a sequence such that $\varepsilon_r \downarrow 0$ and $\sum_{r=1}^{\infty} \varepsilon_r \leq \varepsilon/2$. Then

in view of (2.13) for any $r = 1, 2, \dots$ there exists n_r such that

$$(2.14) \quad \sup_{\alpha \in A} E^{\mathring{P}_\alpha} \left[\sum_{m=n_r}^{\infty} \langle \mathring{w}_m \rangle (T \wedge \gamma) \right] \leq \varepsilon_r^2.$$

Putting

$$\Omega_r = \left\{ \mathring{w} \in C_0(I, l^2); \sum_{m=n_r}^{\infty} \langle \mathring{w}_m \rangle (T \wedge \gamma) > \varepsilon_r \right\},$$

we get from (2.14)

$$(2.15) \quad \mathring{P}_\alpha \left(\bigcup_{r=1}^{\infty} \Omega_r \right) \leq \sum_{r=1}^{\infty} \varepsilon_r \leq \varepsilon/2.$$

We then set $\hat{K} = \left\{ c = (c_1, c_2, \dots) \in l^1; \|c\|_{l^1} \leq 2\beta/\varepsilon, \sum_{m=n_r}^{\infty} |c_m| \leq \varepsilon_r \right.$

for $r = 1, 2, \dots \}$. Obviously \hat{K} is compact and satisfies from (2.12) and (2.15)

$$\sup_{\alpha \in A} \mathring{P}_\alpha(\{\mathring{w}; \text{dia} \langle \mathring{w} \rangle (t \wedge \gamma) \in \hat{K} \text{ for } t \in I\}) \geq 1 - \varepsilon.$$

Consequently we get from (2.10)

$$\sup_{\alpha \in A} \mathring{P}_\alpha(\{\mathring{w}; \text{dia} \langle \mathring{w} \rangle (t) \in \hat{K} \text{ for } t \in I\}) \geq 1 - 2\varepsilon.$$

We can prove the implication $ii) \rightarrow i)$ in the same way as above. \square

We now return to the proof of Theorem A.

Proof of Theorem A. — It is obvious that $i)$ and $ii)$ are equivalent to each other. We show the equivalence between $ii)$ and $iii)$. We may assume that $I = [0, T]$ and $\dim H = +\infty$. The Rebolledo's result [Cor. II, 3.14 in

[II]) implies that the condition (2.8) with $\{\Phi(P_\alpha)\}$ instead of $\{Q_\alpha\}$ is equivalent to the condition (2.8) with $\{\text{dia} \langle \Phi(P_\alpha) \rangle\}$ instead of $\{Q_\alpha\}$. On the other hand we get by Lemma 2.1 that the condition (2.9) with $\{\Phi(P_\alpha)\}$ instead of $\{Q_\alpha\}$ is equivalent to the condition (2.9) with $\{\text{dia} \langle \Phi(P_\alpha) \rangle\}$ instead of $\{Q_\alpha\}$. Consequently *ii*) and *iii*) are equivalent to each other. The proof is completed. \square

3. PRESERVATION OF LOCAL MARTINGALE PROPERTY

First we show that any accumulation point of a sequence of H-valued continuous local martingales under the weak convergence is also a H-valued continuous local martingale.

THEOREM B. — $\mathcal{P}_{lm}(C_0(I, H))$ is closed in $\mathcal{P}(C_0(I, H))$.

Proof. — We may assume $H = l^2$. Let $\{\overset{\circ}{P}_n\}$ converges weakly to $\overset{\circ}{P}(\overset{\circ}{P}_n \in \mathcal{P}_{lm}(C_0(I, l^2)), \overset{\circ}{P} \in \mathcal{P}(C_0(I, l^2)))$. Denote by R_n the probability measure on $C_0(I, l^2) \times C_0(I, l^1)$ induced by $((\overset{\circ}{w}, \text{dia} \langle \overset{\circ}{w} \rangle), \overset{\circ}{P}_n)$. Theorem A implies $\{R_n\}$ is tight. Therefore there exists a subsequence $\{n'\}$ such that $\{R_{n'}\}$ converges weakly to R . For $N = 1, 2, \dots$ set

$$\sigma_N(\hat{w}) = \inf \{ t; t + \|\hat{w}(t)\|_{l^1} > N \}.$$

For each $i = 1, 2, \dots, s_1, s_2, \dots, s_p, t_1, t_2, \dots, t_q \leq s < t$ and bounded continuous function f on $(l^2)^p \times (l^1)^q$, we have

$$(3.1) \quad \begin{aligned} E^{R_{n'}}[\overset{\circ}{w}_i(t \wedge \sigma_N) f(\overset{\circ}{w}(s_1), \dots, \overset{\circ}{w}(s_p), \hat{w}(t_1), \dots, \hat{w}(t_q))] \\ = E^{R_{n'}}[\overset{\circ}{w}_i(s \wedge \sigma_N) f(\overset{\circ}{w}(s_1), \dots, \overset{\circ}{w}(s_p), \hat{w}(t_1), \dots, \hat{w}(t_q))]. \end{aligned}$$

Since σ_N is R -a. s. continuous on

$$C_0(I, l^2) \times C_0(I, l^1) \quad \text{and} \quad \sup_{n'} E^{R_{n'}}[\overset{\circ}{w}_i(t \wedge \sigma_N)^2] < +\infty,$$

the equality (3.1) holds with R instead of $R_{n'}$. Therefore we have

$$\overset{\circ}{P} \in \mathcal{P}_{lm}(C_0(I, l^2)). \quad \square$$

Next we state a continuity property of the mapping $\text{dia} \langle \quad \rangle$ defined by (2.6).

THEOREM C. — $\text{dia} \langle \quad \rangle: \mathcal{P}_{lm}(C_0(I, l^2)) \rightarrow \mathcal{P}(C_0(I, l^1))$ is continuous.

Proof. — Within the situation of the above proof, we can show that

$(\hat{w}_i^2 - \hat{w}_i, \mathbf{R})$ is a continuous martingale for any $i = 1, 2, \dots$. Therefore \mathbf{R} is the probability measure on $C_0(\mathbf{I}, l^2) \times C_0(\mathbf{I}, l^1)$ induced by $((\hat{w}, \text{dia} \langle \hat{w} \rangle), \mathring{\mathbf{P}})$. This implies that $\{\mathbf{R}_n\}$ converges weakly to \mathbf{R} . Hence $\{\text{dia} \langle \mathring{\mathbf{P}}_n \rangle\}$ converges weakly to $\text{dia} \langle \mathring{\mathbf{P}} \rangle$ and the proof is completed. \square

Applying the above theorems, we can easily get the following remark about the central limit theorem for a sequence of Hilbert space valued continuous local martingales.

Remark. — Let $\mathbf{V} = (v_{ij})_{i,j=1,2,\dots}$ be a nonnegative definite, symmetric real matrix such that $\sum_{i=1}^{\infty} v_{ii} < +\infty$ and $\mathring{\mathbf{P}}_w \in \mathcal{P}_{lm}(C_0(\mathbf{I}, l^2))$ be a Wiener

measure such that the mean vector is zero and the covariance function is $((t \wedge s)v_{ij})$. Consider a sequence $\{\mathring{\mathbf{P}}_n\}$ ($\mathring{\mathbf{P}}_n \in \mathcal{P}_{lm}(C_0(\mathbf{I}, l^2))$). Then the following two conditions are equivalent to each other.

- i) $\{\mathring{\mathbf{P}}_n\}$ converges weakly to $\mathring{\mathbf{P}}_w$ in $C_0(\mathbf{I}, l^2)$.
- ii) $\{\text{dia} \langle \mathring{\mathbf{P}}_n \rangle\}$ converges weakly to the distribution of $(v_{11}t, v_{22}t, \dots)$ in $C_0(\mathbf{I}, l^1)$ and for any $i, j = 1, 2, \dots$ and $t > 0$ $\{\text{the distribution of } (\langle \hat{w}_i, \hat{w}_j \rangle(t), \mathring{\mathbf{P}}_n) \text{ on } \mathbb{R}\}$ converges weakly to the δ -distribution at $v_{ij}t$ on \mathbb{R} .

Finally we note that such infinite-dimensional central limit theorem for current valued stochastic processes are investigated by Ochi [10] and Ikeda and Ochi [5].

APPENDIX

As stated in the introduction, in the case of $H = \mathbb{R}$, Theorem A is proved by Rebolledo as a consequence of a general Aldous theorem for right continuous processes and using Lengart's inequality (cf. [7] [11]). The considered processes M^n may be discontinuous and only the limits of $\{M^n\}$ and $\{\langle M^n \rangle\}$ are assumed continuous. The proof is quite heavy. We give here a direct simple proof adapted to the special case of continuous processes.

LEMMA A.1. — Let $M(t)$ ($t \in [0, T]$) be an \mathbb{R} -valued continuous martingale defined on an usual filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ starting at 0 with $E[M(T)^4] < +\infty$. For each finite partition $\Delta = \{0 = s_0 < s_1 < \dots < s_l = T\}$ of $[0, T]$, set

$$(A.1) \quad I_\Delta(M, t) = \sum_{j=0}^{l-1} M(s_j) \{M(s_{j+1} \wedge t) - M(s_j \wedge t)\} \quad (0 \leq t \leq T).$$

Then there exists a universal positive constant C such that

$$(A.2) \quad E \left[\sup_{0 \leq t \leq T} \left| \int_0^t M(s) dM(s) - I_\Delta(M, t) \right|^2 \right] \leq 4\epsilon^2 E[\langle M \rangle(T)] + CE[\langle M \rangle(T)^2]^{1/2} E[\langle M \rangle(T)^2; \Omega_\epsilon^c]^{1/2} \quad \text{for } \epsilon > 0,$$

where $\Omega_\epsilon = \{ \omega \in \Omega : \sup_{\substack{|s-t| \leq |\Delta| \\ s, t \leq T}} |M(t, \omega) - M(s, \omega)| \leq \epsilon \}$

and $|\Delta| = \max_{0 \leq j \leq l-1} (s_{j+1} - s_j)$.

Proof. — It is easy to see that the left hand side of (A.2)

$$\leq 4E \left[\int_0^T (M(s) - M([s]))^2 d\langle M \rangle(s) \right] \leq 4\epsilon^2 E[\langle M \rangle(T)] + 16E \left[\sup_{0 \leq s \leq T} M(s)^2 \langle M \rangle(T); \Omega_\epsilon^c \right],$$

where $[s] = s_j$ for $s_j \leq s < s_{j+1}$ ($j = 0, 1, \dots, l-1$). Applying Schwarz's inequality to the second part of the right hand side of the above inequality, we get (A.2). \square

LEMMA A.2. — Let $M^n(t)$ ($n = 1, 2, \dots$) be an \mathbb{R} -valued continuous martingale defined on a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t^n)$ with $M^n(0) = 0$. Suppose that there exists a stochastic process M on (Ω, \mathcal{F}, P) such that with probability one $\{M^n\}$ converges to M uniformly on each compact interval. Further suppose that there exists a constant $\beta > 0$ such that $\sup_n E[|M^n(t)|^{4+\beta}] < \infty$ ($t > 0$). Then M is a continuous martingale with $E[|M(t)|^{4+\beta}] < \infty$ ($t > 0$) and satisfies

$$(A.3) \quad \lim_{n \rightarrow \infty} E \left[\sup_{0 \leq t \leq T} |\langle M^n \rangle(t) - \langle M \rangle(t)|^2 \right] = 0 \quad \text{for } T > 0.$$

Proof. — It is obvious that M is a continuous martingale with $E[|M(t)|^{4+\beta}] < \infty$ ($t > 0$). Itô formula implies

$$M^n(t)^2 = 2 \int_0^t M^n(s) dM^n(s) + \langle M^n \rangle(t).$$

We then get from the assumption

$$(A.4) \quad \lim_{n \rightarrow \infty} E \left[\sup_{0 \leq t \leq T} |M^n(t)^2 - M(t)^2|^2 \right] = 0.$$

For any finite partition $\Delta = \{0 = s_0 < s_1 < \dots < s_l = T\}$ of $[0, T]$ we have

$$\begin{aligned} E \left[\sup_{0 \leq t \leq T} \left| \int_0^t M^n(s) dM^n(s) - \int_0^t M(s) dM(s) \right|^2 \right] \\ \leq 3E \left[\sup_{0 \leq t \leq T} \left| \int_0^t M^n(s) dM^n(s) - I_\Delta(M^n, t) \right|^2 \right] + 3E \left[\sup_{0 \leq t \leq T} |I_\Delta(M^n, t) - I_\Delta(M, t)|^2 \right] \\ + 3E \left[\sup_{0 \leq t \leq T} \left| \int_0^t M(s) dM(s) - I_\Delta(M, t) \right|^2 \right] = a_n(\Delta, 1) + a_n(\Delta, 2) + a_n(\Delta, 3) \quad \text{say,} \end{aligned}$$

where $I_\Delta(M^n, t)$ and $I_\Delta(M, t)$ are defined in the same way as (A.1). Since $\{\langle M^n \rangle (T)^2\}$ is uniformly integrable, Lemma A.1 implies that for any $\varepsilon > 0$ there exists a positive constant δ such that

$$a_n(\Delta, 1) + a_n(\Delta, 3) < \varepsilon \quad \text{for} \quad |\Delta| < \delta, \quad n = 1, 2, \dots,$$

On the other hand it is easy to see that $\lim_{n \rightarrow \infty} a_n(\Delta, 2) = 0$ for any Δ . Therefore we have

$$(A.5) \quad \lim_{n \rightarrow \infty} E \left[\sup_{0 \leq t \leq T} \left| \int_0^t M^n(s) dM^n(s) - \int_0^t M(s) dM(s) \right|^2 \right] = 0.$$

Combining (A.4) with (A.5), we then get (A.3). □

Proof of Theorem A in case of $H = \mathbb{R}$. — First we prove that iii) implies ii). Let $M^n(t)$ ($n = 1, 2, \dots$) be an \mathbb{R} -valued continuous local martingale with $M^n(0) = 0$ defined on a filtered probability space $(\Omega_n, \mathcal{F}_n, P_n, \mathcal{F}_t^n)$. We assume that $\{\langle M^n \rangle\}$ is tight in $C_0([0, +\infty), \mathbb{R})$. Then for any $n = 1, 2, \dots$ there exist an extension $(\Omega'_n, \mathcal{F}'_t, P'_n, \mathcal{F}_t'^n)$ of $(\Omega_n, \mathcal{F}_n, P_n, \mathcal{F}_t^n)$ and a one-dimensional Brownian motion $B^n(t)$ on $(\Omega'_n, \mathcal{F}'_t, P'_n, \mathcal{F}_t'^n)$ with $B^n(0) = 0$ such that $M^n(t) = B^n(\langle M^n \rangle (t))$. The tightness of $\{(B^n, \langle M^n \rangle)\}$ implies the tightness of $\{M^n\}$.

Next we prove that ii) implies iii). Let $M^n(t)$ ($n = 1, 2, \dots$) be an \mathbb{R} -valued continuous local martingale defined on a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t^n)$ with $M^n(0) = 0$. We assume that with probability one $\{M^n(t)\}$ converges uniformly on each compact interval. It is sufficient to prove that $\{\langle M^n \rangle\}$ is tight in $C_0([0, T], \mathbb{R})$ ($T > 0$). For $N = 1, 2, \dots$ and $n = 1, 2, \dots$ put

$$S_n(N) = \begin{cases} \inf \{t \in [0, T]; |M^n(t)| > N\} \\ T \quad \text{if} \quad \{ \} = \phi. \end{cases}$$

It is easy to see that for any $\varepsilon > 0$ there exists N_0 such that

$$(A.6) \quad P(S_n(N_0) = T) > 1 - \varepsilon \quad \text{for} \quad n = 1, 2, \dots$$

Since $\{M^n(t \wedge S_n(N))\}$ is tight in $C_0([0, T], \mathbb{R})$ for $N = 1, 2, \dots$, we have by Lemma A.2 that $\{\langle M^n \rangle (t \wedge S_n(N))\}$ is tight in $C_0([0, T], \mathbb{R})$ for $N = 1, 2, \dots$. Therefore the tightness in $C_0([0, T], \mathbb{R})$ of $\{\langle M^n \rangle\}$ follows from (A.6). □

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