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A Morse Theory for light rays on stably causal Lorentzian manifolds*

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ABSTRACT. – In this paper a Morse Theory for lightlike geodesics joining a point with a timelike curve is obtained on a stably causal space-times with boundary. Some applications to the multiple image effect are presented. In particular, we give some conditions on the geometry and the topology of space-time, in order that the number of images in the gravitational lens effect is infinite or odd. © Elsevier, Paris

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RÉSUMÉ. – Dans cet article, nous présentons une théorie de Morse pour les rayons lumineux joignant un évènement p à une courbe γ de genre temps

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dans un espace-temps stablement causal avec bords. Quelques applications aux lentilles gravitationnelles sont données. En particulier, nous donnons des conditions suffisantes sur la géométrie et la topologie de l'espace–temps, pour que le nombre d'images gravitationnelles soit infini ou impair.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In 1978 Walsh, Carswell and Weymann announced the discovery of 0957+561, the first candidate to be a *gravitational lens*. Since then, gravitational lensing is a research field developing very fast. Currently, about 60 phenomena of multiple optic images of *quasi stellar objects* have been observed. Such multiple image effects are due to the deflection of light in presence of gravitational fields. We refer to [33] for a detailed physical description of the gravitational lens effect.

A mathematical model of the gravitational lens effect is based on a variational characterization of light rays and, in particular, on an extension of the classical Fermat principle to General Relativity. We recall that the Fermat principle states that the trajectory of a light ray, starting from a source A directed towards a point B in an optical medium, is such that the travel time is minimal, or better stationary.

In General Relativity a gravitational field is described by a four dimensional Lorentzian manifold $(\mathcal{M}, g)$, also called *space-time*. A Lorentzian manifold is a couple $(\mathcal{M}, g)$, where $\mathcal{M}$ is a smooth connected manifold and g is a Lorentzian metric on $\mathcal{M}$. This means that g is a smooth symmetric (0,2) tensor field on $\mathcal{M}$, such that for any $p \in \mathcal{M}$, the bilinear form $g(p): T_p\mathcal{M} \times T_p\mathcal{M} \longrightarrow \mathbf{R}$ is nondegenerate and its index is 1. We refer to classical books as [3,15,26] for the main properties of Lorentzian Geometry. The points of a space-time are often called *events*. The gravitational field is described by means of a (0,2) tensor field $\mathcal{T}$, the *energy-stress* tensor. The metric g is related to the physical properties of the gravitational field by the Einstein equations

$$R_g + \frac{1}{2} S_g g = 8\pi \mathcal{T},$$

where R_g and S_g denote respectively the Ricci tensor and the scalar curvature of the metric g.

Let $p \in \mathcal{M}$. A vector $v \in T_p\mathcal{M}$ is called *timelike* (respectively *lightlike*, *spacelike*), if $g(p)[v,v]$ is negative (respectively null, positive). A vector v is said *causal* if it is nonspacelike. A smooth curve $z\colon]a,b[\longrightarrow \mathcal{M}$ is said *timelike* (respectively *lightlike*, *spacelike*, *causal*), if for any $s \in]a,b[$, the tangent vector $\dot{z}(s)$ to the curve is timelike (respectively lightlike, spacelike, or causal). Timelike curves can be interpreted as world-lines of observers or objects in a space-time. A smooth curve $z\colon]a,b[\longrightarrow \mathcal{M}$ is called *geodesic* if

$$D_s\dot{z} = 0 \qquad \forall s \in]a,b[,$$

where $D_s\dot{z}$ is the covariant derivative of $\dot{z}$ along z induced by the Levi-Civita connection of g. It is well known that if $z\colon]a,b[\longrightarrow \mathcal{M}$ is a geodesic, there exists a constant E_z such that

$$E_z = g(z(s))[\dot{z}(s), \dot{z}(s)] \qquad \forall s \in]a,b[.$$

Then z is said *timelike* (respectively *lightlike*, *spacelike*), if E_z is negative (respectively null, positive).

In a space-time, timelike geodesic represent the trajectories of freely falling particles. Lightlike geodesics represent the trajectories of light rays.

In the following we shall denote the Lorentzian metric g by $\langle \cdot, \cdot \rangle$. A Lorentzian manifold is said to be *time oriented* if there exists a continuous vector field Y on $\mathcal{M}$, such that $Y(z)$ is timelike for all $z \in \mathcal{M}$. A timelike vector field allows to define a time-orientation on a Lorentzian manifold. A causal vector $v \in T_p\mathcal{M}$ is called *future pointing* (respectively *past pointing*) if $\langle Y(z), v\rangle > 0$ (respectively $\langle Y(z), v\rangle < 0$). Observe that with this definition, the vector field Y is past pointing. A causal curve $z\colon]a,b[\longrightarrow \mathcal{M}$ is called *future pointing* if, for any $s \in]a,b[$, $\dot{z}(s)$ is a future pointing causal vector. An analogous definition holds for past pointing causal curves.

A Lorentzian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ is said to be *stably causal*, if it is causal (that is it does not contain closed causal curves), and if this property is preserved by uniformly small perturbations of the metric. Equivalently, (see [15, Prop. 6.4.9]), $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ is stably causal if there exists a smooth function $T\colon \mathcal{M} \longrightarrow \mathbf{R}$ such that the gradient ∇T of T (with respect to the Lorentzian metric) is timelike. The vector field ∇T gives a time-orientation on $\mathcal{M}$.

Let $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ be a time-oriented Lorentzian manifold and consider an event $p \in \mathcal{M}$ and a future pointing timelike curve $\gamma\colon \mathbf{R} \longrightarrow \mathcal{M}$. To set up a framework for a relativistic version of the Fermat principle, we need a set $\mathcal{L}_{p,\gamma}$ of lightlike curves joining p with γ, and a functional $\tau\colon \mathcal{L}_{p,\gamma} \longrightarrow \mathbf{R}$, such that the lightlike geodesics (that is the light rays) joining p and γ are the stationary points of τ.

Consider p as a source of light and γ as the worldline of an observer. Then the lightlike geodesics joining p with γ in the future of p are the images of the source seen by the observer on γ. Conversely, if γ represents the worldline of a light source, then the lightlike geodesics joining p with γ in the past of p are the images of the source seen by an observer at p. Whenever multiple images are observed (that is multiple stationary points of τ), astrophysicists speak of *gravitational lens effect.*

As already observed in some previous papers [8,12,23,36], the multiple image effect is strictly related to the topological and geometrical properties of the manifold. In the papers above, such relations are exploited by developing a Morse Theory for the light rays joining p and γ. We recall that Morse Theory relates the set of the critical points of a smooth functional (satisfying suitable nondegeneracy assumptions, see section 2) on a manifold, to the topological properties of the manifold itself. In particular the number of critical points of such a functional can be estimated by some topological invariants of the manifold. In [36] it is developed a Morse Theory for globally hyperbolic space-times, using a finite dimensional approach. The results of [36] are compared with the gravitational lens effect in [23]. In [8] it is proved a Morse Theory for the lightlike geodesics joining a point p and a timelike curve γ in a conformally stationary Lorentzian manifold with light-convex boundary. In [12] the results of [36] are proved by an infinite dimensional approach. The paper [36] contains some inaccuracies in the proofs, and one of the motivations of the paper [12] was to fill the gaps left unsolved in [36].

An alternative approach to a Morse Theory for lightlike geodesics is announced in [29]. In this paper, the author sets up a variational framework for such geodesics, and proves the related Fermat principle. We would like to mention also the papers [30,31,32], where Morse Theory is applied to the study of the gravitational lens effect in a quasi-Newtonian approximation scheme, often adopted by astrophysicists.

In this paper we develop a Morse Theory for lightlike geodesics joining an event p and a timelike curve γ having image contained in an open connected subset Λ of a stably causal Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$. The results proven in this paper generalize the ones obtained in [8,12,36]. Whenever $\mathcal{M} = \Lambda$ and multiple images are seen, we obtain a mathematical model for the gravitational lens effect. Whenever Λ is a proper subset of $\mathcal{M}$, we can interpretate the set $\mathcal{M} \setminus \Lambda$ as a *nontransparent deflector* (modeled by a hole in $\mathcal{M}$), whose presence can give rise to a multiple image effect.

In section 3 we shall present some examples related to Schwarzschild, Reissner-Nordström and Kerr space-times. Another physical interpretation is the following. The set Λ can be thought as a region of the universe having the property that the light rays starting from p and moving outside of Λ does not reach the observer γ. For instance, this is satisfied if $\mathcal{M} = \mathbf{R}^4$ is the Minkowski space and $\Lambda = \mathcal{M}_0 \times \mathbf{R}$, where $\mathcal{M}_0$ is a convex subset of $\mathbf{R}^3$.

We assume that the curve γ is a closed embedding of $\mathbf{R}$ in $\mathcal{M}$ and $\gamma(\mathbf{R}) \subset \Lambda$. In particular, γ has no *endpoints* in $\mathcal{M}$, that is $\gamma(s)$ is eventually outside every compact subset of $\mathcal{M}$ for $s \to \pm\infty$. The curve γ is future pointing, so $T \circ \gamma : \mathbf{R} \longrightarrow \mathbf{R}$ is strictly increasing.

We will assume that the open connected subset Λ satisfies the following assumptions:

(a) $\partial\Lambda$ is a smooth submanifold of $\mathcal{M}$;

(b) $\partial\Lambda$ is timelike, that is for any $z \in \partial\Lambda$ the normal vector $\nu(z)$ to $\partial\Lambda$ is spacelike;

(c) Λ is light–convex, i.e. all the lightlike geodesics in $\overline{\Lambda} = \Lambda \cup \partial\Lambda$ with endpoints in Λ are entirely contained in Λ.

In order to state the main result of this paper, we need to introduce some Sobolev spaces. For any $k \in \mathbf{N}$ and for any interval $[a, b]$, we denote by $H^{1,2}([a,b], \mathbf{R}^k)$ the Sobolev space of the absolutely continuous curves on $\mathbf{R}^k$, having square integrable derivative, see [1]. The space $H^{1,2}([a,b], \mathbf{R}^k)$ is equipped with a structure of Hilbert space, whose norm is given by

$$\|x\|^2 = \int_a^b |x(s)|^2 ds + \int_0^1 |\dot{x}(s)|^2 ds.$$

Now let $\mathcal{M}$ be a smooth manifold. We denote by $H^{1,2}([0,1], \mathcal{M})$ the set of the curves $z : [0,1] \longrightarrow \mathcal{M}$ such that for any local chart (U, φ) of the manifold satisfying $U \cap z([0,1]) \neq \emptyset$, the curve $\varphi \circ z : z^{-1}(U) \longrightarrow \mathbf{R}^n$, $n = \dim\mathcal{M}$, belongs to the Sobolev space $H^{1,2}(z^{-1}(U), \mathbf{R}^n)$.

It is well known (cf. [27]) that $H^{1,2}([0,1], \mathcal{M})$ is an infinite dimensional manifold modeled on $H^{1,2}([0,1], \mathbf{R}^n)$. For every $z \in H^{1,2}([0,1], \mathcal{M})$, the tangent space $T_z H^{1,2}([0,1], \mathcal{M})$ is given by

$$T_z H^{1,2}([0,1], \mathcal{M}) = \{\xi \in H^{1,2}([0,1], T\mathcal{M}) : \xi(s) \in T_{z(s)}\mathcal{M}, \forall s \in [0,1]\},$$

where $T\mathcal{M}$ is the tangent bundle of $\mathcal{M}$.

Let $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ be a time oriented Lorentzian manifold and consider an open subset Λ of $\mathcal{M}$, we shall denote by $H^{1,2}([0,1], \Lambda)$ the open subset of $H^{1,2}([0,1], \mathcal{M})$ consisting of curves z with image $z([0,1])$ contained in

Λ. Moreover, let p be an event in Λ and $\gamma: \mathbf{R} \longrightarrow \Lambda$ be a smooth timelike curve, such that $p \notin \gamma(\mathbf{R})$. We introduce the following space:

$$\Omega^{1,2}_{p,\gamma} = \Omega^{1,2}_{p,\gamma}(\Lambda) = \{z \in H^{1,2}([0,1],\Lambda) \mid z(0) = p,\ z(1) \in \gamma(\mathbf{R})\}. \quad (1.1)$$

It is not difficult to see that $\Omega^{1,2}_{p,\gamma}$ is a smooth submanifold of $H^{1,2}([0,1],\Lambda)$ (see [17]); for every $z \in \Omega^{1,2}_{p,\gamma}$, the tangent space $T_z\Omega^{1,2}_{p,\gamma}$ is identified with:

$$T_z\Omega^{1,2}_{p,\gamma} = \{\zeta \in T_zH^{1,2}([0,1],\mathcal{M}) \mid \zeta(0) = 0,\ \zeta(1) \parallel \dot\gamma(\gamma^{-1}(z(1)))\}. \quad (1.2)$$

The *Arrival Time* functional on $\Omega^{1,2}_{p,\gamma}$, is defined as:

$$\tau_{p,\gamma}(z) = \gamma^{-1}(z(1)). \quad (1.3)$$

Since γ is an embedding, the functional $\tau_{p,\gamma}$ is smooth. Now we consider the set

$$\mathcal{L}^+_{p,\gamma} = \mathcal{L}^+_{p,\gamma}(\Lambda) = \Big\{z \in \Omega^{1,2}_{p,\gamma} \mid \langle \dot z, \dot z\rangle = 0 \ \text{ and } \ \langle \nabla T(z), \dot z\rangle \geq 0 \ \text{almost everywhere}\Big\}. \quad (1.4)$$

Remark 1.1. – Observe that the definition of $\mathcal{L}^+_{p,\gamma}$ does not depend on the particular choice of a time function T, but only on the orientation of its gradient ∇T.

The set $\mathcal{L}^+_{p,\gamma}$ and the arrival time functional $\tau_{p,\gamma}$ restricted to it, are the natural candidates to develop a variational theory for lightlike geodesics joining p and γ in the future of p and having image in Λ. Indeed, smooth lightlike curves joining p with γ in the future of p are contained in $\mathcal{L}^+_{p,\gamma}$. Moreover $\mathcal{L}^+_{p,\gamma}$ can be considered as the closure of the set of smooth lightlike curves joining p with γ in the future of p, with respect to the Sobolev $H^{1,2}$– topology. We point out that the $H^{1,2}$–topology is quite natural to the applications of Morse Theory to the study of variational problems involving curves (see [27] for the study of Riemannian geodesics).

Unfortunately we shall show in Section 3 that $\mathcal{L}^+_{p,\gamma}$ fails to be a C^1 manifold precisely at those curves z such that $\dot z(s) = 0$ in subsets of the interval $[0,1]$ having positive Lebesgue measure (it can be proved that $\mathcal{L}^+_{p,\gamma}$ can be equipped only with a structure of Lipschitz manifold, see [11]). This fact makes difficult to apply the standard techniques of Calculus of Variations to get a Morse Theory for the light rays joining p with γ. However, using a technique of approximation of $\mathcal{L}^+_{p,\gamma}$ with a family of smooth manifolds consisting of timelike curves (see section 4), we shall relate the set of the lightlike geodesics joining p and γ, to the topology of $\mathcal{L}^+_{p,\gamma}$.

For any $c \in \mathbf{R}$, we denote by $\tau^c_{p,\gamma}$ the c–sublevel of $\tau_{p,\gamma}$ in $\Omega^{1,2}_{p,\gamma}$:

$$\tau^c_{p,\gamma} = \left\{ z \in \Omega^{1,2}_{p,\gamma} \mid \tau_{p,\gamma}(z) \leq c \right\}.$$

In order to get the Morse Relations of light rays, we need the following precompactness condition for $\mathcal{L}^+_{p,\gamma}$:

DEFINITION 1.2. – *Let c be a real number, $\mathcal{L}^+_{p,\gamma}$ is said to be c–precompact if there exists a compact subset $K = K(c)$ of $\overline{\Lambda}$ such that for every $z \in \mathcal{L}^+_{p,\gamma} \cap \tau^c_{p,\gamma}$, $z([0,1]) \subset K$.*

Remark 1.3. – It should be emphasized that, in Definition 1.2, if we give the c-precompactness in Λ rather than $\overline{\Lambda}$, we would basically be in the globally hyperbolic case. Indeed, in this case the set:

$$\left\{ q \in \Lambda : q \in z([0,1]) \text{ for some } z \in \mathcal{L}^+_{p,\gamma} \right\}$$

is a globally hyperbolic set (see [15] for the definition). The compactness condition above is weaker than the global hyperbolicity, because of the presence of the boundary.

Before stating our main result, we recall some definitions.

DEFINITION 1.4. – *Let $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ be a Lorentzian manifold, and $z\colon [0,1] \longrightarrow \mathcal{M}$ a geodesic, a smooth vector field ζ along z is called Jacobi field if it satisfies the equation*

$$D^2_s \zeta + R(\zeta, \dot{z})\dot{z} = 0 \ , \tag{1.5}$$

where R is the curvature tensor of the metric $\langle \cdot, \cdot \rangle$ (cf. [3]). A point $z(s)$, $s \in]0,1]$ is said conjugate to $z(0)$ along z if there exists a Jacobi field ζ along $z_{|[0,s]}$ such that

$$\zeta(0) = \zeta(s) = 0 \ . \tag{1.6}$$

The multiplicity of the conjugate point $z(s)$ is the maximal number of linearly independent Jacobi fields satisfying (1.6).

By (1.5) the set of the Jacobi fields is a vector space of dimension $2\dim\mathcal{M}$. Hence the multiplicity of a conjugate point is finite and by (1.6), it is at most $\dim\mathcal{M}$ (actually is at most $\dim\mathcal{M} - 1$ because $\zeta(s) = s\dot{z}(s)$ is a Jacobi field which is null only at $s = 0$).

DEFINITION 1.5. – *The index $\mu(z)$ is the number of conjugate points $z(s)$, $s\ \in]0,1[$, to $z(0)$, counted with their multiplicity.*

It is well known that the index of a lightlike geodesic is finite (see [3]).

DEFINITION 1.6. – *Let p be a point and γ a timelike curve on a Lorentzian manifold $(\mathcal{M}, g)$, then p and γ are said nonconjugate if for any lightlike geodesic $z:[0,1] \longrightarrow \mathcal{M}$ joining p and γ, $z(1)$ is nonconjugate to p along z. It is well known that such a condition is true except for a residual set of pairs (p,γ).*

Let X be a topological space and $\mathcal{K}$ a field, for any $q \in \mathbf{N}$ let $H_q(X;\mathcal{K})$ be the q-th homology group of X with coefficients in $\mathcal{K}$ (cf. for instance [35]). Since $\mathcal{K}$ is a field, $H_q(X;\mathcal{K})$ is a vector space and its dimension $\beta_q(X;\mathcal{K})$ (eventually $+\infty$) is called q-th *Betti number of X* (with coefficients in $\mathcal{K}$). The *Poincaré polynomial* $\mathcal{P}(X;\mathcal{K})$ is defined as the following formal series:

$$\mathcal{P}(X;\mathcal{K})(r) = \sum_{q\in\mathbf{N}} \beta_q(X;\mathcal{K})r^q \ .$$

THEOREM 1.7. – *Let $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ be a stably causal Lorentzian manifold, Λ an open subset of $\mathcal{M}$, $p \in \Lambda$, $\gamma: \mathbf{R} \longrightarrow \Lambda$ a smooth timelike curve such that γ is a closed embedding and $p \notin \gamma(\mathbf{R})$. Assume that:*

L_1) Λ *satisfies* (a)-(c)*;*

L_2) *$\mathcal{L}^+_{p,\gamma}(\Lambda) \neq \emptyset$ and p and γ are nonconjugate;*

L_3) *For any $c \in \mathbf{R}^+$, $\mathcal{L}^+_{p,\gamma}(\Lambda)$ is c-precompact;*

Then for any field $\mathcal{K}$ there exists a formal series $S(r)$ with coefficients in $\mathbf{N} \cup \{+\infty\}$, such that

$$\sum_{z\in\mathcal{G}^+_{p,\gamma}(\Lambda)} r^{\mu(z)} = \mathcal{P}(\mathcal{L}^+_{p,\gamma};\mathcal{K}) + (1+r)S(r) \ . \tag{1.7}$$

Here $\mathcal{G}^+_{p,\gamma}(\Lambda)$ is the set of the future pointing lightlike geodesics joining p and γ in the future of p and with image in Λ.

Remark 1.8. – The same result holds for the lightlike geodesics joining p and γ in the past of p.

Remark 1.9. – Observe that the Betti numbers $\beta_q(X;\mathcal{K})$ (and the coefficient of the formal series $S(r)$ in (1.7)) depend in a substantial way on the choice of the field $\mathcal{K}$. On the other hand, the left hand side of the equality (1.7) does not depend on $\mathcal{K}$, hence one can obtain more information on $\mathcal{G}^+_{p,\gamma}(\Lambda)$ by letting the coefficient field $\mathcal{K}$ arbitrary in (1.7).

Remark 1.10. – Observe that there are simple examples in which $\mathcal{L}^+_{p,\gamma}(\Lambda)$ is the empty set, see [11].

Assumption L_3) can not be removed. Indeed, let $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_0)$ be a Riemannian manifold such that there exist two points p_1, $p_2 \in \mathcal{M}_0$ which are not joined by any geodesic for the metric $\langle\cdot,\cdot\rangle_0$. Consider the (static) Lorentzian manifold $(\mathcal{M}\langle\cdot,\cdot\rangle)$, where $\mathcal{M} = \mathcal{M}_0 \times \mathbf{R}$ and $\langle\cdot,\cdot\rangle$ is given by

$$\langle\zeta,\zeta\rangle = \langle\xi,\xi\rangle_0 - \tau^2 ,$$

for any $z = (x,t) \in \mathcal{M}_0 \times \mathbf{R}$ and $\zeta = (\xi,\tau) \in T_z\mathcal{M}$. Let $\Lambda = \mathcal{M}$. Consider the point $p = (p_1, 0)$ and the timelike curve $\gamma(s) = (p_2, s)$. Straightforward calculations show that p and γ satisfy assumptions $L_1) - L_2)$, but not L_3). Theorem 1.7 does not hold for p and γ, since there are no lightlike geodesics joining p and γ, while $\mathcal{P}(\mathcal{L}^+_{p,\gamma}, \mathcal{K}) \neq \{0\}$ for any field $\mathcal{K}$.

Remark 1.11. – The result of Theorem 1.7 covers all the results of [8,12,36]. For further results whenever p and γ are nonconjugate, see [9].

Remark 1.12. – Let c_q be the number of the future pointing lightlike geodesics joining p and γ having index q. Then (1.7) can be written in the following way:

$$\sum_{q=0}^{\infty} c_q r^q = \sum_{q=0}^{\infty} \beta_q(\mathcal{L}^+_{p,\gamma}; \mathcal{K}) r^q + (1+r)S(r). \tag{1.8}$$

From (1.8) we deduce that a certain number of future pointing light rays joining p and γ are obtained according to the topology of $\mathcal{L}^+_{p,\gamma}$. In particular, setting $r = 1$ in (1.8), we have the following estimate on the number $\operatorname{card}(\mathcal{G}^+_{p,\gamma})$ of the light rays joining p and γ:

$$\operatorname{card}(\mathcal{G}^+_{p,\gamma}) = \sum_{q=0}^{\infty} \beta_q(\mathcal{L}^+_{p,\gamma}; \mathcal{K}) + 2S(1). \tag{1.9}$$

Since $S(1)$ is nonnegative we get also the classical Morse inequalities

$$c_q \geq \beta_q(\mathcal{L}^+_{p,\gamma}; \mathcal{K}), \qquad \forall q \in \mathbf{N}. \tag{1.10}$$

The other critical points are due to the complementary term $S(r)$. It depends on the geometric properties of $\mathcal{L}^+_{p,\gamma}$ and the coefficient field $\mathcal{K}$. We could define (as in the classical Morse Theory, see [5]) the couple (p,γ) to be *perfect* if there exists a field $\mathcal{K}$ such that $S(r) \equiv 0$.

An example of the influence of the topology of $\mathcal{L}^+_{p,\gamma}$ on the number of future pointing, lightlike geodesics between p and γ is given by the next Theorem.

THEOREM 1.13. – *Under the assumptions of Theorem 1.7 we have:*

(a) *If* $\mathcal{L}^+_{p,\gamma}$ *is contractible* [1], *the number* $\operatorname{card}\mathcal{G}^+_{p,\gamma}$ *of the future pointing light rays joining* p *with* γ *and with image in* Λ *is infinite or odd;*

(b) *if* $\mathcal{L}^+_{p,\gamma}$ *is noncontractible, there exist at least two future pointing light rays joining* p *and* γ.

Remark 1.14. – It is possible to produce some examples where $\mathcal{L}^+_{p,\gamma}$ is contractible, and the number of future pointing lightlike geodesics joining p with γ is arbitrarily large (see Example 3.3).

Actually, the topology of $\mathcal{L}^+_{p,\gamma}$ is not known for any Lorentzian manifold. More information can be obtained if its topology can be related to the topology of the manifold Λ. Let $\Omega(\Lambda)$ be the *based loop space* of all the continuous curves $z: [0,1] \longrightarrow \Lambda$ such that $z(0) = z(1) = \bar{z}$. Since Λ is connected, $\Omega(\Lambda)$ does not depend on $\bar{z}$. We equip $\Omega(\Lambda)$ with the *compact-open topology* (cf. [16]). Since the Poincaré polynomial is a homotopical invariant, as an immediate consequence of Theorem 1.7 we have the following result.

THEOREM 1.15. – *Under the assumptions of* Theorem 1.7, *assume also that*

L_4) $\mathcal{L}^+_{p,\gamma}$ *has the same homotopy type of the based loop space* $\Omega(\Lambda)$.

Then for any field $\mathcal{K}$ *there exists a formal series* $S(r)$ *with coefficients in* $\mathbf{N} \cup \{+\infty\}$, *such that*

$$\sum_{z \in \mathcal{G}^+_{p,\gamma}(\Lambda)} r^{\mu(z)} = \mathcal{P}(\Omega(\Lambda), \mathcal{K}) + (1+r)S(r)\ . \tag{1.11}$$

Assumption L_4) is certainly satisfied if $(\Lambda, \langle\cdot,\cdot\rangle)$ is conformally stationary (for instance if $(\Lambda, \langle\cdot,\cdot\rangle)$ is a Robertson-Walker space-time) or if it is isometric to a globally hyperbolic manifold satisfying the metric growth condition of [36] (see section 3).

THEOREM 1.16. – *Under the assumptions of Theorem 1.15 we have:*

(a) *If* Λ *is contractible, then the number of the future pointing lightlike geodesics joining* p *with* γ *and with image in* Λ *is infinite or odd;*

(b) *if* Λ *is noncontractible, then the number of the future pointing lightlike geodesics joining* p *with* γ *and with image in* Λ *is infinite.*

[1] We recall that a topological space is said to be *contractible* if it is homotopically equivalent to a point.

The oddity of the number of lightlike geodesics has been predicted by astronomers. The result is based on the same abstract principle: the number of the critical points of a Morse function defined on a contractible Riemannian manifold (possibly infinite dimensional), bounded from below, and satisfying the Palais-Smale compactness condition, is infinite or odd, assuming that the Morse index of the critical points is finite (see section 2).

2. SOME RECALLS ON ABSTRACT MORSE THEORY

In this section we shall recall, for the convenience of the reader, the basic results on Morse Theory on Hilbert manifolds (for the proofs, see [5,22,27]). We first recall the Palais-Smale compactness condition, which plays a basic role in Calculus of Variations in the Large.

DEFINITION 2.1. – *Let $f: X \longrightarrow \mathbf{R}$ be a C^1 functional defined on a Riemannian manifold (X, h) and let $c \in \mathbf{R}$. The map f satisfies the Palais-Smale condition at the level c ($(\mathrm{PS})_c$), if every sequence $(x_m)_{m\in\mathrm{N}}$ such that*

$$f(x_m) \to c\ , \qquad \|\nabla f(x_m)\| \to 0\ ,$$

contains a converging subsequence.

Here $\nabla f(x)$ denotes the gradient of f at the point x, with respect to the Riemannian metric h, and $\|\cdot\|$ is the norm on the tangent bundle induced by h.

Let $f: X \longrightarrow \mathbf{R}$ be a smooth functional defined on the Hilbert manifold X, a point $x \in X$ is called *critical point* if $f'(x) = 0$. A number $c \in \mathbf{R}$ is called *critical value for* f if there exists a critical point x of f such that $f(x) = c$, otherwise c is called *regular value.*

We recall now the notion of nondegenerate critical point and Morse index of a critical point. Let x be a critical point of f, where f is twice differentiable, then it is well defined (see [27]) the Hessian $f''(x): T_xX \times T_xX \longrightarrow \mathbf{R}$, by setting for any $\xi \in T_xX$,

$$f''(x)[\xi, \xi] = \frac{d^2 f(\eta(s))}{ds^2}\Big|_{s=0}\ , \tag{2.1}$$

(here $\eta(s)$ is a smooth curve on X such that $\eta(0) = x$, $\dot\eta(0) = \xi$) and extending $f''(x)$ by polarization to any couple of tangent vectors.

The critical point x is said nondegenerate if the bilinear form $f''(x)[\cdot,\cdot]$ is nondegenerate. The functional f is said *Morse function* if all its critical points are nondegenerate. The *Morse index* $m(x, f)$ of the critical point x is the maximal dimension of a subspace W of T_xX, such that the restriction of $f''(x)$ to W is negative definite. Clearly $m(x, f)$ can be infinite if X is infinite dimensional.

For any $c \in \mathbf{R}$, we set

$$f^c = \{x \in X | f(x) \leq c\} , \tag{2.2}$$

and for any $a < b$,

$$f_a^b = \{x \in X | a \leq f(x) \leq b\} . \tag{2.3}$$

Let (X, Y) be a topological pair, that is X is a topological space and Y is a subspace of X. For any field $\mathcal{K}$ and for any $q \in \mathbf{N}$, we denote by $H_q(X, Y; \mathcal{K})$ the q–th relative homology group (cf. [35]). Since $\mathcal{K}$ is a field, $H_q(X, Y; \mathcal{K})$ is a vector space and its dimension, denoted by $\beta_q(X, Y; \mathcal{K})$, is called the q-th Betti number of the couple (X, Y). Finally the Poincaré polynomial of the topological pair (X, Y) is defined by

$$\mathcal{P}(X, Y; \mathcal{K}) = \sum_{q=0}^{+\infty} \beta_q(X, Y; \mathcal{K}) r^q.$$

We shall state now the Morse Relations proved in [27] for a Morse function satisfying (PS) on a (possibly infinite dimensional) Riemannian manifold. For the version stated below see [5,22].

THEOREM 2.2. – *Let (X, h) be a Riemannian manifold of class C^2, $f: X \longrightarrow \mathbf{R}$ a C^2 function on X, $K(f)$ the set of its critical points, and $a < b$ two regular vales of f. Assume that:*

(1) *f_a^b is a complete metric subspace of X;*

(2) *f satisfies $(PS)_c$ for any $c \in [a, b]$;*

(3) *The Morse index of every critical point of f is finite.*

(4) *The set $K(f) \cap f_a^b$ consists of nondegenerate critical points.*

Then for any field $\mathcal{K}$, there exists a polynomial $S_{a,b}(r)$ with positive integer coefficients such that

$$\sum_{x \in K(f) \cap f_a^b} r^{m(x,f)} = \mathcal{P}(f^b, f^a; \mathcal{K}) + (1 + r) S_{a,b}(r) , \tag{2.4}$$

We point out that since nondegenerate critical points are isolated and $(\mathrm{PS})_c$ holds for any $c \in [a, b]$, $K(f; a, b)$ is finite, hence $S_{a,b}(r)$ is a polynomial.

If the Morse function f is bounded from below, choosing $a < \inf f$ we get the following corollary.

COROLLARY 2.3. – *Assume that f is bounded from below and let b a regular value. Under the assumptions* (1)–(4) *of* Theorem 2.2, *for any field $\mathcal{K}$ there exists a polynomial $S_b(r)$ with positive integer coefficients such that*

$$\sum_{x\in k(f)\cap f^b} r^{m(x,f)} = \mathcal{P}(f^b,\mathcal{K}) + (1+r)S_b(r) . \tag{2.5}$$

Finally it is possible to take the limit as $b \to +\infty$ in (2.5) (see for instance [4]), getting the following result.

THEOREM 2.4. – *Assume that f is bounded from below. Under the assumption* (1)-(4) *of* Theorem 2.2, *with $a < \inf f$ and $b = +\infty$, for any field $\mathcal{K}$ there exists a formal series $S(r)$ with coefficients in* $\mathbf{N} \cup \{+\infty\}$, *such that*

$$\sum_{x\in K(f)} r^{m(x,f)} = \mathcal{P}(X,\mathcal{K}) + (1+r)S(r) . \tag{2.6}$$

Remark 2.5. – If the Riemannian manifold (X, h) is not complete, the results of Theorem 2.2, Theorem 2.4 and Corollary 2.3 still hold, assuming that, for every $c \in \mathbf{R}$, the c-sublevels of the functional f are complete metric subspaces of X.

3. FERMAT PRINCIPLES AND SOME EXAMPLES

In this section we shall present some different versions of the Fermat principle, adapted to suitable classes of Lorentzian manifolds. Moreover we shall present some applications to the multiple image effect.

We prefer here to show several versions of the Fermat's principle for two reasons. In first place, the techniques used for proving the Fermat principle and the Morse Relations (1.7) are different, and they are more complicated, according to the class of Lorentzian manifold considered. Secondly, from a historical point of view, the Fermat principle has been formulated gradually in classes of manifolds that became wider and wider. The first formulation of a relativistic version of the Fermat principle is given in a static space-time, and it is due to H. Weyl ([37]); it was extended to stationary space-times by T. Levi-Civita in [19] and by K. Uhlenbeck (see [36]) in space-times of splitting type. Recently, some formulations of the principle for arbitrary space-times have been obtained (see [18,29]).

Let $\mathcal{M}$ be a smooth manifold and consider the Sobolev manifold $H^{1,2}([0,1],\mathcal{M})$ introduced at Section 1. Moreover, let x_0,x_1 be two points of $\mathcal{M}$, we set

$$\Omega^{1,2}(x_0,x_1,\mathcal{M}) = \{x \in H^{1,2}([0,1],\mathcal{M}) : x(0) = x_0, x(1) = x_1\}. \quad (3.1)$$

It is well known that $\Omega^{1,2}(x_0,x_1,\mathcal{M})$ is a smooth submanifold of $H^{1,2}([0,1],\mathcal{M})$ and it is homotopically equivalent to the based loop space $\Omega(\mathcal{M})$ (see [27]).

3.1. Static space-times

Let $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_{(R)})$ be a connected Riemannian manifold and let $\beta: \mathcal{M}_0 \longrightarrow \mathbf{R}^+$ be a smooth positive function on $\mathcal{M}_0$. Consider the (standard) static space-time $(\mathcal{M}, \langle\cdot,\cdot\rangle)$, where $\mathcal{M} = \mathcal{M}_0 \times \mathbf{R}$ and the Lorentzian metric is given by

$$\langle\zeta,\zeta\rangle = \langle\xi,\xi\rangle_{(R)} - \beta(x)\tau^2, \quad (3.2)$$

for any $z = (x,t) \in \mathcal{M}$ and $\zeta = (\xi,\tau) \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbf{R}$. A static Lorentzian manifold is stably causal and we choose the time function given by the projection on the time component,

$$T(x,t) = t.$$

For any $z \in \mathcal{M}$, the gradient of T is given by $\nabla T(z) = (0, -\beta(x)^{-1})$. Since lightlike geodesics are independent (up to reparameterization) on conformal changes of the metric, we can assume that the function $\beta(x)$ is identically equal to 1 and consider the metric

$$\langle\zeta,\zeta\rangle = \langle\xi,\xi\rangle_{(R)} - \tau^2, \quad (3.3)$$

Let $p = (x_0, 0)$ and consider the timelike curve $z(s) = (x(s), t(s)) = (x_1, s)$, with x_0, $x_1 \in \mathcal{M}_0$, and $x_0 \neq x_1$. The manifold $\Omega^{1,2}_{p,\gamma}$ of the curves joining p with γ is given by

$$\Omega^{1,2}_{p,\gamma} = \{z(s) = (x(s),t(s)) \in H^{1,2}([0,1],\mathcal{M}) \\ : x(0) = x_0, x(1) = x_1, t(0) = 0\}$$

and since $\mathcal{M} = \mathcal{M}_0 \times \mathbf{R}$,

$$\Omega^{1,2}_{p,\gamma} = \Omega^{1,2}(x_0,x_1,\mathcal{M}_0) \times H^{1,2}(0,\mathbf{R}), \quad (3.4)$$

where
$$H^{1,2}(0,\mathbf{R}) = \{t(s) \in H^{1,2}([0,1],\mathbf{R}) : t(0) = 0\}.$$

The arrival time functional $\tau_{p,\gamma}: \Omega^{1,2}_{p,\gamma} \longrightarrow \mathbf{R}$ of a curve $z(s) = (x(s), t(s))$ is given by the evaluation of the time component $t(s)$ at $s = 1$, that is

$$\tau_{p,\gamma}(z) = \tau_{p,\gamma}(x,t) = t(1) = \int_0^1 \dot{t}ds = \int_0^1 \langle \nabla T(z), \dot{z}\rangle ds. \tag{3.5}$$

The space $\mathcal{L}^+_{p,\gamma}$ of the lightlike future pointing curves joining p and γ (see (1.4)) is given by

$$\mathcal{L}^+_{p,\gamma} = \{z = (x,t) \in \Omega^{1,2}_{p,\gamma} : \dot{t} \geq 0, \langle \dot{x}, \dot{x}\rangle_{(R)} - \dot{t}^2 = 0 \text{ almost everywhere }\} \tag{3.6}$$

By (3.6), the restriction of the arrival time on $\mathcal{L}^+_{p,\gamma}$ is given by

$$\tau_{p,\gamma}(z) = \tau_{p,\gamma}(x,t) = t(1) = \int_0^1 \langle \nabla T(z), \dot{z}\rangle ds = \int_0^1 \sqrt{\langle \dot{x}, \dot{x}\rangle_{(R)}}ds. \tag{3.7}$$

Hence the arrival time of $z = (x,t)$ is equal to the length of the spatial component with respect to the Riemannian structure $\langle \cdot, \cdot \rangle_{(R)}$. It is well known that the length functional $F: \Omega^{1,2}(x_0, x_1, \mathcal{M}_0) \longrightarrow \mathbf{R}$,

$$F(x) = \int_0^1 \sqrt{\langle \dot{x}, \dot{x}\rangle_{(R)}}ds$$

is nondifferentiable at the curve $x(s)$ such that $\dot{x}(s) = 0$ in a set of positive Lebesgue measure. Since the restriction of the arrival time $\tau_{p,\gamma}$ on $\mathcal{L}^+_{p,\gamma}$ is nonsmooth, we deduce (a fortiori) that $\mathcal{L}^+_{p,\gamma}$ *is not a smooth manifold.* By (3.6) we have that $\mathcal{L}^+_{p,\gamma}$ is the graph of the map $\Phi: \Omega^{1,2}(x_0, x_1, \mathcal{M}_0) \longrightarrow H^{1,2}(0,\mathbf{R})$ given by

$$\Phi(x)(s) = \int_0^s \sqrt{\langle \dot{x}, \dot{x}\rangle_{(R)}}dr. \tag{3.8}$$

It is not difficult to show that Φ is locally Lipschitz continuous, hence $\mathcal{L}^+_{p,\gamma}$ can be equipped by a structure of Lipschitz manifold.

We consider now the *action integral* $Q: \Omega^{1,2}(x_0, x_1, \mathcal{M}_0) \longrightarrow \mathbf{R}$, given by

$$Q(x) = \int_0^1 \langle \dot{x}, \dot{x}\rangle_{(R)} ds.$$

Since in (3.3) there are not mixed terms, we have that a smooth curve $z(s) = (x(s), t(s))$ is a geodesic for the static structure $\langle \cdot, \cdot \rangle$ if and only if $x(s)$ is a geodesic on $\mathcal{M}_0$ for the Riemannian structure $\langle\cdot,\cdot\rangle_{(R)}$ and t is a segment. Moreover, it is well known (see [27]) that Q is smooth and its critical points are the geodesics joining x_0 and x_1. Then we deduce the following Fermat principle for the geodesics joining p with γ:

A smooth lightlike future pointing curve $z(s) = (x(s), t(s))$ is a lightlike geodesic joining p and γ if and only if x is a critical point of Q and $t = \Phi(x)$.

Notice that the variational principle above characterizes only the spatial projections of lightlike geodesics as critical point of a functional.

Assume now that the Riemannian metric $\langle\cdot,\cdot\rangle_{(R)}$ is complete. By virtue of the variational principle above, Morse Theory for the light rays joining p and γ is a consequence of Morse Theory for Riemannian geodesics (cf. [5,25,27]). Indeed, Morse Theory for Riemannian geodesics (see [27]) show that the action integral Q satisfies the assumptions of Theorem 2.4 (whenever x_0 and x_1 are nonconjugate). Then we can prove that the Morse Index of a Riemannian geodesic x is equal to the index $\mu(z)$ (see Definition 1.5) of the corresponding lightlike geodesic (see [20] for the details). Moreover, since $\mathcal{L}^+_{p,\gamma}$ is the graph of the map Φ given by (3.8), we have that $\mathcal{L}^+_{p,\gamma}$ is homotopically equivalent to $\Omega^{1,2}(x_0, x_1, \mathcal{M}_0)$ and also to the based loop space $\Omega(\mathcal{M})$). Hence, assumption L_4) of Theorem 1.15 is satisfied. In particular we have that if $\mathcal{M}$ is noncontractible, there exist infinitely many images, while if $\mathcal{M}$ is contractible, the number of images is odd or infinite.

We point out that the above results hold also for *conformally static Lorentzian manifolds*, for which the metric (3.2) is multiplied by a conformal factor $\alpha(x,t) > 0$. A class of physically relevant conformally static space-times is given by the generalized Robertson-Walker space-times, whose metric is given by

$$\langle\zeta,\zeta\rangle = \alpha(t)\langle\xi,\xi\rangle_{(R)} - \beta(t)\tau^2,$$

where $\alpha(t)$ and $\beta(t)$ are smooth positive functions, depending only on the time variable.

3.2. Stationary space-times

We consider a (standard) stationary Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$, where $\mathcal{M} = \mathcal{M}_0 \times \mathbf{R}$ and the metric is given by

$$\langle\zeta,\zeta\rangle = \langle\xi,\xi\rangle_{(R)} + 2\langle\delta(x),\xi\rangle_{(R)}\tau - \tau^2, \tag{3.9}$$

where $\langle\cdot,\cdot\rangle_{(R)}$ is a Riemannian metric on $\mathcal{M}_0$ and $\delta(x)$ is a smooth vector field on $\mathcal{M}_0$. As in the static case, it is not restrictive to assume that the coefficient of τ^2 is identically equal to 1.

Note that $(\mathcal{M}, g)$ is stably causal. A time function is given by the projection $T(x,t) = t$. In this case the gradient of T is

$$\nabla T(z) = \nabla T(x,t) = \frac{1}{1 + \langle \delta(x), \delta(x) \rangle_{(R)}} (\delta(x), -1).$$

Let $p = (x_0, 0)$ and consider the timelike curve $z(s) = (x(s), t(s)) = (x_1, s)$, with x_0, $x_1 \in \mathcal{M}_0$, and $x_0 \neq x_1$. The manifold $\Omega^{1,2}_{p,\gamma}$ of the curves joining p with γ is given by (3.4), while the arrival time is given by (3.5). The space $\mathcal{L}^+_{p,\gamma}$ is

$$\mathcal{L}^+_{p,\gamma} = \{ z = (x,t) \in \Omega^{1,2}_{p,\gamma} : \dot{t} \geq 0,$$
$$\langle \dot{x}, \dot{x} \rangle_{(R)} + 2 \langle \delta(x), \dot{x} \rangle_{(R)} \dot{t} - \dot{t}^2 = 0 \text{ almost everywhere } \} \tag{3.10}$$

By (3.10), the restriction of the arrival time on $\mathcal{L}^+_{p,\gamma}$ is

$$\tau_{p,\gamma}(z) = \tau_{p,\gamma}(x,t) = t(1) = \int_0^1 \langle \nabla T(z), \dot{z} \rangle ds =$$
$$\int_0^1 \langle \delta(x), \dot{x} \rangle_{(R)} ds + \int_0^1 \sqrt{\langle \dot{x}, \dot{x} \rangle_{(R)} + \langle \delta(x), \dot{x} \rangle^2_{(R)}} ds. \tag{3.11}$$

In this case the arrival time is not equal to the length functional of a Riemannian metric, as in the static case, but to the length functional of a *pseudo-Finsler structure*, recently used to study lightlike trajectories (see [21]). Again the functional $F : \Omega^{1,2}(x_0, x_1, \mathcal{M}_0) \longrightarrow \mathbf{R}$ defined by

$$F(x) = \int_0^1 \langle \delta(x), \dot{x} \rangle_{(R)} ds + \int_0^1 \sqrt{\langle \dot{x}, \dot{x} \rangle_{(R)} + \langle \delta(x), \dot{x} \rangle^2_{(R)}} ds$$

is nondifferentiable and $\mathcal{L}^+_{p,\gamma}$ is not a smooth manifold. Moreover $\mathcal{L}^+_{p,\gamma}$ is the graph of the locally Lipschitz map $\Phi : \Omega^{1,2}(x_0, x_1, \mathcal{M}_0) \longrightarrow H^{1,2}(0, \mathbf{R})$ defined by

$$\Phi(x)(s) = \int_0^s \langle \delta(x), \dot{x} \rangle_{(R)} ds + \int_0^s \sqrt{\langle \dot{x}, \dot{x} \rangle_{(R)} + \langle \delta(x), \dot{x} \rangle^2_{(R)}} ds$$

Using the functional F on the space $\Omega^{1,2}(\mathcal{M}_0, x_0, x_1)$, the Levi-Civita version of the Fermat principle can be obtained.

A different version of the Fermat principle has been obtained in [8]. Consider the functional $\tilde{F}: \Omega^{1,2}(x_0, x_1, \mathcal{M}_0) \longrightarrow \mathbf{R}$ defined by

$$\tilde{F}(x) = \int_0^1 \langle \delta(x), \dot{x} \rangle_{(R)} ds + \sqrt{\int_0^1 \langle \dot{x}, \dot{x} \rangle_{(R)} ds + \int_0^1 \langle \delta(x), \dot{x} \rangle_{(R)}^2 ds}.$$

(Notice that $\tilde{F}$ has not the same meaning as the action integral in the static case). The functional $\tilde{F}$ is smooth and the following variational principle holds.

A smooth lightlike future pointing curve $z(s) = (x(s), t(s))$ is a lightlike geodesic joining p and γ if and only if x is a critical point of $\tilde{F}$ and $t = \Phi(x)$.

As in the static case, this variational principle characterizes only the spatial projections of lightlike geodesics as critical point of a functional.

Assume now that the Riemannian metric $\langle \cdot, \cdot \rangle_{(R)}$ is complete and the vector field δ is bounded, that is

$$\sup_{x \in \mathcal{M}_0} \langle \delta(x), \delta(x) \rangle_{(R)} < +\infty.$$

Using the above variational principle, the Morse Relations (1.7) hold for the lightlike geodesics joining p and γ (cf. [8]). Moreover, as in the static case assumption L_4) is satisfied and Theorems 1.15 and 1.16 hold. All the results are true also for conformally stationary Lorentzian manifolds.

3.3. Multiple image effect can occur even if the topology of the space-time is trivial

We show now an example contained in [6] where even if the Lorentzian manifold $\mathcal{M}$ and the space $\mathcal{L}^+_{p,\gamma}$ are contractible, there is a multiple image effect.

Consider the Euclidean Riemannian manifold $(\mathbf{R}^n, \langle \cdot, \cdot \rangle_{(R)})$, where

$$\langle \zeta, \zeta \rangle_{(R)} = \zeta_1^2 + \zeta_2^2 \dots + \zeta_n^2,$$

for any $\zeta = (\zeta_1, \dots, \zeta_n)$.

Let A be a subset of $\mathbf{R}^n$ such that $\mathbf{R}^n \setminus A$ is noncontractible (for instance A is a ball) and let B,C be open subset of $\mathbf{R}^n$ such that

$$A \subset B \subset C,$$
$$\mathbf{R}^n \setminus C \text{ is noncontractible },$$
$$d(\partial B, A) > 0,$$

where ∂B is the boundary of B and $d(\partial B, A)$ is the Euclidean distance. Moreover, we consider a family of smooth functions $\varphi_\lambda : \mathbf{R}^n \longrightarrow]0, +\infty[$, $\lambda > 0$, such that

$$\lim_{\lambda \to +\infty} m_\lambda = \inf_{x \in B \setminus A} \varphi_\lambda(x) = +\infty.$$

and

$$\sup_{\lambda > 0} \sup_{x \in \mathbf{R}^n \setminus C} \varphi_\lambda(x) < +\infty.$$

Now, consider on $\mathbf{R}^{n+1}$ the following family of static Lorentzian metrics:

$$\langle \zeta, \zeta \rangle_\lambda = \varphi_\lambda(x) \langle \xi, \xi \rangle_{(R)} - \tau^2,$$

for any $z = (x, t) \in \mathbf{R}^{n+1}$ and $\zeta = (\xi, \tau) \in T_z \mathbf{R}^{n+1} \equiv \mathbf{R}^n \times \mathbf{R}$.

Let $x_0, x_1 \in \mathbf{R}^n \setminus C$ and set $p = (x_0, 0)$ and $\gamma(s) = (x_1, s)$. Then the following result holds (see [6]):

For any $m \in \mathbf{N}$ there exists $\lambda_m = \lambda_m(x_0, x_1)$, such that for any $\lambda > \lambda_m$ there exists at least m lightlike geodesics joining p and γ for the metric $\langle \zeta, \zeta \rangle_\lambda$.

In particular we have that we can choose metrics to get an arbitrarily large number of images. Notice that since the metrics $\langle \zeta, \zeta \rangle_\lambda$ are static, assumption L_4) of Theorem 1.15 is satisfied, then $\mathcal{L}^+_{p,\gamma}$ is contractible.

3.4 Multiple image effect in Schwarzschild, Reissner-Nordström and Kerr space-times

We present now some results on multiple image effect in some physically relevant space-times, Schwarzschild, Reissner-Nordström and Kerr space-times. For more details on the physical properties of these space-times, we refer to [15].

The *Schwarzschild space-time* represents the asymptotically flat space-time outside a static, spherically symmetric body. Let $m > 0$ be the mass of the body. The Schwarzschild metric in polar coordinates (r, θ, ϕ) is given by

$$ds^2 = \frac{1}{\beta(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) - \beta(r) dt^2, \tag{3.12}$$

where $\beta(r) = 1 - 2m/r$. The static Lorentzian manifold $(\mathcal{M}, ds^2) = (\mathcal{M}_0 \times \mathbf{R}, ds^2)$, where $\mathcal{M}_0 = \{(r, \theta, \phi) : r > 2m\}$ is called *Schwarzschild space-time*. Let $r_* \in]2m, 3m]$ and consider the open subset of $\mathcal{M}$,

$$\mathcal{M}_* = \{(r, \theta, \phi) : r > r_*\} \times \mathbf{R}.$$

The boundary $\partial\mathcal{M}_*$ is smooth and timelike. Moreover, $\mathcal{M}_*$ is light-convex (see [20]). Let $p = (x_0, 0)$ and $\gamma(s) = (x_1, s)$ contained in $\mathcal{M}_*$. Since $\mathcal{M}_*$ is homotopically equivalent to the unit sphere S^2 embedded in $\mathbf{R}^3$, and assumption L_4) of Theorem 1.15 is satisfied ($\mathcal{M}_*$ is static), there exist infinitely many lightlike geodesics joining p and γ in the future of p, with image in $\mathcal{M}_*$. Taking the limit as $r_* \to 2m$, the same conclusion holds for the Schwarzschild space-time $(\mathcal{M}, ds^2)$.

Consider now the Reissner-Nordström space-time. It represents the asymptotically flat space-time outside a static, spherically symmetric, massive body, carrying an electric charge. Let $m > 0$ be the mass of the body and e its charge. The Reissner-Nordström metric in polar coordinates (r, θ, ϕ) is given by (3.12), with

$$\beta(r) = 1 - 2m/r + e^2/r^2.$$

Whenever $m > e$, the Reissner-Nordström metric is well defined and it is a static metric on the manifold $\mathcal{M}_0 \times \mathbf{R}$, where $\mathcal{M}_0 = \{(r, \theta, \phi) : r > m + \sqrt{m^2 - e^2}\}$.

The same results as the Schwarzschild space-time holds for the Reissner-Nordström space-time, choosing $r_* \in]m + \sqrt{m^2 - e^2}, \frac{1}{2}(3m + \sqrt{9m^2 - 8e^2})[$ (see [20]). Hence we have a multiple image effect also in the Reissner-Nordström space-time.

Finally we consider the *Kerr space-time*, which represents the asymptotically flat space-time outside an axis-symmetric, rotating, massive object. Let m be the mass of the body and a such that ma is the angular momentum as measured from infinity (see [15]). The Kerr metric in polar coordinates is given by

$$ds^2 = \lambda(r,\theta)\left(\frac{dr^2}{\Delta(r)} + d\theta^2\right)$$
$$+(r^2 + a^2)\sin^2\theta d\phi^2 - dt^2 + \frac{2mr}{\lambda(r,\theta)}\left(a\sin^2\theta d\phi - dt\right)^2, \qquad (3.13)$$

where $\lambda(r, \theta) = r^2 + a^2\cos^2\theta$ and $\Delta(r) = r^2 - 2mr + a^2$. Notice that when $a = 0$, the metric reduces to the Schwarzschild one.

When $m^2 > a^2$, (3.13) is a stationary metric on the manifold

$$\tilde{\mathcal{M}}_a = \{(r, \theta, \phi) : r > \sqrt{m^2 - a^2\cos^2\theta}\} \times \mathbf{R}. \qquad (3.14)$$

The manifold (3.14), equipped with the metric (3.13) is called *Kerr space-time outside the stationary limit surface*. If the angular momentum is small, some open subsets of the Kerr space-time outside the limit surface are light convex. Indeed the following result holds (see [20]).

Let r_0 be the smallest root bigger than $2m$ of the equation

$$9m^2r^4(r-m)(r-2m)+(r-3m)[r^3+3m(r-m)(r-2m)]=0.$$

Moreover, let $\epsilon(a)\colon \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ be a decreasing function such that

$$\lim_{a\to 0}\epsilon(a)=\epsilon_0,$$

with

$$0<\epsilon_0<(r_0-m)^2-m^2.$$

Then there exists $a_0>0$ such that for any $|a|\le a_0$, the open subset

$$\mathcal{M}_a=\{(r,\theta,\phi): r>\sqrt{m^2+\epsilon(a)-a^2\cos^2\theta}\}\times\mathbf{R}$$

of $\tilde{\mathcal{M}}_a$ is light-convex.

Since $\mathcal{M}_a$ is homotopically equivalent to S^2, choosing a point $p=(x_0,0)$ in $\mathcal{M}_a$ and a curve $\gamma(s)=(x_1,s)$ with image in $\mathcal{M}_a$, we have the existence of infinitely many lightlike geodesics joining p with γ.

3.5. Orthogonal Splitting Spacetimes

We consider now an orthogonal splitting Lorentzian manifold $(\mathcal{M},\langle\cdot,\cdot\rangle)$, where $\mathcal{M}=\mathcal{M}_0\times\mathbf{R}$ and the metric is given by

$$\langle\zeta,\zeta\rangle=\langle\alpha(x,t)\xi,\xi\rangle_{(R)}-\tau^2, \tag{3.15}$$

for any $z=(x,t)\in\mathcal{M}$ and $\zeta=(\xi,\tau)\in T_z\mathcal{M}$, where $\langle\cdot,\cdot\rangle_{(R)}$ is a Riemannian metric on $\mathcal{M}_0$ and $\alpha(x,t)$ is a positive linear operator on $T_x\mathcal{M}_0$, smoothly depending on $z=(x,t)\in\mathcal{M}$. Moreover, as in the static case, it is not restrictive to assume that the coefficient of τ^2 is identically equal to 1.

An orthogonal splitting manifold $(\mathcal{M},g)$ is stably causal and a time function is given by the projection $T(x,t)=t$. In this case $\nabla T(z)=(0,-1)$ for any $z\in\mathcal{M}$.

Let $p=(x_0,0)$ and consider the timelike curve $z(s)=(x(s),t(s))=(x_1,s)$, with x_0, $x_1\in\mathcal{M}_0$, and $x_0\ne x_1$. The manifold $\Omega^{1,2}_{p,\gamma}$ of the curves joining p with γ is given by (3.4), while the arrival time on $\Omega^{1,2}_{p,\gamma}$ is given by (3.5). The space $\mathcal{L}^+_{p,\gamma}$ is

$$\mathcal{L}^+_{p,\gamma} = \{z = (x,t) \in \Omega^{1,2}_{p,\gamma} : \dot{t} \geq 0,$$
$$\langle \alpha(z)\dot{x}, \dot{x}\rangle_{(R)} - \dot{t}^2 = 0 \text{ almost everywhere } \}. \qquad (3.16)$$

By (3.16), the restriction of the arrival time on $\mathcal{L}^+_{p,\gamma}$ is

$$\tau_{p,\gamma}(z) = \tau_{p,\gamma}(x,t) = t(1) = \int_0^1 \langle \nabla T(z), \dot{z}\rangle ds =$$
$$\int_0^1 \sqrt{\langle \alpha(x,t)\dot{x}, \dot{x}\rangle_{(R)}} ds. \qquad (3.17)$$

Now, for any $x \in \Omega^{1,2}(x_0, x_1, \mathcal{M}_0)$, we consider the Cauchy problem

$$\begin{cases} \dot{t} = \sqrt{\langle \alpha(x,t)\dot{x}, \dot{x}\rangle_{(R)}} \\ t(0) = 0. \end{cases} \qquad (3.18)$$

The problem (3.18) has one and only one solution $t = \phi(x)$. We set

$$X^1 = \{x \in \Omega^{1,2}(x_0, x_1, \mathcal{M}_0) : \phi(x) \text{ is defined in } [0,1] \}. \qquad (3.19)$$

and consider the map $\phi: X^1 \longrightarrow H^{1,2}(0, \mathbf{R})$, $x \to \phi(x)$. By (3.18) and (3.19) we see that $\mathcal{L}^+_{p,\gamma}$ is the graph of ϕ and for any $z = (x,t) = (x, \phi(x)) \in \mathcal{L}^+_{p,\gamma}$,

$$\tau_{p,\gamma}(z) = F(x) = \int_0^1 \sqrt{\langle \alpha(x, \phi(x))\dot{x}, \dot{x}\rangle_{(R)}} ds. \qquad (3.20)$$

Differently from the static and stationary cases, the functional F given by (3.20) can not be interpreted as length functional for some structure, because of the dependence of the Lorentzian metric on the time coordinate.

It can be proved that X^1 is an open subset of $\Omega^{1,2}(x_0, x_1, \mathcal{M}_0)$ and the map ϕ is locally Lipschitz. It follows that again $\mathcal{L}^+_{p,\gamma}$ is a Lipschitz manifold and the functional F is Lipschitz continuous on X^1. In analogy with the static case we can consider the functional $Q: X^1 \longrightarrow \mathbf{R}$ defined by

$$Q(x) = \int_0^1 \langle \alpha(x, \phi(x))\dot{x}, \dot{x}\rangle_{(R)} ds. \qquad (3.21)$$

Differently from the static case, the functional Q is in general nonsmooth (see [11]). The effective dependence of the metric $\langle \cdot, \cdot \rangle$ on the time variable is responsible for the nonsmoothness of Q.

In [12], in order to obtain the Morse Relations as in (1.7), the Lipschitz manifold $\mathcal{L}^+_{p,\gamma}$ is approximated by a family of smooth manifolds consisting of timelike curves. In section 4 an analogous approximation scheme is introduced in the framework of causally stable Lorentzian manifolds with boundary (not necessarily globally hyperbolic).

In order to relate the topology of $\mathcal{L}^+_{p,\gamma}$ to the topology of the manifold $\mathcal{M}$, one can use the following growth condition (similar to that introduced in [36]) for the coefficient of the metric:

There exist two positive continuous functions $a(x), b(x): \mathcal{M}_0 \longrightarrow \mathbf{R}$, such that for any $z = (x,t) \in \mathcal{M}$ and $\xi \in T_x\mathcal{M}_0$, $\langle \xi, \xi \rangle_{(R)} = 1$:

$$|\langle \alpha(x,t)\xi, \xi \rangle_{(R)}| \leq a(x) + b(x)|t|. \tag{3.22}$$

If (3.22) holds, then the open set X^1 defined at (3.19) is equal to $\Omega^{1,2}(x_0, x_1, \mathcal{M}_0)$. Hence $\mathcal{L}^+_{p,\gamma}$ and $\Omega^{1,2}(x_0, x_1 \mathcal{M}_0)$ are homotopically equivalent and assumption L_4) of Theorem 1.15 is satisfied.

3.6. Stably causal Lorentzian manifolds

We present now two Fermat principles for the light rays in a stably causal Lorentzian manifold. In order to state such principles, we introduce a new manifold of lightlike curves. For any $k \in \mathbf{N}$ and for any interval $[a,b]$, we denote by $H^{2,2}([a,b], \mathbf{R}^k)$ the Sobolev space of the C^1 curves in $\mathbf{R}^k$ such that, the first derivative is absolutely continuous and the second derivative is square integrable. $H^{2,2}([a,b], \mathbf{R}^k)$ is equipped with a structure of Hilbert space, whose norm is given by

$$||x||^2_{2,2} = ||x||^2 + ||\dot{x}||^2 + ||\ddot{x}||^2 = \int_0^1 |x(s)|^2 ds + \int_0^1 |\dot{x}(s)|^2 ds + \int_0^1 |\ddot{x}(s)|^2 ds.$$

Now, let $\mathcal{M}$ be a smooth manifold, we denote by $H^{2,2}([0,1], \mathcal{M})$ the set of the curves $x: [0,1] \longrightarrow \mathcal{M}$, such that for any local chart (U, φ) of the manifold such that $U \cap x([0,1]) \neq \emptyset$, the curve $\varphi \circ x: x^{-1}(U) \longrightarrow \mathbf{R}^n$, $n = \dim \mathcal{M}$, belongs to the Sobolev space $H^{2,2}(x^{-1}(U), \mathbf{R}^n)$. As the space $H^{1,2}([0,1], \mathcal{M})$ introduced in Section 1, $H^{2,2}([0,1], \mathcal{M})$ is an infinite dimensional manifold modeled on the Hilbert space $H^{2,2}([0,1], \mathbf{R}^n)$. The tangent space $T_x H^{2,2}([0,1], \mathcal{M})$ to a curve $x \in H^{2,2}([0,1], \mathcal{M})$ is defined similarly to the tangent space for a curve in $H^{1,2}([0,1],\mathcal{M})$, except for the fact that its elements are of class $H^{2,2}([0,1],T\mathcal{M})$.

Now let $(\mathcal{M},\langle\cdot,\cdot\rangle)$ be a stably causal Lorentzian manifold and let T be a smooth time function. Fix $p\in\mathcal{M}$ and $\gamma\colon\mathbf{R}\longrightarrow\mathcal{M}$ timelike future pointing curve such that $T(p)=0$, $p\notin\gamma(\mathbf{R})$. Assume that γ is a closed embedding of $\mathbf{R}$ and it is vertical with respect to ∇T, that is there exists a smooth negative function $\lambda(s)$, such that $\dot\gamma(s)=\lambda(s)\nabla T(\gamma(s))$, for any $s\in\mathbf{R}$.

Consider the set

$$\Omega^{2,2}_{p,\gamma}=\{z\in H^{2,2}([0,1],\Lambda)\mid z(0)=p,\ z(1)\in\gamma(\mathbf{R})\}.$$

The space $\Omega^{2,2}_{p,\gamma}$ is a smooth submanifold of $H^{2,2}([0,1],\Lambda)$ (see [17]) and it is a submanifold of $\Omega^{1,2}_{p,\gamma}$, too. For every $z\in\Omega^{2,2}_{p,\gamma}$, the tangent space $T_z\Omega^{2,2}_{p,\gamma}$ consists of the elements of $T_z\Omega^{1,2}_{p,\gamma}$ belonging to $H^{2,2}([0,1],T\mathcal{M})$.

The *Arrival Time* functional on $\Omega^{2,2}_{p,\gamma}$, is defined as in (1.3) and it is smooth. Now we introduce the set

$$\tilde{\mathcal{L}}^{+}_{p,\gamma}=\left\{z\in\Omega^{2,2}_{p,\gamma}\mid\langle\dot z,\dot z\rangle=0,\langle\nabla T(z),\dot z\rangle>0\text{ for any }s\ \right\}.\tag{3.23}$$

Unlike the case of $\mathcal{L}^{+}_{p,\gamma}$, the condition $\langle\nabla T(z),\dot z\rangle>0$ (for any s) allows to prove that $\tilde{\mathcal{L}}^{+}_{p,\gamma}$ is a submanifold of $\Omega^{2,2}_{p,\gamma}$. For any $z\in\tilde{\mathcal{L}}^{+}_{p,\gamma}$ the tangent space is given by

$$T_z\tilde{\mathcal{L}}^{+}_{p,\gamma}=\{\zeta\in T_z\Omega^{2,2}_{p,\gamma}:\langle\nabla_s\zeta,\dot z\rangle=0\text{ for any }s\ \},$$

where $\nabla_s\zeta$ is the covariant derivative of ζ along z.

Consider now the following functionals $F,Q\colon\tilde{\mathcal{L}}^{+}_{p,\gamma}\longrightarrow\mathbf{R}$,

$$F(z)=T(z(1))-T(z(p))=T(z(1))-T(z(0))=\int_0^1\langle\nabla T(z),\dot z\rangle ds,$$

$$Q(z)=\int_0^1\langle\nabla T(z),\dot z\rangle^2ds.$$

We point out that if γ is parameterized so that

$$\dot\gamma=\frac{-\nabla T(\gamma)}{\sqrt{-\langle\nabla T(\gamma),\nabla T(\gamma)\rangle}},$$

the functional $F(z)$ coincides with the arrival time $\tau_{p,\gamma}(z)$ (up to an additive constant).

The functionals F and Q are smooth and the following versions of the Fermat principle hold (see [2]):

A curve $z \in \tilde{\mathcal{L}}^+_{p,\gamma}$ is a critical point of F if and only if z is a pregeodesic.

A curve $z \in \tilde{\mathcal{L}}^+_{p,\gamma}$ is a critical point of Q if and only if z is a pregeodesic such that $\langle \nabla T(z(s)), \dot{z}(s)\rangle$ is constant.

We recall that a smooth curve $z\colon]a,b[\longrightarrow \mathcal{M}$ is a *pregeodesic* if there exists a reparametrization w of z, such that w is a geodesic. Equivalently z is a pregeodesic if and only if there exists a continuous function $\lambda(s)$ such that

$$\nabla_s \dot{z} = \lambda(s)\dot{z}. \tag{3.24}$$

The two variational principles stated above present some features that we would like to describe. First of all, the critical points of F and Q are only pregeodesic, which is a limitation. In any case it can be seen as an improvement of the results in the static and the stationary case, where the Fermat principles were only stated for the spatial projection of the geodesics.

In spite of the simplicity of their statements, the above principles are not suitable for proving the Morse Relations (1.7). If we work on the manifold $\tilde{\mathcal{L}}^+_{p,\gamma}$, we have the lack of completeness of the sublevels of F and Q, due to the condition $\langle \nabla T(z), \dot{z}\rangle > 0$ for any s. So, there is no real advantage in working on the manifold $\tilde{\mathcal{L}}^+_{p,\gamma}$. On the other hand, the natural topology on the curves space to study the functional Q is the $H^{1,2}$-topology. Thus we shall work on $\mathcal{L}^+_{p,\gamma}$. To overcome the lack of regularity, we approximate $\mathcal{L}^+_{p,\gamma}$ with a family of regular manifolds $\mathcal{L}^+_{p,\gamma,\epsilon}$, consisting of timelike curves. Our strategy will be to prove the Morse Relations on $\mathcal{L}^+_{p,\gamma,\epsilon}$, $\epsilon > 0$, and to pass to the limit as $\epsilon \to 0$. We would like to remark here that the idea of studying light rays as limits of timelike curves is already present in the book of Levi-Civita [19] for stationary space-times.

In the next sections, the Morse Theory will be studied using the functional Q rather than F. There are two main reasons for this choice. The first one is that the set of critical points of Q are in one-to-one correspondence with the set of future pointing lightlike geodesics joining p with γ. This depends on the fact that the critical points of Q come with a specific parameterization ($\langle \nabla T(z), \dot{z}\rangle$ constant). The second fact is that F is homogeneous of degree 1 in $\dot{z}$, and this fact causes several problems. For instance F is invariant by reparametrization, which implies that no critical point of F is isolated. Moreover, working in a space of curves with square integrable derivative, F

does not satisfy good compactness properties (for instance the Palais-Smale condition).

3.7. Fermat principle in arbitrary space-times

We conclude this section with some result in the Fermat principle for arbitrary space-times. The first formulation for the Fermat principle in an arbitrary space-time is contained in [18], with some interesting applications to the gravitational lens effect. In that paper the Fermat principle is stated in the traditional "Gateaux formulation", where variations are formulated in terms of parameters ϵ and stationarity of the functional is characterized in terms of the vanishing of the first derivative with respect to ϵ and with respect to all variations. No differential structure of the functional and of the set of curves (in other words no "Frechet formulation") is required.

The mathematical results of [18] are better clarified and correctly used in [28], still using a Gateaux formalism.

Finally in [29] the first "Frechet" version of the Fermat principle in an arbitrary space-time is proved. Let $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ be an arbitrary Lorentzian manifold, p an event and $\gamma\colon \mathbf{R} \longrightarrow \mathcal{M}$ a timelike curve, such that $p \notin \gamma(\mathbf{R})$ and γ is a closed embedding of $\mathbf{R}$. Consider the set

$$\mathcal{F}^{+}_{p,\gamma} = \{z \in H^{2,2}_{p,\gamma} : \langle \dot z, \dot z\rangle = 0, \langle \dot z(s), U(z(s))\rangle = \int_0^1 \langle \dot z(s), U(z(s))\rangle ds,$$
$$\langle \dot z(1), U(z(1))\rangle < 0\},$$

where $U(s)$ is the vector field along z solving

$$\begin{cases} \nabla_s U = 0 \\ U(1) = \dot\gamma(z(1)). \end{cases}$$

In [29] it is shown that $\mathcal{F}^{+}_{p,\gamma}$ is a submanifold of $H^{2,2}_{p,\gamma}$ and the critical points of $\tau_{p,\gamma}$ on $\mathcal{F}^{+}_{p,\gamma}$ are the lightlike geodesics joining p and γ in the future of p (in the sense that $\langle \dot z(s), U(z(s))\rangle < 0$ for every s).

Notice that variational principle above states that the light rays are the critical points of the arrival time $\tau_{p,\gamma}$, which probably is the most natural functional to produce a Fermat principle. The assumption $\langle \dot z(s), U(z(s))\rangle =$ $const.$ allows to choose the right geodesic parameterization for the critical points of $\tau_{p,\gamma}$.

However, a Morse Theory for the lightlike geodesics joining an event with a timelike curve in an arbitrary space-time has not been developed, yet.

4. THE CRITICAL POINTS OF THE FUNCTIONAL Q ON $\mathcal{L}^+_{p,\gamma}$ AND AN APPROXIMATION SCHEME BY SMOOTH MANIFOLDS

From now on we fix a stably causal Lorentzian manifold $(\mathcal{M}, \langle , \cdot, \cdot \rangle)$ an open subset Λ of $\mathcal{M}$ satisfying assumptions (a)-(c), an event $p \in \Lambda$ and a timelike future pointing curve $\gamma : \mathbf{R} \longrightarrow \mathcal{M}$, such that $p \notin \gamma(\mathbf{R})$, $\gamma(\mathbf{R}) \subset \Lambda$ and γ is a closed embedding of $\mathbf{R}$ in $\mathcal{M}$.

Since Λ is an open set of $\mathcal{M}$ with smooth timelike boundary, there exists a function $\varphi \in C^2(\mathcal{M}, \mathbf{R})$ such that

$$\begin{aligned} \Lambda &= \{z \in \mathcal{M} : \varphi(z) > 0\}; \\ \partial\Lambda &= \{z \in \mathcal{M} : \varphi(z) = 0\}; \\ \langle \nabla\varphi(z), \nabla\varphi(z)\rangle &> 0, \qquad \forall z \in \partial\Lambda. \end{aligned} \tag{4.1}$$

Let

$$\mathcal{N}(p, \gamma, \Lambda) = \{q \in \Lambda : \text{ there exists } z \in \Omega^{1,2}_{p,\gamma}(\Lambda), \langle \dot{z}, \dot{z}\rangle \leq 0 \text{ a.e. }, q \in z([0,1])\}.$$

The following results are proved in [13].

PROPOSITION 4.1. – *There exist a time function $T : \mathcal{M} \longrightarrow \mathbf{R}$ and a smooth function $\varphi : \mathcal{M} \longrightarrow \mathbf{R}$, satisfying the following properties:*
(a) $T(p) = 0$ *and ∇T is normalized, i.e. $\langle \nabla T(z), \nabla T(z)\rangle = -1$, for an $z \in \mathcal{M}$;*
(b) *γ is vertical, that is $\dot{\gamma}(s)$ is parallel to $\nabla T(\gamma(s))$, for any $s \in \mathbf{R}$;*
(c) *T is unbounded on γ and in particular*

$$\lim_{s \to +\infty} T(\gamma(s)) = +\infty;$$

(d) *$H^T(z)[\zeta, \zeta] \leq 0$, for any $z \in \mathcal{N}(p, \gamma, \Lambda)$ and for any causal vector $\zeta \in T_z\mathcal{M}$, where $H^T(z)[\cdot, \cdot]$ is the Hessian of T with respect to the Lorentzian structure of $\mathcal{M}$ (see [21] for the definition of the Hessian);*
(e) *$\langle \nabla T, \nabla\varphi\rangle = 0$ on a neighborhood of $\partial\Lambda$;*
(f) *$\langle \nabla T(z), \nabla\varphi(z)\rangle \geq 0$, for any $z \in \mathcal{N}(p, \gamma, \Lambda)$;*

We fix the time function T found in the proposition above. We introduce an auxiliary Riemannian structure on $\mathcal{M}$ that will be systematically used throughout the rest of the paper. The Riemannian metric, that will be denoted by $\langle \cdot, \cdot \rangle_{(R)}$, is defined by the following formula:

$$\langle \zeta, \zeta \rangle_{(R)} = \langle \zeta, \zeta \rangle + 2\langle \nabla T(z), \zeta\rangle^2, \tag{4.2}$$

for any $z \in \mathcal{M}$ and $\zeta \in T_z\mathcal{M}$. Observe that (4.2) clearly defines a smooth bilinear form on $T_z\mathcal{M}$; the (strict) positivity of $\langle \cdot, \cdot \rangle_{(R)}$ follows easily from the wrong way Schwartz's inequality. We denote by $\|\cdot\|_R$ the norm on $T_z\mathcal{M}$ induced by $\langle \cdot, \cdot \rangle_{(R)}$ and by $\nabla_s^{(R)}$ the covariant derivative induced by the Levi–Civita connection of $\langle \cdot, \cdot \rangle_{(R)}$. We introduce the following Riemannian structure on $\Omega^{1,2}_{p,\gamma}$,

$$\langle \zeta, \zeta \rangle_1 = \int_0^1 \langle \nabla_s^{(R)} \zeta, \nabla_s^{(R)} \zeta \rangle_{(R)} ds, \tag{4.3}$$

for any $z \in \Omega^{1,2}_{p,\gamma}$ and $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$.

Now consider the functional $Q: \Omega^{1,2}_{p,\gamma} \longrightarrow \mathbf{R}$ defined as

$$Q(z) = \int_0^1 \langle \nabla T(z), \dot{z} \rangle^2 ds. \tag{4.4}$$

We have mentioned in section 3 that $\mathcal{L}^+_{p,\gamma}$ is not a smooth manifold (see (1.4)). In spite of this, we introduce now a notion of critical point of Q on $\mathcal{L}^+_{p,\gamma}$.

DEFINITION 4.2. – *A curve $z \in \mathcal{L}^+_{p,\gamma}$ is called a critical point of Q if z is a pregeodesic in Λ parameterized by $\langle \dot{z}(s), \nabla T(z(s)) \rangle$ constant (and different from 0). A real number c is called critical value of Q on $\mathcal{L}^+_{p,\gamma}$ if there exists a critical point z of on $\mathcal{L}^+_{p,\gamma}$, such that $Q(z) = c$. A real number c which is not a critical value of Q is called regular value.*

Remark 4.3. – It is not difficult to verify that there is a bijection between the future pointing lightlike geodesics joining p and γ and the critical points of Q (cf. also [13]).

We set

$$\mathcal{C} = \{ z \in \mathcal{L}^+_{p,\gamma} : z \text{ is a critical point of } Q \}. \tag{4.5}$$

To develop a Morse Theory for light rays, the following notion will be useful.

DEFINITION 4.4. – *Let z be a C^1-curve in $\mathcal{L}^+_{p,\gamma}$ such that $\dot{z}(s) \neq 0$ for any $s \in [0,1]$. A tangent vector $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$ is called admissible variation if ζ is C^1 and*

$$\langle \dot{z}(s), \nabla_s \zeta \rangle = 0, \qquad \text{for any } s \in [0,1]\ . \tag{4.6}$$

Remark 4.5. – Note that by the variational principle proved in [2] and the regularity results in [13], z is a critical point of Q on $\mathcal{L}^+_{p,\gamma}$, if and only if z is of class C^1, $\dot{z}(s) \neq 0$ for any s and $Q'(z)[\zeta] = 0$ for any admissible variation ζ. Here $Q'(z)[\zeta]$ denotes the Gateaux derivative along the direction ζ, whose existence is due to the property $\dot{z}(s) \neq 0$ for any $s \in [0,1]$. Moreover, it is not difficult to see that z is a critical point of Q if and only if $\zeta(1) = 0$ for any admissible variation (cf. [2]).

PROPOSITION 4.6. – *Let z be a critical point of Q on $\mathcal{L}^+_{p,\gamma}$ and $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$ is of class C^1. Then ζ is an admissible variation if and only if*

$$\langle \dot{z}(s), \zeta(s) \rangle = 0, \qquad \text{for any } s \in [0,1]\ . \tag{4.7}$$

Proof. – *Suppose that ζ satisfies (4.7). Since z is a pregeodesic, $\nabla_s \dot{z} = \lambda(s)\dot{z}$ for some real continuous map $\lambda(s)$. Then differentiating in (4.7) gives:*

$$0 = \lambda(s)\langle \dot{z}(s), \zeta \rangle + \langle \dot{z}(s), \nabla_s \zeta \rangle = \langle \dot{z}(s), \nabla_s \zeta \rangle,$$

proving (4.6).

Now assume that ζ is an admissible variation and set $\alpha(s) = \langle \dot{z}, \zeta \rangle(s)$. Recalling that z is a pregeodesic, by (4.6),

$$\alpha'(s) = \lambda(s)\langle \dot{z}(s), \zeta \rangle + \langle \dot{z}(s), \nabla_s \zeta \rangle = \lambda(s)\alpha(s),$$

But $\alpha(0) = \langle \dot{z}(0), \zeta(0) \rangle = 0$. This implies $\alpha(s) = 0$ for any $s \in [0,1]$. □

The family of approximating manifolds is defined as follows. Fix $\epsilon > 0$, and consider the set

$$\mathcal{L}^+_{p,\gamma,\epsilon} = \{ z \in \Omega^{1,2}_{p,\gamma} : \langle \dot{z}, \dot{z} \rangle = -\epsilon^2 \text{ a.e., } T(z(\cdot)) \text{ is strictly increasing } \}. \tag{4.8}$$

Remark 4.7. – As shown in [13], if $\mathcal{L}^+_{p,\gamma}$ is nonempty, then $\mathcal{L}^+_{p,\gamma,\epsilon}$ is nonempty for ϵ small enough.

PROPOSITION 4.8. – *Assume that $\mathcal{L}^+_{p,\gamma,\epsilon}$ is nonempty, then $\mathcal{L}^+_{p,\gamma,\epsilon}$ is a C^1 submanifold of $\Omega^{1,2}_{p,\gamma}$. For any $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$ the tangent space is given by*

$$T_z\mathcal{L}^+_{p,\gamma,\epsilon} = \{ \zeta \in T_z\Omega^{1,2}_{p,\gamma} : \langle \dot{z}, \nabla_s \zeta \rangle = 0 \text{ a.e. } \}, \tag{4.9}$$

The proof of Proposition 4.8 is contained in [13]. We give now a sketch of the proof. Consider the map $\Psi_\epsilon : \Omega^{1,2}_{p,\gamma} \longrightarrow L^2([0,1], \mathbf{R})$ defined as

$$\Psi_\epsilon(z) = \sqrt{2}\langle \nabla T(z), \dot{z} \rangle - \sqrt{\epsilon^2 + \langle \dot{z}, \dot{z} \rangle + 2\langle \nabla T(z), \dot{z} \rangle^2}.$$

Ψ_ϵ is a map of class C^1 map (see [13]) and the differential $\Psi'_\epsilon: T_z\Omega^{1,2}_{p,\gamma} \longrightarrow L^2([0,1],\mathbf{R})$ is given by

$$\Psi'_\epsilon(z)[\zeta] = \sqrt{2}\big(\langle H^T(z)\zeta, \dot z\rangle + \langle \nabla T(z), \nabla_s\zeta\rangle\big) - \frac{\langle \dot z, \nabla_s\zeta\rangle + 2\langle \nabla T(z), \dot z\rangle\big(\langle H^T(z)\zeta, \dot z\rangle + \langle \nabla T(z), \nabla_s\zeta\rangle\big)}{\sqrt{\epsilon^2 + \langle \dot z, \dot z\rangle + 2\langle \nabla T(z), \dot z\rangle^2}}.$$

It is easy to see that $\mathcal{L}^+_{p,\gamma,\epsilon} = \Psi_\epsilon^{-1}(0)$. To show that $\mathcal{L}^+_{p,\gamma,\epsilon}$ is a C^1 submanifold of $\Omega^{1,2}_{p,\gamma}$, it suffices to show that for any $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$, $\Psi'_\epsilon(z)$ is surjective. Towards this goal, we fix $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$ and $\phi \in L^2([0,1],\mathbf{R})$. Since $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$, we only need to find $\zeta \in T_z\Omega^{1,2}_{p,\gamma}$ satisfying

$$\langle \dot z, \nabla_s\zeta\rangle = \sqrt{2}\langle \nabla T(z), \dot z\rangle\phi.$$

We look for a solution of the form

$$\zeta = \mu(s)\nabla T(z(s)). \tag{4.10}$$

The condition for such a ζ to belong to $T_z\Omega^{1,2}_{p,\gamma}$ is $\mu(0) = 0$. Equation (4.10) becomes

$$\langle \dot z, \nabla_s\zeta\rangle = \mu'\langle \nabla T(z), \dot z\rangle + \mu\langle H^T(z)\dot z, \dot z\rangle = \sqrt{2}\langle \nabla T(z), \dot z\rangle\phi.$$

Since $\langle \nabla T(z), \dot z\rangle \geq \epsilon > 0$, we need just to prove that the Cauchy linear problem:

$$\begin{cases} \mu' = -\frac{\langle H^T(z)\dot z, \dot z\rangle}{\langle \nabla T(z), \dot z\rangle}\mu + \sqrt{2}\phi \\ \mu(0) = 0 \end{cases}$$

admits a solution $\mu \in H^{1,2}([0,1],\mathbf{R})$. This can be done exhibiting the exact solution μ. Finally the tangent space $T_z\mathcal{L}^+_{p,\gamma,\epsilon}$ is given by the kernel of $\Psi'_\epsilon(z)$. □

Consider now the functional Q on $\mathcal{L}^+_{p,\gamma,\epsilon}$. Unfortunately, because of the presence of the boundary $\partial\Lambda$, the sublevels

$$Q^c = \{z \in \mathcal{L}^+_{p,\gamma,\epsilon} : Q(z) \leq c\}$$

are not complete. For this reason we need to introduce a further approximation scheme. Let $\varphi(z)$ be the function found in Proposition 4.1. For any $\delta > 0$ consider the family of approximating functionals $Q_\delta: \Omega^{1,2}_{p,\gamma} \longrightarrow \mathbf{R}$,

$$Q_\delta(z) = Q(z) + \delta\int_0^1 \frac{1}{\varphi^2(z)}ds. \tag{4.11}$$

The functional Q_δ is of class C^2 on every local chart of $\mathcal{L}^+_{p,\gamma,\epsilon}$ (see [13]). The following proposition has been proved in [14].

PROPOSITION 4.9. – *Assume that* $\mathrm{L}_1) - \mathrm{L}_3)$ *hold. Then:*

(a) *the sublevels of Q_δ are complete metric subspaces of $\mathcal{L}^+_{p,\gamma,\epsilon}$.*
(b) *Q_δ satisfies the Palais-Smale compactness condition at any level $c \geq 0$;*

The critical points of Q on $\mathcal{L}^+_{p,\gamma}$ (cf. Definition 4.2) are related to the critical points of Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon}$ by the following basic proposition (cf. [14]).

PROPOSITION 4.10. – *Let $(\epsilon_m)_{m\in\mathbf{N}}$ and $(\delta_m)_{m\in\mathbf{N}}$ two infinitesimal sequences and let $(z_m)_{m\in\mathbf{N}}$ be a sequence of curves in $\Omega^{1,2}_{p,\gamma}$ such that:*

(i) $z_m \in \mathcal{L}^+_{p,\gamma,\epsilon_m}$, *for any* $m \in \mathbf{N}$;
(ii) z_m *is a critical point of* Q_{δ_m}, *for any* $m \in \mathbf{N}$;
(iii) $\sup_{m\in\mathbf{N}} \tau_{p,\gamma}(z_m) < +\infty$.

Then the sequence $(z_m)_{m\in\mathbf{N}}$ contains a subsequence strongly converging in the topology of $\Omega^{1,2}_{p,\gamma}$ to a critical point of Q on $\mathcal{L}^+_{p,\gamma}$.

In the next proposition we recall some results on the existence of transition functions between $\mathcal{L}^+_{p,\gamma}$ and $\mathcal{L}^+_{p,\gamma,\epsilon}$ (for the proof see [13]).

PROPOSITION 4.11. – *Assume that* $\mathrm{L}_3)$ *of* Theorem 1.7 *holds and let $c > \inf_{\mathcal{L}^+_{p,\gamma}} Q$. Then there exist a positive number $\epsilon_0 = \epsilon_0(c)$ such that for any $\epsilon \in]0, \epsilon_0]$ there exists two injective maps*

$$\phi_\epsilon \colon Q^c \cap \mathcal{L}^+_{p,\gamma} \longrightarrow \mathcal{L}^+_{p,\gamma,\epsilon}$$
$$\psi_\epsilon \colon \mathcal{L}^+_{p,\gamma,\epsilon} \longrightarrow \mathcal{L}^+_{p,\gamma},$$

satisfying the following properties:

(1) *ϕ_ϵ and ψ_ϵ are continuous;*
(2) *For any $z \in \mathcal{L}^+_{p,\gamma,\epsilon}$ such that $Q(\psi_\epsilon(z)) \leq c+1$, it is $\psi_\epsilon(\phi_\epsilon(z)) = z$;*
(3) *for every $z \in Q^c \cap \mathcal{L}^+_{p,\gamma}$, it is $\psi_\epsilon(\phi_\epsilon(z)) = z$;*
(4) *there exists a positive constant $M = M(c)$ such that*

$$d_1(\phi_\epsilon(z), z) \leq M\epsilon,$$

where d_1 is the metric induced by the Riemannian structure (4.3) *on $\Omega^{1,2}_{p,\gamma}$. In particular*

$$\lim_{\epsilon\to 0} \phi_\epsilon(z) = z,$$

for any $z \in Q^c \cap \mathcal{L}^+_{p,\gamma}$.

5. THE HESSIAN OF Q AT THE CRITICAL POINTS

In this section we shall define and evaluate the Hessian $H^Q(z)$ of a critical point of the functional Q on $\mathcal{L}^+_{p,\gamma}$. First we need the following lemma.

LEMMA 5.1. – *Let z be a critical point of Q on $\mathcal{L}^+_{p,\gamma}$ and let ζ be an admissible variation of class C^2 for z. Then there exists a C^2 surface $\mu\colon]-\sigma,\sigma[\times[0,1] \longrightarrow \mathcal{M}$, satisfying the following properties:*

(a) *For any $r \in]-\sigma,\sigma[$, $\mu(r,\cdot) \in C^2(p,\gamma,\Lambda)$, where $C^2(p,\gamma,\Lambda)$ is the set of the curves of class $C^2([0,1],\Lambda)$ joining p and γ;*

(b) *$\mu(0,s) = z(s)$, $\mu_r(0,s) = \zeta(s)$, where μ_r denote the partial derivative of μ with respect to r;*

(c) *$\langle \mu_s, \mu_r\rangle(s) = 0$, for any $r \in]-\sigma,\sigma[, s \in [0,1]$;*

(d) *$\mu_r(r,1)$ is parallel to $\dot\gamma(\tau_{p,\gamma}(\mu(r,1)))$;*

(e) *$\mu_s(r,s) \neq 0$ for any $r \in]-\sigma,\sigma[, s \in [0,1]$.*

μ is called admissible surface related to the admissible variation ζ.

Proof. – Let $C^2(p,\gamma)$ the space of the C^2 curves $z\colon[0,1] \longrightarrow \Lambda$ joining p and γ. It is well known that it has a structure of Banach manifold. For any $z \in C^2(p,\gamma)$ the tangent space is given by

$$T_zC^2(p,\gamma) = \{\zeta \in C^2([0,1],T\mathcal{M}) : \\ \zeta \text{ is a vector field along } z\ , \zeta(0) = 0, \zeta(1) \parallel \dot\gamma(z(1))\}.$$

Now consider the space

$$\hat{\mathcal{L}}^{2,+}_{p,\gamma} = \{z \in C^2(p,\gamma) : \langle \dot z, \dot z\rangle = 0, \langle \nabla T(z), \dot z\rangle > 0\}.$$

The space $\hat{\mathcal{L}}^{2,+}_{p,\gamma}$ is a submanifold of $C^2(p,\gamma)$: the submanifold of the future pointing lightlike C^2-curves joining p and γ. The proof is similar to that of Proposition 4.8, choosing as function

$$\psi(z) = \sqrt{2}\langle \nabla T(z), \dot z\rangle - \sqrt{\langle \dot z, \dot z\rangle + 2\langle \nabla T(z), \dot z\rangle^2}.$$

The tangent space at $z \in \hat{\mathcal{L}}^{2,+}_{p,\gamma}$ is given by

$$T_z\hat{\mathcal{L}}^{2,+}_{p,\gamma} = \{\zeta \in T_zC^2(p,\gamma) : \langle \dot z, \nabla_s\zeta\rangle = 0\},$$

that is $T_z\mathcal{L}^{2,+}_{p,\gamma}$ is the set of the admissible variations for z (cf. Definition 4.4). Then the results follows by the Implicit Function Theorem. Indeed for any admissible variation ζ, an admissible surface represents a curve on the manifold $\hat{\mathcal{L}}^{2,+}_{p,\gamma}$, having tangent vector ζ for $r = 0$. □

We define the Hessian of Q at z along an admissible variation ζ as

$$H^Q(z)[\zeta,\zeta] = \frac{d^2}{dr^2}(Q(\mu(r,\cdot)))_{|r=0}, \tag{5.1}$$

where $\mu(r,s)$ is an admissible C^2-surface for ζ. We will show in the next proposition that $H^Q(z)[\zeta,\zeta]$ does not depend on μ

Remark 5.2. – Notice that on $\mathcal{L}^+_{p,\gamma}$,

$$Q(z) = \int_0^1 \langle \dot z, \nabla T(z)\rangle^2 ds = \int_0^1 \left(\langle \dot z, \dot z\rangle + \langle \dot z, \nabla T(z)\rangle^2\right) ds \equiv J(z). \tag{5.2}$$

Proposition 5.3. – *Let z be a critical point of Q on $\mathcal{L}^+_{p,\gamma}$ and ζ an admissible variation. Then:*

$$\begin{aligned} &H^Q(z)[\zeta,\zeta] = \\ &\int_0^1 c(s)[\langle D_s\zeta, D_s\zeta\rangle - \langle R(\zeta,\dot z)\dot z,\zeta\rangle]\, ds \\ &+ \int_0^1 \left(\langle \nabla T(z), D_s\zeta\rangle + \langle H^T(z)\dot z,\zeta\rangle\right)^2 ds \\ &= \int_0^1 c(s)[\langle D_s\zeta, D_s\zeta\rangle - \langle R(\zeta,\dot z)\dot z,\zeta\rangle]\, ds + \int_0^1 \dot\tau^2 ds, \end{aligned} \tag{5.3}$$

where $\tau = \langle \nabla T(z), \zeta\rangle$, R is the curvature tensor of the Lorentzian metric of $\mathcal{M}$, H^T is the Hessian of T with respect to the Lorentzian metric on $\mathcal{M}$, and $c(s)$ is the solution of the Cauchy problem

$$\begin{cases} \nabla_s(c(s)\dot z) = 0 \\ c(1) = 1 \end{cases} \tag{5.4}$$

Proof. – Let $z(r,s) \colon]-\sigma,\sigma[\times[0,1] \longrightarrow \mathcal{M}$ be an admissible surface for ζ and J the functional defined at Remark 5.2. By (5.2),

$$Q(z(r,\cdot)) = J(z(r,\cdot)),$$

then

$$H^Q(z)[\zeta,\zeta] = \frac{\partial^2 J(z(r,\cdot))}{\partial^2 r}_{|r=0}.$$

Now, let

$$t(r,s) = \int_0^s \langle \dot z(r,\theta), \nabla T(z(r,\theta))\rangle d\theta,$$

since $T(p) = 0$,

$$t(r,s) = T(z(r,s)).$$

By the above position we have

$$h(r) \equiv J(z(r,\cdot)) = \int_0^1 \left[\langle \partial_s z, \partial_s z\rangle + (\partial_s t)^2\right] ds \ .$$

Since J is a smooth functional on $\Omega^{1,2}_{p,\gamma}$ and $z(r,s)$ is smooth, the function h is smooth and

$$H^Q(z)[\zeta,\zeta] = h''(0) \ . \tag{5.5}$$

By well known formulas in semiriemannian geometry,

$$h'(r) = 2\int_0^1 \left[\langle D_r\partial_s z, \partial_s z\rangle + \partial_s t \partial_r \partial_s t\right] ds \ .$$

Moreover, since $D_r\partial_s = D_s\partial_r$ (see [26]) and $\partial_s\partial_r = \partial_r\partial_s$,

$$h'(r) = 2\int_0^1 \left[\langle D_s\partial_r z, \partial_s z\rangle + \partial_s t \partial_s \partial_r t\right] ds \ .$$

Integrating by parts gives:

$$\begin{aligned}\frac{1}{2}h'(r) = &- \int_0^1 \langle D_s\partial_s z, \partial_r z\rangle \, ds - \int_0^1 \partial_s^2 t \partial_r t \, ds \\ &+ \langle \partial_r z(r,1), \partial_s z(r,1)\rangle - \langle \partial_r z(r,0), \partial_s z(r,0)\rangle \\ &+ \partial_r t(r,1)\partial_s t(r,1) - \partial_r t(r,0)\partial_s t(r,0) \ . \end{aligned} \tag{5.6}$$

Since $z(r,0) = p$ and $t(r,0) = 0$ for any $r \in]-\sigma,\sigma[$, we have:

$$\partial_r t(r,0) = 0 \ , \quad \partial_r z(r,0) = 0 \ , \tag{5.7}$$

hence

$$\begin{aligned}&\langle \partial_r z(r,1), \partial_s z(r,1)\rangle - \langle \partial_r z(r,0), \partial_s z(r,0)\rangle \\ &+ \partial_r t(r,1)\partial_s t(r,1) - \partial_r t(r,0)\partial_s t(r,0) \\ &= \langle \partial_r z(r,1), \partial_s z(r,1)\rangle + \partial_r t(r,1)\partial_s t(r,1).\end{aligned}$$

Moreover, since $\partial_r z(r,1)$ is parallel to $\dot\gamma$ and γ is vertical (cf. Proposition 4.1):

$$\partial_r z = -\langle \partial_r z, \nabla T(z)\rangle \nabla T.$$

Then

$$\langle \partial_r z, \partial_s z\rangle(1) = -\langle \partial_r z, \nabla T(z)\rangle\langle \partial_s z, \nabla T(z)\rangle(1).$$

On the other hand, $t(r,s) = T(r,s)$, hence

$$\partial_r t = \langle \nabla T(z), \partial_r z\rangle \quad \partial_s t = \langle \nabla T(z), \partial_s z\rangle,$$

therefore

$$\langle \partial_r z(r,1), \partial_s z(r,1)\rangle + \partial_r t(r,1)\partial_s t(r,1) = 0.$$

Then, (5.6) gives

$$\frac{1}{2}h'(r) = -\int_0^1 \langle D_s\partial_s z, \partial_r z\rangle\, ds - \int_0^1 \partial_s^2 t\langle \nabla T(z), \partial_r z\rangle\, ds, \tag{5.8}$$

because $\partial_r t = \langle \nabla T(z), \partial_r z\rangle$.

Now, let $Y(r,s)$ the vector field on $z(r,s)$ obtained by the parallel transport along $z(r,\cdot)$ of $\partial_s z(r,1)$, that is $Y(r,s)$ satisfies

$$\begin{cases} D_s Y(r,s) = 0 \\ Y(r,1) = \partial_s z(r,1). \end{cases} \tag{5.9}$$

By regularity results on the solution of linear systems of differential equations, we have that $Y(r,s)$ is a smooth vector field on $z(r,s)$.

Analogously, let $W(r,s)$ be the smooth vector field on $z(r,s)$ satisfying

$$\begin{cases} D_s W(r,s) = \partial_s^2 t(r,s)\nabla T(z(r,s)) \\ W(r,1) = 0. \end{cases} \tag{5.10}$$

Integrating by parts in (5.8) and using (5.9) and (5.10) gives:

$$\frac{1}{2}h'(r) = \int_0^1 \langle \partial_s z + W - Y, D_s\partial_r z\rangle\, ds\ . \tag{5.11}$$

Indeed,

$$\frac{1}{2}h'(r) = -\int_0^1 \langle D_s\partial_s z - D_s Y + D_s W, \partial_r z\rangle\, ds\ ,$$

and integrating by parts we get (5.11) because the boundary terms are null. Hence, setting

$$\Delta(r,s) = \partial_s z(r,s) + W(r,s) - Y(r,s)\ , \tag{5.12}$$

we obtain

$$\frac{1}{2}h'(r) = \int_0^1 \langle \Delta, D_s \partial_r z \rangle \; ds \; .$$

Differentiating again gives

$$\frac{1}{2}h''(r) = \int_0^1 \langle D_r \Delta, D_s \partial_r z \rangle \; ds \; + \int_0^1 \langle \Delta, D_r D_s \partial_r z \rangle \; ds \; . \tag{5.13}$$

By (5.7), integrating by parts the first integral in (5.13) gives

$$\int_0^1 \langle D_r \Delta, D_s \partial_r z \rangle \; ds \; = - \int_0^1 \langle D_s D_r \Delta, \partial_r z \rangle \; ds \; + \langle D_r \Delta(r,1), \partial_r z(r,1) \rangle \; . \tag{5.14}$$

By the well known formula in semiriemannian geometry (see [21, p.123])

$$D_s D_r \Delta = D_r D_s \Delta + R(\partial_s z, \partial_r z)\Delta, \tag{5.15}$$

we have:

$$\int_0^1 \langle D_r \Delta, D_s \partial_r z \rangle \; ds \; =$$

$$- \int_0^1 \langle D_r D_s \Delta + R(\partial_s z, \partial_r z)\Delta, \partial_r z \rangle \; ds + \langle D_r \Delta(r,1), \partial_r z(r,1) \rangle \; . \tag{5.16}$$

Now consider the second integral in (5.13). From (5.15), we get:

$$\int_0^1 \langle \Delta, D_r D_s \partial_r z \rangle \; ds = \int_0^1 \langle \Delta, D_s D_r \partial_r z + R(\partial_r z, \partial_s z)\partial_r z \rangle \; ds \; . \tag{5.17}$$

Combining (5.16) and (5.17) gives:

$$\frac{1}{2}h''(r) = \int_0^1 \langle \Delta, D_s D_r \partial_r z] \; ds$$

$$- \int_0^1 \langle D_r D_s \Delta, \partial_r z \rangle \; ds + \langle D_r \Delta(r,1), \partial_r z(r,1) \rangle \; . \tag{5.18}$$

Indeed, since of the symmetry properties of the curvature tensor (see [26, p.75]),

$$\langle R(\partial_s z, \partial_r z)\Delta, \partial_r z \rangle = \langle R(\partial_r z, \partial_s z)\partial_r z, \Delta \rangle \; .$$

We have to evaluate $h''(0)$. The boundary term in (5.18) is null for $r = 0$. Indeed, $\partial_r z(0,1) = \zeta(1) = 0$, because z is a critical point of Q (cf. Remark 4.5). Moreover by (5.9), (5.10) and (5.12),

$$D_s\Delta = D_s\partial_s z + \partial_s^2 t \cdot \nabla T(z) ,$$

hence

$$\frac{1}{2}H^Q(z)[\zeta,\zeta] = \left(\int_0^1 \langle -D_r D_s\partial_s z - D_r(\partial_s^2 t\nabla T(z)), \partial_r z\rangle \, ds\right)\Big|_{r=0} + \left(\int_0^1 \langle \Delta, D_s D_r \partial_r z\rangle \, ds\right)\Big|_{r=0} . \tag{5.19}$$

Again (5.15) gives

$$-\int_0^1 \langle D_r D_s \partial_s z, \partial_r z\rangle \, ds = -\int_0^1 \langle D_s D_r \partial_s z + R(\partial_r z, \partial_s z)\partial_s z, \partial_r z\rangle \, ds .$$

Integrating by parts in the integral $-\int_0^1 \langle D_s D_r \partial_s z, \partial_r z\rangle \, ds$, since $D_r\partial_s z = D_s\partial_r z$, $\partial_r z(0,0) = 0$ and $\partial_r z(0,1) = 0$ we obtain at $r = 0$:

$$-\int_0^1 \langle D_s D_r \partial_s z, \partial_r z\rangle \, ds = \int_0^1 [\langle D_s\zeta, D_s\zeta\rangle - \langle R(\zeta,\dot{z})\dot{z},\zeta\rangle] \, ds . \tag{5.20}$$

Now we evaluate $-\int_0^1 \langle D_r(\partial_s^2 t \cdot \nabla T(z)), \partial_r z\rangle \, ds$. We have

$$D_r(\partial_s^2 t \cdot \nabla T(z)) = \partial_s^2 t D_r(\nabla T) + (\partial_r \partial_s^2 t)\nabla T .$$

For $r = 0$, $\partial_s t = \langle \dot{z}, \nabla T\rangle$ is constant (since z is a critical point of Q), hence the first term does not give any contribute. About the second term, the Schwartz Theorem gives

$$-\int_0^1 \partial_r\partial_s^2 t \langle \nabla T, \partial_r z\rangle \, ds = -\int_0^1 \partial_r(\partial_s^2 t)\partial_r t \, ds = -\int_0^1 \partial_{srs}^3 t \partial_r t \, ds ,$$

and integrating by parts gives:

$$-\int_0^1 \partial_r\partial_s^2 t\langle \nabla T, \partial_r z\rangle \, ds =$$

$$-\partial_r t(r,1)\partial_r\partial_s t(r,1) + \partial_r t(r,0)\partial_r\partial_s t(r,0) + \int_0^1 (\partial_s\partial_r t)^2 \, ds .$$

Since $\zeta(0) = 0$ and $\zeta(1) = 0$ (see Remark 4.5), evaluating at $r = 0$ gives:

$$-\int_0^1 \partial_r \partial_s^2 t \langle \nabla T, \partial_r z\rangle \, ds = \int_0^1 \left(\langle \nabla T(z), \nabla_s \zeta\rangle + \langle H^T(z)\dot{z}, \zeta\rangle\right)^2 ds, \tag{5.21}$$

because

$$(\partial_s \partial_r t)_{r=0} = \partial_s(\langle \nabla T, \zeta\rangle) = \langle \nabla T(z), \nabla_s \zeta\rangle + \langle H^T(z)\dot{z}, \zeta\rangle.$$

Finally it remains to evaluate

$$\left(\int_0^1 \langle \Delta, D_s D_r \partial_r z\rangle \, ds\right)_{\big| r=0} .$$

Since $z(r, \cdot)$ is a lightlike curve for any r, $\langle \partial_s z, \partial_s z\rangle = 0$. Differentiating with respect to r gives:

$$0 = \langle D_r \partial_s z, \partial_s z\rangle = \langle D_s \partial_r z, \partial_s z\rangle \ .$$

By (5.15), differentiating again gives:

$$0 = \langle D_s \partial_r z, D_r \partial_s z\rangle + \langle D_r D_s \partial_r z, \partial_s z\rangle =$$
$$\langle D_s \partial_r z, D_s \partial_r z\rangle + \langle D_s D_r \partial_r z, \partial_s z\rangle + \langle R(\partial_r z, \partial_s z)\partial_r z, \partial_s z\rangle \ .$$

Taking the limit as $r \to 0$ and using the symmetry properties of the curvature tensor R gives:

$$\langle \dot{z}, \lim_{r\to 0} D_s D_r \partial_r z\rangle = -\langle D_s \zeta, D_s \zeta\rangle + \langle R(\zeta, \dot{z})\dot{z}, \zeta\rangle \ . \tag{5.22}$$

Now, (5.12) gives:

$$\int_0^1 \langle \Delta, D_s D_r \partial_r z\rangle \, ds = \int_0^1 \langle \partial_s z + W - Y, D_s D_r \partial_r z\rangle \, ds \ . \tag{5.23}$$

Integrating by parts gives:

$$\int_0^1 \langle W, D_s D_r \partial_r z\rangle \, ds \ = -\int_0^1 \langle D_s W, D_r \partial_r z\rangle \, ds \ ,$$

because $W(r, 1) = 0$ and $\partial_r z(r, 0) = 0$ for any r (since $z(r, 0) = p$ for any r), giving $D_r \partial_r z(r, 0) = 0$.

Moreover by (5.10), $D_s W = \partial_s^2 t(r,s)\nabla T(z) \longrightarrow 0$ uniformly as $r \to 0$, because if $r = 0$ we have $(\partial_s^2 t) = \partial_s(\langle \dot{z}, \nabla T(z)\rangle) = 0$. Hence, whenever $r = 0$,

$$\int_0^1 \langle W, D_s D_r \partial_r z\rangle \, ds = 0.$$

Then, taking the limit as $r \to 0$ in (5.23) gives:

$$\lim_{r\to 0} \int_0^1 \langle \Delta, D_s D_r \, \partial_r z\rangle \, \mathrm{d}s = \int_0^1 \langle \dot{z} - Y(s,0), \lim_{r\to 0} D_s D_r \partial_r z\rangle \, ds \ . \tag{5.24}$$

Since $Y(s,0)$ is the parallel transport of $\dot{z}(1)$ along z (cf. (5.9)), $Y(s,0) = c(s)\dot{z}(s)$ (notice that $c(s) > 0$ for any $s \in [0,1]$). Collecting (5.22)-(5.24) gives

$$\left(\int_0^1 \langle \Delta, D_s D_r \partial_r z\rangle \, ds\right)_{r=0} = \\ - \int_0^1 (1 - c(s))[\langle D_s\zeta, D_s\zeta\rangle + \langle R(\zeta,\dot{z})\dot{z}, \zeta\rangle] \, ds.$$

Finally, combining (5.19)-(5.21), (5.24) and (5.5) gives (5.3) and the proof is complete. □

Remark 5.4. – From (5.3) we immediately deduce that $H^Q(z)$ is a quadratic form which does not depend on the choice of the admissible surface $z(r,s)$. Moreover $H^Q(z)$ is continuous with respect to the topology of $T_z\Omega_{p,\gamma}^{1,2}$. Since smooth vector fields are dense in $T_z\Omega_{p,\gamma}^{1,2}$, $H^Q(z)$ can be extended to a continuous quadratic form on the Hilbert space L_z of the admissible variations:

$$L_z = \{\zeta \in T_z\Omega_{p,\gamma}^{1,2} : \langle \dot{z}, \zeta\rangle = 0\}. \tag{5.25}$$

Moreover by polarization, it is defined the associate bilinear form $H^Q(z): L_z \times L_z \longrightarrow \mathbf{R}$ setting

$$H^Q(z)[\zeta,\zeta'] = 2\int_0^1 c(s)[\langle D_s\zeta, D_s\zeta'\rangle - \langle R(\zeta,\dot{z})\dot{z}, \zeta'\rangle] \, ds + 2\int_0^1 \dot{\tau}\dot{\tau}' \, ds \ , \tag{5.26}$$

where $\tau = \langle \nabla T(z), \zeta\rangle$ and $\tau' = \langle \nabla T(z), \zeta'\rangle$.

We define the Morse index $m(z,Q)$ as the maximal dimension of a subspace of L_z where $H^Q(z)$ is negative definite. In order to prove that $m(z,Q)$ is finite we can use the following

PROPOSITION 5.5. – *Let z be a critical point of Q and $l(z): L_z \longrightarrow L_z$ be the linear operator associated to the bilinear form $H^Q(z)$. Then there exist two continuous linear operators $A(z)$, $K(z)$ on L_z such that $A(z)$ is positive definite, $K(z)$ is compact and $l(z) = A(z) + K(z)$.*

Proof. – A simple argument shows that there exists a constant $\nu = \nu(z) > 0$, such that

$$c(s)\langle w, w\rangle + \langle \nabla T(z(s)), w\rangle^2 \geq \nu(z(s))\langle w, w\rangle_{(R)},$$

for any vector field $w(s)$ along $z(s)$, such that $\langle w(s), \dot{z}\rangle = 0$. Then, for any admissible variation ζ,

$$c(s)\langle D_s\zeta, D_s\zeta\rangle + \langle \nabla T(z(s)), D_s\zeta\rangle^2 \geq \nu(z(s))\langle D_s\zeta, D_s\zeta\rangle_{(R)}, \quad (5.27)$$

because $\langle D_s\zeta, \dot{z}\rangle = 0$ for any s. Now, since

$$B(X,Y) = \nabla_X Y - \nabla_X^R Y \quad (5.28)$$

is a (1,2) tensor field, by (5.3), (5.27) and (5.28), there exists $A(z)$ such that

$$\langle A(z)\zeta, \zeta\rangle_1 = \int_0^1 \left[c(s)\langle D_s\zeta, D_s\zeta\rangle + \langle \nabla T(z(s)), D_s\zeta\rangle^2\right] ds.$$

Then the proof follows by the compact embedding of $H^{1,2}([0,1], \mathbf{R}^N)$ into $L^2([0,1], \mathbf{R}^N)$, $L^\infty([0,1], \mathbf{R}^N)$ and $H^{1/2,2}([0,1], \mathbf{R}^N)$ (cf. [1]). □

COROLLARY 5.6. – *Let z be a critical point of Q. Then the operator $l(z)$ associated to $H^Q(z)$ is a Fredholm operator of index 0. Moreover, the Morse index $m(z, Q)$ is finite.*

Proof. – By Proposition 5.5, $l(z)$ is a compact perturbation of an invertible, positive definite operator, so the proof follows from well known results in functional analysis. □

6. THE INDEX THEOREM

In this section we shall prove an extension to light rays of the classical Morse Index Theorem for Riemannian geodesics joining the point p with the curve γ. Such an extension relates the index of a lightlike geodesic joining p with γ to the Morse index of the corresponding critical point z of Q on $\mathcal{L}_{p,\gamma}$ (z is a pregeodesic reparameterized by $\langle \nabla T(z), \dot{z}\rangle$ constant).

THEOREM 6.1. – *Let z be a critical point of of Q on $\mathcal{L}_{p,\gamma}$ and let w be the lightlike geodesic joining p and γ such that z is the reparametrization of w with $\langle \nabla T(z), \dot{z} \rangle$ constant. Then*

$$\mu(w) = m(z, Q), \tag{6.1}$$

where $\mu(w)$ is the index of w (cf. Definition 1.5*).*

In order to prove Theorem 6.1 we extend the notion of conjugate point (see Definition 1.4) to s. For any $s \in]0,1]$ let

$$L_z(s) = \{\zeta \in L_z | \zeta = 0 \text{ in } [s,1]\} \ , \tag{6.2}$$

where L_z is defined by (5.25). Note that $L_z(s)$ is a closed subspace of L_z For any $s \in]0,1]$ let $H_s^Q(z)$ be the restriction of $H^Q(z)$ to $L_z(s)$. Then

$$H_s^Q(z)[\zeta,\zeta'] = 2\int_0^s c(r)[\langle D_s\zeta, D_s\zeta'\rangle - \langle R(\zeta,\dot{z})\dot{z},\zeta'\rangle]\, dr + 2\int_0^s \dot{\tau}\dot{\tau}'\, dr \ , \tag{6.3}$$

for any ζ,$\zeta' \in L_z(s)$, where $\tau = \langle \nabla T(z), \zeta \rangle$ and $\tau' = \langle \nabla T(z), \zeta' \rangle$.

Arguing as in the proof of Proposition 5.5, the linear operator associated to $H_s^Q(z)$ is a compact perturbation of a positive operator, and in particular it is a Fredholm operator of index 0.

DEFINITION 6.2. – *A point $z(s)$ is said conjugate to p (along z) if there exists a nonnull vector $\zeta \in L_z(s)$ such that for any $\zeta' \in L_z(s)$,*

$$H_s^Q(z)[\zeta,\zeta'] = 0 \ . \tag{6.4}$$

The multiplicity of the conjugate point $z(s)$ is the maximal number of linearly independent vector fields satisfying (6.4)*. The index $\mu(z)$ of the critical point z is the number of conjugate points $z(s)$, $s \in]0,1[$, counted with their multiplicity.*

Remark 6.3. – Let $l_z(s): L_z(s) \longrightarrow L_z(s)$ be the continuous linear operator associated to $H_s^Q(z)$: the solutions of (6.4) are exactly the elements of the kernel of $l_s(z)$. Then the multiplicity of a conjugate point $z(s)$ is finite, because $l_s(z)$ is a Fredholm operator.

Because of the structure of $H^Q(z)$, it is possible to prove an extension of the classical Morse Index Theorem for Riemannian geodesics for a critical point z of Q.

THEOREM 6.4. – *Let z be a critical point of Q, then*

$$m(z,Q) = \mu(z) \ . \tag{6.5}$$

The proof of Theorem 6.4 is the same as for the classical Morse Index Theorem (cf. [25] and [20,22] for a functional proof).

By Theorem 6.4, in order to prove Theorem 6.1 it suffices to prove the following proposition.

PROPOSITION 6.5. – *Let z be a critical point of Q and w the reparametrization of z which is a lightlike geodesic joining p and γ.*

Then a point $z(s)$, $s\in]0,1]$, is conjugate to p if and only if $w(s)$ is conjugate to p along w. Moreover the multiplicities of $z(s)$ and $w(s)$ are equal. Consequently:

$$\mu(z)=\mu(w)\ . \tag{6.6}$$

Proof. – It suffices to prove the proposition for $s=1$. We first need to write the equation satisfied by the solutions of (6.4).

Let $\zeta\in L_z$ be a solution of (6.4) (with $s=1$). For any $\zeta\in L_z$, integrating by parts gives

$$0=\int_0^1\langle D_s(c(s)D_s\zeta)+c(s)R(\zeta,\dot z)\dot z,\zeta'\rangle\ ds\ +\int_0^1\ddot\tau\tau'\ ds\ , \tag{6.7}$$

where $\tau=\langle\nabla T(z),\zeta\rangle$ and $\tau'=\langle\nabla T(z),\zeta'\rangle$.

Let $\bar\zeta$ be a vector field along z such that $\bar\zeta(0)=0$, $\bar\zeta(1)=0$. By Proposition 4.6,

$$\zeta'=\bar\zeta-\frac{\langle\bar\zeta,\dot z\rangle}{\langle\nabla T(z),\dot z\rangle}\nabla T(z)$$

is an admissible variation. Substituting ζ' in (6.7), since $\bar\zeta$ is arbitrary, we deduce that ζ satisfies

$$D_s(c(s)D_s\zeta)+c(s)R(\zeta,\dot z)\dot z-\ddot\tau\nabla T(z)+\lambda(s)\dot z=0\ , \tag{6.8}$$

where $\lambda(s)$ is given by

$$\lambda(s)=\langle D_s(c(s)D_s\zeta)+c(s)R(\zeta,\dot z)\dot z+\langle\nabla T(z),\frac{1}{\langle\nabla T(z),\dot z\rangle}\nabla T(z)\rangle\ddot\tau. \tag{6.9}$$

Now z is a pregeodesic. Then there exists a real continuous function $b(s)$ such that $D_s\dot z=b(s)\dot z$. Since ζ is an admissible variation, differentiating $\langle\dot z,D_s\zeta\rangle=0$ with respect to s gives

$$0=\langle D_s^2\zeta,\dot z\rangle+b(s)\langle D_s\zeta,\dot z\rangle=\langle D_s^2\zeta,\dot z\rangle\ . \tag{6.10}$$

Moreover, multiplying (6.8) by $\dot{z}$, since the curvature tensor R is antisymmetric and z is lightlike, by (4.10) we deduce

$$0 = -\langle \nabla T(z), \dot{z} \rangle \ddot{\tau} = E_z \ddot{\tau} ,$$

where E_z is a positive constant. Now, $\tau(0) = \tau(1) = 0$, because z is a critical point of Q, so $\zeta(1) = 0$. Then

$$\tau(s) = 0 \qquad \text{for any } s \in [0,1] \ . \tag{6.11}$$

Hence ζ satisfies (6.4) if and only if ζ satisfies the system

$$\begin{cases} c'(s) D_s \zeta + c(s) D_s^2 \zeta + c(s) R(\zeta, \dot{z}) \dot{z} + \lambda(s) \dot{z} = 0 \\ \langle \nabla T(z(s)), \zeta(s) \rangle = 0 \\ \zeta(0) = \zeta(1) = 0 \ . \end{cases} \tag{6.12}$$

We study now the relations between the Jacobi equation (1.5) and (6.8) (recalling that $D_s(c(s)\dot{z}(s)) \equiv 0$ and $c(1) = 1$). Let ζ be a solution of (6.12). Moreover, let $\varphi \colon [0,1] \longrightarrow [0,1]$ be the reparametrization such that $w(s) = z(\varphi(s))$ is a geodesic. Then, φ solves $c(\varphi)\varphi'' + c'(\varphi)(\varphi')^2 = 0$

Consider a vector field $\tilde{\zeta}$ along w of the form

$$\tilde{\zeta}(s) = \zeta(\varphi(s)) + a(s)\dot{w} , \tag{6.13}$$

with $a(s)$ smooth function to be determined in order that $\tilde{\zeta}$ is a Jacobi vector field along z with $\tilde{\zeta}(0) = \tilde{\zeta}(1) = 0$. Since w is a geodesic, differentiating (6.13) gives

$$\begin{aligned} D_s \tilde{\zeta} &= \varphi' D_s \zeta + a'(s) w \\ D_s^2 \tilde{\zeta} &= (\varphi')^2 D_s^2 \zeta + \varphi'' D_s \zeta + a''(s) \dot{w} \ . \end{aligned}$$

Moreover, since $\dot{w} = \varphi' \dot{z}$, by (6.12),

$$\begin{aligned} D_s^2 \tilde{\zeta} = (\varphi')^2 \Big(-\frac{\lambda(\varphi)}{c(\varphi)} \dot{z} - R(\zeta, \dot{z}) \dot{z} - \frac{c'(\varphi)}{c(\varphi)} D_s \zeta \Big) \\ + \varphi'' D_s \zeta + a''(s) \dot{w} \ . \end{aligned}$$

Finally the equation satisfied by φ gives:

$$D_s^2 \tilde{\zeta} = -\frac{\lambda(\varphi)}{c(\varphi)} \varphi' \dot{w} - R(\tilde{\zeta}, \dot{w}) \dot{w} + a''(s) \dot{w} \ .$$

Hence $\tilde{\zeta}$ is a Jacobi field along w with $\tilde{\zeta}(0) = \tilde{\zeta}(1) = 0$ if and only if $a(s)$ satisfies the boundary value problem

$$\begin{cases} a''(s) = \frac{\lambda(\varphi)}{c(\varphi)}\dot{\varphi} \\ a(0) = a(1) = 0 \ . \end{cases}$$

This problem has one and only one solution given by

$$a(s) = \int_0^s \left(\int_0^r \frac{\lambda(\varphi)}{c(\varphi)}\dot{\varphi}\, d\sigma \right) dr - \int_0^1 \left(\int_0^r \frac{\lambda(\varphi)}{c(\varphi)}\dot{\varphi}\, d\sigma \right) dr \ . \tag{6.14}$$

Then it is well defined the map $\zeta \to \tilde{\zeta}$ given by (6.13), between the finite dimensional vector space of solutions of (6.12) and the finite dimensional vector space of the Jacobi fields along w with null boundary condition. By (6.9), (6.13) and (6.14), such map is linear. We show now that it is injective. Indeed, if for some ζ

$$\zeta(\varphi(s)) + a(s)\dot{w} = 0 \ ,$$

there exists a smooth function $\mu(r)$ such that

$$\zeta(r) = \mu(r)\dot{z}(r) \ . \tag{6.15}$$

By (6.12), $\langle \nabla T(z), \dot{z}\rangle(s) = 0$ for any $s \in [0,1]$, hence

$$\mu(r)\langle \nabla T(z), \dot{z}\rangle = 0 \ .$$

Since $\langle \nabla T(z), \dot{z}\rangle \neq 0$, we obtain $\mu(r) \equiv 0$ and by (6.15), $\zeta = 0$. Hence we have shown that if $z(1)$ is a conjugate point to p along z, then $w(1)$ is a conjugate point to p along w, and the multiplicity of $z(1)$ is less than the multiplicity of $w(1)$. Since the result holds for any s, we get

$$\mu(z) \leq \mu(w) \ .$$

So it remains to prove $\mu(w) \leq \mu(z)$. Let $\psi(s)$ be the reparametrization such that $z(s) = w(\psi(s))$. Straightforward calculations show that ψ solves the differential differential equation

$$c(s)\psi''(s) + c'(s)\psi'(s) = 0 \ . \tag{6.16}$$

Let $\tilde{\zeta}$ a Jacobi vector field along w such that $\tilde{\zeta}(0) = \tilde{\zeta}(1) = 0$, we look for a field ζ of the form

$$\zeta(s) = \tilde{\zeta}(\psi(s)) + b(s)\dot{z}(s) \ , \tag{6.17}$$

where $b(s)$ is a smooth function to be determined and satisfying $b(0) = b(1) = 0$. By (6.12) $\langle \nabla T(z), \zeta \rangle(s) = 0$ for any $s \in [0,1]$, hence

$$0 = \langle \nabla T(z), \zeta \rangle = \langle \tilde{\zeta}(\psi) + b(s)\dot{z}, \nabla T(z) \rangle \ .$$

It follows that $b(s)$ is given by

$$b(s) = -\frac{\langle \nabla T(z), \tilde{\zeta}(\psi) \rangle}{\langle \nabla T(z), \zeta \rangle} \ . \tag{6.18}$$

Hence $b(s)$ is univocally determined and we have defined a map $\tilde{\zeta} \to \zeta$ between the finite dimensional vector space of the Jacobi fields along w (with null boundary conditions) and L_z. Such map is injective. Indeed if for some $\tilde{\zeta}$

$$\tilde{\zeta}(\psi(s)) + b(s)\dot{z}(s) = 0 \ ,$$

there exists a smooth function $\mu(r)$ such that $\tilde{\zeta} = \mu(r)\dot{w}$, so that $\tilde{\zeta}$ is proportional to $\dot{w}$. On the other hand it is well known that the only Jacobi field proportional to $\dot{w}$ with null boundary conditions is the null field, so $\tilde{\zeta} = 0$.

It remains to show that the image of the linear map (6.17) is contained in the set of the solutions of (6.12). Let $\tilde{\zeta}$ be a Jacobi vector field along w with null boundary conditions and let ζ be given by (6.17)–(6.18). Since z is a pregeodesic,

$$D_s\zeta = \psi' D_s\tilde{\zeta} + b'(s)\dot{z} + b(s)D_s\dot{z} = \dot{\psi}' D_s\tilde{\zeta} + b_1(s)\dot{z} \ ,$$

where $b_1(s)$ is a smooth function. Differentiating again gives:

$$D_s^2\zeta = \psi'' D_s\tilde{\zeta} + (\psi')^2 D_s^2\tilde{\zeta} + b_2(s)\dot{z} \ ,$$

where $b_2(s)$ is a smooth function. Hence, by (6.16) and the antisymmetric properties of the curvature tensor:

$$\begin{gathered} c'(s)D_s\zeta + c(s)D_s^2\zeta + c(s)R(\zeta,\dot{z})\dot{z} = \\ [c(s)\psi''(s) + c'(s)\psi'(s)]D_s\tilde{\zeta} + (\psi')^2 c(s)D_s^2\zeta \\ +c(s)R(\tilde{\zeta}(\psi) + b(s)\dot{z},\dot{z})\dot{z} + \lambda(s)\dot{z} + b_3(s)\dot{z} \ , \end{gathered}$$

where $b_3(s)$ is a smooth function. Finally, since $\tilde{\zeta}$ is a Jacobi field and $\dot{z} = \psi'\dot{w}$,

$$c'(s)D_s\zeta + c(s)D_s^2\zeta + c(s)R(\zeta,\dot{z})\dot{z} =$$
$$c(s)(\psi')^2\Big(D_s^2\tilde{\zeta} + R(\tilde{\zeta},\dot{w})\dot{w}\Big) + b_3\dot{z} = b_3\dot{z}\ .$$

Multiplying by the vector field $(1/\langle\nabla T(z),\dot{z}\rangle)\nabla T(z)$, from (6.9) we obtain $b_3(s) = -\lambda(s)$, so ζ satisfies (6.12).

Now, since the map (6.17) is injective, we have $\mu(w) \leq \tilde{\mu}(z)$ and the proof is complete. □

According to section 2 we say that Q is a Morse function if all its critical points are nondegenerate, that is the linear operator $l(z)\colon L_z \longrightarrow L_z$ associated to any critical point is an isomorphism. By Proposition 6.5 we obtain the following result.

COROLLARY 6.6. – *Q is a Morse function if and only if p and γ are nonconjugate.*

7. MORSE RELATIONS ON THE SUBLEVELS OF Q ON $\mathcal{L}^+_{p,\gamma}$

The main result of this section are the Morse Relations on the sublevels of Q.

THEOREM 7.1. – *Let $b > \inf\{Q(z), z \in \mathcal{L}^+_{p,\gamma}\}$ be a regular value of Q. Then, for any field $\mathcal{K}$ there exists a polynomial $S_b(r)$ with positive integer coefficients, such that*

$$\sum_{z\in\mathcal{C}\cap Q^b} r^{m(z,Q)} = \mathcal{P}(\mathcal{L}^+_{p,\gamma}\cap Q^b,\mathcal{K}) + (1+r)S_b(r)\ , \tag{7.1}$$

where $\mathcal{C}$ is defined by (4.5) *and $\mathcal{P}(\mathcal{L}^+_{p,\gamma}(\Lambda),\mathcal{K})$ is the Poincaré polynomial of $\mathcal{L}^+_{p,\gamma}\cap Q^b$ with coefficients in the field $\mathcal{K}$.*

In order to prove Theorem 7.1, some preliminary results are needed. From the apriori estimates of Proposition 4.10 we have immediately the following

LEMMA 7.2. – *There exist $\delta(b) > 0$, $\epsilon(b) > 0$ and $\rho(b) > 0$ such that for any $\delta \in]0,\delta(b)]$ and $\epsilon \in]0,\epsilon(b)]$,*

(a) *b is a regular value of Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon}(\Lambda)$;*

(b) *for any critical point z of Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon}(\Lambda)\cap Q^b_\delta$, $d_R(z(s),\partial\Lambda) \geq \rho(b)$, for any $s \in [0,1]$, where $d_R(\cdot,\cdot)$ is the distance on $\mathcal{M}$ induced by the metric* (4.2).

LEMMA 7.3. – *Let z be a critical point of Q on $\mathcal{L}^+_{p,\gamma}$, then there exists a neighborhood U_z of z in $\Omega^{1,2}_{p,\gamma}$, two positive numbers δ_z and ϵ_z and a continuous surface $\eta_z: [-\epsilon_z, \epsilon_z] \times [-\delta_z, \delta_z] \longrightarrow \mathcal{L}^+_{p,\gamma,\epsilon}$, such that:*

(1) $Q'_\delta(\eta_z(\epsilon, \delta)) = 0$ *on* $\mathcal{L}^+_{p,\gamma,\epsilon}$;

(2) *for any* $(\epsilon, \delta) \in [-\epsilon_z, \epsilon_z] \times [-\delta_z, \delta_z]$, $\eta_z(\epsilon, \delta))$ *is the only critical point of* Q_δ *in* $\mathcal{L}^+_{p,\gamma,\epsilon} \cap U_z$;

(3) $\eta_z(0, 0) = z$;

Proof. – If the space $\mathcal{L}^+_{p,\gamma}$ were a smooth submanifold of $H^{1,2}([0,1], \Lambda)$, the result could be obtained by applying the implicit function theorem to the map $H(\epsilon, \delta, y)$ given by the restriction of $Q'_\delta(y)$ to $T_y\mathcal{L}^+_{p,\gamma,\epsilon}$ on a $H^{1,2}$ neighborhood of $(0, 0, z)$. Unfortunately $\mathcal{L}^+_{p,\gamma}$ is nonsmooth, so we can not apply directly the implicit function theorem in $H^{1,2}([0,1], \Lambda)$.

We consider the sets $B^{2,+}_{p,\gamma}$ and $B^{2,+}_{p,\gamma,\epsilon}$ consisting respectively of the C^2 curves in $\mathcal{L}^+_{p,\gamma}$ and $\mathcal{L}^+_{p,\gamma,\epsilon}$. It is not difficult to prove that $B^{2,+}_{p,\gamma}$ and $B^{2,+}_{p,\gamma,\epsilon}$, equipped with the usual C^2-norm, are smooth Banach submanifolds of $H^{1,2}([0,1], \Lambda)$.

By the same arguments used to prove Proposition 4.11, one can show that (in a C^2-neighborhood U_z of z) there exists a smooth map $\psi(\epsilon, y)$ such that $B^{2,+}_{p,\gamma,\epsilon} = \{\psi(\epsilon, y) : y \in B^{2,+}_{p,\gamma} \cap U_z\}$ and $\psi(0, y) = y$ (for any ϵ sufficiently small).

Set $H(\epsilon, \delta, y)(\cdot) = Q'_\delta(\psi(\epsilon, y))\frac{\partial\psi}{\partial y}(\epsilon, y)[\cdot]$ Then the map H is defined on $]-\epsilon_0, \epsilon_0[\times]-\delta_0, \delta_0[\times(B^{2,+}_{p,\gamma} \cap V_z)$ and it is of class C^1. Moreover, since z is a critical point of Q, $H_y(0, 0, z)$ is the Hessian of Q on $\mathcal{L}^+_{p,\gamma}$, restricted to the vector fields of class C^2 along z.

Denote by $l(z)^{-1}$ the inverse of the isomorphism associated to $H^Q(z)$ (as defined by (5.1)). It is not difficult to show that $l(z)^{-1}$ maps the subspace of $L(z)$ consisting of C^1 curves into itself. Then the results follows arguing a s in the proof of the implicit function theorem in [7, p.148]. □

As proved in [14], the set of the critical points z of Q on $\mathcal{L}^+_{p,\gamma}$ with $Q(z) \leq b \in \mathbf{R}^+$ is compact. Then, by Lemma 7.3 the following result follows immediately.

LEMMA 7.4. – *The set $\mathcal{C}$ of the critical points of Q consists of isolated points. Moreover, for any $b \in \mathbf{R}^+$, $\mathcal{C} \cap Q^b$ is finite, and set of the critical values of Q on $\mathcal{L}^+_{p,\gamma}$ is discrete in* $\mathbf{R}$.

Moreover, the following proposition holds.

PROPOSITION 7.5. – *Let b be a regular value of Q on $\mathcal{L}^+_{p,\gamma}$ and let (cf. (4.5))*

$$\mathcal{C} \cap Q^b = \{z_1, \dots, z_k\} .$$

Then there exists positive numbers $\delta_0 = \delta_0(b)$, $\epsilon_0 = \epsilon_0(b)$ *and* $\rho_0 = \rho_0(b)$ *such that denoting by* $B(z, \rho_0)$ *the open ball in* $\Omega^{1,2}_{p,\gamma}$ *centered at* z *and of radius* ρ_0 *we have:*

(i) $B(z_i, \rho_0) \cap B(z_j, \rho_0) = \emptyset$, *for any* $i \neq j \in \{1, \ldots, k\}$;

(ii) *For any* $i \in \{1, \ldots, k\}$, $\delta \in]0, \delta_0]$ *and* $\epsilon \in]0, \epsilon_0[$, *there exists one and only one critical point* $z_{i,\epsilon,\delta}$ *of* Q_δ *contained in* $B(z_i, \rho_0)$ *and*

$$\{z_{1,\epsilon,\delta}, \ldots, z_{k,\epsilon,\delta}\} \text{ is the set of all the critical points of } Q_\delta \text{ on } \mathcal{L}^+_{p,\gamma,\epsilon} \cap Q^b_\delta. \tag{7.2}$$

Moreover $z_{i,\epsilon,\delta}$ *is nondegenerate and*

$$m(z_{i,\epsilon,\delta}, Q_\delta) = m(z_i, Q) . \tag{7.3}$$

Proof. – The existence and the uniqueness of the critical points $z_{i,\epsilon,\delta}$ (for ϵ and δ small enough) is a consequence of Lemmas 7.2–7.4. Finally, (7.3) is a consequence of the continuity of the Morse index with respect of the convergence of bilinear forms. □

By Proposition 7.5 it follows that, if ϵ and δ are small enough, all the critical points of Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon} \cap Q^b_\delta$ are nondegenerate. Since Q_δ satisfies the Palais-Smale condition (cf. Proposition 4.9) and it is bounded from below, applying Corollary 2.3 to Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon}$ gives the following

PROPOSITION 7.6. – *Let* $b > \inf\{Q(z), z \in \mathcal{L}^+_{p,\gamma}\}$ *be a regular value of* Q *and denote by* $\mathcal{C}_{\epsilon,\delta}$ *the set of the critical points of* Q_δ *on* $\mathcal{L}^+_{p,\gamma,\epsilon}$. *Then there exists* $\bar{\epsilon} > 0$, $\bar{\delta} > 0$ *such that for any* $\epsilon \in]0, \bar{\epsilon}]$, $\delta \in]0, \bar{\delta}]$ *and for any field* $\mathcal{K}$, *there exists a polynomial* $S_{\epsilon,\delta,b}(r)$ *such that*

$$\sum_{z \in \mathcal{C}_{\epsilon,\delta} \cap Q^b_\delta} r^{m(z,Q_\delta)} = \mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon} \cap Q^b_\delta; \mathcal{K}) + (1 + r) S_{\epsilon,\delta,b}(r) .$$

We recall now the notion of strong and weak deformation retract. Let X be a topological space, a subspace Y of X is a *strong deformation retract of* X if there exists a continuous map $H: [0, 1] \times X \longrightarrow X$, such that

(a) $H(0, x) = x$, for any $x \in X$;

(b) $H(1, x) \in Y$, for any $x \in X$;

(c) $H(t, y) = y$, for any $y \in Y$.

Y is a *weak deformation retract of* X if (a) and (b) holds, and (c) is replaced by

(c)' $H(t, y) \in Y$, for any $y \in Y$.

Now we are finally ready to prove Theorem 7.1.

Proof of Theorem 7.1. – By Proposition 7.5, if ϵ and δ are sufficiently small, we have:

$$\sum_{z \in \mathcal{C}_{\epsilon,\delta} \cap Q^b_\delta} r^{m(z,Q_\delta)} = \sum_{z \in \mathcal{C} \cap Q^b} r^{m(z,Q)} . \tag{7.4}$$

Then, by (7.4) and Proposition 7.6 we have just to prove that, if ϵ and δ are sufficiently small,

$$\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon} \cap Q^b_\delta; \mathcal{K}) = \mathcal{P}(\mathcal{L}^+_{p,\gamma} \cap Q^b; \mathcal{K}) . \tag{7.5}$$

Towards this goal let, for any $\rho > 0$,

$$\mathcal{L}^+_{p,\gamma,\epsilon}(\Lambda, \rho) = \{z \in \mathcal{L}^+_{p,\gamma,\epsilon} : \varphi(z(s)) \geq \rho, \quad \text{for any } s \in [0,1] \} .$$

Thanks to Lemma 7.2 and using a Lipschitz continuous partition of the unity on the Hilbert manifold $H^{1,2}([0,1], \Lambda)$, for any ρ sufficiently small it is possible to construct a locally Lipschitz continuous vector field $w(z) \in T_z\mathcal{L}^+_{p,\gamma,\epsilon}$ on $\mathcal{L}^+_{p,\gamma,\epsilon} \cap Q^b$, such that:

(i) $w(z)(s) = 0$, if $\varphi(z(s)) \geq 2\rho$;

(ii) $\|w(z)\|_1 \leq 1$ (cf.(4.3));

(iii) $\langle w(z)(s), \nabla\varphi(z)(s)\rangle_{(R)} > 0$, for any $s \in [0,1]$ such that $0 \leq \varphi(z(s)) \leq \rho$;

Now, let $\eta(s, z)$ be the solution of the Cauchy problem

$$\begin{cases} \dot{\eta} = -w(\eta) \\ \eta(0) = z \in \mathcal{L}^+_{p,\gamma,\epsilon} \cap Q^b . \end{cases}$$

Since the integral

$$\int_0^1 \frac{1}{\varphi^2(z)} ds$$

is decreasing along the flow η, the same techniques of the proof of Lemma 4.5 of [10] show that if ϵ, δ and ρ are sufficiently small,

$$\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon} \cap Q^b; \mathcal{K}) = \mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon}(\Lambda, \rho) \cap Q^b; \mathcal{K}) ;$$
$$\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon} \cap Q^b_\delta; \mathcal{K}) = \mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon}(\Lambda, \rho) \cap Q^b_\delta; \mathcal{K}) .$$

Now, using a vector field similar to $w(z)$ on the space $\mathcal{L}^+_{p,\gamma,\epsilon}(\Lambda, \rho)$ and the same technique of Lemma 4.5 of [10] show that if ϵ, δ and ρ are sufficiently small, $\mathcal{L}^+_{p,\gamma,\epsilon}(\Lambda,\rho)\cap Q^b_\delta$ and $\mathcal{L}^+_{p,\gamma,\epsilon}(\Lambda,\rho)\cap Q^b$ can be weakly retracted on the same space. Then:

$$\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^b_\delta;\mathcal{K})=\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^b;\mathcal{K}). \tag{7.6}$$

Now, since b is a regular value of Q on $\mathcal{L}^+_{p,\gamma}$, it is also a regular value of Q_δ on $\mathcal{L}^+_{p,\gamma,\epsilon}$ (if ϵ and δ are sufficiently small, cf. Proposition 4.10). Then, for any θ sufficiently small, by Proposition 4.9 and classical deformation results (see for instance [22]), there exists a continuous map $\Pi\colon[0,1]\times\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^{b+\theta}\longrightarrow\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^{b+\theta}_\delta$ such that $\Pi(0,z)=z$ and $\Pi(1,z)\in\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^{b-\theta}_\delta$, for any $z\in\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^{b+\theta}_\delta$. This allows to conclude that (see also (7.6))

$$\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^b_\delta,\mathcal{K})=\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^{b-\theta}_\delta,\mathcal{K})=\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^{b-\theta},\mathcal{K}).$$

Finally, using the homeomorphism given by Proposition 4.11 gives (if ϵ, δ and θ are sufficiently small):

$$\mathcal{P}(\mathcal{L}^+_{p,\gamma,\epsilon}\cap Q^{b-\theta};\mathcal{K})=\mathcal{P}(\mathcal{L}^+_{p,\gamma}\cap Q^b;\mathcal{K}),$$

and the proof is complete. □

8. PROOF OF THEOREMS 1.7, 1.13, 1.15

The proof of Theorem 1.7 requires a limit process in the Relations (5.1).

Proof of Theorem 1.7. – By Lemma 7.4 there exists a sequence $(b_h)_{n\in\mathbf{N}}$ of regular values of Q on $\mathcal{L}^+_{p,\gamma}$, with $b_h\to+\infty$. Moreover, for any $h\in\mathbf{N}$, by the exact sequence of the homology for the topological pair $(\mathcal{L}^+_{p,\gamma},Q^{b_h})$, there exists a formal series $S_{h,1}(r)$ (see for instance [4]) with coefficients in $\mathbf{N}\cup\{+\infty\}$, such that

$$\mathcal{P}(Q^{b_h};\mathcal{K})+\mathcal{P}(\mathcal{L}^+_{p,\gamma},Q^{b_h};\mathcal{K})=\mathcal{P}(\mathcal{L}^+_{p,\gamma};\mathcal{K})+(1+r)S_{h,1}(r)\ . \tag{8.1}$$

Combining (7.1) and (8.1) gives the existence of a formal series $S_h(r)$ with coefficients in $\mathbf{N}\cup\{+\infty\}$ such that

$$\sum_{z\in\mathcal{C}\cap Q^b} r^{m(z,Q)}+\mathcal{P}(\mathcal{L}^+_{p,\gamma},Q^b;\mathcal{K})=\mathcal{P}(\mathcal{L}^+_{p,\gamma};\mathcal{K})+(1+r)S_h(r)\ , \tag{8.2}$$

where $\mathcal{C}$ is defined by (4.5).

Let, for any $n \in \mathbf{N}$, $m(n,h)$ be the number of critical points of Q of index n and contained in Q^{b_h}. The sequence $(m(n,h))_{h\in\mathbf{N}}$ is nondecreasing and converges to the number $m(n)$ of critical points of Q with Morse index equal to n. Since $\mathbf{N}\cup\{+\infty\}$ is compact (with respect of the usual convergence), a diagunalisation argument shows the existence of a subsequence of integers k_h such that, for any $n \in \mathbf{N}$ the sequences of the coefficients q_{n,k_h} of the formal series in (8.2) converge to $q_n \in \mathbf{N}\cup\{+\infty\}$. For the sake of simplicity, we can assume that $k_h \equiv h$ and we set

$$S(r) = \sum_{n=0}^{\infty} q_n r^n .$$

We shall prove (1.7) showing that for any $n \in \mathbf{N}$,

$$m(n) = \beta_n(\mathcal{L}^+_{p,\gamma}, \mathcal{K}) + q_{n-1} + q_n , \tag{8.3}$$

where $\beta_n(\mathcal{L}^+_{p,\gamma}; \mathcal{K})$ is the n-th Betti number of $\mathcal{L}^+_{p,\gamma}$. We have two cases.

(1) $m(n) = +\infty$.

By (8.2) we have that either $\beta_n(\mathcal{L}^+_{p,\gamma})$, or q_{n-1}, or q_n must be $+\infty$. In any case (8.3) holds, hence (1.7) holds for the coefficient n.

(2) $m(n) < +\infty$.

Let

$$b^* = \max\{Q(z)|\ z \text{ is a critical point of } Q \text{ on } \mathcal{L}^+_{p,\gamma},\ m(z,Q) = n\} .$$

By (8.2), in order to prove (8.3) for the coefficient n, it suffices to show that for any $h \in \mathbf{N}$ such that $b_h > b^*$,

$$H_n(\mathcal{L}^+_{p,\gamma}, Q^{b_h}; \mathcal{K}) = \{0\} . \tag{8.4}$$

Assume by contradiction that (8.4) does not hold for some $h \in \mathbf{N}$ such that $b_h > b^*$. Let α be a nontrivial element of $H_n(\mathcal{L}^+_{p,\gamma}, Q^{b_h}; \mathcal{K})$ and let K_α be its support. Since the singular homology has compact support, K_α is a compact subset of $\mathcal{L}^+_{p,\gamma}$. Let

$$c > \sup\{Q(z), z \in K_\alpha\} \tag{8.5}$$

be a regular value for Q (by the definition, c is greater than b_h) and consider the long exact sequence in homology of the topological triple $(\mathcal{L}^+_{p,\gamma}, Q^c, Q^{b_h})$,

$$\dots \longrightarrow H_n(Q^c, Q^{b_h}; \mathcal{K}) \xrightarrow{i^1_n} H_n(\mathcal{L}^+_{p,\gamma}, Q^{b_h}; \mathcal{K}) \xrightarrow{i^2_q} H_n(\mathcal{L}^+_{p,\gamma}, Q^c; \mathcal{K}) \longrightarrow \dots, \tag{8.6}$$

where i_n^1 and i_n^2 are the homomorphisms induced in homology respectively by the inclusions $i_1 : (Q^c, Q^{b_h}) \longrightarrow (\mathcal{L}^+_{p,\gamma}, Q^{b_h})$ and $i_2 : (\mathcal{L}^+_{p,\gamma}, Q^{b_h}) \longrightarrow (\mathcal{L}^+_{p,\gamma}, Q^c)$. By (8.5) $i_n^2(\alpha) = 0$. Then, by the exactness of (8.6) there exists $\beta \in H_n(Q^c, Q^{b_h}; \mathcal{K}) \setminus \{0\}$ such that

$$i_n^1(\beta) = \alpha \ .$$

Hence,

$$H_n(Q^c, Q^{b_h}; \mathcal{K}) \neq \{0\}. \tag{8.7}$$

Now, consider the shortening flow constructed in [14]. Thanks to Lemma 7.3 and the homeomorphisms of Proposition 4.11, we can also construct a flow in a neighborhood of any critical point of Q on $\mathcal{L}^+_{p,\gamma}$, so that we can repeat the proofs of classical Morse Theory (cf. [22]). In this way, by (8.7), we obtain the existence of a critical point z of Q such that $Q(z) \in [b_h, c]$, and $m(z, Q) = n$. Since $b_h > b^*$, we get a contradiction.

Hence (8.3) holds for any $n \in \mathbf{N}$. Finally, by Theorem 6.1 we can replace the Morse index $m(z, Q)$ of a critical point z with the index $\mu(w_z)$ of the lightlike geodesic (joining p and γ) obtained by the reparametrization of z, completing the proof. □

Proof of Theorem 1.13. – Assume that $\mathcal{L}^+_{p,\gamma}$ is contractible, then the Poincaré polynomial of $\mathcal{L}^+_{p,\gamma}$ with respect to any field $\mathcal{K}$ is given by

$$\mathcal{P}(\mathcal{L}^+_{p,\gamma}, \mathcal{K}) = 1.$$

Let $\mathcal{G}^+_{p,\gamma}$ the set of the future pointing lightlike geodesics joining p with γ, (1.7) gives:

$$\text{card } \mathcal{G}^+_{p,\gamma} = 1 + 2S(1).$$

Then $\text{card}\mathcal{G}^+_{p,\gamma}$ is odd or infinite, according $S(1)$ is finite or infinite.

Assume now that $\mathcal{L}^+_{p,\gamma}$ is noncontractible, by the critical point theory of Ljusternik-Schnirelmann, the functional Q has at least two critical points on $\mathcal{L}^+_{p,\gamma}$ (see [14]). □

The proof of Theorem 1.15 is an obvious consequence of Theorem 1.7 and L_4), since the Poincaré polynomial is a homotopical invariant.

Proof of Theorem 1.16. – Assume that Λ is contractible, then $\Omega(\Lambda)$ is contractible and by L_4), $\mathcal{L}^+_{p,\gamma}$ is contractible. Then the proof follows by the first part of Theorem 1.13. Assume now that Λ is noncontractible, then, by a result of Serre (see [34]), for a suitable field $\mathcal{K}$, the set

$$\mathcal{Q} = \{q \in \mathbf{N} : \beta_q(\Omega(\Lambda), \mathcal{K}) \neq 0\}$$

is infinite. Finally, by the Morse inequalities (1.7), for any $q \in \mathcal{Q}$ there exists a future pointing light ray joining p and γ, with index $q \in \mathcal{Q}$. □

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