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Large volume asymptotics of brownian integrals and orbital magnetism

by

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ABSTRACT. – We study the asymptotic expansion of a class of Brownian integrals with paths constrained to a finite domain as this domain is dilated to infinity. The three first terms of this expansion are explicitly given in terms of functional integrals. As a first application we consider the finite size effects in the orbital magnetism of a free electron gas subjected to a constant magnetic field in two and three dimensions. Sum rules relating the volume and surface terms to the current density along the boundary are established. We also obtain that the constant term in the pressure (the third term) of a two dimensional domain with smooth boundaries is purely topological, as in the non magnetic case. The effects of corners in a polygonal shape are identified, and their contribution to the zero field susceptibility is calculated in the case of a square shaped domain. The second application concerns the asymptotic expansion of the statistical sum

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for a quantum magnetic billiard in the semiclassical and high temperature limits. In the semiclassical expansion, the occurence of the magnetic field is seen in the third term, whereas in the high temperature expansion, it appears only in the fifth term.

RÉSUMÉ. – On étudie le développement asymptotique de certaines intégrales browniennes dont les chemins sont astreints à demeurer dans une région bornée de l'espace lorsque cette région est étendue à l'infini par dilatation. Les trois premiers termes de ce développement sont explicites. La première application concerne les effets diamagnétiques de taille finie dans un gaz électronique libre soumis à un champ magnétique homogène. La densité de courant au voisinage des parois obéit à des règles de somme qui mettent en jeu les contributions volumiques et superficielles de la pression thermodynamique. En présence du champ magnétique le terme d'ordre 1 dans le développement de la pression d'un système bidimensionnel avec frontière régulière est purement topologique. On identifie également l'effet des coins dans des domaines polygonaux, et on calcule leur contribution à la susceptibilité en champ nul dans le cas du carré. Dans la seconde application, on établit le développement asymptotique semiclassique et de haute température de la fonction de partition d'un billiard magnétique. Dans le développement semiclassique, le champ magnétique apparaît dans le troisième terme, alors que dans le développement de haute température, celui-ci n'apparaît que dans le cinquième terme.

Mots clés : Billiard magnétique, magnétisme orbital, courants de bords, intégrales fonctionnelles, développement de Weyl.

I. INTRODUCTION

In a previous work [1] we have studied the diamagnetism of quantum charges in thermal equilibrium subjected to a uniform magnetic field by means of the Feynman-Kac-Ito representation of the Gibbs state. It is well known that diamagnetic effects result from an induced current localized at the surface of the sample, and we were mainly concerned by establishing exact relations between the magnetisation and the surface current in the thermodynamic limit.

In the present work, we continue our investigation by the same methods as in [1] focusing attention on finite size effects in orbital magnetism. The understanding of finite size effects in this context is of interest both for physical and mathematical reasons. In a recent paper [2], Kunz presented a study of the surface corrections to orbital magnetism. The surface correction to the zero-field Landau susceptibility is given by an explicit formula for a sample of general shape and could contribute to the magnetic properties of small metallic aggregates. At the mathematical level the problem can be viewed as a natural generalisation with a magnetic field of the famous Kac problem [3] of calculating the asymptotics of the statistical sum $\mathrm{Tr}\exp(\beta\Delta) = \sum_i \exp(-\beta E_i)$ of eigenvalues E_i of the Laplacian $-\Delta$ in a bounded domain as $\beta \to 0$.

Let us formulate our questions and results in more precise terms. We consider a quantum particle of charge e and mass m subjected to a constant magnetic field $\mathbf{B}$ and confined in a finite region $\Sigma_R = \{R\mathbf{r} \mid \mathbf{r} \in \Sigma\}$ in $\mathbf{R}^\nu, \nu = 2, 3$, that is the dilation of some fixed domain Σ. We consider domains which are simply connected, having smooth boundaries $\partial\Sigma$ (with finite curvature), but we do not require convexity. The Hamiltonian is ($\hbar =$ Plank constant, $c =$ velocity of light)

$$H_R = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 \tag{1.1}$$

where the potential vector $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \wedge \mathbf{r}$ is chosen in the symmetric gauge, and H_R is defined with Dirichlet boundary conditions on $\partial\Sigma_R$. Using the Feynman-Kac-Ito representation in terms of the Brownian bridge, the statistical sum reads (*see* [1], section II)

$$\mathrm{Tr}e^{-\beta H_R}$$
$$= \left(\frac{1}{\sqrt{2\pi}\lambda}\right)^\nu \int_{\Sigma_R} d\mathbf{r} \int D\boldsymbol{\alpha} \exp\left(-i\frac{\mu}{2}\mathbf{b}\cdot\int \boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}\right)\chi_{\Sigma_R}(\mathbf{r}+\lambda\boldsymbol{\alpha}) \tag{1.2}$$
$$= \left(\frac{1}{\sqrt{2\pi}\varepsilon}\right)^\nu \int_{\Sigma} d\mathbf{r} \int D\boldsymbol{\alpha} \exp\left(-i\frac{\mu}{2}\mathbf{b}\cdot\int \boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}\right)\chi_{\Sigma}(\mathbf{r}+\varepsilon\boldsymbol{\alpha}) \tag{1.3}$$

with $\varepsilon = \frac{\lambda}{R}$ and λ given in (1.6). In the functional integrals (1.2) and (1.3), $\boldsymbol{\alpha}(s) = (\alpha_1(s), \ldots, \alpha_\nu(s))$ is the ν- dimensional Brownian bridge process, $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(1) = 0$, with covariance

$$\int D\boldsymbol{\alpha}\alpha^i(s)\alpha^j(t) = \delta_{ij}(\min(s,t) - st) \tag{1.4}$$

and $\int \boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}$ is to be understood as an Ito integral.

$$\chi_\Sigma(\mathbf{r}+\lambda\boldsymbol{\alpha}) = \begin{cases} 1, & \text{if } \mathbf{r}+\lambda\boldsymbol{\alpha}(s) \in \Sigma \text{ for } 0 \le s \le 1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

is the indicator function of the paths that remain in Σ. We denote by $\mathbf{b}$ the unit vector in the direction of $\mathbf{B}$ in dimension $\nu = 3$; for $\nu = 2$, $\mathbf{b}$ is chosen orthogonal to the planar surface Σ, or Σ_R.

With the physical constants occurring in the Hamiltonian (1.1) we can form two lengths, the thermal de Broglie length λ and the magnetic length l

$$\lambda = \hbar\left(\frac{\beta}{m}\right)^{\frac{1}{2}}, \qquad l = \left(\frac{\hbar c}{eB}\right)^{\frac{1}{2}} \tag{1.6}$$

The functional integral (1.3) involves the two independent dimensionless parameters

$$\varepsilon = \frac{\lambda}{R} = \frac{\hbar}{R}\left(\frac{\beta}{m}\right)^{\frac{1}{2}}, \qquad \mu = \left(\frac{\lambda}{l}\right)^2 = \frac{\beta\hbar eB}{mc} \tag{1.7}$$

Setting

$$C_1 = \frac{\lambda R}{l^2}, \qquad C_2 = \frac{R^2}{l^2}$$

one can single out three limiting regimes of interest:

(i) *large volume limit* $R \to \infty$ *corresponding to* $\varepsilon \to 0$, μ *fixed,*

(ii) *semiclassical limit* $\hbar \to 0$ *corresponding to* $\varepsilon \to 0$, $\mu\varepsilon^{-1} = C_1$ *fixed,*

(iii) *high temperature limit* $\beta \to 0$ *corresponding to* $\varepsilon \to 0$, $\mu\varepsilon^{-2} = C_2$ *fixed.*

In the absence of magnetic field ($\mu = 0$), the three limits (i) - (iii) coincide with the usual asymptotics of the statistical sum for the Laplacian in Σ_R, which can therefore be equivalently viewed as a large volume, semiclassical or high temperature limit (high temperature limit equals small time limit in the language of stochastic processes). However, when $\mu \neq 0$, the three limits (i) - (iii) correspond to distinct physical situations, and each of them can be considered as a possible generalisation of the Kac problem. In this paper we shall mainly be interested in the case (i) (large volume limit with fixed magnetic field). This limit is the appropriate one to exhibit finite size effects in orbital magnetism. It can also be phrased as a semiclassical limit $\hbar \to 0$ in which the magnetic field is rescaled so that the product $\hbar B$ is kept constant. To our knowledge, the existing literature (with the exception of [2, 4]) has been more concerned with the two limits (ii) and (iii) in the sense that one considers strictly confined systems, where the confinement is insured either by an external potential or by an inhomogeneous magnetic field growing at large distances. Then one studies the high temperature and semiclassical asymptotics of the statistical sum (*see* [5] for recent references as well as the discussion of section VI).

In section II, we establish the asymptotic behaviour of the quantity (1.2) as $R \to \infty$ in a more general setting, allowing for a general functional $F(\boldsymbol{\alpha})$ in place of the magnetic phase factor and an arbitrary dimension $\nu \geq 2$. The expansion is established up to order $R^{\nu-2}$, and this constitutes the main mathematical results of the paper. In section III, we apply these results to the magnetic situation. In particular, the functional integrals occurring in the calculation of the zero-field magnetic susceptibility can be computed explicitely, thus recovering Kunz formulae [2]. Section IV is devoted to some geometrical aspect of the surface correction to the magnetization and its relation to the current density ($\nu = 3$). In [1] we showed that in $\nu = 2$ dimensions the bulk magnetization can be expressed as the integral of the current density along a plane wall. In [2] Kunz has shown that the surface correction to the bulk magnetization has a similar form: it is the first moment of the current along a plane wall. In this section we give the generalization of these results to three dimensions by expressing the surface correction of the pressure as a surface integral of the planar interface pressures associated with magnetized half spaces, as well as the relation of the latter quantities with surface current. We study in section V the case when the boundary is not smooth (polygonal shapes in dimension $\nu = 2$). The contribution of corners is isolated, and the corner zero-field susceptibility is explicitly computed for a square shaped domain. The section VI discusses some conclusions that can be drawn on the two other limits (ii) and (iii) from our analysis. We show that in the semiclassical limit, the effect of the magnetic field is only seen in the third term of the asymptotic expansion of the statistical sum, and in its fifth term in the high temperature limit. Finally, concluding remarks are presented in section VII.

II. LARGE VOLUME EXPANSIONS

As explained in the intoduction the main purpose of this paper is to obtain a generalization of Kac's result which can give amongst other things the surface corrections to the pressure and magnetization. These physical quantities both are of the form:

$$\mathcal{I}_R \equiv \int_{\Sigma_R} d^\nu r \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \chi_{\Sigma_R}(\mathbf{r} + \boldsymbol{\alpha}) \tag{2.1}$$

where Σ is a region in $\mathbb{R}^\nu$ with boundary $\partial\Sigma$, $\Sigma_R = \{R\mathbf{r} \,|\, \mathbf{r} \in \Sigma\}$, and χ_{Σ_R} is the indicator function (1.5) of the paths that remain in Σ_R (in this section, we set $\lambda = 1$). We shall assume that F in (2.1) is square-integrable in the sense that

$$\|F\|^2 \equiv \int D\alpha |F(\alpha)|^2 < \infty$$

We will obtain expansions for $\mathcal{I}_R$ of the form

$$\mathcal{I}_R = \mathcal{I}_R^{(0)} + O(R^{\nu-1}), \tag{2.2}$$

$$\mathcal{I}_R = \mathcal{I}_R^{(0)} + \mathcal{I}_R^{(1)} + O(R^{\nu-2}), \tag{2.3}$$

and

$$\mathcal{I}_R = \mathcal{I}_R^{(0)} + \mathcal{I}_R^{(1)} + \mathcal{I}_R^{(2)} + R^{\nu-2}\epsilon_R \tag{2.4}$$

where $\epsilon_R \to 0$ as $R \to \infty$, by taking successively stricter smoothness conditions on $\partial\Sigma$.

Assumption A0

At every point of $\partial\Sigma$ the principal curvatures, $\kappa_1, \ldots, \kappa_{\nu-1}$, and the principal directions are defined, and

$$\kappa_0 \equiv \sup_i \sup_{r\in\partial\Sigma} |\kappa_i(\mathbf{r})| < \infty \tag{2.5}$$

Since near the surface we shall be using Gaussian coordinates in which a point is labelled by the nearest point to it on the surface and its distance from the surface we shall need also the following assumption.

Assumption A1

There exists $\delta > 0$ satisfying $0 < \delta < 1/\kappa_0$ such that if $\mathbf{r} \in \Sigma$ and $d(\mathbf{r}, \partial\Sigma) < \delta$ then there is a unique point $\mathbf{s}(\mathbf{r}) \in \partial\Sigma$ for which $|\mathbf{r} - \mathbf{s}| = d(\mathbf{r}, \partial\Sigma)$.

At each $\mathbf{s}$ of $\partial\Sigma$ let $\mathbf{n}$ be the inward drawn unit normal to $\partial\Sigma$ and $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{\nu-1}$ unit vectors in the tangent plane of $\partial\Sigma$ along the principal directions (directions of principal curvatures). Set up coordinates $x, x_1, \ldots, x_{\nu-1}$ at each point $\mathbf{s}$ of $\partial\Sigma$ so that x is along $\mathbf{n}$ and $x_1, \ldots x_{\nu-1}$ are along $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{\nu-1}$. In this local frame the surface $\partial\Sigma$ is given by

$$x = f_{\mathbf{r}}(x_1, x_2, \ldots, x_{\nu-1}) = f_{\mathbf{r}}(\tilde{\mathbf{x}}) \tag{2.6}$$

with $\tilde{\mathbf{x}} = (x_1, x_2, ..., x_{\nu-1})$. The interior, Σ, is given by

$$x > f_{\mathbf{r}}(x_1, x_2, \ldots, x_{\nu-1}) = f_{\mathbf{r}}(\tilde{\mathbf{x}}) \tag{2.7}$$

Let

$$\Sigma^* = \{\mathbf{r} \in \Sigma \mid d(\mathbf{r}, \partial\Sigma) < \delta\}$$

The volume element in Σ^* is given by

$$\prod_{i=1}^{\nu-1}(1 - x\kappa_i) d\sigma dx \tag{2.5}$$

In Appendix E we give a proof for this formula.

We start by rewriting $\mathcal{I}_R$ in the form

$$\mathcal{I}_R = R^\nu \int_\Sigma d^\nu r \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \chi_\Sigma(\mathbf{r} + R^{-1}\boldsymbol{\alpha}) \tag{2.8}$$

In this discussion we shall call $\mathbf{r} + R^{-1}\boldsymbol{\alpha}(s)$ a path starting at $\mathbf{r}$ so that the integral (2.8) is constrained to the set of paths starting at $\mathbf{r}$ and staying in Σ. We can write the integral with respect to $\boldsymbol{\alpha}$ in $\mathcal{I}_R$ as an integral over all paths starting at $\mathbf{r}$ and then take away the integral over the paths which leave Σ

$$\begin{aligned}\mathcal{I}_R &= R^\nu \int_\Sigma d^\nu r \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) - R^\nu \int_\Sigma d^\nu r \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha})(1 - \chi_\Sigma(\mathbf{r} + R^{-1}\boldsymbol{\alpha})) \\ &= I_1 - I_2 \end{aligned} \tag{2.9}$$

The idea here is first to estimate I_2 and to show that it is of order $R^{\nu-1}$, then we approximate I_2 by the integral over the paths which cross the tangent plane and show that the difference between I_2 and this integral is of order $R^{\nu-2}$. Finally we approximate the remainder by the integral over the paths which cross the parabolic surface tangent to $\partial\Sigma$ with the same curvatures as $\partial\Sigma$ and show that the correction term is bounded by $R^{\nu-2}\epsilon_R$ where $\epsilon_R \to 0$ as $R \to \infty$.

We have

$$I_1 = R^\nu |\Sigma| \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \tag{2.10}$$

and

$$\begin{aligned} I_2 &= R^\nu \int_{\Sigma\setminus\Sigma^*} d^\nu r \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha})(1 - \chi_\Sigma(\mathbf{r} + R^{-1}\boldsymbol{\alpha})) \\ &\quad + R^\nu \int_{\Sigma^*} d^\nu r \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha})(1 - \chi_\Sigma(\mathbf{r} + R^{-1}\boldsymbol{\alpha}))(1 - \Delta_R(\boldsymbol{\alpha})) \\ &\quad + R^\nu \int_{\Sigma^*} d^\nu r \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha})(1 - \chi_\Sigma(\mathbf{r} + R^{-1}\boldsymbol{\alpha}))\Delta_R(\boldsymbol{\alpha}) \\ &= I_3 + I_4 + I_5 \end{aligned} \tag{2.11}$$

where Δ_R is the indicator function of the set $\{\boldsymbol{\alpha} | \sup_{s\in[0,1]} |\boldsymbol{\alpha}(s)| < \eta R\}$, and η is to be defined later. In the last equation we have split the paths which leave Σ into those which start in $\Sigma \setminus \Sigma^*$ and those which start in Σ^* and the latter according to whether $\sup |\boldsymbol{\alpha}|$ is greater than or less than ϵR. It is easy to show by using the bound (*see* sect. II.7 in [6])

$$\int_{\sup |\boldsymbol{\alpha}|>a} D\boldsymbol{\alpha} \leq Ce^{-Ca^2} \tag{2.12}$$

that $I_3 + I_4 = O\left(e^{-CR^2}\right)$. Indeed

$$\begin{aligned} I_3 &\leq R^\nu |\partial\Sigma| \int_{\sup |\boldsymbol{\alpha}|>\delta R} D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| \\ &\leq R^\nu |\partial\Sigma| \|F\| \left(\int_{\sup |\boldsymbol{\alpha}|>\delta R} D\boldsymbol{\alpha} \right)^{\frac{1}{2}} \leq Ce^{-CR^2} \end{aligned} \tag{2.13}$$

To simplify the notation, we shall denote all constants by C. Similarly, using assumptions A0 and A1

$$\begin{aligned} I_4 &\leq R^\nu \int_{\partial\Sigma} d\sigma \int_0^\delta dx \prod_{i=1}^{\nu-1} (1 - x\kappa_i) \int D\boldsymbol{\alpha}\ (1 - \Delta_R(\boldsymbol{\alpha}))\ |F(\boldsymbol{\alpha})| \\ &\leq R^\nu 2^{\nu-1} \int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}\ (1 - \Delta_R(\boldsymbol{\alpha}))\ |F(\boldsymbol{\alpha})| \\ &\leq R^\nu 2^{\nu-1} \|F\| \left(\int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}\ (1 - \Delta_R(\boldsymbol{\alpha})) \right)^{\frac{1}{2}} \\ &\leq R^\nu 2^{\nu-1} \|F\| \left(\int_{\partial\Sigma} d\sigma \int_{\sup \boldsymbol{\alpha}>\epsilon R} D\boldsymbol{\alpha} \right)^{\frac{1}{2}} \leq Ce^{-CR^2} \end{aligned} \tag{2.14}$$

We now estimate the last term I_5. We have

$$\begin{aligned} I_5 =& R^\nu \int_{\partial\Sigma} d\sigma \int_0^\delta dx \prod_{i=1}^{\nu-1} (1 - x\kappa_i) \\ &\times \int_{x+\inf(R^{-1}\boldsymbol{\alpha}\cdot\mathbf{n} - f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}))<0} D\boldsymbol{\alpha}\ \Delta_R(\boldsymbol{\alpha})\ F(\boldsymbol{\alpha}) \end{aligned} \tag{2.15}$$

We have used the notation $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} - (\boldsymbol{\alpha} \cdot \mathbf{n})\mathbf{n}$, that is, $\tilde{\boldsymbol{\alpha}}$ is the projection of $\boldsymbol{\alpha}$ onto the tangent plane at $\mathbf{r}$. To bound I_5 by $R^{\nu-1}$ we need only that $f_{\mathbf{r}}$ be linearly bounded on each neighbourhood of the surface.

Assumption A2

There exists $\eta > 0$ *and* $C > 0$ *such that for each* $\mathbf{r} \in \partial\Sigma$,

$$|f_{\mathbf{r}}(\tilde{\mathbf{x}})| < C|\tilde{\mathbf{x}}| \tag{2.16}$$

whenever $|\tilde{\mathbf{x}}| < \eta$.

We then have

$$\begin{aligned} |I_5| &\leq R^{\nu-1}2^{\nu-1}\int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}|F(\boldsymbol{\alpha})|\Delta_R(\boldsymbol{\alpha})\{\sup|\boldsymbol{\alpha}.\mathbf{n}|+R\sup|f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})|\} \\ &\leq R^{\nu-1}2^{\nu-1}\int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}|F(\boldsymbol{\alpha})|\{\sup|\boldsymbol{\alpha}\cdot\mathbf{n}| + C\sup|\tilde{\boldsymbol{\alpha}}|\} \\ &\leq R^{\nu-1}2^{\nu-1}\|F\|\int_{\partial\Sigma} d\sigma \left(\int D\boldsymbol{\alpha}\{\sup|\boldsymbol{\alpha}\cdot\mathbf{n}| + C\sup|\tilde{\boldsymbol{\alpha}}|\}^2\right)^{\frac{1}{2}} \leq CR^{\nu-1} \end{aligned} \tag{2.17}$$

We have thus proved.

PROPOSITION 1. – *Under the assumptions A0, A1 and A2*

$$\mathcal{I}_R = \mathcal{I}_R^{(0)} + O(R^{\nu-1}) \tag{2.18}$$

where

$$\mathcal{I}_R^{(0)} = R^\nu|\Sigma| \int D\boldsymbol{\alpha}F(\boldsymbol{\alpha}) \tag{2.19}$$

This result has already been obtained in [1].

We now strengthen the assumption on the surface to obtain the second expansion (2.3). In this case we want to approximate the integral over the paths which leave Σ by the integral over the paths which cross the tangent plane. We thus require that $|f_{\mathbf{r}}(\tilde{\mathbf{x}})|$ be bounded by $|\tilde{\mathbf{x}}|^{1+\gamma}$ on each neighbourhood of the surface; for simplicity we take $\gamma = 1$.

Assumption A3

There exists $\eta > 0$ *and* $C > 0$ *such that for each* $\mathbf{r} \in \partial\Sigma$,

$$|f_{\mathbf{r}}(\tilde{\mathbf{x}})| < C|\tilde{\mathbf{x}}|^2 \tag{2.20}$$

whenever $|\tilde{\mathbf{x}}| < \eta$.

PROPOSITION 2. – *Under the assumptions A0- A3*

$$\mathcal{I}_R = \mathcal{I}_R^{(0)} + \mathcal{I}_R^{(1)} + O(R^{\nu-2}) \tag{2.21}$$

where

$$\mathcal{I}_R^{(1)} = R^{\nu-1} \int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \inf(\boldsymbol{\alpha} \cdot \mathbf{n}) \tag{2.22}$$

Proof. – Let

$$\begin{aligned} I_6 &= -R^{\nu-1} \int_{\partial\Sigma} d\sigma \int \Delta_R(\boldsymbol{\alpha})\ D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \inf(\boldsymbol{\alpha} \cdot \mathbf{n}) \\ &= -R^{\nu-1} \int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \inf(\boldsymbol{\alpha} \cdot \mathbf{n})) + O(e^{-CR^2}) \end{aligned} \tag{2.23}$$

Note that the first term in (2.23) can be written as

$$R^{\nu-1} \int_{\partial\Sigma} d\sigma \int_0^\infty dx \int_{x+\inf(\boldsymbol{\alpha}\cdot\mathbf{n})<0} D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \tag{2.24}$$

and so corresponds to the integral over the paths which cross the tangent plane. With this definition we have

$$\begin{aligned} I_5 = I_6 &+ R^{\nu-1} \int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}\ \Delta_R(\boldsymbol{\alpha})\ F(\boldsymbol{\alpha}) \\ &\times \left[\min\{\delta R, -\inf(\boldsymbol{\alpha} \cdot \mathbf{n} - R f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}))\} + \inf(\boldsymbol{\alpha} \cdot \mathbf{n}))\right] \\ &+ R^\nu \int_{\partial\Sigma} d\sigma \int_0^\delta dx \left(\prod_{i=1}^{\nu-1}(1 - x\kappa_i) - 1\right) \\ &\times \int_{x+\inf(R^{-1}\boldsymbol{\alpha}\cdot\mathbf{n}-f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}))<0} D\boldsymbol{\alpha}\ \Delta_R(\boldsymbol{\alpha})\ F(\boldsymbol{\alpha}) \\ &= I_6 + I_7 + I_8. \end{aligned} \tag{2.25}$$

We want to show that $I_7 \leq CR^{\nu-2}$. Choose a time $\tau_{\mathbf{n}} \in [0, 1]$, depending on $\boldsymbol{\alpha}$, for which $\boldsymbol{\alpha}(s) \cdot \mathbf{n}$ attains its minimum, *i.e.* $(\boldsymbol{\alpha} \cdot \mathbf{n})(\tau_{\mathbf{n}}) = \inf(\boldsymbol{\alpha} \cdot \mathbf{n})$. Then

$$\inf(\boldsymbol{\alpha}\cdot\mathbf{n} - R f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})) - \inf(\boldsymbol{\alpha}\cdot\mathbf{n}) \leq -R f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}(\tau_{\mathbf{n}})) \leq R^{-1} C \sup|\tilde{\boldsymbol{\alpha}}|^2 \tag{2.26}$$

Choose τ_R such that

$$\inf(\boldsymbol{\alpha} \cdot \mathbf{n} - R f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})) = \boldsymbol{\alpha} \cdot \mathbf{n}(\tau_R) - R f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}(\tau_R)) \tag{2.27}$$

Then by assumption A3

$$\inf(\boldsymbol{\alpha} \cdot \mathbf{n} - R f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})) - \inf(\boldsymbol{\alpha}\cdot\mathbf{n}) \geq -R f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}(\tau_R)) \geq -R^{-1} C \sup|\tilde{\boldsymbol{\alpha}}|^2 \tag{2.28}$$

Therefore

$$|\inf(\boldsymbol{\alpha}\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})) - \inf(\boldsymbol{\alpha}\cdot\mathbf{n})| \le R^{-1}C\sup|\tilde{\boldsymbol{\alpha}}|^2 \tag{2.29}$$

For $\boldsymbol{\alpha}$ such that $\sup|\boldsymbol{\alpha}| \le R^{\frac{1}{4}}$ we have

$$|\inf(\boldsymbol{\alpha}\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})) - \inf(\boldsymbol{\alpha}\cdot\mathbf{n})| \le CR^{-\frac{1}{2}} \tag{2.30}$$

and so for R sufficiently large

$$\min\{\delta R, -\inf(\boldsymbol{\alpha}\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}))\} = -\inf(\boldsymbol{\alpha}\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})) \tag{2.31}$$

Thus by (2.29) the contribution to I_7 from these paths is bounded by

$$CR^{\nu-2}\int_{\partial\Sigma} d\sigma|F(\boldsymbol{\alpha})|\sup|\tilde{\boldsymbol{\alpha}}|^2 \le CR^{\nu-2} \tag{2.32}$$

The contribution to I_7 from $\boldsymbol{\alpha}$'s such that $\sup|\boldsymbol{\alpha}| > R^{\frac{1}{4}}$ is clearly bounded by $Ce^{-CR^{\frac{1}{2}}}$.

To estimate I_8, we expand the product

$$\prod_{i=1}^{\nu-1}(1 - x\kappa_i) = 1 + \sum_{k=1}^{\nu-1} a_k x^k \tag{2.33}$$

and observe that

$$\left|\prod_{i=1}^{\nu-1}(1 - x\kappa_i) - 1\right| \le x\sum_{k=1}^{\nu-1}|a_k|\delta^{k-1} \le Cx \tag{2.34}$$

since curvatures are uniformly bounded by κ_0. Hence

$$\begin{aligned} |I_8| &\le \frac{1}{2}CR^{\nu-2}\int_{\partial\Sigma} d\sigma\int D\boldsymbol{\alpha}\ \Delta_R(\boldsymbol{\alpha})\ F(\boldsymbol{\alpha})(\sup(|\boldsymbol{\alpha}\cdot\mathbf{n}| + R|f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})|))^2 \\ &\le \frac{1}{2}CR^{\nu-2}\int_{\partial\Sigma} d\sigma\int D\boldsymbol{\alpha}F(\boldsymbol{\alpha}) \\ &\quad\times\left(\sup|\boldsymbol{\alpha}\cdot\mathbf{n}|^2 + 2CR^{-1}\sup|\boldsymbol{\alpha}\cdot\mathbf{n}|\sup|\tilde{\boldsymbol{\alpha}}|^2 + R^{-2}C^2\sup|\tilde{\boldsymbol{\alpha}}|^4\right) \\ &\le CR^{\nu-2} \end{aligned} \tag{2.35}$$

and this concludes the proof of proposition 2.

To obtain our final expansion (2.4) of $\mathcal{I}_R$ we approximate the integral over the paths which leave Σ by the integral over the paths which cross a parabolic surface tangent to $\partial\Sigma$. To do this we shall assume that $f_{\mathbf{r}}(\tilde{\mathbf{x}}) \sim \frac{1}{2}\sum_{i=1}^{\nu-1}\kappa_i x_i^2$ for small $|\tilde{\mathbf{x}}|$:

Assumption A4

There exists $\eta > 0$ *and* $C > 0$ *such that for each* $\mathbf{r} \in \partial\Sigma$,

$$|f_{\mathbf{r}}(\tilde{\mathbf{x}}) - \frac{1}{2}\sum_{i=1}^{\nu-1}\kappa_i x_i^2| < C|\tilde{\mathbf{x}}|^3 \tag{2.36}$$

whenever $|\tilde{\mathbf{x}}| < \eta$.

PROPOSITION 3. – *Under the assumptions A1- A4*

$$\mathcal{I}_R = \mathcal{I}_R^{(0)} + \mathcal{I}_R^{(1)} + \mathcal{I}_R^{(2)} + R^{\nu-2}\epsilon_R \tag{2.37}$$

with

$$\mathcal{I}_R^{(2)} = -R^{\nu-2}\int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}F(\boldsymbol{\alpha})\frac{1}{2}\sum_{i=1}^{\nu-1}\kappa_i\big((\boldsymbol{\alpha}\cdot\mathbf{t}_i)^2(\tau_{\mathbf{n}}) - (\boldsymbol{\alpha}\cdot\mathbf{n})^2(\tau_{\mathbf{n}})\big) \tag{2.38}$$

where $\tau_{\mathbf{n}}$ *is such that* $(\boldsymbol{\alpha}\cdot\mathbf{n})(\tau_{\mathbf{n}}) = \inf\boldsymbol{\alpha}\cdot\mathbf{n}$ *and* $\epsilon_R \to 0$ *as* $R \to \infty$.

Proof. – We set $\boldsymbol{\alpha}\cdot\mathbf{t_i} = \alpha_i$. We write

$$I_7 = I_9 + I_{10} \tag{2.39}$$

where

$$\begin{aligned} I_9 &= R^{\nu-2}\int_{\partial\Sigma} d\sigma \int d\boldsymbol{\alpha}F(\boldsymbol{\alpha})\Delta_R(\boldsymbol{\alpha})\left(\frac{1}{2}\sum_{i=1}^{\nu-1}\kappa_i\alpha_i^2(\tau_{\mathbf{n}})\right) \\ &= R^{\nu-2}\int_{\partial\Sigma} d\sigma \int d\boldsymbol{\alpha}F(\boldsymbol{\alpha})\left(\frac{1}{2}\sum_{i=1}^{\nu-1}\kappa_i\alpha_i^2(\tau_{\mathbf{n}})\right) + O(e^{-CR^2}) \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} I_{10} = {} & -R^{\nu-1}\int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}F(\boldsymbol{\alpha})\Delta_R(\alpha) \\ & \times\left(\frac{1}{2}R^{-1}\sum_{i=1}^{\nu-1}\kappa_i\alpha_i^2(\tau_{\mathbf{n}}) + \min\{\delta R,\ \inf(\boldsymbol{\alpha}\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\boldsymbol{\alpha})) - \inf(\boldsymbol{\alpha}\cdot\mathbf{n})\}\right) \end{aligned} \tag{2.41}$$

and show that $I_{10} = o(R^{\nu-1})$. Arguing as we did from (2.26) to (2.31), we conclude that the dominant contribution in (2.41) is obtaind when (2.31) holds. Now

$$\begin{aligned}\frac{1}{2}R^{-1}\sum_{i=1}^{\nu-1}\kappa_i\alpha_i^2(\tau_{\mathbf{n}}) + \inf(\boldsymbol{\alpha}\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})) - \inf(\boldsymbol{\alpha}\cdot\mathbf{n}) \\ \leq R\left(\frac{1}{2}\sum_{i=1}^{\nu-1}\kappa_i R^{-2}\alpha_i^2(\tau_{\mathbf{n}}) - f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}(\tau_{\mathbf{n}}))\right) \\ \leq CR^{-2}\sup|\tilde{\alpha}|^3\end{aligned} \tag{2.42}$$

On the other hand

$$\begin{aligned}\frac{1}{2}R^{-1}\sum_{i=1}^{\nu-1}\kappa_i\alpha_i^2(\tau_{\mathbf{n}}) + \inf(\boldsymbol{\alpha}\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}})) - \inf(\boldsymbol{\alpha}\cdot\mathbf{n}) \\ \geq R\left(\frac{1}{2}\sum_{i=1}^{\nu-1}\kappa_i R^{-2}\alpha_i^2(\tau_R) - f_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}(\tau_R))\right) \\ + \frac{1}{2}R^{-1}\sum_{i=1}^{\nu-1}\kappa_i(\alpha_i^2(\tau_{\mathbf{n}}) - \alpha_i^2(\tau_R)) \\ \geq -CR^{-2}\sup|\tilde{\alpha}|^3 - \frac{1}{2}\kappa_0 R^{-1}\sum_{i=1}^{\nu-1}|\alpha_i^2(\tau_{\mathbf{n}}) - \alpha_i^2(\tau_R)|\end{aligned} \tag{2.43}$$

Therefore

$$\begin{aligned}R^{2-\nu}|I_{10}| \leq & R^{-1}\int_{\partial\Sigma} d\sigma C\int D\boldsymbol{\alpha}F(\boldsymbol{\alpha})\sup|\tilde{\boldsymbol{\alpha}}|^3 \\ & + \frac{1}{2}\kappa_0\int_{\partial\Sigma} d\sigma\int D\boldsymbol{\alpha}F(\boldsymbol{\alpha})\sum_{i=1}^{\nu-1}|\alpha_i^2(\tau_{\mathbf{n}}) - \alpha_i^2(\tau_R)|\end{aligned} \tag{2.44}$$

The inequality (2.27) implies

$$|(\boldsymbol{\alpha}\cdot\mathbf{n})(\tau_R) - (\boldsymbol{\alpha}\cdot\mathbf{n})(\tau_{\mathbf{n}})| \leq 2R^{-1}C\sup|\tilde{\boldsymbol{\alpha}}|^2 \tag{2.45}$$

Therefore if τ_∞ is a limit point of $\{\tau_R\}$

$$(\boldsymbol{\alpha}\cdot\mathbf{n})(\tau_\infty) = (\boldsymbol{\alpha}\cdot\mathbf{n})(\tau_{\mathbf{n}}) \tag{2.46}$$

Now $\tau_{\mathbf{n}}$ is unique (*see* sect. 2.8 D in [7]), therefore $\tau_R \to \tau_{\mathbf{n}}$ as $R \to \infty$. Thus $R^{2-\nu}I_{10} \to 0$ as $R \to \infty$.

Note that a_1 in (2.33) is equal to $(\nu - 1)\kappa_m$ where κ_m is the mean curvature,

$$\kappa_m = (\nu - 1)^{-1} \sum_{i=1}^{\nu-1} \kappa_i \tag{2.47}$$

and break up I_8 into three parts

$$I_8 = I_{11} + I_{12} + I_{13} \tag{2.48}$$

where

$$\begin{aligned} I_{11} =& -\frac{1}{2}R^{\nu-2}(\nu - 1) \int_{\partial\Sigma} d\sigma\kappa_m \int D\boldsymbol{\alpha}F(\boldsymbol{\alpha})\ \Delta_R(\boldsymbol{\alpha})(\inf(\boldsymbol{\alpha}\cdot\mathbf{n}))^2 \\ =& -\frac{1}{2}R^{\nu-2}(\nu - 1) \int_{\partial\Sigma} d\sigma\kappa_m \int D\boldsymbol{\alpha}F(\boldsymbol{\alpha})(\inf(\boldsymbol{\alpha}\cdot\mathbf{n}))^2 + O(e^{-CR^2}) \end{aligned} \tag{2.49}$$

$$\begin{aligned} I_{12} =& \frac{1}{2}R^{\nu-2}(\nu - 1) \int_{\partial\Sigma} d\sigma\kappa_m \\ & \times \int D\boldsymbol{\alpha}F(\boldsymbol{\alpha})\Delta_R(\boldsymbol{\alpha}) \\ & ((\min\{\delta R, -\inf(\boldsymbol{\alpha}\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\tilde{\boldsymbol{\alpha}}))\})^2 - (\inf(\boldsymbol{\alpha}\cdot\mathbf{n}))^2) \end{aligned} \tag{2.50}$$

and

$$\begin{aligned} I_{13} =& -R^{\nu} \int_{\partial\Sigma} d\sigma \int_0^{\delta} dx \left(\sum_{k=2}^{\nu-1} a_k x^k\right) \\ & \times \int_{x+\inf(R^{-1}\boldsymbol{\alpha}\cdot\mathbf{n}-f_{\mathbf{r}}(R^{-1}\boldsymbol{\alpha}))<0} D\boldsymbol{\alpha}F(\boldsymbol{\alpha})\Delta_R(\boldsymbol{\alpha}) \end{aligned} \tag{2.51}$$

We deal with I_{12} as we dealt with I_7 (*see* (2.26) to (2.31)). The contribution to I_{12} from those α's such that $\sup|\alpha| \leq R^{\frac{1}{4}}$ is bounded by

$$\begin{aligned} & C\frac{1}{2}R^{\nu-3}(\nu - 1)\kappa_0 \int_{\partial\Sigma} d\sigma c \\ & \quad \times \int D\boldsymbol{\alpha}F(\boldsymbol{\alpha}) \sup|\tilde{\alpha}|^2 (2\sup|\boldsymbol{\alpha}\cdot\mathbf{n}| + CR^{-1}\sup|\tilde{\boldsymbol{\alpha}}|^2) \\ & \leq CR^{\nu-3} \end{aligned} \tag{2.52}$$

and the contribution to I_{12} from α's such that $\sup|\alpha| > R^{\frac{1}{4}}$ is clearly bounded by $Ce^{-CR^{\frac{1}{2}}}$. In I_{13}

$$\left|\sum_{k=2}^{\nu-1} a_k x^k\right| \leq x^2 \sum_{k=2}^{\nu-1} |a_k|\delta^{k-2} \leq Cx^2 \tag{2.53}$$

since curvatures are uniformly bounded by κ_0. Therefore

$$|I_{13}| \leq \frac{1}{3} R^{\nu-3} C \int_{\partial\Sigma} d\sigma \times \int D\alpha F(\alpha)\Delta_R(\alpha)|\inf(\alpha\cdot\mathbf{n} - Rf_{\mathbf{r}}(R^{-1}\tilde{\alpha})|^3 \leq CR^{\nu-3} \tag{2.54}$$

This completes the proof of proposition 3.

We remark that the conditions A2-A4 that we assume here can be weakened, for example, in the linear bound (2.16) we can make both η and C depend on $\mathbf{r}$ with the condition that $\int_{\partial\Sigma} d\sigma C^2 < \infty$ and $\int_{\partial\Sigma} d\sigma e^{-K\eta^2 t^2} < e^{-|t|}|\partial\Sigma|$ for all $t \in$ R and similarly for the other two conditions (2.20) and (2.36). However for the sake of making the exposition simple we stated the results with the stronger conditions η and C independent of $\mathbf{r}$.

If the functional $F(\alpha)$ is rotation invariant, the corrections $\mathcal{I}_R^{(1)}$ and $\mathcal{I}_R^{(2)}$ take a simpler form. Since $D\alpha$ is also rotation invariant, $\int D\alpha F(\alpha)\inf(\alpha\cdot\mathbf{n})$ is independent of the orientation of $\mathbf{n}$ at the surface, and thus can be evaluated in (2.22) once for all for a fixed unit vector $\mathbf{k}_1$. In the same way, the functional integral occurring in (2.38) is independent of the point on the surface and can all be computed for an arbitrary fixed pair of unit orthogonal vectors $\mathbf{k}_1, \mathbf{k}_2$. Thus we have

COROLLARY. – *Assume that $F(\alpha)$ is invariant under rotations and let $\mathbf{k}_1, \mathbf{k}_2$ be any fixed pair of orthogonal unit vectors, then*

$$\mathcal{I}_R^{(1)} = R^\nu|\partial\Sigma| \int D\alpha F(\alpha)\inf(\alpha\cdot\mathbf{k}_1) \tag{2.55}$$

$$\mathcal{I}_R^{(2)} = -R^{\nu-2}\left(\int_{\partial\Sigma} d\sigma \frac{1}{2}\sum_{i=1}^{\nu-1}\kappa_i\right)\int D\alpha F(\alpha)\big((\alpha\cdot\mathbf{k}_2)^2(\tau_1) - (\alpha\cdot\mathbf{k}_1)^2(\tau_1)\big) \tag{2.56}$$

with $(\alpha\cdot\mathbf{k}_1)(\tau_1) = \inf(\alpha\cdot\mathbf{k}_1)$.

In particular, in two dimensions, the term $\mathcal{I}_R^{(2)}$ is purely topological since by the Gauss-Bonnet theorem $\int_{\partial\Sigma} d\sigma\kappa = 2\pi(1-m)$ if Σ has m holes.

Finally, the familiar case of the "small time" β expansion of the statistical sum for the Laplacian is obtained by setting $F(\boldsymbol{\alpha}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{\nu}$ and $R = \beta^{-\frac{1}{2}}$ in (2.55) and (2.56) (*see* (1.3) and (1.7), and the discussion of scalings in the introduction). Then the functional integral occurring in (2.55) equals (using $\inf(-g) = -\sup g$)

$$-\left(\frac{1}{\sqrt{2\pi}}\right)^{\nu} \int D\alpha_1 \sup \alpha_1 = -\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{2\pi}}\right)^{\nu} \tag{2.57}$$

and that occurring in (2.56) equals, using (C.2) and (C.3)

$$\begin{aligned}&\left(\frac{1}{\sqrt{2\pi}}\right)^{\nu}\left(\int D\alpha_1 \int D\alpha_2\, \alpha_2^2(\tau_1) - \int D\alpha_1 (\sup \alpha_1)^2\right)\\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^{\nu} \int D\alpha_1 (\tau_1(1-\tau_1) - (\sup \alpha_1)^2) = -\frac{1}{3}\left(\frac{1}{\sqrt{2\pi}}\right)^{\nu}\end{aligned} \tag{2.58}$$

With (2.57) and (2.58) one recovers the usual first terms of the "small time" expansion of the heat kernel.

III. APPLICATION TO THE MAGNETIC SYSTEM

We apply the results of the previous section to a system of free particles with Maxwell-Boltzmann statistics in presence of a constant field $\mathbf{B}$ in dimensions $\nu = 2, 3$. The grand-canonical potential $\mathcal{Q}_R$ corresponding to the Hamiltonian (1.1) is given by

$$\begin{aligned}\mathcal{Q}_R &= \frac{z}{\beta}\mathrm{Tr}\exp(-\beta H_R) = \frac{z}{\beta}\mathcal{I}_{\lambda^{-1}R}\\ &= \frac{z}{\beta}(\mathcal{I}^{(0)}_{\lambda^{-1}R} + \mathcal{I}^{(1)}_{\lambda^{-1}R} + \mathcal{I}^{(2)}_{\lambda^{-1}R} + o(R^{\nu-2}))\end{aligned} \tag{3.1}$$

In view of (1.2), $\mathcal{I}_{\lambda^{-1}R}$ is an integral of the form (2.1) (with R replaced by $\lambda^{-1}R = \varepsilon^{-1}$) where the functional $F(\boldsymbol{\alpha})$ is now

$$F(\boldsymbol{\alpha}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{\nu} \exp\left(-i\frac{\mu}{2}\mathbf{b}\cdot\int \boldsymbol{\alpha}\wedge \mathbf{d}\boldsymbol{\alpha}\right) = F(-\boldsymbol{\alpha}) \tag{3.2}$$

and it can be expanded according to the Propositions 1-3. The volume term of Proposition 1 is the well known Landau bulk pressure

$$p^{(0)} = \frac{z}{\beta} \lim_{R\to\infty} \frac{1}{|\Sigma_R|} \mathcal{I}^{(0)}_{\lambda^{-1}R} = \frac{z}{\beta}\left(\frac{1}{\sqrt{2\pi}\lambda}\right)^{\nu} \int D\alpha \exp\left(-\frac{\mu^2}{2}G(\alpha)\right) \tag{3.3}$$

where, for a one dimensional Brownian bridge α

$$G(\alpha) = \int_0^1 ds\alpha^2(s) - \left(\int_0^1 ds\alpha(s)\right)^2$$

It will not be further discussed here (*see* (A.1), as well as Proposition 1 and formula (2.19) in [1]).

To express conveniently the surface contribution $(z/\beta)\mathcal{I}^{(1)}_{\lambda^{-1}R}$, let us define, for any unit vector $\mathbf{k}$, the following functional integral, depending on the two vectors $\mu\mathbf{b}$ and $\mathbf{k}$

$$\Pi^{(1)}(\mathbf{k}) = \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \sup(\boldsymbol{\alpha}\cdot\mathbf{k}) = -\int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \inf(\boldsymbol{\alpha}\cdot\mathbf{k}) \tag{3.4}$$

The second equality follows from the fact that $F(\alpha)$ is an even function of α (together with $\sup(-g) = -\inf g$), and for the same reason $\Pi^{(1)}(\mathbf{k}) = \Pi^{(1)}(-\mathbf{k})$. Since $\Pi^{(1)}(\mathbf{k})$ is a real quantity (*see* for instance (3.6) bellow), it depends on the magnetic phase only through $\cos\left(\frac{\mu}{2}\mathbf{b}\cdot\int\boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}\right)$, so $\Pi^{(1)}(\mathbf{k})$ is also an even function of μ. Morover, because of the rotational invariance of the measure $D\boldsymbol{\alpha}$, $\Pi^{(1)}(\mathbf{k})$ depends only on μ and on the projection $\mu\mathbf{b}\cdot\mathbf{k}$ of the magnetic field on the direction $\mathbf{k}$. Then, according to the result of Proposition 2, one can define a surface pressure $p^{(1)}$, expressed as a surface integral

$$p^{(1)} = \frac{z}{\beta} \lim_{R\to\infty} \frac{1}{|\partial\Sigma_R|} \mathcal{I}^{(1)}_{\lambda^{-1}R} \tag{3.5}$$

$$= \frac{1}{|\partial\Sigma|} \int_{\partial\Sigma} d\sigma p^{(1)}(\mathbf{n}) \tag{3.6}$$

with

$$p^{(1)}(\mathbf{n}) = -\frac{z}{\beta\lambda^{\nu-1}} \Pi^{(1)}(\mathbf{n}) \tag{3.7}$$

and $\mathbf{n}$ is the inward unit normal at the point σ of the surface; $p^{(1)}(\mathbf{n})$ is the interface pressure between a magnetized half-space and empty space. The following simple interpretation of $\Pi^{(1)}(\mathbf{k})$ can be given: it is (up to the factor $z/\beta\lambda^{\nu-1}$) the excess interface pressure obtained by cutting the infinitely extended system into two half-spaces with a Dirichlet wall having normal $\mathbf{k}$. To see this we choose an orthonormal system of axis $\mathbf{k_j}, j = 1,\ldots,\nu$, such that $\mathbf{k_1} = \mathbf{k}$, and consider two cubic boxes $\Lambda_L^{\pm}$ of volume L^ν

$$\Lambda_L^{\pm} = \left\{\mathbf{x} \mid 0 \leq \pm x_1 \leq L,\ -\frac{L}{2} \leq x_j \leq \frac{L}{2},\ j = 2,\ldots,\nu\right\}$$

The excess interface pressure is then defined as

$$\Pi^{(1)}(\mathbf{k}) = \lim_{L\to\infty} \frac{1}{2L^{\nu-1}}\left(\mathcal{I}_{\Lambda_L^+\bigcup\Lambda_L^-} - \mathcal{I}_{\Lambda_L^+} - \mathcal{I}_{\Lambda_L^-}\right) \tag{3.8}$$

$\mathcal{I}_{\Lambda_L^+\bigcup\Lambda_L-}$ can be split into the sum of two contributions: (i) the paths that start in Λ_L^+ or in Λ_L^- and remain in the same box; (ii) the paths that start in one box and visit the other box. The contribution (i) is exactly $\mathcal{I}_{\Lambda_L^+} + \mathcal{I}_{\Lambda_L^-}$, thus the quantity (3.7) is given by the contribution (ii).The paths starting in Λ_L^- and visiting Λ_L^+ give the contribution

$$\int D\boldsymbol{\alpha} F(\alpha) \int_{-L}^{0} dx_1 \theta(x_1 + \sup\alpha_1)$$
$$\times \prod_{j=2}^{\nu} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx_j \theta\left(x_j + \inf\alpha_j + \frac{L}{2}\right)\theta\left(\frac{L}{2} - x_j - \sup\alpha_j\right)$$

As $L \to \infty$, the constraints on the components $\alpha_j, j = 2,\ldots,\nu$, of the path become irrelevant, and this quantity behaves asymptotically as $L^{\nu-1}\int D\alpha F(\alpha)\sup\alpha_1$. Since the paths starting in Λ_L^+ give the same contribution, the excess interface pressure (3.8) is indeed given by (3.4). Hence the surface pressure $p^{(1)}$ can be calculated from (3.6) once the planar interface pressure is known for general orientations of $\mathbf{b}$ and $\mathbf{k}$.

Let us consider in dimension $\nu = 3$ the two special cases $\Pi_{\|}^{(1)}(\mu)$ and $\Pi_{\perp}^{(1)}(\mu)$ obtained by taking $\mathbf{b}$ parallel or perpendicular to $\mathbf{k}$ (say $\mathbf{k} = \mathbf{b} = \mathbf{k_1}$ or $\mathbf{k} = \mathbf{k_1},\ \mathbf{b} = \mathbf{k_2}$) giving

$$\begin{aligned}\Pi_{\|}^{(1)}(\mu) &= \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int D\boldsymbol{\alpha} \exp\left(-i\mu\int \alpha_2 d\alpha_3\right)\sup\alpha_1 \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int D\alpha_1 \sup\alpha_1 \int D\alpha_2 \exp\left(-\frac{\mu^2}{2}G(\alpha_2)\right)\end{aligned} \tag{3.9}$$

$$\begin{aligned}\Pi_{\perp}^{(1)}(\mu) &= \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int D\boldsymbol{\alpha} \exp\left(i\mu\int \alpha_1 d\alpha_3\right)\sup\alpha_1 \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int D\alpha_1 \sup\alpha_1 \exp\left(-\frac{\mu^2}{2}G(\alpha_1)\right)\end{aligned} \tag{3.10}$$

The expressions (3.9) and (3.10) where obtained by performing the Gaussian integration over α_3 with the help of (A.1). The parallel component can be computed explicitly (*see* (C.3) and sect.15 in [6])

$$\Pi_{\parallel}^{(1)}(\mu) = \frac{\mu}{16\pi \sinh \frac{\mu}{2}} = \frac{1}{\pi} - \frac{\mu^2}{192\pi} + O(\mu^4) \qquad (3.11)$$

We can also compute the perpendicular component up to second order in μ (*see* appendix D)

$$\begin{aligned}\Pi_{\perp}^{(1)}(\mu) =&\left(\frac{1}{\sqrt{2\pi}}\right)^3 \left(\int D\alpha_1 \sup \alpha_1 - \frac{\mu^2}{2}\int D\alpha_1 \sup \alpha_1 G(\alpha_1) + O(\mu^4)\right)\\ =&\frac{1}{8\pi} - \frac{3}{512\pi}\mu^2 + O(\mu^4) \end{aligned} \qquad (3.12)$$

This enables us to compute $\Pi^{(1)}(\mathbf{k})$ to second order in μ. Because of the symmetries of $\Pi^{(1)}(\mathbf{k})$ (even in μ and depending only on μ and $\mu\mathbf{b}\cdot\mathbf{k}$), its expansion to second order in μ is necessarily of the form

$$\Pi^{(1)}(\mathbf{k}) = a + b\mu^2 + c\mu^2(\mathbf{b}\cdot\mathbf{k})^2 + O(\mu^4) \qquad (3.13)$$

The coefficients a, b and c in (3.13) can then be found from (3.11) and (3.12) giving

$$\Pi^{(1)}(\mathbf{k}) = \frac{1}{8\pi}\left(1 - \frac{3}{64}\mu^2 + \frac{1}{192}\mu^2(\mathbf{b}\cdot\mathbf{k})^2\right) + O(\mu^4) \qquad (3.14)$$

and hence, from (3.6), (3.7) one finds the surface pressure to second order in μ. In particular, the surface magnetic susceptibility $\chi^{(1)}$ at $B = 0$ is

$$\chi^{(1)} = \frac{\partial^2 p^{(1)}}{\partial B^2}\,|_{B=0}= \frac{ze^2}{mc^2}\frac{1}{256\pi}\left(3 - \frac{1}{3|\partial\Sigma|}\int_{\partial\Sigma} d\sigma(\mathbf{b}\cdot\mathbf{n})^2\right) \qquad (3.15)$$

The Fermi-Dirac statistics can be incorporated by replacing (3.1) by

$$\mathcal{Q}_R^{FD} = \frac{1}{\beta}\mathrm{Trln}(1 + z\exp(-\beta H_R)) = \sum_{n=1}^{\infty}(-z)^{n-1}\mathcal{Q}_R(n\beta) \qquad (3.16)$$

Thus the Fermi-Dirac susceptibility is

$$\chi^{FD(1)} = \sum_{n=1}^{\infty}(-z)^{n-1}\chi^{(1)}(n\beta) = \frac{1}{1+z}\chi^{(1)} \tag{3.17}$$

since the Maxwell-Boltzmann susceptibility (3.15) is independent of β. The expression (3.17) multiplied by 2 to take into account of the two spin states has been found by Kunz (formula (5.27) in [2]).

In two dimensions, $\mathbf{b}$ and $\mathbf{n}$ are orthogonal, thus $\Pi^{(1)}(\mathbf{n})$ is independent of $\mathbf{n}$ and reduces to the perpendicular component (3.10) with $(\sqrt{2\pi})^{-3}$ replaced by $(\sqrt{2\pi})^{-2}$. Therefore, we see from (3.6), (3.7) that the two-dimensional pressures $p^{(1)}$ and $p^{(1)}(\mathbf{n})$ are identical, with

$$p^{(1)} = p^{(1)}(\mathbf{n}) = -\sqrt{2\pi}\frac{z}{\beta\lambda}\Pi_{\perp}^{(1)}(\mu) \tag{3.18}$$

with $\Pi_{\perp}^{(1)}(\mu)$ given in (3.10).The associated susceptibilities are

$$\chi^{(1)} = \frac{3}{256}\frac{z\hbar e^2}{c^2}\left(\frac{2\beta}{\pi m^3}\right)^{\frac{1}{2}} \tag{3.19}$$

$$\chi^{FD(1)} = z^{-1}f_{-\frac{1}{2}}(z)\chi^{(1)}, \quad f_\rho(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^\rho}z^n \tag{3.20}$$

The next term $(z/\beta)\mathcal{I}^{(2)}_{\lambda^{-1}R}$ in the expansion of the grand-canonical potential can be expressed by the functional integral of Proposition 3. It can be calculated for various shapes when the quantities

$$\Pi^{(2)}(\mathbf{k_1},\mathbf{k_2}) = \int D\alpha F(\alpha)\left[(\alpha\cdot\mathbf{k_2})^2(\tau_1) - (\alpha\cdot\mathbf{k_1})^2(\tau_1)\right]$$
$$\inf(\alpha\cdot\mathbf{k_1}) = (\alpha\cdot\mathbf{k_1})(\tau_1) \tag{3.21}$$

are known for any pair of orthogonal unit vectors $\mathbf{k_1},\mathbf{k_2}$. In two dimensions, since $F(\alpha)$ is invariant under rotations around $\mathbf{b}$, this correction is again universal in the sense that it depends only on the topology of the sample, but not on its shape (*see* end of section 2). At the moment, $\Pi^{(2)}(\mathbf{k_1},\mathbf{k_2})$ is only known explicitly at zero field (*see* (2.58)).

To conclude this section, we emphasize that we have worked in the grand-canonical ensemble. If we want to compute physical quantities at fixed averaged density ρ

$$\rho = \beta z\frac{\partial}{\partial z}\mathcal{Q}_R \tag{3.22}$$

we must first find the activity z_R as a function of the size R by solving (3.22) at fixed ρ, and then expand jointly the potential $\mathcal{Q}_R(z_R)$ for large R.

IV. RELATION BETWEEN MAGNETIZATION AND CURRENT IN A HALF SPACE

In this section, we establish various relations between the current density and the thermodynamical magnetization for matter contained in a half space system in three dimensions, limited by a plane whose normal is $\mathbf{n}$ pointing into the system. In the grand canonical ensemble, the average of the quantum mechanical current operator density $\frac{e}{2m}(\mathbf{v}\delta(\mathbf{q}-\mathbf{r})+\delta(\mathbf{q}-\mathbf{r})\mathbf{v})$, with $\mathbf{q}$ the position and $\mathbf{v}=-i\hbar\nabla-\frac{e}{c}\mathbf{A}(\mathbf{r})$ the velocity operator, is expressed by

$$\mathbf{j}(\mathbf{r})=\frac{e}{m}\mathcal{R}e\Big[\Big(-i\hbar\nabla-\frac{e}{c}\mathbf{A}(\mathbf{r})\Big)\rho(\mathbf{r},\mathbf{r}')\Big]_{\mathbf{r}=\mathbf{r}'} \qquad (4.1)$$

where $\rho(\mathbf{r},\mathbf{r}')$ is the one particle reduced density matrix. For the half space system, it has the functional integral representation (sect. II in [1])

$$\rho(\mathbf{r},\mathbf{r}')=\frac{z}{(2\pi)^{3/2}}\exp\Big[-\frac{|\mathbf{r}-\mathbf{r}'|^2}{2}-i\frac{\mu}{2}\mathbf{b}\cdot(\mathbf{r}\wedge\mathbf{r}')\Big]$$
$$\times\int_{\mathbf{r}\cdot\mathbf{n}+\inf(\boldsymbol{\alpha}\cdot\mathbf{n})>0} D\boldsymbol{\alpha}$$
$$\times\exp\Big[-i\frac{\mu}{2}\mathbf{b}\int\boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}-i\mu\mathbf{b}\cdot\Big((\mathbf{r}-\mathbf{r}')\wedge\int_0^1 ds\boldsymbol{\alpha}(s)\Big)\Big] \qquad (4.2)$$

In (4.2) and the rest of this section, we set $\lambda=1$, *i.e.* we measure all lengths in the unit of the de Broglie length. Because of translation invariance, the current at a point inside the half space depends only on the distance $x=\mathbf{r}\cdot\mathbf{n}$ of the point from the limiting plane, and the current has no component orthogonal to the plane. Working out the components of $\mathbf{j}$ that are parallel to the plane, one finds from (4.1) and (4.2) that the current in the semi-infinite system with inward normal $\mathbf{n}$ is given by

$$\mathbf{j}(x,\mathbf{n})=-\gamma\mathcal{R}e\int_{x+\inf(\boldsymbol{\alpha}.\mathbf{n})>0}D\boldsymbol{\alpha}\,F(\boldsymbol{\alpha})\,\mathbf{b}\wedge\int_0^1 ds\boldsymbol{\alpha}(s),\qquad \gamma=\frac{\mu\hbar ez}{m} \qquad (4.3)$$

where $F(\boldsymbol{\alpha})$ is the functional (3.2).

The direction of $\mathbf{j}(x,\mathbf{n})$ is along $\mathbf{n}\wedge\mathbf{b}$ in the plane (*see* (4.6) below). There is no current in the bulk : if one removes the constraint $x+\inf(\boldsymbol{\alpha}.\mathbf{n})>0$ in (4.3) by letting $x\to\infty$, the integral vanishes since the integrand is an odd function of $\boldsymbol{\alpha}$. In fact, $\mathbf{j}(x,\mathbf{n})=O(e^{-Cx^2})$, $x\to\infty$, as in Proposition 2 of ref. [1].

The following propositions 4 and 5 relate the total integral of the current to the bulk magnetization.

PROPOSITION 4. – *Let* $\mathbf{m}^{(0)} = \nabla_{\mathbf{B}} p^{(0)}$ *be the bulk (Landau) magnetisation, then*

$$\int_0^\infty dx\, \mathbf{j}(x, \mathbf{n}) = c\, \mathbf{n} \wedge \mathbf{m}^{(0)} \tag{4.4}$$

Proof. – Let $\mathbf{k}_j$, $j = 1, 2, 3$ be the orthonormal right handed triad of unit vectors with $\mathbf{k}_1 = \mathbf{b}$ and $\mathbf{k}_2$ along $\mathbf{b} \wedge \mathbf{n}$, and set $n_j = \mathbf{n} \cdot \mathbf{k}_j$, $\alpha_j = \boldsymbol{\alpha} \cdot \mathbf{k}_j$. Then

$$\mathbf{j}(x, \mathbf{n}) = \frac{-\gamma}{(2\pi)^{3/2}} \mathcal{R}e \int_{x > -\inf(n_1\alpha_1 + n_3\alpha_3)} D\boldsymbol{\alpha}\, e^{i\mu \int \alpha_3 d\alpha_2} \times \left(\mathbf{k}_2 \int_0^1 ds \alpha_3(s) - \mathbf{k}_3 \int_0^1 ds \alpha_2(s) \right) \tag{4.5}$$

$$= \frac{-\gamma}{(2\pi)^{3/2}} \mathbf{k}_2 \int_{x > -\inf(n_1\alpha_1 + n_3\alpha_3)} D\alpha_1 D\alpha_3\, e^{-\frac{\mu^2}{2} G(\alpha_3)} \int_0^1 ds \alpha_3(s) \tag{4.6}$$

The $\mathbf{k}_3$ component of $\mathbf{j}(x, \mathbf{n})$ in (4.5) vanishes because of the antisymmetry in α_2, and we can perform the α_2 integral by (A.1). Moreover, since the integral (4.6) without restriction vanishes because of the antisymmetry in α_3, we can replace the constraint by $x < -\inf(n_1\alpha_1 + n_3\alpha_3)$ and change the sign of the integral. Therefore

$$\int_0^\infty dx\, \mathbf{j}(x, \mathbf{n}) = \frac{-\gamma}{(2\pi)^{3/2}} \mathbf{k}_2 \int D\alpha_1 D\alpha_3 \times \inf(n_1\alpha_1 + n_3\alpha_3) \int_0^1 ds \alpha_3(s) e^{-\frac{\mu^2}{2} G(\alpha_3)} \tag{4.7}$$

We make the change of variable (B.1) $\boldsymbol{\alpha}(s) \to \boldsymbol{\alpha}(\widetilde{s+u}) - \boldsymbol{\alpha}(u)$ and integrate with respect to u; $G(\alpha_3)$ is invariant and $\inf(n_1\alpha_1 + n_3\alpha_3) \int_0^1 ds \alpha_3(s)$ changes to

$$\left(\inf(n_1\alpha_1 + n_3\alpha_3) - (n_1\alpha_1(u) + n_3\alpha_3(u)) \right) \left(\int_0^1 ds \alpha_3(s) - \alpha_3(u) \right)$$

When integrated on u, α_1 and α_3, the term involving $\inf(n_1\alpha_1 + n_3\alpha_3)$ does not contribute and the term linear in α_1 vanishes by antisymmetry. Noting also that $n_3 \mathbf{k}_2 = \mathbf{n} \wedge \mathbf{b}$ one finds

$$\int_0^\infty dx\, \mathbf{j}(x, \mathbf{n}) = \mathbf{b} \wedge \mathbf{n} \frac{\gamma}{(2\pi)^{3/2}} \int D\alpha_3 G(\alpha_3) e^{-\frac{\mu^2}{2} G(\alpha_3)} \tag{4.8}$$

By differentiating (3.3) with respect to $\mathbf{B}$, we recognize that the right hand side in (4.8) is precisely $c\mathbf{n} \wedge \nabla_{\mathbf{B}} p^{(0)} = c\mathbf{n} \wedge \mathbf{m}^{(0)}$.

COROLLARY 1. – *Let the surface $\partial\Sigma$ be parametrized (in $\mathbf{R}^3$) by $\mathbf{r}(\sigma)$, $\sigma \in \partial\Sigma$. Then*

$$\mathbf{m}^{(0)} = \frac{1}{2c|\Sigma|} \int_{\partial\Sigma} d\sigma \int_0^\infty dx\, \mathbf{r}(\sigma) \wedge \mathbf{j}(x, \mathbf{n}(\sigma)) \tag{4.9}$$

Proof. – In view of (4.4), the right hand side of (4.9) is equal to

$$\begin{aligned}
&\frac{1}{2|\Sigma|} \int_{\partial\Sigma} d\sigma\, \mathbf{r}(\sigma) \wedge (\mathbf{n}(\sigma) \wedge \mathbf{m}^{(0)}) \\
&= \frac{1}{2|\Sigma|} \int_{\partial\Sigma} d\sigma\, \mathbf{n}(\sigma)(\mathbf{r}(\sigma) \cdot \mathbf{m}^{(0)}) - \frac{1}{2|\Sigma|} \int_{\partial\Sigma} d\sigma (\mathbf{n}(\sigma) \cdot \mathbf{r}(\sigma))\mathbf{m}^{(0)} \\
&= \frac{1}{2|\Sigma|} \int_{\Sigma} d\mathbf{r}(-\nabla(\mathbf{r} \cdot \mathbf{m}^{(0)}) + (\nabla \cdot \mathbf{r})\mathbf{m}^{(0)}) = \mathbf{m}^{(0)}
\end{aligned}$$

by Gauss divergence theorem.

In two dimensions, $\mathbf{n}$ is always orthogonal to $\mathbf{b}$, $\mathbf{k}_2 = \mathbf{b} \wedge \mathbf{n}$, and the current magnitude $j(x) = \mathbf{k}_2 \cdot \mathbf{j}(x, \mathbf{n})$ is independent of $\mathbf{n}$. Then it is easy to check that both Proposition 4 and its corollary reduce to the relation $\int_0^\infty j(x) = -cm^0$, already found in [1], where $m^0 = \mathbf{b} \cdot \mathbf{m}^{(0)}$ is the scalar value of the two dimensional bulk magnetization $\mathbf{m}^{(0)}$.

The formulae (4.4) and (4.9) give the proper relations between the bulk magnetization and the current density flowing in the neighborhood of a point σ of the surface in the three dimensional system. The total integral of the current is still related to the bulk magnetization by (4.4). The equality (4.9) is the precise form that the familiar relation $M_\Sigma = \frac{1}{2c} \int_\Sigma d\mathbf{r}\mathbf{r} \wedge \mathbf{j}_\Sigma(\mathbf{r})$ between finite volume magnetization and current takes in the thermodynamic limit.

PROPOSITION 5. – *Let $\mathbf{m}^{(1)}(\mathbf{n}) = \nabla_{\mathbf{B}} p^{(1)}(\mathbf{n})$ be the magnetization associated with the planar interface pressure (3.7). Then*

$$\int_0^\infty dx\, x\mathbf{j}(x, \mathbf{n}) = -c\mathbf{n} \wedge \mathbf{m}^{(1)}(\mathbf{n}) \tag{4.10}$$

Proof. – Proceeding with the same arguments used in (4.5)-(4.7), one obtains

$$\begin{aligned}
\int_0^\infty dx\, x\mathbf{j}(x, \mathbf{n}) = \frac{\gamma}{2(2\pi)^{3/2}} \mathbf{k}_2 \int D\alpha_1 D\alpha_3 (\inf(n_1\alpha_1 + n_3\alpha_3))^2 \\
\times \int_0^1 ds \alpha_3(s) e^{-\frac{\mu^2}{2} G(\alpha_3)}
\end{aligned} \tag{4.11}$$

After the transformation $\boldsymbol{\alpha}(s) \to \boldsymbol{\alpha}(\widetilde{s+u}) - \boldsymbol{\alpha}(u)$, $(\inf(n_1\alpha_1 + n_3\alpha_3))^2 \int_0^1 ds\alpha_3(s)$ changes to

$$\left[(\inf(n_1\alpha_1 + n_3\alpha_3))^2 - 2\inf(n_1\alpha_1 + n_3\alpha_3)(n_1\alpha_1(u) + n_3\alpha_3(u))\right.$$
$$\left. + (n_1\alpha_1(u) + n_3\alpha_3(u))^2\right]\left(\int_0^1 ds\alpha_3(s) - \alpha_3(u)\right) \tag{4.12}$$

The first term in (4.12) vanishes once integrated on u, and the third term will not contribute because it is odd in α. Thus, introducing the definition (A.3)

$$\int_0^\infty dx\, x\mathbf{j}(x, \mathbf{n}) = \frac{\gamma}{(2\pi)^{3/2}}\mathbf{k}_2 \int D\alpha_1 D\alpha_3 \inf(n_1\alpha_1 + n_3\alpha_3)$$
$$\times G(\alpha_3, n_1\alpha_1 + n_3\alpha_3)e^{-\frac{\mu^2}{2}G(\alpha_3)} \int_0^1 ds\alpha_3(s) \tag{4.13}$$

It remains to identify $\nabla_{\mathbf{B}} p^{(1)}(\mathbf{n})$ with the right hand side of (4.13). From (3.7) one finds

$$\mathbf{m}^{(1)}(\mathbf{n}) = -\frac{i\gamma}{2\mu c}\int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha}) \inf(\boldsymbol{\alpha}\cdot\mathbf{n}) \int \boldsymbol{\alpha} \wedge d\boldsymbol{\alpha}$$
$$= -\frac{i\gamma}{\mu c}\frac{1}{(2\pi)^{3/2}}\int D\boldsymbol{\alpha}\, e^{i\mu\int \alpha_3 d\alpha_2} \inf(n_1\alpha_1 + n_3\alpha_3)$$
$$\times \left(\mathbf{k}_1 \int \alpha_2 d\alpha_3 + \mathbf{k}_2 \int \alpha_3 d\alpha_1 + \mathbf{k}_3 \int \alpha_1 d\alpha_2\right) \tag{4.14}$$

Since $\mathbf{m}^{(1)}(\mathbf{n})$ is a real quantity only $\sin\left(\int_0^1 \alpha_3(s) d\alpha_2\right)$ enters in the integral (4.14). This implies that the $\mathbf{k}_2$ component vanishes, the corresponding integrand being odd in α_2. We can perform the $D\alpha_2$ integration (calculating the first moment with the help of (A.2))

$$\mathbf{m}^{(1)}(\mathbf{n}) = -\frac{\gamma}{c}\frac{1}{(2\pi)^{3/2}}\int D\alpha_1 D\alpha_3\, e^{-\frac{\mu^2}{2}G(\alpha_3)} \inf(n_1\alpha_1 + n_3\alpha_3)$$
$$\times (\mathbf{k}_1 G(\alpha_3) - \mathbf{k}_3 G(\alpha_3, \alpha_1)) \tag{4.15}$$

Comparing (4.15) with (4.13) the result of the proposition follows from the observation that $\mathbf{n} \wedge \mathbf{k}_1 = n_3\mathbf{k}_2$, $\mathbf{n} \wedge \mathbf{k}_3 = -n_1\mathbf{k}_2$

It is worth noting that the magnetization $\mathbf{m}^{(1)}(\mathbf{n})$ at a point of the surface, although lying in the plane subtended by $\mathbf{n}$ and $\mathbf{b}$, is not directed along $\mathbf{b}$, but depends on the orientation of the surface element with respect to $\mathbf{b}$. However, the surface magnetization of the whole sample

$$\mathbf{m}^{(1)} = \nabla_{\mathbf{B}} p^{(1)} = \frac{1}{|\partial\Sigma|}\int_{\partial\Sigma} d\sigma \mathbf{m}^{(1)}(\mathbf{n})$$

is along $\mathbf{b}$.

In 2 dimensions, since there is no distinction between $p^{(1)}(\mathbf{n})$ and $p^{(1)}$ (*see* (3.18)), we have also the equality $\mathbf{m}^{(1)}(\mathbf{n}) = \mathbf{m}^{(1)}$. Then, the proposition 5 reduces to

$$\int_0^{\infty} dx\, xj(x) = cm^{(1)} \tag{4.16}$$

with $m^{(1)} = \mathbf{b}\cdot\mathbf{m}^{(1)}$ is the scalar surface magnetization, a relation already obtained in [2]. The first moment sum rule (4.10) is the proper generalization of (4.16) to three dimensions. In this case, it bears no direct relation to the three dimensional surface pressure $p^{(1)}$ given by (3.6).

We add that the equalities (4.4), (4.9) and (4.10) remain true if one includes Fermi statistics since an average quantity $\mathcal{O}_B(\beta)$ in a free gas with Boltzman statistics is related to the corresponding quantity $\mathcal{O}_{FD}(\beta)$ with Fermi-Dirac statistics by

$$\mathcal{O}_{FD}(\beta) = \sum_{n=1}^{\infty}(-z)^{n-1}\mathcal{O}_B(n\beta) \tag{4.17}$$

V. CORNERS

In this section we briefly discuss the case of non smooth boundaries constituted of polygons. For simplicity, we restrict the analysis to two dimensional regions Σ_R limited by convex polygonal contours with k faces of length L_j and obtuse angles θ_j, $j = 1, ..., k$, $\frac{\pi}{2} \leq \theta_j \leq \pi$. The orientation of the faces are given by their inward unit normals $\mathbf{n}_j$ with $\mathbf{n}_j \cdot \mathbf{n}_{j+1} = |\cos\theta_j|$, $j = 1, ..., k$ ($k+1$ identified to 1).

We write for a general square integrable functional $F(\boldsymbol{\alpha})$

$$\begin{aligned}\mathcal{I}_R &= \int_{\Sigma_R} d^2\mathbf{r}\int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha})\prod_{j=1}^{k}\theta(\mathbf{n}_j\cdot\mathbf{r} + \inf(\mathbf{n}_j\cdot\boldsymbol{\alpha}))\\ &= \int_{\Sigma_R} d^2\mathbf{r}\int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha})\prod_{j=1}^{k}(1-\theta(-\mathbf{n}_j\cdot\mathbf{r} - \inf(\mathbf{n}_j\cdot\boldsymbol{\alpha}))) \quad (5.1)\\ &= I_0 + I_1 + I_2 + \tilde{I} \quad (5.2)\end{aligned}$$

with

$$I_0 = \int_{\Sigma_R} d^2\mathbf{r} \int D\alpha\, F(\alpha) = R^\nu |\Sigma| \int D\alpha\, F(\alpha) \tag{5.3}$$

$$I_1 = -\sum_{j=1}^{k} \int_{\Sigma_R} d^2\mathbf{r} \int D\alpha\, F(\alpha)\, \theta(-\mathbf{n}_j \cdot \mathbf{r} - \inf(\mathbf{n}_j \cdot \alpha)) \tag{5.4}$$

$$I_2 = \sum_{j=1}^{k} \int_{\Sigma_R} d^2\mathbf{r} \int D\alpha\, F(\alpha)\, \theta(-\mathbf{n}_j \cdot \mathbf{r} - \inf(\mathbf{n}_j \cdot \alpha)) \times \theta(-\mathbf{n}_{j+1} \cdot \mathbf{r} - \inf(\mathbf{n}_{j+1} \cdot \alpha)) \tag{5.5}$$

In I_0, all restrictions on paths have been removed, so $I_0 = \mathcal{I}_R^{(0)}$ is equal to the bulk contribution (2.19) of proposition 1. The integrals I_1 and I_2 result of expanding the product in (5.1) and keeping the linear θ-constraints in (5.4) and quadratic θ-constraints in (5.5). In I_1, paths have to cross one face, in I_2 they have to cross two adjacent faces (*i.e.* to encircle a corner). Paths that contribute to the remainder $\tilde{I}$ have to cross at least two non adjacent faces, thus they must extend over distances of order R, and their contribution is $O(e^{-CR^2})$.

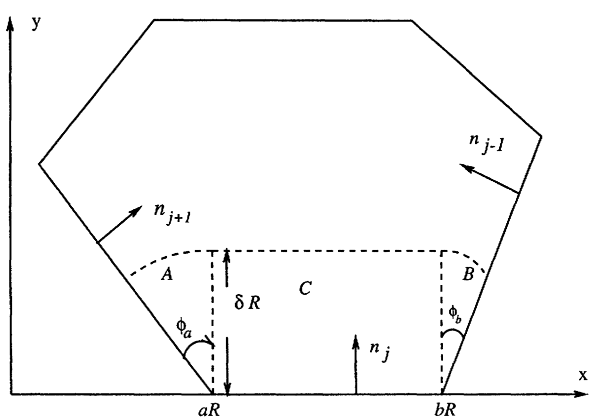

Figure 1.

We now consider one face of length $L = (b - a)R$ in between two corners of angles θ_a and θ_b. We set coordinates such that this face is along the x-axis and its normal along the y-axis. We associate to it the rectangle C of height δR, δ fixed, as well as the two sectors A and B of radii δR and angles $\phi_a = \theta_a - \frac{\pi}{2}$ and $\phi_b = \theta_b - \frac{\pi}{2}$ (*see* fig. 1). We decompose the term in (5.4) corresponding to this face, say I_{ab}, into the sum of the integrals of paths starting in the three regions plus a reminder $O(e^{-CR^2})$ consisting of paths that have to travel a distance at least δR,

$$I_{ab} = I_C + I_A + I_B + O(e^{-CR^2}) \tag{5.6}$$

Clearly

$$\begin{aligned} I_C &= -\int_{aR}^{bR} dx \int_0^{\delta} Rdy \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha})\, \theta(-y - \inf(\mathbf{n}\cdot\boldsymbol{\alpha})) \\ &= L \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha}) \inf(\mathbf{n}\cdot\boldsymbol{\alpha}) + O(e^{-CR^2}) \end{aligned} \tag{5.7}$$

and, in polar coordinates

$$\begin{aligned} I_A &= -\int_{\frac{\pi}{2}}^{\theta_a} d\phi \int_0^{\delta R} rdr \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha})\, \theta(-r\sin\phi - \inf(\mathbf{n}\cdot\boldsymbol{\alpha})) \\ &= -\int_{\frac{\pi}{2}}^{\theta_a} d\phi \int_0^{\infty} rdr \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha})\, \theta\big(-r - (\sin\phi)^{-1}\inf(\mathbf{n}\cdot\boldsymbol{\alpha})\big) \\ &\quad + O(e^{-CR^2}) \\ &= \frac{1}{2}\cot\theta_a \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha})\, (\inf(\mathbf{n}\cdot\boldsymbol{\alpha}))^2 + O(e^{-CR^2}) \end{aligned} \tag{5.8}$$

One has the same result for I_B with θ_b in place of θ_a.

We now consider a corner of angle $\theta_j = \theta$ measured from a face oriented along the x-axis, with normal $\mathbf{n}_j$ along the y-axis. Then the normal to the next face has coordinates $\mathbf{n}_{j+1} = (\sin\theta, -\cos\theta)$ (*see* fig. 2).

The dominant contribution to the term I_θ in the sum (5.5) corresponding to the corner θ will be given by the paths starting in the parallelogram Λ_R with sides of length δR

$$\begin{aligned} I_\theta &= \int_{\Lambda_R} d^2\mathbf{r} \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha})\, \theta(-y - \inf(\mathbf{n}_j\cdot\boldsymbol{\alpha})) \\ &\quad \times \theta(-x\sin\theta + y\cos\theta - \inf(\mathbf{n}_{j+1}\cdot\boldsymbol{\alpha})) \\ &= (\sin\theta)^{-1} \int_0^{\delta R\sin\theta} dx' \int_0^{\delta R\sin\theta} dy' \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha})\, \theta(-y' - \inf(\mathbf{n}_j\cdot\boldsymbol{\alpha})) \\ &\quad \times \theta(-x' - \inf(\mathbf{n}_{j+1}\cdot\boldsymbol{\alpha})) + O(e^{-CR^2}) \\ &= (\sin\theta)^{-1} \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha}) \inf(\mathbf{n}_j\cdot\boldsymbol{\alpha}) \inf(\mathbf{n}_{j+1}\cdot\boldsymbol{\alpha}) + O(e^{-CR^2}) \end{aligned} \tag{5.9}$$

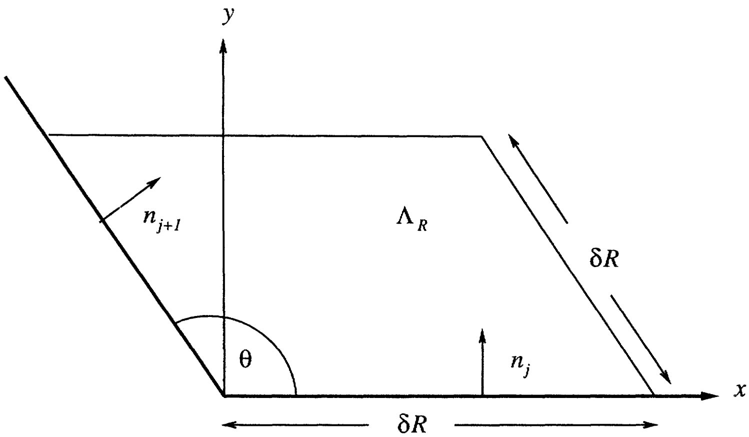

Figure 2.

We have made the change of variables $x' = x \sin\theta - y\cos\theta$, $y' = y$ with Jacobian $(\sin\theta)^{-1}$.

The sum on all faces of the terms (5.7) gives the usual surface contribution (2.22)

$$\mathcal{I}_R^{(1)} = \sum_{j=1}^{k} L_j \int D\boldsymbol{\alpha}\, F(\boldsymbol{\alpha}) \inf(\mathbf{n}_j \cdot \boldsymbol{\alpha}) \tag{5.10}$$

We can identify the contribution of a corner $c(\theta_j)$ (of order 1) as the term (5.9) plus twice the term (5.8), *i.e.*

$$c(\theta_j) = (\sin\theta_j)^{-1} \int D\boldsymbol{\alpha} F(\boldsymbol{\alpha}) \inf(\mathbf{n}_j \cdot \boldsymbol{\alpha}) \Big[\inf(\mathbf{n}_{j+1} \cdot \boldsymbol{\alpha}) + \cos\theta_j \inf(\mathbf{n}_j \cdot \boldsymbol{\alpha})\Big] \tag{5.11}$$

Thus we arrive at.

Proposition 6. – *Let Σ_R be a two dimensional domain enclosed by a polygon with k corner with angles θ_j, $\frac{\pi}{2} \le \theta_j < \pi$ then*

$$\mathcal{I}_R = \mathcal{I}_R^{(0)} + \mathcal{I}_R^{(1)} + \mathcal{I}_R^{(2)} + O(e^{-CR^2}) \tag{5.12}$$

where $\mathcal{I}_R^{(0)}$ and $\mathcal{I}_R^{(1)}$ are as in (5.3) *and* (5.10), *and $\mathcal{I}_R^{(2)} = \sum_{j=1}^{k} c(\theta_j)$, $c(\theta_j)$ defined by* (5.11).

It is interesting to note that, contrary to (2.4), the remainder in (5.12) is exponentially small; so $\mathcal{I}_R$ for a polygon in two dimensions has only the three non exponential terms of Proposition 6 in its large volume expansion (a similar situation occurs in [8]). We expect that when approximating a smooth curve by polygonal lines, $\mathcal{I}_R^{(2)}$ in (5.12) should tend to the expression (2.38) given in Proposition 3 for $\nu = 2$, but we have not verified this by a direct calculation.

In the magnetic case, with $F(\boldsymbol{\alpha})$ defined in (3.2), because of rotational invariance of $F(\boldsymbol{\alpha})$, we can always choose axis such that the contribution of a corner reads (using also $\inf(-\alpha) = -\sup\alpha$)

$$c(\theta) = \frac{1}{2\pi} \int D\boldsymbol{\alpha}\, e^{-i\mu \int \alpha_1 d\alpha_2} \sup\alpha_2 \times (\sup(\alpha_1 \sin\theta - \alpha_2 \cos\theta) + \cos\theta \sup\alpha_2) \qquad (5.13)$$

For a right angle $c(\frac{\pi}{2})$ can be computed up to second order in the magnetic field strength (*see* appendix D)

$$c\left(\frac{\pi}{2}\right) = \frac{1}{16} - \left(\frac{1}{128} - \frac{2}{45\pi^2}\right)\mu^2 + O(\mu^4) \qquad (5.14)$$

Since the coefficient of μ^2 is negative, this corner contribution to the zero field susceptibility is diamagnetic (as the bulk term), in contrast to the surface contribution which is always paramagnetic according to (3.19).

VI. SEMICLASSICAL AND HIGH TEMPERATURE ASYMPTOTICS

The propositions proved in section 2 are sufficiently general to allow us to derive expansions in powers of $\hbar$, keeping all other variables fixed (semiclassical) or in powers of β, keeping all other variables fixed (high temperature). In what follows we use the dimensionless parameters μ and ϵ of formula (1.3) in the introduction. We recall that in these limits μ tends to zero with ϵ (*see* (ii) and (iii) in section I), so that we have also to expand the phase factor in (1.3). Thus the basic quantities which will enter in our expansion are

$$W_n(\epsilon) = \int_\Sigma d\mathbf{r} \int D\boldsymbol{\alpha} \left(\mathbf{b} \cdot \int \boldsymbol{\alpha} \wedge d\boldsymbol{\alpha}\right)^n \chi_\Sigma(\mathbf{r} + \epsilon\boldsymbol{\alpha}) \qquad (6.1)$$

for $n = 0, 2$. From the problem without a magnetic field it is well known that $W_0(\epsilon)$ admits an asymptotic expansion in powers of ϵ

$$W_0(\epsilon) = \sum_{i=0}^{N} d_i \epsilon^i + O(\epsilon^{N+1}) \qquad (6.2)$$

where the coefficients d_i are explicitly known up to $i = 6$ in two dimensions and $i = 2$ in three dimensions. For example $d_0 = |\Sigma|$, $d_1 = -\frac{1}{2}\sqrt{\frac{\pi}{2}}|\partial\Sigma|$ and in two dimensions $d_2 = \frac{\pi}{3}(1-m)$ if Σ has m holes (*see* (2.57) (2.58)). We refer to [9] and references therein for these properties, and precise assumptions on Σ.

The quantity $W_2(\epsilon)$ is of the form $W_2(\epsilon) = \epsilon^\nu \mathcal{I}_{\epsilon^{-1}}$ where $\mathcal{I}_{\epsilon^{-1}}$ is the functional integral (2.1) with R replaced by ϵ^{-1} and $F(\boldsymbol{\alpha}) = (\mathbf{b}\cdot\int \boldsymbol{\alpha}\wedge d\boldsymbol{\alpha})^2$. Therefore proposition 3 yields

$$W_2(\epsilon) = h_0 + h_1\epsilon + h_2\epsilon^2 + o(\epsilon^2) \tag{6.3}$$

with

$$h_0 = |\Sigma| \int D\boldsymbol{\alpha}\left(\mathbf{b}\cdot\int \boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}\right)^2 \tag{6.4}$$

$$h_1 = \int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}\,\inf(\boldsymbol{\alpha}\cdot\mathbf{n})\left(\mathbf{b}\cdot\int \boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}\right)^2 \tag{6.5}$$

$$\begin{aligned} h_2 = &-\int_{\partial\Sigma} d\sigma \int D\boldsymbol{\alpha}\left(\mathbf{b}\cdot\int \boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}\right)^2 \\ &\times\left(\frac{1}{2}\sum_{i=1}^{\nu}\kappa_i\big((\boldsymbol{\alpha}\cdot\mathbf{t}_i)^2(\tau_{\mathbf{n}}) - (\boldsymbol{\alpha}\cdot\mathbf{n})^2(\tau_{\mathbf{n}})\big)\right) \end{aligned} \tag{6.6}$$

The functional integral in (6.4) can be computed exactly and yields $h_0 = \frac{1}{3}|\Sigma|$. The one appearing in (6.5) can be read off from the magnetic susceptibility (3.15)

$$h_1 = -\frac{1}{2^6\pi}\int_{\partial\Sigma} d\sigma\left(3 - \frac{1}{3}(\mathbf{b}\cdot\mathbf{n})^2\right) \tag{6.7}$$

Note that in two dimensions $h_1 = -(3/2^6\pi)|\partial\Sigma|$. Similarly for the special case of a square we have from (5.14) $h_2 = \frac{8}{45\pi^2} - 1/32$. In the semiclassical expansion (i), defined in the introduction, ϵ is proportional to $\hbar$ and $\mu = C_1\epsilon$. Expanding the phase in (1.3) to order μ^2 gives

$$\mathrm{Tr}\, e^{-\beta H_R} = (\sqrt{2\pi}\epsilon)^{-\nu}\left(W_0(\epsilon) - \frac{\mu^2}{8}W_2(\epsilon) + R(\mu,\epsilon)\right) \tag{6.8}$$

where the rest $R(\mu,\epsilon)$ satisfies

$$\begin{aligned} |R(\mu,\epsilon)| &\le C\mu^4\int_\Sigma d\mathbf{r}\int D\boldsymbol{\alpha}\left(\mathbf{b}\cdot\int\boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}\right)^4 \chi_\Sigma(\mathbf{r}+\epsilon\boldsymbol{\alpha}) \\ &\le CC_1^4\epsilon^4|\Lambda|\int D\boldsymbol{\alpha}\left(\mathbf{b}\cdot\int\boldsymbol{\alpha}\wedge d\boldsymbol{\alpha}\right)^4 \end{aligned} \tag{6.9}$$

The first inequality in (6.9) follows from $|\cos x - (1 - \frac{x^2}{2})| \leq Cx^4$, for some $C > 0$ and all x. Thus $R(\mu, \epsilon) = O(\epsilon^4)$. From (6.2), (6.3) and (6.8)

$$\operatorname{Tr} e^{-\beta H_R} = (\sqrt{2\pi}\epsilon)^{-\nu}\left[d_0 + d_1\epsilon + \left(d_2 - \frac{C_1^2}{8}h_0\right)\epsilon^2 + \left(d_3 - \frac{C_1^2}{8}h_1\right)\epsilon^3 + O(\epsilon^4)\right] \quad (6.10)$$

We see that the first two terms are the usual ones and the effects of the magnetic field enters only in the third term of the expansion.

We note that a similar fact holds in the situation considered in [5] and references therein where, as explained in the introduction, the particles are confined by a smooth potential well instead of Dirichlet boundary conditions. As $\hbar \to 0$, the leading term is identified and seen to be independent of the magnetic field. This result holds even for inhomogeneous fields.

In the high temperature expansion (ii), defined in the introduction, ϵ is proportional to $\sqrt{\beta}$ and $\mu = C_2\epsilon^2$. Using again (6.2), (6.3) and (6.8) we get

$$\begin{aligned}\operatorname{Tr} e^{-\beta H_R} =& (\sqrt{2\pi}\epsilon)^{-\nu}\Big[d_0 + d_1\epsilon + d_2\epsilon^2 + d_3\epsilon^3 \\ &+ \left(d_4 - \frac{C_2^2}{8}h_0\right)\epsilon^4 + \left(d_5 - \frac{C_2^2}{8}h_1\right)\epsilon^5 \\ &+ \left(d_6 - \frac{C_2^2}{8}h_2\right)\epsilon^6 + O(\epsilon^7)\Big]\end{aligned} \quad (6.11)$$

Now the effects of the magnetic field enter only in the fifth term of the expansion. In low temperature physics or studies in quantum chaos the density of states of a particle confined in a finite region ("billiard") is of central importance. Since in (6.11) we evaluate the asymptotic behaviour of its Laplace transform, an immediate consequence of the Tauberian theorem is that the leading term in the high energy behaviour of the density of states (all other variables being kept fixed), is the usual Weyl term. Unfortunately (6.11) doesn't give a rigorous information on the next terms of the high energy asymptotics, but it suggests that the effects of the magnetic field enter only in higher orders.

These facts are again consistent with the situation in [5]. Indeed, it is proved that when the confining potential grows faster than the magnetic field at infinity, the leading term in the $\beta \to 0$ asymptotics is independent of the field. On the other hand if the magnetic field grows faster than the potential at infinity, the leading term depends only on the field. The latter case, however cannot be compared with our situation where the Dirichlet boundary condition always "dominates" any possible growth of the magnetic field.

VII. CONCLUDING REMARKS

The main result of the paper is the expansion of the Brownian integrals presented in Section II, valid for a general class of functionals $F(\alpha)$. Many works have considered the expansion of such integrals when $F(\alpha)$ has the special Feynman-Kac form, *i.e.* when the powerful tools provided by the associated differential equation are available. However, in several circumstances, one is interested in more general functionals : this will be the case if one wants to expand average values of observables (such as the diamagnetic current), or generalize such finite size expansions to interacting particle systems (as in Section IV of [1] for a low density gas), where there is no one body Schroedinger equation at hand.

In our formalism, we were not able to push the expansion (2.4) beyond the third term. We do not know if this limitation is technical or a price paid for generality, in the sense that existence and computation of further terms in power of R would only be possible when there is an underlying differential equation obtained from the Feynman-Kac formula.

As far as diamagnetism is concerned, we mention some open problems. In Section III, we have studied the large volume expansion of thermodynamical quantities (recovering some existing results). Using the propositions of Section II, one could easily compute low activity corrections to these quantities in the interacting electron gas. We conjecture also that the basic relations given in Section IV (the sum rules of Proposition 4 and 5) remain true in presence of interactions. These sum rules involve the current density of the semi-infinite system. An interesting question, not treated here, would be to discuss the finite size corrections to the boundary current itself (for hard and smooth walls) in order to gain a better understanding of the current distribution in the finite sample.

We plan to come back to these questions in future work.

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APPENDIX A

Gaussian integrals

Let α be a one dimensional Brownian bridge. For continuous functions f and g (sect. 15 in [6])

$$\int D\alpha \exp\left(i\int f d\alpha\right) = \exp\left(-\frac{1}{2}G(f)\right) \tag{A.1}$$

$$\int D\alpha \exp\left(i\int f d\alpha\right)\left(\int g d\alpha\right) = iG(f,g)\exp\left(-\frac{1}{2}G(f)\right) \tag{A.2}$$

with

$$G(f,g) = \int ds f(s)g(s) - \left(\int ds f(s)\right)\left(\int ds g(s)\right), \quad G(f) = G(f,f) \tag{A.3}$$

Equation (A.2) follows from (A.1) replacing f by λf and differentiating at $\lambda = 0$.

APPENDIX B

Change of variable

The measure $D\alpha$ is invariant under the change (Lemma 2 in [1])

$$\alpha(s) \longrightarrow \alpha(\widetilde{s+u}) - \alpha(u) \tag{B.1}$$

where

$$\widetilde{s+u} = \begin{cases} s+u, & \text{if} \quad s+u \leq 1; \\ s+u-1, & \text{if} \quad s+u > 1 \end{cases} \qquad 0 \leq s, u \leq 1.$$

APPENDIX C

Distribution of $\sup\alpha$ and τ_α

For a one dimensional Brownian bridge $\alpha(s)$, $0 \le s \le 1$, $\alpha(0) = \alpha(1) = 0$, the normalized joint distribution of $\sup\alpha$ and the time τ_α at which $\alpha(s)$ attains its maximum is from Proposition (8.15) in [7]

$$P(\sup\alpha \in db\,;\ \tau_\alpha \in ds) = \sqrt{\frac{2}{\pi}}\frac{b^2}{\sqrt{s^3(1-s)^3}}\exp\left(-\frac{b^2}{2s(1-s)}\right)dbds \tag{C.1}$$

In particular, the distribution of τ_α is found to be uniform in the interval $[0,1]$

$$P(\tau_\alpha \in ds) = ds \tag{C.2}$$

One has also $P(\sup\alpha \in db) = 4b\exp(-2b^2)db$, and hence

$$\int D\alpha \sup\alpha = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \qquad \int D\alpha(\sup\alpha)^2 = \frac{1}{2} \tag{C.3}$$

APPENDIX D

Integrals involving $\sup\alpha$ and $\alpha(u)$

To calculate integrals of the form $\int D\alpha(\sup\alpha)^k(\alpha(u))^n$, we consider Brownian motion on the positive half-line with absorbing barrier at the origin. The probability for a path starting at x to reach y within the time u is

$$G_u^D(x|y) = \frac{1}{\sqrt{2\pi u}}\left(\exp\left(-\frac{(x-y)^2}{2u}\right) - \exp\left(-\frac{(x+y)^2}{2u}\right)\right), \quad x,\ y \ge 0 \tag{D.1}$$

i.e. the solution of the diffusion equation with Dirichlet boundary condition at the origin. From the Feynman-Kac formula, we have for closed paths $x + \alpha(s),\ 0 \le s \le 1,\ \alpha(0) = \alpha(1) = 0,$

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int D\alpha\theta(x + \inf\alpha)(x + \alpha(u))^n \\
&= \frac{1}{\sqrt{2\pi}}\int D\alpha\theta(x - \sup\alpha)(x - \alpha(u))^n \\
&= \int_0^\infty G_u(x|y)y^n G_{1-u}(y|x), \quad 0 \le u \le 1
\end{aligned} \tag{D.2}$$

This formula is equivalent to

$$\int D\alpha\theta(x-\sup\alpha)(\alpha(u))^n = \sqrt{2\pi}\int_0^{\infty} G_u(x|y)(x-y)^n G_{1-u}(y|x) \quad (D.3)$$

Multiplying by x^{k-1} and integrating on x, one obtains

$$\int D\alpha(\sup\alpha)^k(\alpha(u))^n = k\sqrt{2\pi}\int_0^{\infty} dx\int_0^{\infty} x^{k-1}G_u(x|y)(x-y)^n G_{1-u}(y|x) \quad (D.4)$$

reducing such functional integrals to Gaussian integrals. One finds

$$\int D\alpha \sup\alpha\, \alpha(u) = \frac{1}{2}\int D\alpha\, \alpha^2(u) = \frac{1}{2}u(1-u) \quad (D.5)$$

$$\int D\alpha \sup\alpha\ \alpha^2(u) = \int D\alpha(\sup\alpha)^2\alpha(u) = f(u)$$

$$\begin{aligned} f(u) = & \frac{\sqrt{2\pi}}{8} - \frac{8}{3\sqrt{2\pi}}(u(1-u))^{3/2} \\ & - \frac{1}{\sqrt{2\pi}}(1-2u)^2(u(1-u))^{1/2} \\ & - \frac{1}{2\sqrt{2\pi}}(1-2u)\arctan\left(\frac{1-2u}{(4u(1-u))^{1/2}}\right) \end{aligned} \quad (D.6)$$

The first equalities in (D.5) and (D.6) result of an application of the change of variable (B.1) in $\int D\alpha(\sup\alpha)^2$ and $\int D\alpha(\sup\alpha)^3$ respectively. The details of the calculation leading to (D.6) can be found in [10].

To calculate the two times integral

$$g(u,v) = \int D\alpha \sup\alpha\, \alpha(u)\alpha(v) \quad (D.7)$$

we could proceed like we did to obtain (D.4). We rather apply the change of variable (B.1) to the integral $\int D\alpha \sup\alpha\, \alpha^2(u)$ giving for any $v \in [0,1]$ (note that the odd moments of the Brownian bridge vanish)

$$\int D\alpha \sup\alpha\, \alpha^2(u) = \int D\alpha \sup\alpha(\alpha^2(\widetilde{u+v}) + \alpha^2(v) - 2\alpha(\widetilde{u+v})\alpha(v)) \quad (D.8)$$

Using $f(u) = f(1-u)$, (D.8) can be rewritten as

$$g(u,v) = \frac{1}{2}(f(u) + f(v) - f(|u-v|)) \quad (D.9)$$

We are now in position to calculate the terms of the expansion (3.12). Using (D.6) and (D.9), or equivalently, integrating (D.9) on u and v gives

$$\int D\alpha \, \mathrm{sup}\alpha G(\alpha) = \frac{1}{2}\int_0^1 du f(u) = \frac{3\sqrt{2\pi}}{128} \tag{D.10}$$

This, together with (C.3), establishes (3.12). To establish (5.14), we expand (5.13) with $\theta = \frac{\pi}{2}$ to second order in μ (since $c(\theta)$) is real, only cos $\left(\int \alpha_1 d\alpha_2\right)$ enters in the integral), and introduce the function (D.7)

$$c\left(\frac{\pi}{2}\right) = \frac{1}{2\pi}\left(\int D\alpha_1 \mathrm{sup}\alpha_1\right)^2 - \frac{\mu^2}{4\pi}\int_0^1 du \int_0^1 dv g(u,v)\frac{\partial^2}{\partial u \partial v} g(u,v) \tag{D.11}$$

Using (D.9) and $f(0) = f(1) = 0$, $f(u) = f(1-u)$, we can transform the coefficient of the μ^2 term in (D,11) by partial integrations to

$$\begin{aligned} -\frac{1}{16\pi}\int_0^1 du \int_0^1 dv f(|u-v|)\frac{\partial^2}{\partial u \partial v} f(|u-v|) \\ = \frac{1}{16\pi}\int_0^1 du f(u)\frac{d^2}{du^2} f(u) = \frac{2}{45\pi^2} - \frac{1}{128} \end{aligned} \tag{D.12}$$

The result follows from a direct integration of $f(u)$ multiplied by $\frac{d^2}{du^2} f(u) = -8(\sqrt{2\pi})^{-1}(u(1-u))^{\frac{1}{2}}$, and this, together with (C.3) leads to (5.14).

APPENDIX E

Calculation of volume element (2.5)

Let the $\nu - 1$-dimensional hypersurface $\partial\Sigma$ be given by $\mathbf{s} = \mathbf{s}(u_1, u_1, \ldots u_{\nu-1})$. Let $\mathbf{n}$ be the unit inward drawn normal at $\mathbf{s}$. The *Shape Operator* A on the tangent space at $\mathbf{s}$ is defined by (*see* chap. V in [11])

$$A\left(\frac{\partial \mathbf{s}}{\partial u_i}\right) = -\frac{\partial \mathbf{n}}{\partial u_i} \tag{E.1}$$

The eigenvalues of A are the principal curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{\nu-1}$. The area element on $\partial\Sigma$ is given by

$$d\sigma = \left|\det\left(\frac{\partial \mathbf{s}}{\partial u_1}, \frac{\partial \mathbf{s}}{\partial u_2}, \ldots, \frac{\partial \mathbf{s}}{\partial u_{\nu-1}}\right)\right| du_1 \ldots du_{\nu-1} \tag{E.2}$$

Let $\mathbf{r}$ be a point in Σ^* and s the point on $\partial\Sigma$ nearest to $\mathbf{r}$ and x the distance from $\mathbf{r}$ to s, then

$$\mathbf{r} = \mathbf{s} + x\mathbf{n} \tag{E.3}$$

The Jacobian of the transformation is given by

$$\begin{aligned} J &= \det\left(\mathbf{n}, \frac{\partial \mathbf{s}}{\partial u_1} + x\frac{\partial \mathbf{n}}{\partial u_1}, \frac{\partial \mathbf{s}}{\partial u_2} + x\frac{\partial \mathbf{n}}{\partial u_2}, \ldots, \frac{\partial \mathbf{s}}{\partial u_{\nu-1}} + x\frac{\partial \mathbf{n}}{\partial u_{\nu-1}}\right) \\ &= \det\left(\mathbf{n}, (I - xA)\frac{\partial \mathbf{s}}{\partial u_1}, (I - xA)\frac{\partial \mathbf{s}}{\partial u_2}, \ldots, (I - xA)\frac{\partial \mathbf{s}}{\partial u_{\nu-1}}\right) \end{aligned} \tag{E.4}$$

Since $(I - xA)\frac{\partial \mathbf{s}}{\partial u_i}$ are in the tangent space at s we can write J in block form

$$\begin{aligned} J &= \det\begin{pmatrix} 1 & 0 \\ 0 & (I - xA)\frac{\partial \mathbf{s}}{\partial u_1}, (I - xA)\frac{\partial \mathbf{s}}{\partial u_2}, \ldots, (I - xA)\frac{\partial \mathbf{s}}{\partial u_{\nu-1}} \end{pmatrix} \\ &= \det(I - xA)\det\left(\frac{\partial \mathbf{s}}{\partial u_1}, \frac{\partial \mathbf{s}}{\partial u_2}, \ldots, \frac{\partial \mathbf{s}}{\partial u_{\nu-1}}\right) \end{aligned} \tag{E.5}$$

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