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## « Minimal length » multi-channel

by

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**SUMMARY.** — We give here an approach to a construction of a stationary multi-channel queueing system in which the incoming customer chooses one of the queues of minimal length.

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### INTRODUCTION

What follows is an approach to a construction of a stationary multi-channel queueing system in which the incoming customer chooses one of the queues of minimal length. The construction of a stationary model in the case of the choice of the minimal waiting time is now classical (see [1]), but the replacement of a minimal waiting time by a minimal length makes the problem more delicate.

In order to attack this question, we use the techniques developed in [6]. Let us recall briefly what the spirit of these results is: the construction of the stationary model is based on an iteration of the evolutionary model and a fixed point argument. In order to describe these models we use semi-groups (representing the spontaneous evolution of the system from a certain state without any exterior incoming customer) perturbed at the entrance time of customers.

In part one we summarize the results of [6] and in part two we first

deal with the « minimal length » system and then briefly quote the results we obtain in the case of a « minimal waiting time » system. For the questions concerning stationary point processes the reader is referred to [5].

## I. NOTATION

Let  $M$  (resp.  $M_+$ ) be the space of simple point measures doubly infinite on  $\mathbb{R}$  (resp. infinite on  $\mathbb{R}_+$ , with no mass in 0) endowed with the borelian vague sigma-field and the canonical flow  $(\tau_t, t \in \mathbb{R})$  (resp.  $\mathbb{R}_+$ ) of translations. A space of type  $S$  (resp.  $E$ ) is a measurable space  $(\Omega, \mathcal{A})$  endowed with a measurable flow  $\theta = (\theta_t, t \in \mathbb{R})$  (resp. semi-flow  $(\theta_t, t \in \mathbb{R}_+)$ ) and a stationary point process  $N$  from  $\Omega$  to  $M$  (resp.  $M_+$ ), moreover such a space will be called probability space of type  $S$  if it is endowed with a stationary probability.

We define a transport from a space  $(\Omega_1, \theta^1, N_1)$  of type  $S$  into a space  $(\Omega_2, \theta^2, N_2)$  of type  $S$  or  $E$  to be a measurable map  $\phi$  from  $\Omega_1$  to  $\Omega_2$  such that  $\theta_t^2 \circ \phi = \phi \circ \theta_t^1$  for every  $t \in \mathbb{R}$  or  $\mathbb{R}_+$  and such that  $N^1 = N^2 \circ \phi$  on  $\mathbb{R}$  or  $\mathbb{R}_+^*$ ; if these spaces are endowed with probabilities we will suppose besides that  $\phi \circ P_1 = P_2$ . ( $\mathbb{R}_+^* = \mathbb{R}_+ - \{0\}$ ).

For a space  $(\Omega, \theta, N)$  of type  $S$  we define  $\hat{\Omega} = \{\omega \in \Omega; N(\omega, \{0\}) = 1\}$ . If  $(\Omega, \theta, N)$  is of type  $S$  (resp.  $E$ ) and if  $T_1(\omega) = \inf\{t > 0, N(\omega, t) = 1\}$  then  $\hat{\theta}$  is the measurable map  $\theta_{T_1}$  from  $\hat{\Omega}$  to  $\hat{\Omega}$  (resp. from  $\Omega$  to  $\Omega$ ), finally we call  $C$  the set of bounded positive measures on  $\hat{\Omega}$  (resp.  $\Omega$ ) such that

$$\hat{\theta} \circ m = m \text{ and } \int T_1 dm = 1.$$

The proof of the following results are to be found in [6]. Let  $\Pi_1$  be an invariant closed subset of  $M \times D^Z$  ( $D$  is a polish space,  $M \times D^Z$  has a natural structure of type  $S$ , and is endowed with the product topology) such that  $\Omega_1 = p(\Pi_1)$  is polish in  $M \times D^{N^*}$ .

Let  $E$  be a polish space and  $\theta^e$  be an extension of the semi-flow on  $\Omega_1$  to  $\Omega_e = \Omega_1 \times E$ .  $\Pi_1 \times E^Z$  then has a natural structure of type  $S$ . We define  $\Pi_s$  to be the measurable invariant subset (which may be empty) of  $\bar{\omega} = (\omega, (e_n, n \in \mathbb{Z}))$  belonging to  $\Pi_1 \times E^Z$  such that if  $k \in \mathbb{N}$  and  $n \in \mathbb{Z}$ ,  $e_{n+k} = \text{proj}_E \circ \theta_{T_k}^e \circ \psi \circ \theta_{T_n}(\bar{\omega})$ , where  $\psi$  is the measurable map from  $\Pi_1 \times E^Z$  to  $\Omega_e$  which maps  $(\omega, (e_q, q \in \mathbb{Z}))$  onto  $(p(\omega), e_0)$ . We then have

**PROPOSITION 1.** — If the class  $C$  attached to  $\Omega_e = \Omega_1 \times E$  is not empty, then  $\Pi_s$  is not empty and if  $\bar{p}$  is the transport from  $\Pi_s$  to  $\Omega_e$  that sends  $\bar{\omega}$  onto  $(\text{proj}_1(\bar{\omega}), X_0(\bar{\omega}))$ , (where  $X_0(\bar{\omega}) = \text{proj}_E \circ \theta^e(-T_0(\bar{\omega}), \psi \circ \theta_{T_0}(\bar{\omega}))$ ;  $(T_n, n \in \mathbb{Z})$  being the « jump times » of the canonical point process on  $\Pi_s$ )

we have the following property: for any bounded positive measure  $m$  on  $\Omega_e$  belonging to  $C$ , there exists one and only one stationary probability on  $\Pi_S$  having a Palm measure whose image through  $\bar{p}$  is  $m$ .

Let us give an example of extension to  $\Omega_e$  of a semi-flow on  $\Omega_1$ . If  $S$  is a measurable semi-group on  $E$  and  $A$  a measurable map from  $D \times E$  to  $E$  then the mapping  $F$  defined by:  $F: R_+ \times \Omega_1 \times E \rightarrow E$   $F(0, \cdot, e) = e$ , on  $\{T_n \leq t < T_{n+1}\}$  ( $T_0 = 0$  by convention on a space of type  $E$ )

$$F_t = S_{t-T_n} \circ F_{T_n} \quad \text{and} \quad F_{T_{n+1}} = A(d_{n+1}, S_{T_{n+1}-T_n} \circ F_{T_n})$$

is such that:

$$(t, \omega_1, e) \rightarrow (\theta_t(\omega_1), F(t, \omega_1, e))$$

is a measurable extension to  $\Omega_e$  of the semi-flow  $\theta$  on  $\Omega_1$ .

We finish this part with a result concerning the class  $C$  attached to a space of type  $E$ .

**PROPOSITION 2.** — Let  $\Omega$  be a separable metric space and  $T$  be a continuous transformation on  $\Omega$ . If  $P$  is a probability measure on  $\Omega$  such that the  $(T^n \circ P, n \in \mathbb{N})$  are tight, then there exists an invariant probability measure with respect to  $T$  in the weakly closed convex hull of the  $T^n \circ P$ .

## II. THE « MINIMAL LENGTH » SYSTEM

In this part we are going to consider an approach to a stationary model of a system of  $K$  parallel channels, in which the incoming customer chooses one of the queues of minimal length, each of these queues having discipline « first come first served ». Because of the lack of continuity implied by this criterion, we will change it slightly.

In the markovian setting, one can find a study of the Kolmogorov equations, for the model with two channels in [3]. Let us describe first the state space  $E$  of the model and the semi-group  $S$  of evolution on  $E$ : we consider the set  $\tilde{E}$  of polynomials (that is of sequences vanishing except on a finite index set) with coefficients in  $R_+$  having the following property:  $P = (a_k; k \geq 0)$ , then  $a_n > 0$  and  $n > 0$  implies  $a_{n-1} > 0$ , endowed with the sigma-field generated by the coefficients (which is the borelian sigma-field of the polish topology defined by the canonical bijection between  $\tilde{E}$  and  $\bigcup_0^\infty R_{+,*}^n$  endowed with the sum topology).

From the topological standpoint, we will imbed  $\tilde{E}$  in  $L^1(\mathbb{N})$ ,  $\tilde{E}$  then has the structure of a separable metric space.

In the following  $\tilde{E}$  will be the model for the state at a given time of a queue of « first come first served » type, and when  $P = (a_0, \dots, a_n)$ , where  $a_n$  is the leading coefficient of  $P$ ,  $a_n$  represents the time of service having to be performed before the customer being served leaves the system,  $a_{n-1}$  the service time of the first person in the queue,  $\dots$ ,  $a_0$  the service time of the last person in the queue.

On  $\tilde{E}$  we consider the semi-group  $\tilde{T} : (\tilde{T}_t, t \in 0)$  which maps

$$P = (a_0, \dots, a_n)$$

(with  $a_i = 0$  for  $i \in n$ , and without requiring  $a_n$  to be the leading coefficient of  $P$ ) on  $\tilde{T}_t$ ,  $P = (u_0(t), \dots, u_n(t))$  where

$$u_n(t) = (a_n - t)_+$$

.....

$$u_k(t) = \left( a_k - \left( t - \|P\| + \sum_0^k a_i \right)_+ \right)_+ \quad \| \cdot \| \text{ denotes the } L^1(N) \text{ norm}$$

.....

$$u_0(t) = (a_0 - (t - \|P\| + a_0)_+)_+.$$

Then  $\tilde{T}$  is a continuous function from  $E \times R_+$  into  $E$ . To prove this we use that  $\|\tilde{T}_t P - P\| \leq t$ , which means we only need to show that  $\tilde{T}_s$  is continuous for  $s \in R_+$ . Choose  $0 < \alpha < b$  ( $b$  is the leading coefficient of  $P$ ), if  $P'$  is in a sufficiently small neighbourhood of  $P$ , then  $\tilde{T}_\alpha P$  and  $\tilde{T}_\alpha P'$  have the same degree, and the continuity of the coefficients gives the conclusion.

We next construct a continuous function  $L$  on  $\tilde{E}$  which will replace the length of the queue in the client's choice (the function  $d^0 + 1$  if one adopts the convention: degree of  $O = -1$ ).

Let  $\phi$  be a positive, measurable function on  $R_+$  with Lebesgue integral equal to 1, and let  $\phi$  be null on a neighbourhood of 0.

*Property.* — If  $L$  is the function from  $\tilde{E}$  into  $R_+$  which maps  $P$  on

$$\int \phi(t) \cdot (d^0(T_t P) + 1) dt,$$

then  $L$  is continuous on  $\tilde{E}$  and  $L \leq d^0 + 1$ .

Let  $1 > m > 0$  and let  $\tilde{E}_m$  be the subset of  $\tilde{E}$  consisting of polynomials with coefficients all less than  $1/m$  and greater than  $m$ , except for the leading coefficient which may be less than  $m$ . If  $\phi$  is null outside of  $(0, m)$  one has further that  $d^0 \leq L \leq d^0 + 1$  on  $\tilde{E}_m$ .

*Sketch of proof.* — To show the continuity of  $L$  we use that the integral defining  $L$  is carried on the interval  $(\alpha, +\infty)$ , (where  $\alpha > 0$ ) and hence for  $P'$  close to  $P$  we have that  $\tilde{T}_\alpha P'$  and  $\tilde{T}_\alpha P$  are close and  $d^0(\tilde{T}_\alpha P') \leq d^0(P)$ ; clearly if  $P'$  is close to  $P$  then

$$\{t \in \mathbb{R}_+ ; d^0(\tilde{T}_t P) = d^0(\tilde{T}_t P')\}$$

will have small Lebesgue measure; one obtains the continuity because the function  $d^0(\tilde{T}_t P')$  is bounded by  $d^0(P)$  on  $(\alpha, +\infty)$ . Finally if  $\phi$  is null outside  $(0, m)$  and if  $P \in \tilde{E}_m$  we have :

$$L(P) = \int_\alpha^m \phi(t) \cdot (d^0(T_t P) + 1) dt \quad \text{and for } t \in (0, m) d^0 P - 1 \leq d^0(T_t P) \leq d^0 P$$

from which we deduce  $d^0 P \leq L(P) \leq d^0 P + 1$ .

In what follows  $m$  will be a constant in  $]0, 1[$  so small that the client's service time  $\sigma$  satisfies  $m \leq \sigma \leq M = 1/m$ .

We fix a function  $\phi$  satisfying the conditions given in the preceding « property » and we take  $E = \tilde{E}_m^K$  as the state space, where  $\tilde{E}_m$  is a stable subset of the semi-group  $\tilde{T}$  on  $\tilde{E}$ ,  $E$  is then endowed with the product of the restrictions of  $\tilde{T}$  to  $\tilde{E}_m$ . Take  $D = (m, 1/m = M)$  and consider the continuous function  $A$  from  $D \times (1, K) \times E$  into  $E$  given by

$$A((\sigma, k, (P_i, i \in (1, K)))) = (P'_i, i \in (1, K))$$

with  $P'_i = P_i$  for  $i \neq k$  and  $P'_k = \sigma + X \cdot P_k$  (which belongs to  $\tilde{E}_m$ ). Let  $\Pi_0$  be a closed invariant subset of  $M \times (m, 1/m)^Z$  (with respect to the canonical flow) whose elements are noted  $\omega_0 = (m, (\sigma_n, n \in \mathbb{Z}))$ . If  $\Pi_1 = \Pi_0 \times (1, K)^Z$ , itself closed and stable for the flow on  $M \times ((m, M) \times (1, K))^Z$  and with elements noted  $\omega = (\omega_0, (X_n, n \in \mathbb{Z}))$ , we also suppose that  $p(\Pi_1) = \Omega_1$  (canonical image of  $\Pi_1$  in  $M_+ \times ((m, M) \times (1, K))^{N^*}$ ) is polish in  $M_+ \times ((m, M) \times (1, K))^{N^*}$ .

Using the remark following proposition 1 one can construct an extension  $\theta^e$  of the semi-flow  $\theta$  on  $\Omega_1$ , where  $\theta^e$  is on  $\Omega_e = \Omega_1 \times E$ .

We now study the following question :

*Let  $P_0$  be a stationary ergodic law on  $\Pi_0$ , admitting the Palm probability  $Q_0$ . We want a triple  $(\Pi, \psi, (E_t, t \in \mathbb{R}))$  where  $\Pi$  is a probability space of type S,  $\psi$  is a transport of  $\Pi$  towards  $\Pi_1$ , and  $(E_t, t \in \mathbb{R})$  is an  $E$ -valued stationary process on  $\Pi$  which is solution of the perturbation of the evolution  $S$  by  $A$  at times  $T_n$  (see the remark following proposition 1).*

*This triple must satisfy :*

— *The image of  $P$  on  $\Pi_0$  is  $P_0$  ( $P$  is defined on  $\Pi$ ).*

—  $L \circ E_{T_n}^-(X_n) = \min \{ k \in (1, K) L \circ E_{T_n}^-(k) \}$  P-a. s.

(and equivalently Q-a. s., where Q is the Palm probability of P). Here  $E_t(k)$  is the  $k^{\text{th}}$  component of the vector  $E_t$  belonging to  $E_m^K$ .

**PROPOSITION 3.** — Suppose that  $E_{Q_0}(\sigma) < K$ ,  $E_{Q_0}(\tau)$ , then the preceding problem has a solution ( $\tau_i$  equals  $T_{i+1} - T_i$ ).

*Proof.* — It is similar to the proof in the third part of [6]. Let

$$\hat{\Pi}_1^n = \hat{\Pi}_0 \times (1, K)^{-\infty, n}.$$

Then for  $n \geq 0$  define  $D'_n$  on  $\hat{\Pi}_1^n$  by  $D'_0 = S_{\tau_0}(\underline{O})$  where  $\underline{O} = (o, \dots, o)E$  and  $D'_{n+1} = S_{\tau_{n+1}} \circ A(\sigma_{n+1}, X_{n+1}, D'_n)$  and on  $\hat{\Pi}_1$  define  $D_n = D'_n \circ p_n$  where  $p_n$  is the canonical projection from  $\hat{\Pi}_1$  to  $\hat{\Pi}_1^n$ .

This allows us to construct laws  $Q^n$  on  $\hat{\Pi}_1^n$  by  $Q^0 = Q_0 \otimes \bigotimes_{n \leq 0} \varepsilon_0$  and  $Q_{n+1} = \int Q_n(d\omega_n) R(D'_n, dx)$  where  $R$  is a kernel from  $E$  to  $(1, K)$  having the property :

**HYPOTHESIS (M).** — If  $e \in E$  then

$$R(e, 1 \{ k \in (1, K) L(P_k) = \min_{l \in (1, K)} L(P_l) \}) = 1$$

(i. e.,  $R(e, dx)$  is carried by the set of indices of  $(1, K)$  where  $L(P_k)$  is minimal).

Thus one can take

$$R(e, dx) = \left( \sum_1^K 1 \{ L(P_k) = \min_j L(P_j) \} \right)^{-1} \sum_1^K \varepsilon_k(dx) 1(L(P_k) = \min_j L(P_j))$$

(The kernel respects the « isotropy » of the space) or as well

$$R(e, dx) = \sum_1^K \varepsilon_k(dx) h_k(e)$$

where

$$h_k(e) = 1 \{ L(P_k) = \min_j L(P_j) \text{ and } L(P_i) = \min_j L(P_j) \text{ if } i < k \}$$

(one chooses the first index where the minimum is attained). Kolmogorov's theorem assures the existence of a unique law  $Q$  on  $\hat{\Pi}_1$ , such that  $p_n \circ Q = Q^n$ .

One has (as in section three of (6)) that for  $n \geq 0$   $\phi_n \circ \hat{\theta}_n \circ Q = \hat{\theta}_n^e \circ \phi_0 \circ Q$ , where  $\phi_n$  is the measurable function given by  $\phi_n(\omega) = (p(\omega), Z_n(\omega))$ . Here

the sequence of random variables  $(Z_l, l \in \mathbb{N})$  is  $E$ -valued. On  $\hat{\Pi}_1$  we define  $Z_0 = \underline{O}$  and  $Z_{n+1} \circ \hat{\theta} = A(\sigma_1, X_1, S_{\tau_0} \circ Z_n)$ . We denote  $Q_e : \phi_0(Q)$ .

We utilise: 1) the continuity of  $\hat{\theta}^e$  on  $\Omega_2$ ; 2) that the set defined by  $L \circ E_{T_1}^-(X_1) = \min_k L \circ E_{T_1}^-(k)$  is closed in  $\Omega_e$  (by the continuity of  $L$ ) and carries all of the  $\hat{\theta}_n^e \circ Q_e$  (by the hypothesis (M) on  $R$ ); hence if the  $\hat{\theta}_n^e \circ Q_e$  are tight,  $\Omega_e$  being metrisable and separable and its Borel sigma-field being induced by a Polish topology, we conclude, using propositions 1 and 2, that the problem has a solution.

Since  $Q_0$  is stationary on  $\hat{\Pi}_0$  and since the space  $(1, K)^{\mathbb{N}^*}$  is compact, to obtain this result it suffices to prove that the laws of  $(E_{T_n}, n \in \mathbb{N})$  on  $\Omega_e$  are tight for  $Q_e$ .

We begin by noting that in  $\tilde{E}_m$  the set  $\|P\| \leq C$  ( $C$  positive constant) is compact. Indeed, we have  $P \in \tilde{E}_m$  and  $\|P\| \leq C$  implies that  $d^0 P \leq C/m$ , and thus we deduce that  $\tilde{E}_m \cap \{\|P\| \leq C\}$  is a finite union of compact sets of  $L^1(\mathbb{N})$  having the following form:

$$\begin{aligned} m \leq a_0 \leq M, \dots, m \leq a_{n-1} \leq M, \quad a_n \leq M \\ \text{and} \quad a_0 + \dots + a_n \leq C \quad \text{and} \quad a_i = 0 \quad \text{for} \quad i \geq n+1. \end{aligned}$$

It remains to verify that for  $(W_t, t \in \mathbb{R}_+)$  on  $\Omega_e$  with values in  $\mathbb{R}_+^K$  defined by:

$$W_t = (\|E_t(k)\|, k \in (1, K)),$$

the variables  $(W_{T_n}, n \geq 0)$  form a tight sequence for  $Q_e$ .

We suppose the number of channels  $K$  superior to 2 (else we have the result at once) and we define  $R_n = R(W_{T_n}^-)$  the reordered vector of waiting times (decreasing) for  $n \geq 1$ , and we define for  $l \in (1, K)$

$$C_n^l = mR_n(l) - M \sum_{j=l+1}^K R_n(j)$$

We are going to use the two following lemmas:

LEMMA A. —  $Q_e$ -a. s. we have  $C_{n+1}^l \leq \sup (3K + m\sigma_n + (M(K-l) - m)\tau_n, C_n^l + (M(K-l) - m)\tau_n - \sigma_n)$  for  $n \geq 1$ .

Proof. — Let  $n$  be an integer  $\geq 1$ , then

1) If  $W_{T_n}^-(X_n) = R_n(j)$  for  $j > l$ :

then  $C_{n+1}^l \leq m(R_n(l) + \sigma_n - \tau_n)_+$

$$- M \sum_{i=1}^K (R_n(j) - \tau_n)_+ \leq (C_n^l + m\sigma_n + (M(K-l) - m)\tau_n)_+$$

2) If there exists  $j_0 > l$  such that  $W_{T_n}^-(X_n) = R_n(j_0)$  :

a) If  $R_n(j_0) + \sigma_n > R_n(l)$  :

$$\begin{aligned} \text{then} \quad C_{n+1}^l &\leq m(R_n(j_0) + \sigma_n - \tau_n)_+ - M \sum_{\substack{u \in (l, K) - \{j_0\} \\ \text{since}}} (R_n(u) - \tau_n)_+ \\ &\leq m\sigma_n + (M(K - l) - m)\tau_n \quad mR_n(j_0) - R_n(l) \leq 0. \end{aligned}$$

b) If  $R_n(j_0) + \sigma_n \leq R_n(l)$  :

$$\begin{aligned} \text{then} \quad C_{n+1}^l &\leq m(R_n(l) - \tau_n)_+ - M \sum_{\substack{i > l \\ i \neq j_0}} (R_n(i) - \tau_n)_+ \\ &\quad - M(R_n(j_0) + \sigma_n - \tau_n)_+ \\ &\leq (C_n^l + (M(K - l) - m)\tau_n)_+. \end{aligned}$$

The set  $\{C_n^l \leq 3K\}$  is included  $Q_e$ -a. s. in the set

$$\{\exists j_0 > l \ W_{T_n}^-(X_n) = R_n(j_0)\} \quad (\text{case 2})$$

since  $C_n^l \geq 3K$  implies  $mR_n(l) - MR_n(K) \geq 3$ , consequently for  $j \leq l$ ,  $mR_n(j) - MR_n(K) \geq 3$  and therefore  $\frac{R_n(j)}{M} - \frac{R_n(K)}{m} \geq 3$  (since  $m \cdot M = 1$ ).

On  $\tilde{E}_m$  we have the following inequalities:

$$\frac{\|P\|}{m} - 1 \leq d^0 P \leq L(P) \leq d^0 P + 1 \leq \frac{\|P\|}{m} + 1$$

(for we have  $md^0 P \leq \|P\|$  and  $M(d^0 P + 1) \geq \|P\|$ ).

Then if  $j \leq l$  and  $p, k \in (1, K)$  are such that

$$W_{T_n}^-(k) = R_n(j) \quad \text{and} \quad W_{T_n}^-(p) = R_n(K)$$

then:

$$L \circ E_{T_n}^-(k) \geq \frac{W_{T_n}^-(k)}{M} - 1 = \frac{R_n(j)}{M} - 1 \quad \text{and} \quad L \circ E_{T_n}^-(p) \leq \frac{R_n(K)}{m} + 1$$

and consequently:

$$\frac{R_n(j)}{M} - \frac{R_n(K)}{m} \geq 3$$

implies  $L \circ E_{T_n}^-(k) > L \circ E_{T_n}^-(p)$  and using hypothesis (M) on the kernel R we have  $Q_e$ -a. s.  $L \circ E_{T_n}^-(X_n) < L \circ E_{T_n}^-(k)$  if  $W_{T_n}^-(k) = R_n(j)$  for a  $j \leq l$ .

From this we deduce the result of lemma A.

**LEMMA B.** — Let  $(X, \theta, P)$  be an ergodic system, A and B random variables respectively positive and integrable such that:

$$\limsup \frac{A \circ \theta_{-p}}{p} < -E(B) \quad P\text{-a. s.}$$

If  $H = \sup_{i \geq 1} (A_{-i} + B_{-i+1} + \dots + B_{-1})$  then

$$\limsup \frac{H \circ \theta_{-p}}{p} \leq \limsup \frac{A \circ \theta_{-p}}{p}.$$

*Proof.* — Note that  $\limsup \frac{A \circ \theta_{-k}}{k}$  is a. s. constant (using ergodicity).

Let  $\rho \in ]0, 1[$  and  $B^\rho = B_+ - \rho B_-$ ; then if  $\rho$  is such that

$$-E(B^\rho) > \limsup \frac{A \circ \theta_{-k}}{k} \quad \text{P-a. s.,}$$

$H \leq H^\rho = \sup_{i \geq 1} (A_{-i} + B_{-i+1}^\rho + \dots + B_{-i}^\rho)$ . For  $n \geq 1$  we have

$$H^\rho \circ \theta_{-n} + B_{-n}^\rho + \dots + B_{-1}^\rho \leq H^\rho$$

and then  $\limsup \frac{H \circ \theta_{-n}}{n} \leq -E(B^\rho)$  for  $\rho$  satisfying the preceding conditions, and hence we obtain lemma B.

We now resume the proof of proposition 3. Let  $p \in (0, 1)$  such that

$$pE_{Q_0}(\tau) \leq E_{Q_0}(\sigma) < (p+1)E_{Q_0}(\tau).$$

The case  $p = 0$  presents no difficulties (it reduces to majorizing as in the classical stationary single waiting queue), and we consider here the case  $p \geq 1$ .

Because of the form of the random variables  $C_n^h$  there exist positive coefficients  $b$  and  $b^h$  ( $h \in (K-p, K-1)$ ) such that:

$$R_n(K-p) - R_n(K) \cdot b = \sum_{k=p}^{K-1} b^h \cdot C_n^h \quad \text{for } n \geq 1.$$

For  $N \geq 1$  we have:

$$(I) \quad H_N = \sum_{l \geq K-p} R_N(l) \leq \sup_{k \in (1, N)} \left( p \cdot \sum_{k=p}^{K-1} b_h C_k^h + x_k + \dots + x_{N-1} \right)$$

$Q_e$ -a. s. where  $x_q = \sigma_q - (p+1)\tau_q$ ,  $q \geq 1$ .

We obtain (I) as follows: let  $T$  denote the largest index  $m \in (1, N)$  such that  $R_m(K)$  is zero (this set is  $Q_e$ -a. s. non empty since  $R_1(K) = 0$   $Q_e$ -a. s.). We then have:

$$\sum_{l \geq K-p} R_N(l) \leq \sum_{l \geq K-p} R_T(l) + (\sigma_T - (p+1)\tau_T) + \dots + (\sigma_{N-1} - (p+1)\tau_{N-1})$$

$Q_e$ -a. s. and (I) follows.

On the other hand if  $M \geq 2$ :

$$(II) \quad R_M(1) \leq \sup_{N \in (1, M-1)} \left( \frac{3}{m} + \frac{M}{m} H_N + \sigma_N - (\tau_N + \dots + \tau_{M-1}) \right)_+$$

Indeed on  $A_k = \{ W_{T_M}^-(k) = R_M(1) > 0 \}$  if  $N$  (random) is the largest index of  $(1, M-1)$  such that  $X_N = k$  (this set is  $Q_e$ -a. s. non empty on  $A_k$ ) we have  $W_{T_M}^-(k) \leq W_{T_N}^-(k) + \sigma_N - (\tau_N + \dots + \tau_{M-1})$  while also  $Q_e$ -a. s.  $mW_{T_N}^-(k) - MR_N(K) = mW_{T_N}^-(X_N) - MR_N(K) \leq 3$  on  $A_k$ , therefore

$$W_{T_N}^-(k) \leq \frac{M}{m} H_N + \frac{3}{m},$$

from which (II) follows.

Further let us set  $Y_q^l = (M(K-l) - m)\tau_q - M\sigma_q$  for  $q \geq 1$  and  $l \in (1, K)$  then  $E(Y_q^l) < 0$  for  $l \geq K-p$ .

Combining lemma A, the bounds (I) and (II) and lemma B, we can show that there exists an a. s. finite random variable  $F$  on  $(\hat{\Pi}_0, Q_0)$  such that  $\lim_{n \rightarrow \infty} \frac{F \circ \hat{\theta}_{-n}}{n} = 0$   $Q_0$ -a. s. and  $R_M(1)$  is dominated in law by

$$\sup_{n \in (1, M-1)} (F \circ \hat{\theta}_n - \tau_n - \dots - \tau_{M-1})_+.$$

From this we conclude that  $R_M(1)$  is dominated by

$$\sup (O, F \circ \hat{\theta}_{-1} - \tau_{-1}, \dots, F \circ \hat{\theta}_{-n} - \tau_{-n} - \dots - \tau_{-1}, \dots)$$

which is an a. s. finite and positive random variable on  $(\hat{\Pi}_0, Q_0)$ . Thus the laws of the  $(W_{T_n}^-, n \geq 1)$  are tight for  $Q_e$ , and hence the laws of the  $(W_{T_n}, n \geq 1)$  are tight as well, and this is sufficient to conclude the proof, as we have previously remarked. ■

In ending this section we are going to examine rapidly the result one obtains in the more classical situation of a multi-channel with minimal waiting-time.

We begin by describing this multi-channel: it is a system of  $K$  parallel waiting queues, each one of which follows a « first come first served » policy, and where the arriving client chooses one of the queues with minimal waiting-time. (This choice can correspond to each client using an etiquette indicating the service time he will need, thus allowing a new client to choose a queue with minimal waiting time; alternatively this model arises in the case where the arriving client joins one large common waiting queue for the  $K$  servers, and patronises the free server when his turn arrives. In the first case one physically has  $K$  servers and  $K$  queues, while in the latter one physically has one common queue and  $K$  servers.)

One can find in [4] and [2] a study of this system within a renewal framework, and in [1] within the ergodic framework. In general, these models give a construction of the reordered vector of waiting-times where the information of which file the client chooses is lost whereas the stationary model we construct here retains this information.

This result, which is simpler than the one at the beginning of this section, is obtained in a similar fashion: the state space  $E$  is none other than  $\mathbb{R}_+^K$ , the continuous semi-flow on  $E$  is the function assigning  $(x - ta)_+$  to  $(t, x) \in \mathbb{R}_+ \times E$  (where  $a = a_1 + \dots + a_K$  if  $(a_i, i \in (1, K))$  is the canonical basis of  $\mathbb{R}^K$ ),  $D$  is equal to  $\mathbb{R}_+ \times (1, K)$  and finally  $A$  is the function assigning  $x + \sigma \cdot a_i$  to  $(\sigma, i, x)$ .

With the above definitions we can pose the following problem:

Let  $P_0$  be a stationary, ergodic probability law on  $\Pi_0$ , with Palm probability  $Q_0$ . We seek a triple  $(\Pi, \psi, (W_t, t \in \mathbb{R}))$ , where  $\Pi$  is a probability space of type S,  $\psi$  is a transport of  $\Pi$  towards  $\Pi_1$ ,  $(W_t, t \in \mathbb{R})$  is an  $E$ -valued stationary process on  $\Pi$  which is a solution of the perturbation of  $S$  by  $A$  at the times  $T_n$ . This triple must verify:

- the image on  $\Pi_0$  of the law  $P$  on  $\Pi$  is  $P_0$ ;
- $P$ -a. s. (and in the same manner  $Q$ -a. s., if  $Q$  is the Palm probability of  $P$ ) we have:

$$W_{T_n}^-(X_n) = \inf \{ k \in (1, K) \mid W_{T_n}^-(k) \}$$

(where  $W_t(l)$  is the  $l$ -th component of the vector  $W_t$  and  $X_n$  is the index of the queue that the client arriving at time  $T_n$  chooses).

We then have the following result:

**PROPOSITION 4.** — If  $E_{Q_0}(\sigma) < K \cdot E_{Q_0}(\tau)$ , then the preceding problem has a solution.

Finally let us add a remark on some paradoxical points a propos of the minimal waiting-time system;

*Remark.* — 1) It is possible to choose  $\Pi_0$  and  $Q_0$  such that one can construct on the one hand a cyclical model of two queues, and on the other hand a minimal waiting-time model on  $(\Pi_0, P_0)$  and obtain:

$$E^c(W^-(X_0)) < E^m(W^-(X_0))$$

$$E^c(\|W^-\|) < E^m(\|W^-\|)$$

(the expectations are taken with respect to both probabilities of Palm, noted  $Q^c$  and  $Q^m$ , and  $\| \cdot \|$  denotes the norm on  $\mathbb{R}^2$  given by

$$\|x\| = |x_1| + |x_2|)$$

Thus for example :

|                              | $\tau_0=1$                                                                                                                                                                                                                    | $\tau_1=1$ | $\tau_2=1$ | $\tau_3=16$ | $\tau_4=1$ | $\tau_5=1$ |         |
|------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|------------|-------------|------------|------------|---------|
| customer n°                  | 0                                                                                                                                                                                                                             | I          | II         | III         | IV         | V          | VI etc. |
| service time of the customer | 13                                                                                                                                                                                                                            | 8          | 5          | 14          | 5          | 2          | 13      |
| cyclical queue               | $\left\{ \begin{array}{l} \text{queue I} \\ \text{queue II} \end{array} \right. \begin{array}{cccccc} \times 0 & 12 & \times 11 & 15 & \times 0 & 4 & \times 0 \\ 0 & \times 0 & 7 & \times 6 & 4 & \times 3 & 0 \end{array}$ |            |            |             |            |            |         |

( $\times$  indicates the queue chosen by the  $i$ -th customer who sees the waiting time vector of the  $i$ -th column as he enters the system).

|                            |                                                                                                                                                                                                                                |  |  |  |  |  |  |
|----------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--|--|--|--|--|--|
| minimal waiting time queue | $\left\{ \begin{array}{l} \text{queue I} \\ \text{queue II} \end{array} \right. \begin{array}{cccccc} \times 0 & 12 & 11 & \times 10 & 8 & 7 & \times 0 \\ 0 & \times 0 & \times 7 & 11 & \times 0 & \times 4 & 0 \end{array}$ |  |  |  |  |  |  |
|----------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--|--|--|--|--|--|

$(\hat{\Pi}_0, Q_0)$  is an ergodic space of six points and we have :

$$E^c(W^-(X_0)) = \frac{1}{6}(0 + 0 + 11 + 6 + 0 + 3) = 20/6$$

$$E^m(W^-(X_0)) = \frac{1}{6}(0 + 0 + 7 + 10 + 0 + 4) = 21/6$$

$$E^c(\|W^-\|) = 62/6 \text{ and } E^m(\|W^-\|) = 70/6.$$

2) Unlike the case of the waiting queue with one server, one does not have in general that the laws of the vectors of waiting times are increasing when one considers the evolutionary model starting from 0 (no client in the system). On the other hand, one cannot always build the stationary model on  $\Pi_0$  (i. e. find the  $(1, K)$ -valued and  $R^K$ -valued  $X$  and  $W$  on  $\hat{\Pi}_0$  which would allow one to solve the problem we posed.

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