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Frame functions and completeness of inner product spaces

by

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ABSTRACT. – Using a notion of theory of wavelets, the frame condition, we show that a real or complex inner product space S (not necessarily separable) is complete iff there is a unit vector x in S (or in $\bar{S}$) such that the sum of squares of absolute values of Fourier coefficients of x through any maximal orthonormal system in S is uniformly separated from 0. In addition, S is complete iff there is at least one weak frame function. These criteria generalize ones of Gudder, Gudder and Holland, and the author.

Key words: Inner product space, Hilbert space, wavelets, frame condition, orthonormal basis, frame function, weak frame function.

RÉSUMÉ. – Grâce à la notion de « Frame Conditions », issue de la théorie des ondelettes, nous montrons qu'un espace vectoriel réel ou complexe S (non nécessairement séparable) muni d'un produit scalaire est complet si et seulement si on peut trouver un vecteur x de norme un dans S (ou dans son complété) dont la somme des modules carrés des coordonnées dans toute base orthonormale complète de S soit uniformément séparée de zéro. De plus, S est complet si et seulement si il existe au moins une fonction « frame » faible. Ces critères généralisent ceux de Gudder, Gudder et Holland, ainsi que ceux de l'auteur.

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1. INTRODUCTION

Real or complex inner product spaces play important role in mathematics. Hilbert spaces form a special class of inner product spaces, which frequently encounter in different areas of mathematical investigations and their applications. Today there are plenty of completeness criteria using functional, topological, algebraic, and measure-theoretic aspects. In [4], 38 different completeness criteria are presented. Very important is that of Gudder [7, 8], and Gudder and Holland [9] saying that a real or complex inner product spaces S is complete iff any *maximal orthonormal system* (MONS, for short)[1] in S is an *orthonormal basis* (ONB, for short) in S, that is,

$$\forall x \in S, \quad \forall \, \mathrm{MONS}\, \{x_i\} \text{ of } S, \qquad x = \sum_i (x,\, x_i)\, x_i. \tag{1.1}$$

In [3], it was shown that S is complete, iff

$$\left.\begin{array}{c} \exists\, 0 \neq x \in S\, (x \in \bar{S}), \quad \forall \, \mathrm{MONS}\, \{x_i\} \text{ of } S, \\ x = \sum_i (x,\, x_i)\, x_i. \end{array}\right\} \tag{1.2}$$

The criteria (1.1) and (1.2) can be rewritten in the equivalent forms

$$\forall x \in S, \quad \forall \, \mathrm{MONS}\, \{x_i\} \text{ of } S, \qquad \| x \|^2 = \sum_i \, |(x,\, x_i)|^2, \tag{1.3}$$

and

$$\left.\begin{array}{c} \exists\, 0 \neq x \in S\, (x \in \bar{S}), \quad \forall \, \mathrm{MONS}\, \{x_i\} \text{ of } S, \\ \| x \|^2 = \sum_i |(x,\, x_i)|^2. \end{array}\right\} \tag{1.4}$$

Both criteria (1.3) and (1.4) have been generalized using a frame function [3]: S is complete iff there is a non-trivial *frame function*, *i.e.* a mapping $f : \mathcal{S}(S) := \{x \in S : \|x\| = 1\} \to R_+$ such that, for some constant W (called the weight of f) and any MONS $\{x_i\}$ in S,

$$\sum_i f(x_i) = W \tag{1.5}$$

holds.

[1] A system of orthonormal vectors $\{x_i\}$ in H is said to be a MONS in S if $x \in H$, $x \perp x_i$ for any i imply $x = 0$.

In theory of wavelets [1], "the frame condition", *i.e.* a system of vectors $\{x_i\}$ in S such that, for any $x \in S$,

$$0 < a \leq \sum_i |(x, x_i)|^2 \leq b \tag{1.6}$$

holds for some constants a and b, plays an important role. This notion has been adopted, for example, in [9], for informational completeness in physics as well as for numerical stability in applied mathematics.

Motivating (1.6), we show that S is complete iff there is a unit vector x in S such that (1.6) holds for any MONS $\{x_i\}$ in S, which will generalize both criteria (1.1) and (1.2). In addition, using a *weak frame function*, we give a criterion generalizing (1.5).

2. COMPLETENESS CRITERIA

We present two completeness criteria generalizing (1.1) and (1.2).

THEOREM 2.1. – *A real or complex inner product space S is complete iff there is a unit vector $x \in S$ and a positive number $W > 0$ such that, for any* MONS $\{x_i\}$ *in S,*

$$0 < W \leq \sum_i |(x, x_i)|^2 \tag{2.1}$$

holds.

Proof. – If S is complete, the statement is evident, and in this case we can put $W = 1$.

Suppose the converse. First we assert that (2.1) holds for any unit vector y in S. Indeed, let $M = \mathrm{sp}\,(x, y)$, where sp denotes the span in S, and define a unitary operator $U : S \to S$ such that $U x = y$ and $U z = z$ for $M^\perp$. Let $\{x_i\}$ be a MONS in S. Then

$$\sum_i |(y, x_i)|^2 = \sum_i |(U x, x_i)|^2 = \sum_i |(x, U^{-1} x_i)|^2 \geq W,$$

when we have used the fact that $\{U^{-1} x_i\}$ is a MONS in S.

Choose a MONS $\{x_i\}$ in S, and motivating by Schroeck [9], we define a mapping $F : S \to l_2\,(I)$, where I is the cardinal number of the MONS $\{x_i\}$ (and of all MONSs in S), and $l_2\,(I)$ is over the same field as S, via

$$F\,(z) = \{(z, x_i)\}_i, \qquad z \in S. \tag{2.2}$$

In view of the Bessel inequality $\sum_i |(z, x_i)|^2 \leq \|z\|^2$, we see that F is a continuous operator from S into $l_2(I)$. Then F can be extended to a continuous operator $\bar{F} : \bar{S} \to l_2(I)$, where $\bar{S}$ denotes the completion of S. It is evident that

$$\bar{F}(z) = \{(z, x_i)\}_i$$

for any $z \in \bar{S}$.

We claim that S is complete. Since by [6], [7], [8], S is complete iff any MONS in S is an ONB, there exists a MONS $\{x_i\}$ which is not ONB. That is, there is a unit vector x_0 in $\bar{S}$ which is orthogonal with to $\{x_i\}$. We can find a sequence of unit vectors $\{x_n\}$ in S such that $x_n \to x_0$. Then $F(x_n) \to \bar{F}(x_0)$. From the continuity of $\bar{F}$ and (2.1), we have

$$\|F(x_n)\|_{l_2} \to \|\bar{F}(x_0)\|_{l_2}$$

and

$$0 < W \leq \|F(x_n)\|_{l_2}^2 \to \|\bar{F}(x_0)\|_{l_2}^2 = 0,$$

which is a contradiction, so that S is complete. □

THEOREM 2.2. – *A real or complex inner product space S is complete iff there is a unit vector $x \in \bar{S}$ and a positive number $W > 0$ such that, for any* MONS $\{x_i\}$ *in S,*

$$0 < W \leq \sum_i |(x, x_i)|^2 \tag{2.3}$$

holds.

Proof. – The necessity is evident. For sufficiency, let $\{x_i\}$ be any MONS in S and similarly as in the proof of Theorem 2.1, define a continuous linear operator $F : \bar{S} \to l_2(I)$ via

$$F(z) = \{(z, x_i)\}_i$$

for any $z \in \bar{S}$. Find a unit vector $x_0 \in S$ such that $\|x - x_0\| < \varepsilon$, where $\varepsilon < \sqrt{W}/2$. Then $\|F(x_0)\|_{l_2} \leq \|F(x) - F(x_0)\|_{l_2} + \|F(x)\|_{l_2}$, so that

$$|\|F(x)\|_{l_2} - \|F(x_0)\|_{l_2}| \leq \|F(x) - F(x_0)\|_{l_2} \leq \|x - x_0\| < \varepsilon$$

which means that

$$\|F(x_0)\|_{l_2} \geq \|F(x)\|_{l_2} - \varepsilon \geq \sqrt{W} - \varepsilon > \sqrt{W}/2 > 0.$$

This proves that there is a unit vector $x_0 \in S$ and a non-zero constant $W_0 = W/4$ such that, for any MONS $\{x_i\}$ in S,

$$0 < W_0 \leq \sum_i |(x_0, x_i)|^2.$$

Calling Theorem 2.1, this entails the completeness of S. □

COROLLARY 2.3. – *If S is an incomplete inner product space, then for any $x \in S$ $(x \in \bar{S})$*

$$\inf_{\{x_i\}} \sum_i |(x, x_i)|^2 = 0,$$

where $\{x_i\}$ is any MONS *in S.*

Proof. – It follows immediately from Theorems 2.1 and 2.2. □

3. WEAK FRAME FUNCTIONS

A mapping $f : \mathcal{S}(S) := \{x \in S : \|x\| = 1\} \to R_+$ is said to be a *frame function* iff there is a constant W (called the *weight* of f) such that

$$\sum_i f(x_i) = W \tag{3.1}$$

holds for any MONS $\{x_i\}$ in S.

A mapping $f : \mathcal{S}(S) \to R_+$ is said to be a *weak frame function* iff (i) there is a positive constant W such that

$$0 < W \leq \sum_i f(x_i) < \infty \tag{3.2}$$

holds for any MONS $\{x_i\}$ in S, and (ii) $f \,|\, \mathcal{S}(M)$ is a frame function for any finite-dimensional subspace M of S. From (ii) we have that $f(\lambda x) = f(x)$ for any $|\lambda| = 1$.

It is evident that any frame function is a weak frame function, and the converse holds, too, as we shall see below, which will prove the completeness. We recall that if x is a unit vector in S or in $\bar{S}$, then $f : z \mapsto |(z, x)|^2$, $z \in \mathcal{S}(S)$, is sometimes (iff S is complete) a special type of a frame function or a weak frame function.

THEOREM 3.1. – *An inner product space S is complete iff there is at least one weak frame function f on $\mathcal{S}(S)$.*

Proof. – The necessity is evident if we put $f(x) = |(x, x_0)|^2, x \in \mathcal{S}(S)$, where x_0 is a unit vector in S.

Suppose that f is a weak frame function. If M is any finite-dimensional subspace of S, then $f \mid \mathcal{S}(M)$ defines a unique finitely additive measure m_M on $L(M)$, the system of all subspaces of M, such that $m_M(N) = \sum_i f(x_i)$, where $\{x_i\}$ is an ONB in $N \in L(M)$. Using the Gleason theorem [4], there is a unique bilinear form t_M on M such that $t_M(x, x) = f(x)$, $x \in \mathcal{S}(M)$. Now we shall define a bilinear form t on $S \times S$ as follows: Let x, y be two vectors of S and let M be any finite-dimensional subspace of S, $\dim M \geq 3$, containing x and y. Put $t(x, y) = t_M(x, y)$. It is easy to show that t is a well-defined bilinear form on $S \times S$, moreover,

$$t(x, x) = f(x), \qquad x \in \mathcal{S}(S).$$

Hence, t may be uniquely extended to a bounded, positive, bilinear form $\bar{t}$ defined on the whole completion $\bar{S}$ of S. At any rate, there exists a unique positive Hermitian operator $T : \bar{S} \to \bar{S}$ such that $t(x, x) = (Tx, x)$, $x \in \bar{S}$.

Now we claim to show that T is a trace class operator on S. Suppose that $\{x_1, \ldots, x_n\}$ is an arbitrary finite system of orthonormal vectors in S. Then

$$\sum_{i=1}^{n} (Tx_i, x_i) < \infty,$$

so that $\sum_i (Tx_i, x_i) < \infty$ for any system of orthonormal vectors $\{x_i\}$ of S.

Let $\{x_1, \ldots, x_n\}$ be any finite system of orthonormal vectors in $\bar{S}$. For any i, $1 \leq i \leq n$, there is a sequence $\{x_k^i\}_k$ in S such that $x_i = \lim_k x_k^i$. Applying the Gram-Schmidt orthogonalization process to $\{x_k^i\}_{i,k}$, we obtain an orthonormal sequence $\{y_j\}_j$ in S which is an ONB in $M_0 = \mathrm{cl}(\mathrm{sp}(\{y_j\})) \in L(\bar{S})$, where cl denotes the closure in S, and $L(\bar{S})$ is the system of all closed subspaces of $\bar{S}$.

Complete $\{x_1, \ldots, x_n\}$ to $\{x_i\}$ to be an ONB in $\overline{M_0}$ and calculate

$$\sum_i (Tx_i, x_i) = \sum_{i,j} (x_i, y_j)(Ty_j, x_i) = \sum_{i,j,k} (x_i, y_j)(Ty_j, y_k)(y_k, x_i)$$
$$= \sum_{j,k} (Ty_j, y_k)(y_k, y_j).$$

On the other hand, for M_0 (similarly as above) there exists a unique positive Hermitian operator $T_{M_0} : M_0 \to M_0$ such that $(T_{M_0} x, y) = (T x, y)$ for all $x, y \in M_0 \cap S$. Therefore,

$$\sum_{j,k} (T y_j, y_k)(y_k, y_j) = \sum_{j,k} (T_{M_0} y_j, y_k)(y_k, y_j) = \sum_j (T_{M_0} y_j, y_j)$$
$$= \sum_j (T y_j, y_j) < \infty.$$

The last inequality shows that T is of trace class operator on $\bar{S}$.

Hence, there is a finite or infinite sequence of orthonormal vectors, $\{e_n\}$, and a sequence of positive proper values of T, $\{\lambda_n\}$, such that $T = \sum_n \lambda_n e_n \otimes \bar{e}_n$. We assert that there is n_0 such that, for any MONS $\{x_i\}$ in S, we have

$$0 < W_1 \leq \sum_i |(e_{n_0}, x_i)|^2, \tag{3.3}$$

where $W_1 = W / \sum_n \lambda_n$. Indeed, if not that, for any n, $\lambda_n \sum_i |(e_n, x_i)|^2 < \lambda_n W_1$, which gives $\sum_n \lambda_n |(e_n, x_i)|^2 < W$, which contradicts (3.2). Applying Theorem 2.2 to (3.3), we see that S is complete. □

COROLLARY 3.2. – *Any weak frame function is a frame function.*

Proof. – It follows from the proof of Theorem 3.1. □

In the rest of this section, we generalize the notions of a frame function and a weak frame function, which allow us to present another completeness criterion.

A mapping $f : \mathcal{S}(S) := \{x \in S : \|x\| = 1\} \to R$ is said to be a *signed frame function* iff there is a constant W (called the *weight* of f) such that

$$\sum_i f(x_i) = W \tag{3.4}$$

holds for any MONS $\{x_i\}$ in S.

A mapping $f : \mathcal{S}(S) \to R$ is said to be a *weak signed frame function* iff (i) for any MONS $\{x_i\}$ is S, $\{f(x_i)\}_i$ is summable; (ii) there is a positive constant W such that

$$0 < W \leq \sum_i |f(x_i)| < \infty \tag{3.5}$$

holds for any MONS $\{x_i\}$ in S, and (iii) $f \mid \mathcal{S}(M)$ is a signed frame function for any finite-dimensional subspace M of S. From (iii) we have that $f(\lambda x) = f(x)$ for any $|\lambda| = 1$.

It is evident that if f is a signed frame function with a non-zero weight, then f a weak signed frame function. We recall that in [3], it has been shown that S is complete iff there is at least one non-zero signed measure frame function on $\mathcal{S}(S)$.

THEOREM 3.3. – *An inner product space S is complete iff there exists at least one weak signed frame function f on $\mathcal{S}(S)$.*

Proof. – If S is complete, the assertion is clear.

Suppose thus that f is a weak signed frame function. If $\dim S < \infty$, then S is evidently complete, so we can assume that $\dim S = \infty$. Due to the definition of f, f is a frame-type function, *i.e.* (i) for any orthonormal system (ONS, for short) $\{x_i\}$ in S, $\{f(x_i)\}_i$ is summable, and (ii) $f \mid \mathcal{S}(M)$ is a signed frame function for any finite-dimensional subspace M of S. By the result of Dorofeev and Sherstnev [2], or [4], Thm. 3.2.20, f is bounded, *i.e.* $\sup\{|f(x) : x \in \mathcal{S}(S)\} < \infty$. [2]

Similarly as in the proof of Theorem 3.1, we can prove, applying the Gleason theorem for bounded signed measures on finite-dimensional Hilbert spaces [4], that there is a bounded bilinear from t on $S \times S$ such that $t(x, x) = f(x)$, $x \in \mathcal{S}(S)$. The boundedness of f entails that t can be uniquely extended to a bounded symmetric bilinear form $\bar{t}$ on $\bar{S} \times \bar{S}$ consequently, there is a Hermitian operator T on $\bar{S}$ such that $f(x) = (Tx, x)$, $x \in \mathcal{S}(S)$.

We claim to show that T is a trace class operator on $\bar{S}$. Suppose the converse. Then there is an ONS $\{f_1, \ldots, f_{n_1}\}$ in $\bar{S}$ such that $\sum_{k=1}^{n_1} |(T f_k, f_k)| > 1$. Choose an $\varepsilon > 0$ such that $\sum_{k=1}^{n_1} |(T f_k, f_k)| > 1+\varepsilon$. It is easy to see that for $\{f_1, \ldots, f_{n_1}\}$ we can find an ONS $\{h_1, \ldots, h_{n_1}\}$ in S such that $\| h_k - f_k \| < \varepsilon/(2 n_1 \| T \|)$, $k = 1, \ldots, n_1$. Then

$$\begin{aligned} |f(h_k) - (T f_k, f_k)| &\le |(T(h_k - f_k), f_k)| + |(T h_k, h_k - f_k)| \\ &\le 2 \| T \| \, \| h_k - f_k \| < \varepsilon/n_1, \end{aligned}$$

[2] If $\dim S < \infty$, there are unbounded signed frame functions, *see* e.g. [4], Prop. 3.2.4.

so that

$$\sum_{k=1}^{n_1} |f(h_k)| \geq \sum_{k=1}^{n_1} |(T f_k, f_k)| - \sum_{k=1}^{n_1} |(T f_k, f_k) - f(h_k)| > 1.$$

Put $H_1 = \{h_1, \ldots, h_{n_1}\}^{\perp_{\bar{S}}}$, where $\perp_{\bar{S}}$ denotes the orthogonalization in $\bar{S}$, then $S_1 = S \cap H_1$ is a dense submanifold in H_1, so that, $f \mid \mathcal{S}(S_1)$ is a frame type function on S_1. Therefore, as in the beginning of the present proof, there is a Hermitian operator $T_1 (= P_{H_1} T P_{H_1})$ on H_1, where P_{H_1} is the orthoprojector from $\bar{S}$ onto H_1, such that $f(x) = (T_1 x, x) = (T x, x)$, $x \in \mathcal{S}(S_1)$. Here T_1 is not any trace class operator on H_1 since T is not such one on $\bar{S}$.

Repeating the same reasonings as above, we find an ONS $\{f_{n_1+1}, \ldots, f_{n_2}\}$ in H_1 such that $\sum_{k=n_1+1}^{n_2} |(T f_k, f_k)| > 1$, and for it we find and ONS $\{h_{n_1+1}, \ldots, h_{n_2}\}$ in S_1 with $\sum_{k=n_1+1}^{n_2} |f(h_k)| > 1$. Continuing this process, we find a countable family of orthonormal vectors $\{h_1, h_2, \ldots\} \subset S$ such that $\sum_{k=1}^{\infty} |f(h_k)| = \infty$ which contradicts our assumption, so that, T is a trace class operator on $\bar{S}$.

Put $T = T^+ - T^-$, where T^+ and T^- are the positive and negative parts of T (which both are trace class operators on $\bar{S}$), and define $f^+(x) := (T^+ x, x)$, $f^-(x) = (T^- x, x)$, $x \in \mathcal{S}(S)$. Then $f = f^+ - f^-$ and, in view of (3.5), for $f_0 := f^+ + f^-$, we have $0 < W \leq \sum_i |f(x_i)| \leq \sum_i f_0(x_i) < \infty$ for any MONS $\{x_i\}$. Since $f_0 \mid \mathcal{S}(M)$ is evidently a signed frame function on any finite-dimensional subspace M of S, f_0 is a positive weak frame function, which by Theorem 3.1 entails the completeness of S. □

COROLLARY 3.4. – *Any weak signed frame function is a signed frame function.*

Proof. – If $\dim S < \infty$, the statement is evident, and, for $\dim S = \infty$, it follows from the proof of Theorem 3.3 □

COROLLARY 3.5. – *If S is an incomplete inner product space, then, for any Hermitian trace class operator T on $\bar{S}$, we have*

$$\inf_{\{x_i\}} \sum_i |(T x_i, x_i)| = 0,$$

where $\{x_i\}$ *is any* MONS *on* S.

Proof. – In the opposite case, S would be by Theorem 3.4 complete. □

Remark. – The presented proofs are quite different from those for frame functions in [3], [4]. In addition, any measure-theoretic criterion using a non-zero completely additive (signed) measure m (for details, *see* [4]) follows from Theorems 3.1 and 3.3, because $f(x) := m(\mathrm{sp}(x))$, $x \in \mathcal{S}(S)$, is a (non-zero) frame function [3] on $\mathcal{S}(S)$.

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[3] If f is a non-zero signed frame function with a non-zero weight, it is clear that f is a weak signed frame function; if the weight of f is zero, there is a unite vector $x \in S$ with $f(x) \neq 0$. Put $S_1 = \mathrm{sp}(x)^{\perp}$. Then $f_1 := f \mid \mathcal{S}(S_1)$ is a signed frame function with a non-zero weight, so that f_1 is a weak signed frame function, and S_1, consequently S, is complete.