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On an example of phase-space tunneling

by

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ABSTRACT. – A typical example of phase space tunneling, which is related to the Born-Oppenheimer approximation and has been studied by Martinez, is considered. An upper bound on the width of the resonance, which seems to be optimal, is proved. The main idea is to construct a suitable canonical transformation, and then to use the standard Agmon-type exponential estimate. In order to define resonances, we use a local distortion method.

RÉSUMÉ. – Nous considérons un exemple type d'effet tunnel dans l'espace des phases. Cet exemple est relié à l'approximation de Born-Oppenheimer et a été étudié par Martinez. Nous donnons une borne supérieure qui semble être optimale, sur la largeur des résonances.

L'idée principale consiste à construire une transformation canonique convenable puis d'utiliser une estimation exponentielle du type d'Agmon. Pour définir les résonances nous utilisons la méthode de distorsion locale.

1. INTRODUCTION

The purpose of this paper is to study a class of two-channel Schrödinger operators as an example of phase-space tunneling phenomena. In [12], Martinez studied resonances for the operator:

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$$H = \begin{pmatrix} -\hbar^2 \dfrac{d^2}{dx^2} + x^2 & 0 \\ 0 & -\hbar^2 \dfrac{d^2}{dx^2} - 1 - x \end{pmatrix} + \hbar R$$

$$\text{on} \quad \mathcal{H} = L^2(\mathbf{R}) \oplus L^2(\mathbf{R}) \tag{1.1}$$

in the semiclassical limit: $\hbar \downarrow 0$, where R is an $\hbar$-pseudodifferential operator of $O(1)$. The principal symbol of H (as an $\hbar$-pseudodifferential operator) is given by $\begin{pmatrix} \xi^2 + x^2 & 0 \\ 0 & \xi^2 - 1 - x \end{pmatrix}$, $x, \xi \in \mathbf{R}$, and the 0-energy surface is

$$\begin{aligned} &\{(x, \xi) | x^2 + \xi^2 = 0\} \cup \{(x, \xi) | \xi^2 - 1 - x = 0\} \\ &= \{(0, 0)\} \cup \{x = -1 + \xi^2\}. \end{aligned} \tag{1.2}$$

This consists of two connected components. The first diagonal elements of the principal symbol is a harmonic oscillator, and the spectrum is pure point: $\{(2n+1)\hbar | n = 0, 1, 2, \cdots\}$. The second element is a Stark Hamiltonian and the spectrum is absolutely continuous, and is the whole real line $\mathbf{R}$. Intuitively, at least, each eigenfunction of the harmonic oscillator is supported (microlocally) in a very small neighborhood of the 0-energy surface $\{(0, 0)\}$, whereas generalized eigenfunctions of the Stark Hamiltonian is supported in a small neighborhood of $\{(x, \xi) | x = -1 + \xi^2\}$. By the effect of the lower order perturbation $\hbar R$, the eigenvalue vanishes generically, and it becomes a quasi-bound state, or a *resonance*. This phenomenon may be considered as an interaction between the two 0-energy surfaces, or so-called *tunneling* between them. Thus we can expect, by an analogy of the WKB analysis, that the imaginary part of each resonance (which is usually called *width*), is $O(e^{-c/\hbar})$ with some $c > 0$ as $\hbar \downarrow 0$. Martinez showed that this is true under certain conditions on R, using FBI-transform, or more specifically, Bargman transform, and the resonance theory of Helffer and Sjöstrand [7].

In this paper, we mainly consider special cases where R has the form

$$R = \begin{pmatrix} 0 & \alpha x + \beta p + \gamma \\ \overline{\alpha} x + \overline{\beta} p + \overline{\gamma} & 0 \end{pmatrix}, \tag{1.3}$$

where x is the multiplication operator by x, $p = -i\hbar (d/dx)$, and α, β, $\gamma \in \mathbf{C}$. We will prove an exponential bound on the widths of the resonances using certain canonical transform and Agmon-type estimates. The bound is sharper than the one by Martinez [12], and seems to be optimal, though we have not been able to verify. Generalization to a larger class is considered in the last section.

In order to state our result explicitly, we first introduce a definition of resonances (*cf.* [4], Chapter 8 and references therein). More precise discussion is given in Section 2. We use a class of *analytic vectors*, defined by

$$\mathcal{A} = \{\varphi \in \mathcal{H} | \hat{\varphi} \in C_0^\infty(\mathbf{R}) \oplus C_0^\infty(\mathbf{R})\} \tag{1.4}$$

where $\hat{\varphi}$ denotes the Fourier transform of φ:

$$\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = (2\pi\hbar)^{-1/2} \int e^{-ix\xi/\hbar} \varphi(x)\, dx. \tag{1.5}$$

If R satisfies certain analytic conditions, in particular if R has the form (1.3), then the function

$$\begin{aligned} &g_{\varphi\psi}(z) = \langle \psi, (H - z)^{-1} \varphi \rangle, \\ &z \in \mathbf{C}_+ = \{z | \operatorname{Im} z > 0\}, \qquad \varphi, \psi \in \mathcal{A}, \end{aligned} \tag{1.6}$$

is extended meromorphically to a neighborhood of $\mathbf{R}$. We will prove this fact using local distortion method in Section 2. A pole of this function is called *resonances*, in other words, the set of resonances is defined by

$$\mathcal{R} = \{z \in \mathbf{C} | z \text{ is a pole of } g_{\varphi\psi} \text{ for some } \varphi, \psi \in \mathcal{A}\}. \tag{1.7}$$

$\mathcal{R} \cap \mathbf{R}$ is the eigenvalues of H, and thus resonance is a generalization of eigenvalue. Usually, resonance is defined as a *non-real* pole, but here we define the resonances as a superset of the eigenvalues for the convenience.

THEOREM 1.1. – *Let H be defined by* (1.1) *and* (1.3). *Let n be a non-negative integer. Then there is a resonance $E_n(\hbar) = (2n+1)\hbar + O(\hbar^2)$. Moreover, for any* $0 < \delta < 5/12$,

$$|\operatorname{Im} E_n(\hbar)| \le C_{n\delta}\, e^{-2\delta/\hbar}, \qquad \hbar > 0. \tag{1.8}$$

Remark. – In [12], Martinez proved (1.8) for $0 < \delta < (3-\sqrt{5})/4 < 5/12$, for a larger class of R.

The tunneling estimates for Schrödinger operators in the semiclassical limit has been studied extensively. See, for example, [17], [6], [3], etc. In these papers, they studied tunneling effects in the configuration space, *i.e.*, for the case when the energy surfaces of the principal symbol are separated spatially. Recently, the tunneling effects in momentum space has been studied by several authors ([1], [12], [13], [15], [16]). In particular, Martinez studied operators of the form (1.1), which appear in the theory of Born-Oppenheimer approximation of two-atomic molecules (*cf.* [11], [14]). In order to formulate micro-local exponential estimates, he used the Bargman transform, which is also called coherent wave packet expansion in the physics literature (*cf.* [5], *see* also [16]). By this expansion, $L^2(\mathbf{R}^d)$ is represented as a subspace of L^2-space on the phase space: $L^2(\mathbf{R}^d \times \mathbf{R}^d)$.

This method is symmetric in configuration variable x and momentum variable ξ, and seems quite natural to study phase-space tunneling. However, it is not clear if the estimates obtained by this method is optimal, since the results depend on the choice of the wave packet with which the transform is defined. moreover, the method is *not* invariant with respect to canonical transform, since the Bargman transform is coherent in x in ξ.

In this paper, we first use a canonical transform to make our 0-energy surfaces separated in x-variable. The main technical step is the exponential decay estimate for the eigenfunctions of the transformed operators, which is a fourth order differential operator. It is proved using a generalized Agmon estimate.

The paper is constructed as follows: In Section 2, we construct the canonical transform explained above, and then define several operators necessary in the proof of Theorem 1.1. Then the local distortion method is introduced to define resonances. In Section 3, an exponential decay estimate for the eigenfunctions is proved, and theorem 1.1 is proved in Section 4. Section 5 is devoted to discussion on the general scheme and generalizations. Some simple results on $\hbar$-pseudodifferential operators are explained in Appendix.

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2. PRELIMINARY CONSTRUCTIONS

2.0. Self-adjointness

It is well-known that $p^2 + x^2$ and $p^2 - 1 - x$ are essentially self-adjoint on $C_0^\infty(\mathbf{R})$, where $p = -i\hbar\,(d/dx)$ and x denotes the multiplication operator by x. Hence $H_0 = \begin{pmatrix} p^2 + x^2 & 0 \\ 0 & p^2 - 1 - x \end{pmatrix}$ is essentially self-adjoint on $C_0^\infty \oplus C_0^\infty$. It is easy to see that R is H_0-bounded symmetric operator, hence $H = H_0 + \hbar R$ is self-adjoint with $D(H) = D(H_0)$ if $\hbar$ is small.

2.1. Canonical transform

If we use the canonical transform

$$(x, p) \longrightarrow (x + p^2, p), \tag{2.1}$$

then our principal symbol is transformed to $\begin{pmatrix} p^2 + (x + p^2)^2 & 0 \\ 0 & -1 - x \end{pmatrix}$. The 0-energy surfaces of the diagonal elements are then $\{(0, 0)\}$ and

$\{x = -1\}$, respectively, and they are separated in the x-variable. The generating function of the transform is $xp+p^3/3$, and hence the quantization is given by

$$W = e^{ip^3/3\hbar} = \mathcal{F}^* e^{i\xi^3/3\hbar} \mathcal{F}, \tag{2.2}$$

where $\mathcal{F}$ is the Fourier transform, and ξ is the conjugate variable of x. Then it is easy to see that W is unitary and

$$WpW^* = p; \qquad WxW^* = x + p^2. \tag{2.3}$$

Thus we have

$$H' \equiv WHW^* = -H_0' + \hbar R' \tag{2.4}$$

where

$$H_0' = \begin{pmatrix} p^2 + (x+p^2)^2 & 0 \\ 0 & -1-x \end{pmatrix};$$

and

$$R' = \begin{pmatrix} 0 & \alpha(x+p^2) + \beta p + \gamma \\ \overline{\alpha}(x+p^2) + \overline{\beta} p + \overline{\gamma} & 0 \end{pmatrix}.$$

It is easy to see that H_0' is self-adjoint on $D(p^2 + (x+p^2)^2) \oplus D(|x|)$, and R' is relatively bounded perturbation of H_0'. Hence H' is a self-adjoint operator with the same domain.

2.2. Approximation system

Here we construct an operator on $\mathcal{H} \oplus \mathcal{H} = L^2(\mathbf{R})^4$, which is a good approximation of H' in order to estimate $\operatorname{Im} E_n(\hbar)$. Let $\varepsilon > 0$ be a small constant specified later. We choose $j_i(x) \in C_0^\infty(\mathbf{R})$, $i = 1, 2$, so that $0 \le j_i(x) \le 1$ and

$$j_1(x) = \begin{cases} 1 & \text{if} \quad x \ge -1+\varepsilon, \\ 0 & \text{if} \quad x \le -1+\varepsilon/2; \end{cases} \tag{2.5}$$

$$j_1(x)^2 + j_2(x)^2 = 1 \quad \text{for} \quad x \in \mathbf{R}. \tag{2.6}$$

Then we define $J_i \in B(\mathcal{H})$, $i = 1, 2$, by

$$J_i(\varphi_1 \oplus \varphi_2) = (j_i\varphi_1) \otimes (j_i\varphi_2), \quad \varphi_1 \oplus \varphi_2 \in \mathcal{H}. \tag{2.7}$$

Now let

$$\mathcal{H}_1 = \mathcal{H} \oplus \mathcal{H} = [L^2(\mathbf{R}) \oplus L^2(\mathbf{R})] \oplus [L^2(\mathbf{R}) \oplus L^2(\mathbf{R})], \tag{2.8}$$

and let $J \in B(\mathcal{H}, \mathcal{H}_1)$ defined by

$$J\varphi = J_1\varphi \oplus J_2\varphi, \qquad \varphi \in \mathcal{H}. \tag{2.9}$$

J is an isometry, *i.e.*, $J^*J = 1$ on $\mathcal{H}$. In fact,

$$\begin{aligned} \|J(\varphi_1 \oplus \varphi_2)\|^2 &= \int (j_1(x)^2 + j_2(x)^2 (|\varphi_1(x)|^2 + |\varphi_2(x)|^2)\, dx \\ &= \|\varphi_1\|^2 + \|\varphi_2\|^2 = \|\varphi_1 \oplus \varphi_2\|^2 \end{aligned} \tag{2.10}$$

for $\varphi_1 \oplus \varphi_2 \in \mathcal{H}$.

Then we set $K = K_1 \oplus K_2$ on $\mathcal{H}_1$ as follows: Let $h(x) \in C_0^\infty(\mathbf{R})$ be a non-negative function such that $h(0) > 0$ and $\operatorname{supp} h \subset \{x \mid |x| \leq 1/2\}$. Let $k(x) \in C^\infty(\mathbf{R})$ be a negative function such that $k_0 = \sup k(x) < -1/4$, and $k(x) = -|1 + x|$ if $|x + 1| > \varepsilon/2$. $K_{i,0}$, $i = 1, 2$, are then defined by

$$\begin{aligned} K_{1,0} &= \begin{pmatrix} p^2 + (x + p^2)^2 & 0 \\ 0 & k(x) \end{pmatrix}; \\ K_{2,0} &= \begin{pmatrix} p^2 + (x + p^2)^2 + h(x) & 0 \\ 0 & -1 - x \end{pmatrix}, \end{aligned} \tag{2.11}$$

and we let $K_i = K_{i,0} + \hbar R'$. As in Subsection 2.1, it is easy to see that K_i, $i = 1, 2$, are self-adjoint on $D(H_0')$.

Since $h(x) = 0$ on $\operatorname{supp} j_2$ and $k(x) = -1 - x$ on $\operatorname{supp} j_1$, we have

$$H' J_i = K_i J_i, \quad i = 1, 2. \tag{2.12}$$

Hence we obtain

$$JH' - KJ = JH' - (H' \oplus H')J = [J_1, H'] \oplus [J_2, H']. \tag{2.13}$$

Note that $[J_i, H']$, $i = 1, 2$, are differential operators supported in $[-1 + \varepsilon/2, -1 + \varepsilon]$.

LEMMA 2.1. – *The spectrum of* $K_{i,0}$, $i = 1, 2$, *are given as follows:*

$$\begin{aligned} \sigma(K_{1,0}) &= (-\infty, k_0] \cup \{(2n + 1)\hbar \mid n = 0, 1, 2, \cdots\}; \\ \sigma(K_1) &= (-\infty, k_0'] \cup \{F_n(\hbar) \mid n = 0, 1, 2, \cdots\}, \end{aligned}$$

where $k_0' = k_0 + O(\hbar)$ and $F_n(\hbar) = (2n + 1)\hbar + O(\hbar^2)$; $\sigma(K_2) = \sigma(K_{2,0}) = \mathbf{R}$.

2.3. Local distortion

In this subsection, we explain a method of local distortion to define resonances. This formulation is formally very close to the one by Hunziker [9], but is also close to the energy distortion method by Babbit and Baslev [2] in the spirit, since we essentially distort the energy of the Stark hamiltonian locally.

Let $\chi(x) \in C_0^\infty(\mathbf{R})$ be a smooth cut-off function such that $0 \leq \chi(x) \leq 1$,

$$\chi(x) = \begin{cases} 1 & \text{if} \quad |x+1| \leq \varepsilon/8, \\ 0 & \text{if} \quad |x+1| \geq \varepsilon/4, \end{cases} \tag{2.14}$$

and let $M = \sup|\chi'(x)|$. If $\theta \in \mathbf{R}$, $|\theta| < M^{-1}$, then the transform

$$T_\theta : x \to x + \theta\chi(x) \tag{2.15}$$

is a diffeomorphism in $\mathbf{R}$, and it induces a unitary transform in $L^2(\mathbf{R})$:

$$U_\theta \varphi(x) = J_\theta(x)^{1/2} \varphi(T_\theta(x)), \qquad \varphi \in L^2(\mathbf{R}), \tag{2.16}$$

where $J_\theta(x) = |\det(\partial T_\theta/\partial x)| = 1 + \theta\chi'(x)$ is the Jacobian. We denote $U_\theta \oplus U_\theta$ in $L^2(\mathbf{R}) \oplus L^2(\mathbf{R})$ by the same symbol U_θ (we will use such notations without further remarks). It is easy to see that by this transform operators x and p are transformed as follows:

$$U_\theta\, x\, U_\theta^{-1} = x + \theta\chi(x) = T_\theta(x), \tag{2.17}$$

$$U_\theta\, p\, U_\theta^{-1} = J_\theta^{-1/2}\, p\, J_\theta^{-1/2}. \tag{2.18}$$

Hence $H'(\theta) = U_\theta H' U_\theta^{-1}$ is an analytic family of type A in a neighborhood of 0. Also, $K_2(\theta) = U_\theta K_2 U_\theta^{-1}$ is an analytic family of type A since $h(x) = 0$ in a neighborhood of 0.

LEMMA 2.2. – *There is $\delta > 0$ such that $H'(\theta)$ and $K_2(\theta)$ are analytic families of type* A *in $B_\delta = \{z \in \mathbf{C} \mid |z| < \delta\}$. Moreover, the point spectrum of $H'(\theta)$ and $K_2(\theta)$ are invariant in θ.*

LEMMA 2.3. – *$H_0'(\theta) = U_\theta H_0' U_\theta^{-1}$ and $K_{2,0}(\theta) = U_\theta K_{2,0} U_\theta^{-1}$ are analytic families of type* A *in B_δ. Moreover,*

$$\sigma(H_0'(\theta)) = \{-1 - x - \theta\chi(x) \mid x \in \mathbf{R}\} \\ \cup \{(2n+1)\hbar \mid n = 0, 1, 2, \cdots\}; \tag{2.19}$$

$$\sigma(K_{2,0}(\theta)) = \{-1 - x - \theta\chi(x) \mid x \in \mathbf{R}\} \tag{2.20}$$

if δ and $\hbar$ are sufficiently small.

The proofs are standard or easy, possibly except for (2.20). In order to show (2.20), we use the following consequence of the Gårding inequality.

LEMMA 2.4. – *Let* $\ell(x) = x^2$ *if* $x > -1/2$; $\ell(x) = -x - 1/4$ *if* $x \leq -1/2$. *Then*

$$(x + p^2)^2 + p^2 \geq \frac{1}{2}\ell(x) - C\hbar, \quad \hbar > 0 \tag{2.21}$$

as an operator inequality.

Proof. – If $\varepsilon > -1/2$ then we have

$$\begin{aligned}(x + \xi^2)^2 + \xi^2 &= (x + \xi^2 - \varepsilon)^2 + (1 + \varepsilon^2)\xi^2 + 2\varepsilon x - \varepsilon^2 \\ &\geq 2\varepsilon(x - \varepsilon/2).\end{aligned} \tag{2.22}$$

If $x \geq -1/2$, we set $\varepsilon = x$ to obtain

$$(x + \xi^2)^2 + \xi^2 \geq 2x(x - x/2) = x^2. \tag{2.23}$$

If $x < -1/2$, we set $\varepsilon = -1/2$ and

$$(x + \xi^2)^2 + \xi^2 \geq -x - 1/4. \tag{2.24}$$

Thus $(x + \xi^2)^2 + \xi^2 \geq \ell(x)$ for $x, \xi \in \mathbf{R}$. Then we apply the Gårding inequality (Appendix, Theorem 5) to

$$(x + \xi^2)^2 + \xi^2 - \frac{1}{2}\ell(x) \geq \frac{1}{2}[(x + \xi^2)^2 + \xi^2] \geq 0 \tag{2.25}$$

to obtain (2.20). ■

By Lemma 2.4, we have

$$(x + p^2)^2 + p^2 + h(x) \geq \alpha > 0 \tag{2.26}$$

if $\hbar$ is sufficiently small, which we always assume implicitly. Since the spectrum of $U_\theta((x + p^2)^2 + p^2 + h(x))U_\theta^{-1}$ is discrete, and hence is invariant in θ, (2.26) and

$$\sigma(U_\theta(-1 - x)U_\theta^{-1}) = \{-1 - x - \theta\chi(x) | x \in \mathbf{R}\} \tag{2.27}$$

imply (2.20).

We let $\theta = i\lambda$ with sufficiently small $\lambda > 0$, which we will fix in the next section. Then the spectrum of $K_{2,0}(\theta)$ avoids the origin. Since $K_2(\theta)$ is a small perturbation $K_{2,0}(\theta)$, we can expect 0 is in the resolvent set of $K_2(\theta)$.

LEMMA 2.5. – *There exist* c, $C > 0$ *such that if* $\hbar$ *is sufficiently small then* $B_c = \{z \in \mathbf{C} | \, |z| < c\} \subset \rho(K_2(\theta))$ *and*

$$\|(K_2(\theta) - z)^{-1}\| \le C, \qquad z \in B_c, \quad \hbar > 0. \tag{2.28}$$

Proof. – Note that the principal symbol of $K_{2,0}(\theta)$ is

$$\begin{pmatrix} (T_\theta(x) + J_\theta(x)^{-2}\xi^2)^2 + J_\theta(x)^{-2}\xi^2 + h(x) & 0 \\ 0 & -1 - T_\theta(x) \end{pmatrix}. \tag{2.29}$$

If $|\theta|$ is sufficiently small, then

$$\mathrm{Re}\,[(T_\theta(x) + J_\theta(x)^{-2}\xi^2)^2 + J_\theta(x)^{-2}\xi^2 + h(x) - z] \ge \alpha > 0 \tag{2.30}$$

for z in a small neighborhood of 0 by the argument above. Hence by the Gårding inequality (or by the construction of parametrix),

$$\|[U_\theta((x+p^2)^2 + p^2 + h(x))\,U_\theta^{-1} - z]^{-1}\| \le C, \quad z \in B_c \tag{2.31}$$

if $\hbar$ and c are sufficiently small. On the other hand, it is clear that

$$\|(-1 - x - \theta\chi(x))^{-1}\| \le (\inf|1 - x - \theta\chi(x)|)^{-1} < C \tag{2.32}$$

independent of $\hbar$. Then (2.28) follows by the standard perturbation argument. ■

Similarly, $H'(\theta)$ is a small perturbation of $H'_0(\theta)$ and the spectrum is very close:

LEMMA 2.6. – *There exist* $E_n(\hbar) \in \mathbf{C}$, $n = 0, 1, 2, \cdots$, *such that* $E_n(\hbar) = (2n+1)\hbar + O(\hbar^2)$ *as* $\hbar \downarrow 0$, *and for any* $C > 0$,

$$\sigma(H'(\theta)) \cap B_{(C\hbar)} = \{E_n(\hbar)\, n = 0, 1, 2, \cdots\} \cap B_{(C\hbar)}. \tag{2.33}$$

Proof. – By the standard perturbation argument as in the proof of Lemma 2.5 (*cf.* [10]), (2.33) holds with $E_n(\hbar) = (2n+1)\hbar + O(\hbar)$. Here $E_n(\hbar)$ is a perturbation of the eigenvalue $(2n+1)\hbar$ of the harmonic oscillator. If we note that R has no diagonal component by the assumption, we see

$$\langle \psi_n^0, R\psi_n^0 \rangle = 0, \quad \text{for any } n, \tag{2.34}$$

where ψ_n^0 is an eigenfunction of $H'_0(\theta)$. Hence the $O(\hbar)$-term in the perturbation expansion vanishes, and we have the desired estimate. ■

2.4. Resonances

Since the set of analytic vectors $\mathcal{A}$ is invariant under the transform $W = e^{ip^3/3\hbar}$, it suffices to consider analytic continuations of

$$g'_{\psi\varphi}(z) = \langle \psi, (H' - z)^{-1} \varphi \rangle, \qquad z \in \mathbf{C}_+, \; \varphi, \psi \in \mathcal{A}. \tag{2.35}$$

On the other hand, $\varphi \in \mathcal{A}$ is an entire function by the Paley-Wiener theorem. Hence $U_\theta \varphi$ is an $\mathcal{H}$-valued analytic function in θ in a neighborhood of 0. For $\theta \in \mathbf{R}$, we have

$$\begin{aligned} g'_{\psi\varphi}(z) &= \langle U_\theta \psi, U_\theta (H' - z)^{-1} U_\theta^{-1} U_\theta \varphi \rangle \\ &= \langle U_\theta \psi, (H'(\theta) - z)^{-1} U_\theta \varphi \rangle. \end{aligned} \tag{2.36}$$

By the analyticity, this equality holds for any θ in a neighborhood of 0. Hence we may set $\theta = i\lambda$ in (2.36) as in the last subsection. Then $g'_{\psi\varphi}(z)$ is analytic in the domain $z \notin \sigma(H'(\theta))$ and it has poles at the eigenvalues of $H'(\theta)$. In particular,

$$\begin{aligned} (\text{resonances of } H \text{ in } B_{(C\hbar)}) &= \sigma(H'(\theta)) \cap B_{(C\hbar)} \\ &= \{E_n(\hbar) | n = 0, 1, 2, \cdots\} \cap B_{(C\hbar)}, \end{aligned} \tag{2.37}$$

where $C > 0$ and $E_n(\hbar)$ are as in Lemma 2.6. Thus we have shown:

PROPOSITION 2.7. – *There exist $\hbar_0 > 0$ and $E_n(\hbar) \in \mathbf{C}$, $n = 0, 1, 2, \cdots$, such that for any $C > 0$, the set of resonances for H in $B_{(C\hbar)} = \{z \in \mathbf{C} | \, |z| < C\hbar\}$ is $\{E_n(\hbar) | n = 0, 1, 2, \cdots\} \cap B_{(C\hbar)}$, where $E_n(\hbar) = (2n + 1)\hbar + O(\hbar^2)$.*

From now on, we will study the eigenvalues and eigenfunctions of $H'(\theta)$ in order to prove Theorem 1.1.

3. EXPONENTIAL DECAY ESTIMATES

Here we prove that the eigenfunction decay exponentially in $\hbar^{-1}$ in *classically forbidden region, i.e.*, in an area with which the projection of the energy surface does not intersect.

We fix $n \geq 0$, and let $F_n = F_n(\hbar) = (2n + 1)\hbar + O(\hbar^2)$ be the $(n+1)$-th eigenvalue of K_1. Let $\psi_n^1(x)$ be the corresponding (normalized) eigenfunction.

It is well-known that for any smooth function $\rho(x)$ on $\mathbf{R}$,

$$\begin{aligned} e^{\rho/\hbar} p e^{-\rho/\hbar} &= (-i\hbar)\, e^{\rho/\hbar} \frac{d}{dx}\, e^{-\rho/\hbar} \\ &= -(-i\hbar)\, \frac{\rho'(x)}{\hbar} + p = p + i\rho'(x). \end{aligned} \tag{3.1}$$

Hence, if $a(\hbar;\, x,\, p)$ is a differential operator: $a(\hbar;\, x,\, p) = \sum_{j=0}^{m} a_j(\hbar;\, x)\, p^j$, then

$$e^{\rho/\hbar}\, a(\hbar;\, x,\, p)\, e^{-\rho/\hbar} = a(\hbar;\, x,\, p + i\rho'(x)). \tag{3.2}$$

In particular,

$$e^{\rho/\hbar}\, K_{1,0}\, e^{-\rho/\hbar} = \begin{pmatrix} (p + i\rho'(x))^2 + (x + (p + i\rho'(x))^2)^2 & 0 \\ 0 & k(x) \end{pmatrix}. \tag{3.3}$$

If $\rho'(x) = 0$, then the principal symbol of the diagonal elements are positive except for $x = \xi = 0$. Let us consider under what conditions on $\rho(x)$ it holds. If we solve the equation: $\xi^2 + (x + \xi^2)^2 = 0$ in $\xi \in \mathbf{C}$, we have

$$\xi = \pm\sqrt{-x - \frac{1}{2} \pm \sqrt{x + \frac{1}{4}}}. \tag{3.4}$$

We let $\eta(x)$ be the least absolute value of the imaginary part of the solution ξ. Namely, $\eta(x) = 1/2$ if $x \leq 1/4; = 1/2 - \sqrt{x + 1/4}$ if $-1/4 < x \leq 0; = \sqrt{x + 1/4} - 1/2$ if $x > 0$. If $0 \leq |\rho'(x)| < \eta(x)$, then

$$|(\xi + i\rho'(x))^2 + (x + (\xi + i\rho'(x))^2)^2|^2 > 0 \quad \text{for} \quad (x,\, \xi) \neq 0. \tag{3.5}$$

We will take a smooth function $\rho(x)$ so that $\rho(x) \geq 0$; $\rho(x) = 0$ in a neighborhood of 0; and $-\rho'(x) \geq 0$ is slightly smaller than $\eta(x)$. Since

$$\int_{-1}^{0} \eta(x)\, dx = \int_{-1}^{-1/4} \frac{1}{2} + \int_{-1/4}^{0} \left(\frac{1}{2} - \sqrt{x + \frac{1}{4}}\right) dx = \frac{5}{12}, \tag{3.6}$$

$\rho(x)$ should be slightly smaller than 5/12 in a small neighborhood of (–1). Now let $I \equiv [-1 + \varepsilon/2,\, -1 + \varepsilon] \supset \operatorname{supp}[T]$, and let $\gamma_0 = 5/12$.

PROPOSITION 3.1. – *There exists* $C > 0$ *such that*

$$\|\chi_I \psi_n^1\| \leq C e^{-(\gamma_0 - \varepsilon')/\hbar}, \qquad \hbar > 0, \tag{3.7}$$

where χ_I *is the characteristic function of I, and* $\varepsilon' = (3/4)\varepsilon$.

Proof. – Motivated by the above argument, we set $\rho \in C^\infty(\mathbf{R})$ so that $\rho(x) = 0$ for $x \geq -\delta$ with some $\delta > 0$; $\rho(x) \geq 0$ for any $x \in \mathbf{R}$; $|\rho'(x)| < \eta(x)$ for $x < 0$; $\rho(x) \geq \int_n^0 \eta(y)\, dy - \varepsilon/4$ for $x \in (-3/2, 0)$; $\rho(x)$ is constant for $x \leq -2$. Let $\psi = \psi_n^1$. Then we compute

$$\begin{aligned} 0 &= \|e^{\rho/\hbar}(K_1 - F_n)\psi\|^2 \\ &= \langle e^{\rho/\hbar}(K_1 - F_n) e^{-\rho/\hbar} \cdot e^{\rho/\hbar}\psi,\ e^{\rho/\hbar}(K_1 - F_n) e^{-\rho/\hbar} \cdot e^{\rho/\hbar}\psi\rangle \\ &= \langle e^{\rho/\hbar}\psi,\ |e^{\rho/\hbar}(K_1 - F_n) e^{-\rho/\hbar}|^2 e^{\rho/\hbar}\psi\rangle. \end{aligned} \tag{3.8}$$

The principal symbol of $|\cdots|$ is given by

$$\begin{pmatrix} |(\xi + i\rho'(x))^2 + (x + (\xi + i\rho'(x))^2)^2|^2 & 0 \\ 0 & k(x)^2 \end{pmatrix} \tag{3.9}$$

and it is positive for $x \neq 0$ by the choice of $\rho(x)$. Note that $F_n = O(\hbar)$. Now let $a(x)$ be a smooth function such that: $a(x)$ is supported in a small neighborhood of 0; $\operatorname{supp} a \subset \{x | \rho(x) = 0\}$; $a(x) \geq 0$ for any $x \in \mathbf{R}$; and $a(0) > 0$. Then it is easy to see

$$\begin{aligned} &|(\xi + i\rho'(x))^2 + (x + (\xi + i\rho'(x))^2)^2|^2 + a(x)^2 \\ &\quad \geq c(\xi^2 + (x + \xi^2)^2 + 1)^2 \end{aligned} \tag{3.10}$$

for any $x, \xi \in \mathbf{R}$ with some $c > 0$. Then by the Gårding inequality, we have

$$\begin{aligned} &\left\langle \varphi, \left[\left|e^{\rho/\hbar}(K_1 - F_n) e^{-\rho/\hbar}\right|^2 + a(x)^2\right] \varphi \right\rangle \\ &\quad \geq \frac{c}{2}(\|\varphi\|^2 + \|K_{1,0}\varphi\|^2) - C\hbar\|\varphi\|^2 \end{aligned} \tag{3.11}$$

for $\varphi \in D(K_1)$. Thus, by letting $\varphi = e^{\rho/\hbar}\psi$, we obtain

$$\|e^{\rho/\hbar}\psi\|^2 + \|K_{1,0}\, e^{\rho/\hbar}\psi\| \leq C\|a e^{\rho/\hbar}\psi\|^2 \leq C\|\psi\|^2 \tag{3.12}$$

if $\hbar$ is sufficiently small. This implies, in particular, $\|e^{\rho/\hbar}\psi\| \leq C$. Noting that

$$\rho(x) \geq \int_x^0 \eta(y)\, dy - \frac{\varepsilon}{4} \geq \frac{5}{12} - \frac{3}{4}\varepsilon \quad \text{for} \quad x \in [-1+\varepsilon/2,\, -1+\varepsilon], \tag{3.13}$$

we conclude (3.7). ■

The same argument holds for $H'(\theta)$. Then we need to use the positivity of

$$|(1+\theta\chi'(x))^{-2}(\xi + i\rho'(x))^2 + ((x+\theta\chi(x)) + (1+\theta\chi'(x))^{-2}(\xi+i\rho'(x))^2)^2|^2, \tag{3.14}$$

instead of (3.9). If $|\theta|$ is sufficiently small, however, we can employ the same $\rho(x)$ to obtain the estimate:

$$\left\langle \varphi, \left[\left|e^{\rho/\hbar}(H'(\theta) - E_n)\, e^{-\rho/\hbar}\right|^2 + a(x)^2\right]\varphi\right\rangle \geq \frac{c}{2}(\|\varphi\|^2 + \|H_0'\varphi\|^2) - C\hbar\|\varphi\|^2 \tag{3.15}$$

for $\varphi \in D(H_0')$, as well as (3.11). We now fix $\theta = i\lambda$ with $\lambda > 0$ so small that the above estimate holds. Then we obtain

PROPOSITION 3.2. – *Let ψ_n be an eigenfunction of $H_0'(\theta)$ with $H'(\theta)\psi_n = E_n\psi$. Then there exists $C > 0$ such that*

$$\|\chi_I \psi_n\| \leq C e^{-(\gamma_0 - \varepsilon')/\hbar}, \; \hbar > 0, \tag{3.16}$$

where $\varepsilon' = (3/4)\varepsilon$.

4. ESTIMATES FOR RESONANCES

We first show that $E_n = F_n + O(\hbar^N)$ for any $N \geq 0$.

LEMMA 4.1. – *Let $n \geq 0$ and $N \geq 1$. Let $\Gamma = \{z \in \mathbf{C} \,|\, |F_n - z| = \hbar^N\}$. If $\hbar$ is sufficiently small, then $\Gamma \subset \rho(K_1)$ and*

$$\|\chi_I (K_1 - z)^{-1}\| \leq C, \qquad z \in \Gamma, \tag{4.1}$$

where C is independent of $\hbar$.

Proof. – The proof is standard and we only explain the case $N = 1$, which is in fact enough for our purpose. We set

$$L = \begin{pmatrix} p^2 + (x+p^2)^2 + h(x) & 0 \\ 0 & k(x) \end{pmatrix} + \hbar R'. \tag{4.2}$$

L has no spectrum in a neighborhood of 0 (if h is sufficiently small), and it is easily shown that

$$\|(L-z)^{-1}\| \le C, \quad z \in \Gamma. \tag{4.3}$$

Let $a(x)$ be a smooth function such that $a(x) = 1$ on I and $a(x) = 0$ on $\operatorname{supp} h(x)$. Then by easy computations,

$$\chi_I (K_1 - z)^{-1} = \chi_I (L-z)^{-1} a - \chi_I (L-z)^{-1} [a, K_1] (K_1 - z)^{-1}. \tag{4.4}$$

But since $[a, K_1] = O(\hbar)$ as an operator from $D(K_1)$ to $D(L)^*$, the last term is $O(\hbar)$ in norm, and clearly $\|(K_1 - z)^{-1}\| = O(\hbar^N)$ for $z \in \Gamma$. Combined with (4.3), these completes the proof for $N = 1$. For the general case, we use a series of functions $a_j(x)$, $j = 1, 2, \cdots, N$, with $\operatorname{supp} a_j \subset \{x | a_{j-1}(x) = 1\}$ to carry out the above argument iteratively. ■

We set

$$T = H'J^* - J^*K = [H', J_1] \oplus [H', J_2]. \tag{4.5}$$

Then the right hand side has a symbol of $O(\hbar)$ supported in I, and we have

COROLLARY 4.2. – *Let* Γ *be as in Lemma* 4.1. *Then*

$$\|T[(K_1 - z)^{-1} \oplus (K_2(\theta) - z)^{-1}]\|_{B(D(K)^*, D(H')^*)} \le C\hbar, \quad z \in \Gamma, \tag{4.6}$$

is $\hbar$ *is sufficiently small.*

LEMMA 4.3. – *Let* C *be as in Lemma* 4.1. *If* $\hbar$ *is sufficiently small,* $\Gamma \subset \rho(H'(\theta))$ *and* $\dim P(\theta) = 1$, *where* $P(\theta)$ *is the projection:*

$$P(\theta) = \frac{1}{2\pi i} \int_\Gamma (H'(\theta) - z)^{-1} dz. \tag{4.7}$$

Proof. – As well as (2.13), we have

$$JH'(\theta) - K(\theta) J = [J_1, H'] \oplus [J_2, H']$$

where $K(\theta) = K_1 \oplus K_2(\theta)$. Hence

$$(H'(\theta) - z)^{-1} = J(K(\theta) - z)^{-1} J^* - (H'(\theta) - z)^{-1} T(K(\theta) - z)^{-1} J^*, \tag{4.8}$$

and

$$(H'(\theta) - z)^{-1} [1 + T(K(\theta) - z)^{-1} J^*] = J(K(\theta) - z)^{-1} J^*. \tag{4.9}$$

By Corollary 4.2, $[\cdots]$ is invertible in $D(H')$ if $\hbar$ is sufficiently small and $z \in \Gamma$, and

$$(H'(\theta) - z)^{-1} = J(K(\theta) - z)^{-1} \times J^* [1 + T(K(\theta) - z)^{-1} J^*]^{-1} \in B(\mathcal{H}). \tag{4.10}$$

Moreover, (4.6) implies

$$\|(H'(\theta) - z)^{-1} - J(K(\theta) - z)^{-1} J^*\| \leq C\hbar. \tag{4.11}$$

On the other hand, the eigenprojection to the F_n-eigenspace of K_1 is given by

$$P_1 = \frac{1}{2\pi i} \int_\Gamma (K_1 - z)^{-1} dz = \langle \psi_n^1, \cdot \rangle \psi_n^1, \tag{4.12}$$

and since $K_2(\theta)$ has no spectrum in Γ,

$$J(P_1 \oplus 0) J^* = \frac{1}{2\pi i} \int_\Gamma J(K(\theta) - z)^{-1} J^* dz. \tag{4.13}$$

Combining them with the definition of $P(\theta)$, we obtain

$$\|J(P_1 \oplus 0) J^* - P(\theta)\| \leq C\hbar^{N+1} \tag{4.14}$$

since $|\Gamma| = C\hbar^N$.

By the exponential decay estimate (Proposition 3.1), we also have

$$\|(1 - J_1^* J_1) \psi_n^1\| \leq Ce^{-(\gamma_0 - \varepsilon)/\hbar} \tag{4.15}$$

and hence

$$\|P_1 - J_1^* J_1 P_1 J_1^* J_1\| \leq Ce^{-(\gamma_0 - \varepsilon)/\hbar}. \tag{4.16}$$

Then (4.14) and (4.16) imply

$$\|J^* P(\theta) J - (P(\theta) \oplus 0)\| \leq C\hbar^{N+1} + Ce^{-(\gamma_0 - \varepsilon)/\hbar}. \tag{4.17}$$

If $\hbar$ is so small that the right hand side is less than 1, then

$$\dim P(\theta) = \dim [J^* P(\theta) J] = \dim P_1 = 1. \tag{4.18}$$

Here we have use the fact that J^* is isometry. ■

Then E_n is contained in Γ since E_n is the unique eigenvalue of $H'(\theta)$ of the form $(2n+1)\hbar + O(\hbar^2) = F_n + O(\hbar^2)$.

COROLLARY 4.4. – *For any* N, $|E_n - F_n| \leq C\hbar^N$. *In particular,* $|\mathrm{Im}\, E_n| \leq C\hbar^N$.

Proof of Theorem. 1.1. – Let φ be the F_n-eigenfunction of K_1, and let

$$\psi = P(\theta) J_1 \varphi. \tag{4.19}$$

Then as in the proof of Lemma 4.3, we have

$$\|\psi - J_1 \varphi\| \leq C\hbar^N \tag{4.20}$$

for any N, since $\|P(\theta) - J(P_1 \oplus 0) J^*\| \leq C\hbar^N$ by (4.11). Combining this with (4.15), we have

$$\langle \psi, J_1 \varphi \rangle = \|J_1 \varphi\|^2 + O(\hbar^N) = \|\varphi\|^2 + O(\hbar^N) > 1/2 \tag{4.21}$$

if $\hbar$ is small. Now we compute the difference of the eigenvalues $E_n - F_n$. Since ψ is an E_n-eigenfunction of $H'(\theta)$,

$$\langle (H'(\theta) - i)^{-1} \psi, J_1 \psi \rangle = (\overline{E_n} + i)^{-1} \langle \psi, J_1 \varphi \rangle. \tag{4.22}$$

On the other hand, since φ is an F_n-eigenfunction of K_1, we also have

$$\begin{aligned} \langle (H'(\theta) - i)^{-1} \psi, J_1 \psi \rangle &= \langle \psi, H'(\overline{\theta}) + i)^{-1} J_1 \varphi \rangle \\ &= \langle \psi, J_1 (K_1(\overline{\theta}) + 1)^{-1} \varphi \rangle \\ &\quad - \langle \psi, H'(\overline{\theta}) + i)^{-1} (H'(\overline{\theta}) J_1 - J_1 K_1)(K_1(\overline{\theta}) + 1)^{-1} \varphi \rangle \\ &= (F_n + i)^{-1} \langle \psi, J_1 \varphi \rangle - (\overline{E_n} + i)^{-1} (F_n + i)^{-1} \langle \psi, T_1 \varphi \rangle. \end{aligned} \tag{4.23}$$

Note that $T_1 \equiv (H'(\overline{\theta}) J_1 - J_1 K_1)$ is a third order differential operator with the coefficients supported in $\mathrm{supp}\, J_1' \subset [-1 - \varepsilon/2, -1 + \varepsilon]$. Then by Propositions 3.1 and 3.2,

$$|\langle \psi, T_1 \varphi \rangle| \leq C e^{-2(\gamma_0 - \varepsilon)/\hbar}. \tag{4.24}$$

Combining these, we conclude

$$|((\overline{E_n} + i)^{-1} - (F_n + i)^{-1} \langle \psi, J_1 \varphi \rangle| \leq C e^{-2(\gamma_0 - \varepsilon)/\hbar}. \tag{4.25}$$

By (4.11), this implies $|E_n - F_n| \leq C e^{-2(\gamma_0 - \varepsilon)/\hbar}$. ■

5. DISCUSSION

Here we discuss what this example suggests us about phase-space tunneling. If we compare our result with the Martinez' result, our estimate is more than twice as sharp. This may suggest that in order to obtain sharp decay estimation on, e.g., eigenfunctions, we have to decide the direction of the decay in the phase space. The *direction* is not necessarily linear, but may be one coordinate of a canonical coordinate system. It seems impossible to obtain a sharp decay estimate for the all directions in the phase space simultaneously. This can be considered as a consequence of the uncertainty principle. Thus a general scheme of the phase space tunneling might be as follows: In order to estimate tunneling effect between two microlocal wells, *i.e.*, two disjoint classically allowed areas in the phase space, we first find a canonical coordinate which separates the wells in a maximum way (in some sense). Then construct a unitary transformation corresponding the canonical change of coordinate. Finally we use the Agmon-type estimates to prove the exponential decay of (generalized) eigenfunctions. However, situations may be greatly varied both geometrically (*i.e.*, in the behavior of classical mechanics) and analytically (*i.e.*, in the choice of symbol classes), and hence it seems hardly possible to construct a general theory which is applicable to all the problems of phase space tunneling phenomena.

At last, we sketch an idea of a generalization of our theorem in the lower order term R. As is seen from the proof, the form of R (1.3) is not really relevant. We only need some analyticity conditions and upper bounds on the symbol. Let $\delta > 0$ and suppose $R' = WRW^*$ has a symbol $a(\hbar;\, x,\, \xi) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and $a(\hbar;\, x,\, \xi)$ satisfies the following conditions: $a_{ij}(\hbar;\, x,\, \xi)$ is analytic in

$$\Omega = \{(z,\, \zeta) \in \mathbf{C}^2 \,|\, |\mathrm{Im}\, z| < \delta,\ |\mathrm{Im}\, \zeta| < 1/2 + \delta |\mathrm{Re}\, \zeta|\}, \tag{5.1}$$

and satisfies

$$\begin{aligned} &|\partial_z^k \partial_\zeta^l a_{11}(z,\, \zeta)| \le C_{kl}(1 + |\zeta|^2 + |z + \zeta^2|^2)^2, \\ &|\partial_z^k \partial_\zeta^l a_{12}(z,\, \zeta)| \le C_{kl}(1 + |\zeta|^2 + |z + \zeta^2|^2), \\ &|\partial_z^k \partial_\zeta^l a_{21}(z,\, \zeta)| \le C_{kl}(1 + |\zeta|^2 + |z + \zeta^2|^2), \\ &|\partial_z^k \partial_\zeta^l a_{22}(z,\, \zeta)| \le C_{kl}, \quad (z,\, \zeta) \in \Omega, \end{aligned}$$

for any $k,\, l \ge 0$. Then the analytic continuation (or distortion) arguments in the proof work, and R' is H_0^*-bounded. One technical problem may be the fact that the operator $[J_i,\, R']$ is not necessarily supported in $\mathrm{supp}\, j_i$, even though the symbol is. But the distribution kernel decay exponentially away from the support of the symbol, by virtue of the analyticity in ξ. We refer [15] Section 4 for the detail of this argument.

APPENDIX: SYMBOL CALCULUS

We need to use calculus with symbol class which includes functions like $((x+\xi^2)^2+\xi^2+1)^{-1}$, which is not included in the standard symbol classes, e.g., $S(\langle\xi\rangle^N, dx^2+d\xi^2/\langle\xi\rangle^2)$.

Here we set the metric g on $\mathbf{R}^d\times\mathbf{R}^d$ by $g = dx^2+d\xi^2$. If $m(x,\xi)\in C^\infty(\mathbf{R}^d\times\mathbf{R}^d)$ satisfies

$$\left|\frac{\partial m(x,\xi)}{\partial(x,\xi)}\right| \le Cm(x,\xi),\ (x,\xi)\in\mathbf{R}^d\times\mathbf{R}^d \tag{A.1}$$

and

$$1\le m(x,\xi)\le C\langle(x,\xi)\rangle^N,\ (x,\xi)\in\mathbf{R}^d\times\mathbf{R}^d \tag{A.2}$$

with some $N\ge 0$, then it is easy to see that m is g continuous (*cf.* [8], Definition 18.4.2), and σ, g temperate (*cf.* [8], Definition 18.5.1). Then the symbol class $S(m,g)$ is well behaved under the condition (A.1), and we can use the theory of Weyl calculus with the quantization:

$$\begin{aligned} a(\hbar;x,p)\varphi(x) = (2\pi\hbar)^{-d}\int & e^{i(x-y)\xi/\hbar} \\ & \times a(\hbar;(x+y)/2,\xi)\varphi(y)\,dyd\xi, \\ & \varphi\in\mathcal{S}(\mathbf{R}^d). \end{aligned} \tag{A.3}$$

In particular,

$$m_0(x,\xi) = (x+\xi^2)^2+\xi^2+1 \tag{A.4}$$

satisfies (A.1). Thus we can now construct the parametrix for $a\in S(m_0,g)$ such that it is elliptic in the following sense:

$$a(\hbar;x,\xi)\ge\alpha\, m_0(x,\xi),\qquad (x,\xi)\in\mathbf{R}^d\times\mathbf{R}^d,\quad \alpha>0. \tag{A.5}$$

In particular, we have the Gårding inequality:

THEOREM A.1. – *Suppose* $a(\hbar;x,\xi)\in S(m_0,g)$ *satisfies* (A.5). *Then for any* $\varepsilon>0$ *there is* $C>0$ *such that*

$$\begin{aligned}\langle\varphi, a(\hbar;x,p)\varphi\rangle \ge (\alpha-\varepsilon)\langle\varphi, m_0(x,p)\varphi\rangle - C\hbar\|\varphi\|^2, \\ \varphi\in\mathcal{S}(\mathbf{R}^d).\end{aligned} \tag{A.6}$$

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