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States and representations of CQ*-algebras

by

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ABSTRACT. — A class of quasi *-algebras which exhibits some analogy with C*-algebras is studied. The extension of some properties of C*-algebras which are relevant for physical applications (such as the GNS-representation) is discussed.

Quasi *-algebras of linear operators in rigged Hilbert space are shown to be typical examples of the developed framework.

RÉSUMÉ. — Nous étudions une classe de quasi *-algèbres présentant une analogie avec les C*-algèbres.

Nous étendons à ces algèbres quelques-unes des propriétés des C*-algèbres les plus familières en physique, par exemple la représentation GNS.

Un exemple typique de ces algèbres est fourni par les quasi *-algèbres d'opérateurs linéaires dans les échelles d'espaces de Hilbert.

1. INTRODUCTION

In the Haag-Kastler [1] algebraic approach to quantum systems, with infinitely many degrees of freedom, a relevant role is played, as is known, by C*-algebras: what is normally done is to associate to each bounded region V the C*-algebra A_V of local observables in V. The uniform completion A of the algebra generated by the A_V's is then considered as the C*-algebra of observables of the system.

There are, however, many physical models that do not fit into the Haag-Kastler setup. In many quantum statistical systems, in fact, the thermodynamical limit of some local observables, for instance the local Heisenberg dynamics, does not exist in the uniform topology and thus it is not an element of the observables algebra as defined before. This is the case, for instance, of the BCS model ([2], [3]), and, in general, of any mean field model. This kind of behavior has been discussed by the authors also for easy spin models with an 'almost' mean field interaction ([4], [5]). Also the long range interactions, like the Coulomb one ([6], [7]), give rise to a similar behavior. The key of this phenomenon is essentially the fact that the interaction between any 'particle' (spins, electrons, ...) of the above systems with any other particle of the same system does not go to zero fast enough: actually, for mean field spin models the finite volume hamiltonian

$$H_V = \frac{1}{|V|} \sum_{i, j \in V} \sigma_i \sigma_j$$

shows that any spin interact with any other independently of the mutual distance. Therefore the time evolution of a spin located at the origin is really affected from the behavior of infinitely far spins.

Also, in the Wightman formulation of quantum field theory, point-like fields are not, in general, elements of any C*-algebra: the field $A(x)$ at a point $x \in \mathbb{R}^4$ is, in fact, an (unbounded) sesquilinear form on the domain $\mathcal{D}$ where all *smeared* fields $A(f), f \in \mathcal{S}(\mathbb{R}^4)$ are defined. If the field $A(x)$ is regular enough [8], then it is the limit of a sequence of observables localized in a shrinking sequence of space-time regions and, therefore, it belongs to a certain completion of the C*-algebra $\mathcal{A}_0$ of local observables ([9], [10]).

As a matter of fact Haag-Kastler approach is often too narrow to cover a lot of models of physical interest. On the other hand, it is rather clear that this algebraic formulation does not depend crucially on the assumption that the observable algebra $\mathcal{A}_0$ is a C*-algebra. Therefore it is, in principle, possible to try to weaken this hypothesis without affecting the elegant Haag-Kastler construction.

Two possibilities are then at hand. The first one occurs if there exists on $\mathcal{A}_0$ a weaker norm such that the completion of $\mathcal{A}_0$ with respect to this norm contains all 'objects' of physical interest. If this possibility fails, it could still happen that these 'objects' can be recovered by taking the completion of $\mathscr{A}_0$ with respect to the locally convex topology generated by a suitable family of seminorms.

In both cases we are led to consider *quasi *-algebras*, introduced by Lassner ([11], [12]), as the most natural mathematical structure where this sort of problems can be successfully handled. Actually, the completion of a locally convex algebra, where the multiplication is not jointly continuous, is the most typical instance of quasi *-algebra. In this case the multiplication in the completion is defined for pairs of elements one of which lies in the original algebra (for detailed studies on quasi*-algebras, *see* [11]-[17]).

Quasi *-algebras are a particular class of *partial *-algebras*: this mathematical object, introduced originally by Antoine and Karwowski [18], has been studied by Antoine, Inoue, Mathot and one of us in a series of papers ([19], [20], [21], [22]). From this point of view, however, quasi *-algebras presents as the simplest example: the lattice of multipliers consists, in fact, of two elements only. One of the first steps of this paper is just to consider quasi *-algebras with a *finer* multiplication structure. We call them *rigged* quasi *-algebras. Roughly speaking, they contains two algebras, which turns out to be, respectively, the sets of *right-* and *left universal multipliers* and which are changed one into the other by the involution.

For what concerns applications to quantum theories (*see* [11], [12], [14] and [8]), the most relevant rol. seems to be played by the quasi *-algebra $(\mathscr{L}(\mathscr{D}, \mathscr{D}'), \mathscr{L}^+(\mathscr{D}))$. Here $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ denotes the space of all continuous linear operators from $\mathscr{D} = \mathscr{D}^\infty(\mathrm{T})$, with T an unbounded self-adjoint operator on Hilbert space $\mathscr{H}$, into its conjugate dual $\mathscr{D}'$ and $\mathscr{L}'(\mathscr{D})$) is the maximal O*-algebra on $\mathscr{D}$ ([15], [16], [23]). This quasi *-algebra is, in some sense, the *unbounded* analogous of C*-algebras [13] and for this reason it has been called a CQ*-algebra.

In this paper we will try to give a purely algebraic description of this structure and study some of its properties. With respect to Reference [13], we change a little the names. We reserve the name CQ*-algebra simply for the (roughly speaking) completion of a C*-algebra with respect to a certain weaker norm. The structure introduced by Lassner will be called here LCQ*-algebra since it is an inductive *limit* of CQ*-algebras as defined here.

In a sense, CQ*-algebras appear as a possible extension of the notion of C*-algebra. Other possible extensions of the notion of C*-algebra (or that of W*-algebra) to unbounded operator algebras have been explored in several papers ([24]-[22]). Our approach follows, however, a different path and works in a different frame.

It is not in the spirit of the present paper to carry out a full mathematical analysis of the introduced structures. This will be undertaken in further publications.

Just to begin with, we will fix here the basic aspects of the theory and discuss non-trivial examples. We will focus our attention, in particular, on those mathematical statements that frequently occur in applications to quantum theories, such as the GNS-representation. This latter is shown to be possible for CQ*-algebras with respect to certain families of states. In this case, however, the representation does not live in a Hilbert space but in a *scale* of Hilbert spaces [28].

The paper is organized as follows.

Section 2 is devoted to preliminaries.

In section 3, we introduce the class of CQ*-algebras and discuss some simple aspects of this structure. The example of the CQ*-algebra of bounded operators in a scale of Hilbert spaces is shown in details.

Section 4 is divided into two parts: in the first one we consider representations of CQ*-algebras and give our variant of the GNS-construction; in the second, we examine the abelian case, trying to extend the famous Gel'fand-Naimark theorem (*see*, e. g. [29]) concerning the representation of an abelian C*-algebra as a C*-algebra of continuous functions.

In section 5, we introduce the notion of LCQ*-algebra and show that it can be made into a partial *-algebra, in natural way, by refining the multiplicative structure it carries as quasi *-algebra.

Finally, we consider a quasi *-algebra $(O(\mathscr{D}'), O(\mathscr{D}))$ of operators in a *nested Hilbert space* $\{\mathscr{D}', \mathscr{H}_r, I\}$ [30] and show that it fulfills, in the most general case, all the algebraic aspects of our definition.

If the nested Hilbert space reduces to the chain of Hilbert spaces generated by an unbounded self-adjoint operator T (this is, as we already mentioned, one of the most significant cases for applications) then also the topological requirements are satisfied, so it turns out that, in this case, $(O(\mathscr{D}'), O(\mathscr{D}))$ is a LCQ*-algebra and it coincides with $(\mathscr{L}(\mathscr{D}, \mathscr{D}'), \mathscr{L}^+(\mathscr{D}))$.

In Section 6 we will discuss two possible physical applications of the developed ideas.

2. BASIC DEFINITIONS AND NOTATIONS

The basic notion we will deal with throughout the paper is that of partial *-algebra [21].

A partial *-algebra is a vector space $\mathscr{A}$ with involution $A \to A^*$ [*i. e.* $(A+\lambda B)^* = A^* + \bar{\lambda} B^*$; $A = A^{**}$] and a subset $\Gamma \subset \mathscr{A} \times \mathscr{A}$ such that (i) $(A, B) \in \Gamma$ implies $(B^*, A^*) \in \Gamma$; (ii) (A, B) and $(A, C) \in \Gamma$ imply

$(A, B+\lambda C)\in\Gamma$; and (iii) if $(A, B)\in\Gamma$, then there exists an element $AB\in\mathcal{A}$ and for this multiplication the distributive property holds in the following sense: if $(A, B)\in\Gamma$ and $(A, C)\in\Gamma$ then

$$AB+AC=A(B+C)$$

Furthermore $(AB)^*=B^*A^*$.

The product is not required to be associative.

The partial *-algebra $\mathcal{A}$ is said to have a unit if there exists an alement $\mathbb{I}$ (necessarily unique) such that $\mathbb{I}^*=\mathbb{I}$, $(\mathbb{I}, A)\in\Gamma$, $\mathbb{I}A=A\mathbb{I}$, $\forall A\in\mathcal{A}$.

If $(A, B)\in\Gamma$ then we say that A is a left multiplier of B [and write $A\in L(B)$] or B is a right multiplier of $A\,[B\in R(A)]$. For $\mathcal{S}\subset\mathcal{A}$ we put $L\mathcal{S}=\bigcap L(A):A\in\mathcal{S}$; the set $R\mathcal{S}$ is defined in analogous way. The set $M\mathcal{S}=L\mathcal{S}\bigcap R\mathcal{S}$ is called the set of universal multipliers of $\mathcal{S}$.

The most typical instance is obtained when $M\mathcal{A}$ is a *-algebra. In this case, following Lassner ([11], [12]) we speak of a quasi *-algebra. We give a detailed definition for reader's convenience.

Let $\mathcal{A}$ be a linear space and $\mathcal{A}_0$ a *-algebra contained in $\mathcal{A}$. We say that $\mathcal{A}$ is a quasi *-algebra with distinguished *-algebra $\mathcal{A}_0$ (or, simply, over $\mathcal{A}_0$) if (i) the right and left multiplications of an element of $\mathcal{A}$ and an element of $\mathcal{A}_0$ are always defined and linear; and (ii) an involution * (which extends the involution of $\mathcal{A}_0$) is defined in $\mathcal{A}$ with the property $(AB)^*=B^*A^*$ whenever the multiplication is defined.

A quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ is said to have a unit $\mathbb{I}$ if there exists an element $\mathbb{I}\in\mathcal{A}_0$ such that $A\mathbb{I}=\mathbb{I}A\ \forall A\in\mathcal{A}$.

A quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ is said to be a topological quasi *-algebra if in $\mathcal{A}$ is defined a locally convex topology ξ such that (a) the involution is continuous and the multiplications are separately continuous; and (b) $\mathcal{A}_0$ is dense in $\mathcal{A}[\xi]$.

Following [13], if $(\mathcal{A}[\xi], \mathcal{A}_0)$ is a topological quasi *-algebra, by ξ_0 we will denote the weakest locally convex topology on $\mathcal{A}_0$ such that for every bounded set $\mathcal{M}\subset\mathcal{A}[\xi]$ the family of maps $B\to AB$, $B\to BA$; $A\in M$ from $\mathcal{A}_0[\xi_0]$ into $\mathcal{A}[\xi]$ is equicontinuous. In this case $\mathcal{A}_0[\xi_0]$ is a locally convex *-algebra. The topology ξ_0 will be called the *reduced topology* of ξ.

Some spaces of continuous linear maps provide the most interesting examples of quasi *-algebras.

Let $\mathcal{D}$ be a dense linear manifold of Hilbert space $\mathcal{H}$. Let us endow $\mathcal{D}$ with a topology t, stronger than that induced on $\mathcal{D}$ by the Hilbert norm and let $\mathcal{D}'[t']$ be its topological dual endowed with the strong dual topology t'. In this fashion we get the familiar triplet

$$\mathcal{D}\subset\mathcal{H}\subset\mathcal{D}'$$

called a *rigged Hilbert space* (RHS).

Given a RHS $\mathcal{D}\subset\mathcal{H}\subset\mathcal{D}'$, we will denote by $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ the set of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}'[t']$. The space $\mathcal{L}(\mathcal{D}', \mathcal{D}')$ carries a natural involution $A \to A^\dagger$ defined by

$$\langle A^\dagger f, g\rangle = \overline{\langle A g, f\rangle} \qquad \forall f, g \in \mathscr{D}$$

If $\mathscr{D}$ is a dense linear manifold of Hilbert space $\mathscr{H}$ we will denote by $\mathscr{L}^+(\mathscr{D})$ the *-algebra of all closable operators in $\mathscr{H}$ with the properties $D(A)=\mathscr{D}$, $D(A^*)\supseteq\mathscr{D}$ and both A and A^* leave $\mathscr{D}$ invariant (* denotes here the usual Hilbert adjoint). The involution in $\mathscr{L}^+(\mathscr{D})$ is defined by $A \to A^+ = A^*/\mathscr{D}$.

If $\mathscr{D}\subset\mathscr{H}\subset\mathscr{D}'$ is a RHS, the space $\mathscr{L}^+(\mathscr{D})$ is not, in general a subset of $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ but, for instance, when t is the so called *graph-topology* defined by a closed O*-algebra $\mathscr{A}$ on $\mathscr{D}$ ([11], [13]) then $\mathscr{L}^+(\mathscr{D})\subset\mathscr{L}(\mathscr{D}, \mathscr{D}')$ and $(\mathscr{L}(\mathscr{D}, \mathscr{D}'), \mathscr{L}^+(\mathscr{D}))$ is a quasi *-algebra and $A^\dagger = A^+ \ \forall A \in \mathscr{L}^+(\mathscr{D})$.

Throughout the paper we will make extensive use of the theory of C*-algebras: in general, we follow the terminology and the notations of classical textbooks such as those by Dixmier [31], Sakai [32], Kadison and Ringrose [33], Bratteli and Robinson [34].

3. CQ*-ALGEBRAS: ELEMENTARY THEORY AND EXAMPLES

DEFINITION 3.1. – *Let* $(\mathscr{A}, \mathscr{A}_0)$ *be a quasi *-algebra. We say that* $(\mathscr{A}, \mathscr{A}_0)$ *is a* BQ**-algebra if*

(i) $\mathscr{A}$ *is a Banach space under the norm* $\|\,.\,\|$

(ii) $\|A^*\| = \|A\|$, $\forall A \in \mathscr{A}$

(iii) *for* $B \in \mathscr{A}_0$ *the maps* $A \to AB$, $A \to BA$ *are continuous with respect to* $\|\,.\,\|$

(iv) $\mathscr{A}_0$ *is* $\|\,.\,\|$*-dense in* $\mathscr{A}$

(v) $\mathscr{A}_0$ *is a Banach algebra with respect to the norm*

$$\|B\|_0 = \frac{1}{2}\sup\{(\|AB\| + \|BA\|),\ A \in \mathscr{A},\ \|A\| \leqq 1\} \tag{1}$$

or, equivalently

$$\|B\|_0 = \sup\{\max(\|AB\|, \|BA\|),\ A \in \mathscr{A},\ \|A\| \leqq 1\} \tag{2}$$

Equations 1, or 2, defines, as is easily seen, in $\mathscr{A}_0$ the reduced topology introduced in Section 2.

Remark. – It is clear that if $(\mathscr{A}, \mathscr{A}_0)$ is a BQ*-algebra then $\mathscr{A}$ is a bilateral Banach $\mathscr{A}_0$-module ([22], [32]).

From the definition of $\|\,.\,\|_0$ it is clear that $\|B^*\|_0 = \|B\|_0$, $\forall B \| \in \mathscr{A}_0$

From now forth $\|\,.\,\|_0$ will denote the norm defined by Eq. 2.

Finally, notice that in the C*-case, *i.e.* $\mathscr{A} = \mathscr{A}_0$, Equation 2 gives exactly the C*-norm ([31], 1.3.5).

The proof of the following lemma is straightforward.

LEMMA 3.1. – *Let* $(\mathcal{A}, \mathcal{A}_0)$ *be a* BQ*-*algebra. The following inequalities hold*

$$\|AB\| \leq \|A\| \|B\|_0 \qquad \forall A \in \mathcal{A}, \quad \forall B \in \mathcal{A}_0 \tag{3}$$

$$\|BC\|_0 \leq \|B\|_0 \|C\|_0 \qquad \forall B, C \in \mathcal{A}_0 \tag{4}$$

Remark. – Inequality 3 plays a quite crucial role, as we shall see below (Proposition 3.2), for the construction of examples of BQ*-algebras.

Making use of (3) it is easy to prove that

$$\|B\|'_0 = \sup\{\max(\|AB\|, \|BA\|), \|A\| \leq 1, A \in \mathcal{A}_0\}$$

defines the same topology as $\|\cdot\|_0$. Also Equation 2 admits an analogous generalization.

Inequality 4 means that $\mathcal{A}_0$, with $\|\cdot\|_0$, is in any case a normed algebra. Therefore condition (v) in Definition 3.1 could be weakened by requiring that $\mathcal{A}_0$ is a complete normed space with respect to $\|\cdot\|_0$.

DEFINITION 3.2. – *A* rigged *quasi *-algebra is a partial *-algebra for which there exist two vector subspaces* $\mathcal{A}_\flat$ *and* $\mathcal{A}_\sharp$ *such that*

(i) $\mathcal{A}_\flat^* = \mathcal{A}_\sharp$

(ii)

$$\Gamma = \{(A, B) \in \mathcal{A} \times \mathcal{A} : A \in \mathcal{A}_\sharp; \textit{ or } B \in \mathcal{A}_\flat\}$$

(iii) *both* $\mathcal{A}_\flat$ *and* $\mathcal{A}_\sharp$ *are algebras with respect to the partial multiplication* $(A, B) \in \Gamma \to AB \in \mathcal{A}$ *defined in* $\mathcal{A}$.

The multiplication $(A, B) \in \Gamma \to AB \in \mathcal{A}$ *is supposed to be* (weakly) semi-associative; i. e. $(AB)C = A(BC)\ \forall A \in \mathcal{A}$ *and* $\forall B, C \in \mathcal{A}_\flat$.

Obviously, we get $R\mathcal{A} = \mathcal{A}_\flat$ and $L\mathcal{A} = \mathcal{A}_\sharp$. We set $\mathcal{A}_0 = M\mathcal{A}$. $\mathcal{A}_0$ is, clearly, a *-algebra; thus $(\mathcal{A}, \mathcal{A}_0)$ is a quasi *-algebra. With a usual terminology, $\mathcal{A}$ is an $R\mathcal{A}$-left-module and an $L\mathcal{A}$-right-module.

Of course, the structure of a rigged quasi *-algebra is fully determined if we know $\mathcal{A}$ with its involution * and the set of right multipliers $R\mathcal{A}$ with its involution . For this reason we will denote a rigged quasi *-algebra by $(\mathcal{A}, *, R\mathcal{A}, \ )$.

DEFINITION 3.3. – *Let* $(\mathcal{A}, *, R\mathcal{A}, {}^\flat)$ *be a rigged quasi *-algebra. We say that* $(\mathcal{A}, *, R\mathcal{A}, \ )$ *is a* CQ*-*algebra if*

(i) $(\mathcal{A}, \mathcal{A}_0)$ *is a* BQ*-*algebra*

(ii) *the map* $A \to AB$, $B \in R\mathcal{A}$ *is continuous in* $\mathcal{A}$;

(iii) $R\mathcal{A}$ *carries a norm* $\|\cdot\|_\flat$ *and an involution* $A \to A^\flat$ *with respect to which* $R\mathcal{A}$ *is a* C*-*algebra*; i. e.

$$\|A^\flat A\|_\flat = \|A\|_\flat^2, \qquad \forall A \in \mathcal{A}_\flat = R\mathcal{A} \tag{5}$$

(iv) *Setting* $\|A\|_\# = \|A^*\|_\flat$ *for* $A \in L\mathcal{A} = \mathcal{A}_\#$, $\|B\|_0 = \max\{\|B\|_\flat, \|B\|_\#\}$, *results;*

(v) $\mathcal{A}_0$ *is dense in* $R\mathcal{A}$ *with respect to* $\|\,.\,\|_\flat$.

LEMMA 3.2. – *In* $L\mathcal{A}$ *let us define an involution* $A \to A^\#$ *by* $A^\# = A^{*\flat *}$ *and the norm* $\|\,.\,\|_\#$ *as above; then* $L\mathcal{A}$ *is a* C*-*algebra with respect to this norm and this involution.*

The proof is straightforward

DEFINITION 3.4. – *A* CQ*-*algebra* $(\mathcal{A}, *, R\mathcal{A},\ )$ *will be said* proper *if* $RA = L\mathcal{A}$ *and* $A^\# = A^\flat$, $\forall A \in R\mathcal{A}$.

LEMMA 3.3. – *Let* $(\mathcal{A}, *, R\mathcal{A},\ )$ *be a proper* CQ*-*algebra; then* $\|A\|_\# = \|A\|_\flat$, $\forall A \in R\mathcal{A}$,

Proof. – Since $\flat = \#$, by lemma 3.2, $R\mathcal{A}$ is a C*-algebra with respect to both norms $\|\,.\,\|_\flat$ and $\|\,.\,\|_\#$. Then the statement follows from the uniqueness of the C*-norm. □

PROPOSITION 3.1. – *Let* $(\mathcal{A}, *, R\mathcal{A}, {}^\flat)$ *be an* abelian CQ*-*algebra* (i.e. $R\mathcal{A} = L\mathcal{A}$ *and* $AB = BA$, $\forall A \in \mathcal{A}$ *and* $\forall B \in R\mathcal{A}$)*; then* $(\mathcal{A}, *, R\mathcal{A},\ )$ *is proper.*

Proof. – The space X of characters of $R\mathcal{A}$ is independent of both norm and involution. If $A \in R\mathcal{A}$ we denote with $\hat{A}$ its Gel'fand transform. Then if $\omega \in X$ we get

$$\hat{A}^\flat(\omega) = \omega(A^\flat) = \overline{\omega(A)}$$
$$\hat{A}^\#(\omega) = \omega(A^\#) = \overline{\omega(A)}$$

This implies $\omega(A^\flat) = \omega(A^\#), \forall \omega \in X$ and so $\omega(A^\flat - A^\#) = 0$, $\forall \omega \in X$. Hence

$A^\# A^\# = 0$. This, in turn, implies that $A^\flat = A^\#$ since the Gel'fand transform is an isometric isomorphism of $R\mathcal{A}$ into the C*-algebra C(X) of continuous functions (possibly, vanishing at infinity) on X. □

Remark. – For a proper CQ*-algebra (even in the abelian case) it not necessarily true that also the involution * coincides with both and #. This is due to the fact that a C*-algebra may have another involution for which it is only a Banach *-algebra, with respect to the same norm. We will come back to this point in Example 3.1.

The next lemma shows that, under stronger assumptions, the involution of $R\mathcal{A}$ can be extended to the whole $\mathcal{A}$, preserving the nature of CQ*-algebra.

LEMMA 3.4. – *Let* $(\mathcal{A}, *, R\mathcal{A},\ )$ *be a proper CQ*-algebra. Let* $\|\,.\,\|$ *be the norm of* $\mathcal{A}$ *and the involution of* $R\mathcal{A}$. *If*

$$\|A^\flat\| = \|A\|, \qquad \forall A \in R\mathcal{A}$$

then the involution can be extended to the whole $\mathscr{A}$ *and* ($\mathscr{A}$, , R$\mathscr{A}$,) *is again a proper* CQ*-*algebra.*

The proof is very simple and will be omitted.

Remark. — In a rigged quasi *-algebra ($\mathscr{A}$, $*$, R$\mathscr{A}$,), one could find an unit $\mathbb{I}_R$ of R$\mathscr{A}$ (and therefore a unit $\mathbb{I}_L = \mathbb{I}_R^*$ of L$\mathscr{A}$) without ($\mathscr{A}$, $*$, R$\mathscr{A}$,) have a unit. But if ($\mathscr{A}$, $*$, R$\mathscr{A}$,) is a CQ*-algebra the existence of $\mathbb{I}_R$ implies that $\mathbb{I}_R = \mathbb{I}_L = \mathbb{I}$ is a unit for the whole $\mathscr{A}$, due to the continuity of the multiplication and to the fact that R$\mathscr{A}$ is dense in $\mathscr{A}$.

CQ*-algebras will be, as announced, the main object of our study. To begin with, let us give some examples that we consider quite enlightening.

Example 3.1. — Let X be a compact space and μ a (positive) Baire measure on X; let C(X) be the C*-algebra of continuous functions on X, endowed with the usual sup norm, denoted here simply as $\|\,.\,\|$ and C′(X) the topological dual of C(X), endowed, as customary, with the norm

$$\|F\|' = \sup\{|F(f)|,\ \|f\| \leqq 1\}, \qquad F \in C'(X) \tag{6}$$

As is known, C′(X) is a Banach space of all complex Baire measures on X and C(X) can be identified, by means of μ, with a subspace of it in the following way:

$$\phi \in C(X) \to T_\phi \in C'(X)$$

with

$$T_\phi(\psi) = \int_X \psi(x)\,\phi(x)\,d\mu(x), \qquad \psi \in C(X)$$

The usual involution of C′(X) extends clearly the involution of C(X). It is not difficult to prove conditions (i), (ii) and (iii) of Definition 3.1. For what concerns condition (v) it is very easy to show that $\|f\|_0 \leqq \|f\|$, $\forall f \in C(X)$, where $\|f\|_0 = \sup_{\|F\|' \leqq 1} \|Ff\|'$. The converse inequality follows from the fact that $\forall f \in C(X)$ there exists $F \in C'(X)$ such that $|F(f)| = \|f\|$: F is just the Dirac δ functional centered in one of the points where $|f(x)|$ is maximum. This implies that $\|f\| \leqq \|f\|_0$, $\forall f \in C(X)$. As a result, the closure $\overline{C(X)}$ of C(X) in C′(X) is a proper CQ*-algebra over C(X).

If we choose $X = [-1, 1]$ and define in C(X) a new involution $\dagger$ by $f^\dagger(x) = \overline{f(-x)}$, we get an example of the situation described in the Remark after Proposition 3.1.

Example 3.2. — The above example is a particular case of a more general situation. Let $\mathscr{A}_0$ be a C*-algebra which can be identified with a subspace of its dual Banach space $\mathscr{A}'_0$, by means of a one-to-one linear map I [for shortness, we set I(A) = A for A $\in \mathscr{A}_0$]. If we define in $\mathscr{A}'_0$ the Arens multiplication [29].

$$\phi\,.\,A(B) = \phi(A, B), \qquad A\,.\,\phi(B) = \phi(BA)$$

and the involution ϕ^* by $\phi^*(A)=\overline{\phi(A^*)}$ then $(\mathscr{A}, \mathscr{A}_0)$ is a proper CQ*-algebra (with $=\#=*$), where $\mathscr{A}$ denotes the closure of $\mathscr{A}_0$ in $\mathscr{A}'_0$. The proof is similar to that of the previous example. The existence, for any $B\in\mathscr{A}$, $B\neq 0$, of a continuous linear form ϕ on $\mathscr{A}$ such that $\|\phi\|'=1$ and $\phi(B)=\|B\|$ is a well known property of normed spaces.

Example 3.3. – Let $\mathscr{H}$ be a Hilbert space with scalar product $\langle .,. \rangle$ and $\lambda(.,.)$ a positive sesquilinear closed form defined on a dense domain $\mathscr{D}_\lambda\subset\mathscr{H}$. Then $\mathscr{D}_\lambda$ becomes a Hilbert space, that we denote by $\mathscr{H}_\lambda$, with respect to the scalar product

$$\langle f, g\rangle_\lambda=\langle f, g\rangle+\lambda(f, g). \tag{7}$$

Let $\mathscr{H}_{\bar\lambda}$ be the Hilbert space of conjugate linear forms on $\mathscr{H}_\lambda$.

This is the canonical way to get a scale of Hilbert spaces ([28], VIII.6)

$$\mathscr{H}_\lambda \overset{i}{\to} \mathscr{H} \overset{j}{\to} \mathscr{H}_{\bar\lambda} \tag{8}$$

where i and j are continuous embeddings with dense range. In fact, the identity map i embeds $\mathscr{H}_\lambda$ in $\mathscr{H}$ and the map $j:\psi\in\mathscr{H}\to j(\psi)\in\mathscr{H}_{\bar\lambda}$, where $j(\psi)(\phi)=\langle\phi, \psi\rangle$, $\forall\phi\in\mathscr{H}_\lambda$ is a linear imbedding of $\mathscr{H}$ into $\mathscr{H}_{\bar\lambda}$. Identifying $\mathscr{H}_\lambda$ and $\mathscr{H}$ with their respective images in $\mathscr{H}_{\bar\lambda}$ we can read (8) as a chain of inclusions

$$\mathscr{H}_\lambda\subset\mathscr{H}\subset\mathscr{H}_{\bar\lambda}$$

Because of this identification, the bilinear form which puts $\mathscr{H}_\lambda$ in duality with (the complex conjugate of) $\mathscr{H}_{\bar\lambda}$ reduces, for pairs (ϕ, ψ) such that $\phi\in\mathscr{H}_\lambda$, $\psi\in\mathscr{H}$, to the scalar product $\langle ., . \rangle$ of $\mathscr{H}$. For this reason we will adopt the same notation. If $f\in\mathscr{H}_\lambda$, then $\|f\|_{\bar\lambda}\leqq\|f\|\leqq\|f\|_\lambda$.

As is known the Riesz lemma implies the existence of a unitary operator U from $\mathscr{H}_\lambda$ onto $\mathscr{H}_{\bar\lambda}$ and therefore $\mathscr{H}_\lambda$ and $\mathscr{H}_{\bar\lambda}$ are isometrically isomorphic);

On the other hand, by the representation theorem for sesquilinear forms ([28], Ch. VIII), there exists a selfadjoint positive operator H such that $D((1+H)^{1/2})=\mathscr{D}_\lambda=\mathscr{H}_\lambda\subseteqq\mathscr{H}$ and

$$\langle f, g\rangle_\lambda=\langle (1+H)^{1/2} f, (1+H)^{1/2} g\rangle, \qquad \forall f, g\in\mathscr{D}_\lambda \tag{9}$$

The operator $R=(1+H)^{1/2}$ has a bounded inverse R^{-1} which maps $\mathscr{H}$ into $\mathscr{H}_\lambda$; therefore (by a standard argument making use of the transposed map) R^{-1} has an extension (denoted here with the same symbol) to $\mathscr{H}_{\bar\lambda}$ and $R^{-1}\mathscr{H}_{\bar\lambda}\subseteqq\mathscr{H}$. Taking these facts into account, we have

$$\langle f, g\rangle_\lambda=\langle Rf, Rg\rangle=\langle Uf, Ug\rangle_{\bar\lambda}, \qquad \forall f, g\in\mathscr{H}_\lambda$$

It turns also out that $R^2=U$ and then $\langle f, g\rangle_\lambda=\langle f, Ug\rangle=\langle Uf, Ug\rangle_{\bar\lambda}$, $\forall f, g\in\mathscr{H}_\lambda$.

Let $\mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$ be the Banach space of bounded operators from $\mathscr{H}_\lambda$ into $\mathscr{H}_{\bar\lambda}$ and let us denote with $\|A\|_{\lambda\bar\lambda}$ the natural norm of $A\in\mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$.

In $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ we can introduce an involution in the following way: to each element $A \in B(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ we associate the linear map A^* from $\mathcal{H}_\lambda$ into $\mathcal{H}_{\bar\lambda}$ defined by the equation

$$\langle A^* f, g \rangle = \overline{\langle A g, f \rangle}, \qquad \forall f, g \in \mathcal{H}_\lambda$$

As can be easily proved $A^* \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and

$$\| A^* \|_{\lambda\bar\lambda} = \| A \|_{\lambda\bar\lambda} \forall A \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda}).$$

Let $\mathcal{B}(\mathcal{H}_\lambda)$ denotes the C*-algebra of bounded operators on $\mathcal{H}_\lambda$ (the usual involution of $\mathcal{B}(\mathcal{H}_\lambda)$ will be denoted here as) and $\mathcal{B}(\mathcal{H}_{\bar\lambda})$ the C*-algebra of bounded operators on $\mathcal{H}_{\bar\lambda}$ (the natural involution of $\mathcal{B}(\mathcal{H}_{\bar\lambda})$ is denoted as $\sharp$). As is easily seen both $\mathcal{B}(\mathcal{H}_\lambda)$ and $\mathcal{B}(\mathcal{H}_{\bar\lambda})$ are contained in $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and $A \in \mathcal{B}(\mathcal{H}_\lambda)$ if, and only if, $A^* \in \mathcal{B}(\mathcal{H}_{\bar\lambda})$. Moreover $B^{\flat *} = B^{*\sharp}$, $\forall B \in \mathcal{B}(\mathcal{H}_\lambda)$.

If $A \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and $B \in \mathcal{B}(\mathcal{H}_\lambda)$ the product AB is defined by $(AB) f = A(B f)$ for $f \in \mathcal{H}_\lambda$. It is readily checked that $AB \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$. If $A \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and $B \in \mathcal{B}(\mathcal{H}_{\bar\lambda})$ the product $BA \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ is defined in similar way.

Then $(\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda}), *, \mathcal{B}(\mathcal{H}_\lambda), \ )$ is a rigged quasi *-algebra. The distinguished *-algebra of $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ is

$$\mathcal{B}^+(\mathcal{H}_\lambda) = \{ A \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda}) : A, A^* \in \mathcal{B}(\mathcal{H}_\lambda) \}$$

Clearly, if $B \in \mathcal{B}(\mathcal{H}_\lambda)$ then $UB \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and the map $B \to UB$ is one-to-one and onto. Moreover $\| B \|_{\lambda\lambda} = \| UB \|_{\lambda\bar\lambda}$ if $\| \cdot \|_{\lambda\lambda}$ denotes the C*-norm of $\mathcal{B}(\mathcal{H}_\lambda)$. Therefore $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and $\mathcal{B}(\mathcal{H}_\lambda)$ are isometrically isomorphic. An analogous statement holds for $\mathcal{B}(\mathcal{H}_{\bar\lambda})$.

We will now prove that $(\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda}), *, \mathcal{B}(\mathcal{H}_\lambda), \ )$ is a CQ*-algebra (clearly, not proper) if $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and $\mathcal{B}(\mathcal{H}_\lambda)$ carry their natural norms.

We need only to show that $\mathcal{B}^+(\mathcal{H}_\lambda)$ is dense in $\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$.

Let $R = (1 + H)^{1/2}$ be the operator defined in Equation (9) and $R = \int_1^\infty \mu \, dE(\mu)$ its spectral resolution. If $\Delta \subset [1, \infty)$ is a bounded Borel set, then $E(\Delta)$ also has a continuous extension (again denoted with the same symbol) to $\mathcal{H}_{\bar\lambda}$; since $E(\Delta)$ is idempotent, it turns out that $E(\Delta)$ maps $\mathcal{H}_{\bar\lambda}$ into $\mathcal{H}_\lambda$. From this it follows that $E(\Delta) A E(\Delta) \in \mathcal{B}^+(\mathcal{H}_\lambda)$, $\forall A \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ and for any pair Δ, Δ' of bounded Borel sets.

By the definition of R it follows easily that $\| A \|_{\lambda, \bar\lambda} = \| R^{-1} A R^{-1} \|$ (the norm in $\mathcal{B}(\mathcal{H})$.

Following [11], we identify $A \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar\lambda})$ with an operator matrix $(A_{m,n})$, where $A_{m,n} = P_m A P_n$ with $P_m = E([m, m+1))$. Then the norm $\| \ \|_{\lambda, \bar\lambda}$ is equivalent to the following one

$$\left(\sum_{m,n=1}^{\infty} \| A_{m,n} \|^2 \frac{1}{m^2 n^2} \right)^{1/2}$$

From this it is clear that any $A \in \mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda})$ can be approximated by means of *finite matrices*. These latter, represent, as it can be easily shown, operators of $\mathscr{B}^+(\mathscr{H}_\lambda)$.

In similar way, the density of $\mathscr{B}^+(\mathscr{H}_\lambda)$ in $\mathscr{B}(\mathscr{H}_\lambda)$ can be proved. One has just to take into account that $\|A\|_{\lambda,\lambda} = \|RAR^{-1}\|$ $\forall A \in \mathscr{B}(\mathscr{H}_\lambda)$ and then adopt a procedure analogous to the above one. The equivalent norm in term of operator matrices is, in this case

$$\left(\sum_{m,n=1}^{\infty} \|A_{m,n}\|^2 \frac{m^2}{n^2}\right)^{1/2}.$$

Finally, let us check that the norm $\|\,.\,\|_1$ of $\mathscr{B}^+(\mathscr{H}_\lambda)$, where

$$\|B\|_1 = \max\{\|B\|_{\lambda,\lambda}, \|B\|_{\bar\lambda,\bar\lambda}\}$$

coincides with the $\|\,.\,\|_0$ defined in Equations 1 or 2.

To show this it is enough to observe that

$$\begin{aligned}\|B\|_{\lambda,\lambda} &= \sup\{\|CB\|_{\lambda,\lambda}, C \in \mathscr{B}(\mathscr{H}_\lambda), \|C\|_{\lambda,\lambda} \leqq 1\}\\ &= \sup\{\|UCB\|_{\lambda,\bar\lambda}, C \in \mathscr{B}(\mathscr{H}_\lambda), \|UC\|_{\lambda,\bar\lambda} \leqq 1\}\\ &= \sup\{\|AB\|_{\lambda,\bar\lambda}, A \in \mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda}), \|A\|_{\lambda,\bar\lambda} \leqq 1\}\end{aligned}$$

So the statement is proved.

As we mentioned beforehand, the completion of any locally convex *-algebra where the multiplication is not jointly continuous is the typical instance of quasi *-algebra. Now the question is: if we endow a given C*-algebra with a weaker norm, is its completion a CQ*-algebra? The answer is provided by the following

PROPOSITION 3.2. – *Let $\mathscr{C}$ be a* C*-*algebra, with norm* $\|\,.\,\|_1$ *and involution* *; *let* $\|\,.\,\|$ *be another norm on* $\mathscr{C}$, *weaker than* $\|\,.\,\|_1$ *and such that*

(i) $\|A\| = \|A^*\|$, $\forall A \in \mathscr{C}$,

(ii) $\|AB\| \leqq \|A\|\,\|B\|_1$, $\forall A, B \in \mathscr{C}$,

then the completion $\hat{\mathscr{C}}$ *of* $\mathscr{C}$, *with its natural norm, is a proper* CQ*-*algebra over* $\mathscr{C}$ (*with* $\;=*$).

Proof. – Conditions (i) and (ii) imply that $\mathscr{C}$ is a topological *-algebra whose topology is defined by $\|\,.\,\|$, so its completion $\hat{\mathscr{C}}$ is a quasi *-algebra and also a Banach space. Let $\|\,.\,\|_0$ denote the norm defined on $\mathscr{C}$ as in Equation (2); then (ii) implies easily that $\|\,.\,\|_0 \leqq \|\,.\,\|_1$. To show the converse inequality, we recall that by, (4), $\mathscr{C}$ with $\|\,.\,\|_0$ is a normed algebra. Let $X = X^* \in \mathscr{C}$ and let $M(X)$ denote the abelian C*-algebra generated by X. Since every norm that makes an abelian C*-algebra into a normed algebra is necessarily stronger than the C*-norm ([32], Theorem 1.2.4), we get the equality $\|X\|_0 = \|X\|_1$, $\forall X = X^* \in \mathscr{C}$. For an arbitrary element $Y \in \mathscr{C}$ we have

$$\| Y \|_2^{1} = \| Y^* Y \|_1 = \| Y^* Y \|_0 \leqq \| Y \|_0^2$$

This concludes the proof. □

Example 3.4. – Let X be a compact space and μ a Baire measure on X. Let C(X) be the Banach space of continuous functions on X. Then the Banach space $L^p(X, d\mu)$ of all (equivalence classes of) measurable functions $f(x)$ such that

$$\int_X |f(x)|^p \, d\mu(x) < \infty \tag{10}$$

is the completion of C(X) with respect to the norm (10) which, when restricted to C(X), fulfills the requirements of the above Proposition (with respect to the natural involution). Thus $(L^p(X, d\mu), C(X))$ is a (commutative) proper CQ*-algebra with $^\flat = * = \#$.

4. REPRESENTATIONS OF CQ*-ALGEBRAS

In the previous Section we have described the basic example of a CQ*-algebra as the set of bounded operators in a scale of Hilbert spaces. It is then natural to look for *representations* of an abstract CQ*-algebra into such a CQ*-algebra of operators.

DEFINITION 4.1. – *Let* $(\mathscr{A}, *, R\mathscr{A}, \ )$ *and* $(\mathscr{B}, *, R\mathscr{B}, \ )$ *be rigged quasi *-algebras.* A *-bimorphism of $(\mathscr{A}, *, R\mathscr{A}, \ )$ *into* $(\mathscr{B}, *, R\mathscr{B}, \ )$ *is a* pair (π, π_R) *of linear maps* $\pi : \mathscr{A} \to \mathscr{B}$ *and* $\pi_R : R\mathscr{A} \mapsto R\mathscr{B}$ *such that*

(i) π_R *is a homomorphism of algebras with* $\pi_R(A^\flat) = \pi_R(A)^\flat$, $A \in R\mathscr{A}$,

(ii) $\pi(A^*) = \pi(A)^*$, $\forall A \in \mathscr{A}$,

(iii) $\pi(AB) = \pi(A)\pi_R(B)$, $\forall A \in \mathscr{A}$, $B \in R\mathscr{A}$.

In general, the restriction of π to $R\mathscr{A}$ is different from π_R. Of course, if $\mathscr{A}$ has a unit then $\pi(B) = \pi(\mathbb{I})\pi_R(B)$; but in contrast with what happens for π_R, $\pi(\mathbb{I})$ need not necessarily be equal to $\mathbb{I}$.

LEMMA 4.1. – *If* (π, π_R) *is a *-bimorphism of* $(\mathscr{A}, *, R\mathscr{A}, \ )$ *into* $(\mathscr{B}, *, R\mathscr{B}, \ )$ *then*

$$\pi(BA) = \pi_L(B)\pi(A), \qquad \forall A \in \mathscr{A}, \quad \forall B \in L\mathscr{A}$$

where $\pi_L(B) = \pi_R(B^*)^*$. *Moreover* π_L *ia a homomorphism of* $L\mathscr{A}$ *into* $L\mathscr{B}$ *preserving the involution* $\#$ *of* $L\mathscr{A}$.

The proof is straightforward.

DEFINITION 4.2. – *A* *-representation *of a* CQ*-*algebra* $(\mathscr{A}, *, R\mathscr{A}, \ )$ *in the scale of Hilbert spaces* $\mathscr{H}_\lambda \subset \mathscr{H} \subset \mathscr{H}_{\bar\lambda}$ *is a *-bimorphism* (π, π_R) *of* $(\mathscr{A}, *, R\mathscr{A}, \ )$ *into the* CQ*-*algebra* $(\mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar\lambda}), *, \mathscr{B}(\mathscr{H}_\lambda). \ )$ *of bounded* *operators in the scale. The representation* π *is said to be* faithful *if* Ker π = Ker $\pi_R = 0$.

We say that the representation π *is* regular *if the linear manifold*

$$\mathscr{D}(\pi)=\{\phi\in\mathscr{H}_\lambda : \pi(A)\phi\in\mathscr{H},\ \forall A\in\mathscr{A}\}$$

is dense in $\mathscr{H}_\lambda$. *In this case* $\mathscr{D}(\pi)$ *is called the* domain of regularity *of* π. U.

If π is a regular *-representation, then $\pi(A)$ when restricted to $\mathscr{D}(\pi)$ can be considered as an element of the *partial *-algebra* $\mathscr{L}^+(\mathscr{D}(\pi), \mathscr{H})$ of all closable operators X in $\mathscr{H}$ such that $D(X)=\mathscr{D}(\pi)$ and $D(X^*)\supseteq\mathscr{D}(\pi)$ (for a detailed study of this partial *-algebra, *see* [21] and [22]). This makes clear the unbounded nature, in $\mathscr{H}$, of a *-representation of a CQ*-algebra and allows the use, which we will make later, of structures and techniques developed for unbounded operator families.

We will now prove, as well as is usually done for C*-algebras, the existence of *-representations. This is in fact the content of the following subsection where the GNS-construction is performed starting from certain *states* called *admissible*.

4.1. The GNS-construction. – Following an idea of Antoine [35], applied also in Refs. [21] and [36] to partial *-algebras, we will make use here, in view of a GNS-construction for CQ*-algebras, of sesquilinear forms, instead of linear forms. This appears to be quite natural whenever the multiplication is not defined for arbitrary elements. Of course the two approaches are fully equivalent for C*-algebras with unit.

As already discussed in Ref. [17], it is not reasonable to expect the GNS-representation to be possible starting from an arbitrary positive sesquilinear form, since our definition of representation requires the *continuity* of the operators $\pi(A)$. Thus we must select a class of well-behaved states.

This fact will suggest how to choose forms which allow a GNS-representation.

As usual, if $\mathscr{A}$ is a complex vector space, we say that a sesquilinear form Ω on $\mathscr{A}\times\mathscr{A}$ is positive if $\Omega(A, A)\geqq 0$, $\forall A\in\mathscr{A}$.

Any positive sesquilinear form on complex linear space is hermitean and fulfills the Cauchy-Schwarz inequality; *i.e.*

$$\Omega(A, B)=\overline{\Omega(B, A)}, \qquad \forall A, B\in\mathscr{A}, \tag{11}$$

$$|\Omega(A, B)|^2\leqq\Omega(A, A)\,\Omega(B, B), \qquad \forall A, B\in\mathscr{A} \tag{12}$$

Before going forth, we need the following.

LEMMA 4.2. – *Let* $\mathscr{H}_R$ *be a Hilbert space, whose scalar product is denoted as* $\langle\, .,\, .\,\rangle_R$ *and* $\mathscr{H}_{\bar{R}}$ *the space of continuous conjugate linear forms on* $\mathscr{H}_R$.

The following statements are equivalent.

(i) *There exists a Hilbert space* $\mathcal{H}$ *such that*

$$\mathcal{H}_R \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{\bar{R}}$$

where $\hookrightarrow$ *means continuous imbedding with dense range.*

(ii) *There exists a positive sesquilinear form* ξ *on* $\mathcal{H}_R$ *fulfilling the following conditions:*

(a) ξ *is non-degenerate*, i.e. $\xi(\phi, \phi)=0 \Leftrightarrow \phi=0$,

(b) ξ *is dominated by the scalar product of* $\mathcal{H}_R$, i.e. *there exists* $C>0$ *such that*

$$\xi(\phi, \phi) \leqq C\langle \phi, \phi\rangle_R = C\|\phi\|_R^2, \qquad \forall \phi \in \mathcal{H}_R. \tag{13}$$

If either (i) or (ii) hold true, then the scalar product $\langle .,. \rangle_\xi$ *of* $\mathcal{H}$ *is an extension of* ξ.

Moreover, for each $\psi \in \mathcal{H}$ *there exists an element* $F_\psi \in \mathcal{H}_{\bar{R}}$ *such that*

$$\langle \phi, \psi \rangle_\xi = F_\psi(\phi)$$

and therefore the conjugate bilinear form which makes $\mathcal{H}_R$ *and* $\mathcal{H}_{\bar{R}}$ *into a dual pair can be identified with an extension of* $\langle .,. \rangle_\xi$.

Proof. – The necessity is clear. For the sufficient part, let ξ be a sesquilinear form fulfilling conditions (a) and (b). Let $\mathcal{H}$ the completion of $\mathcal{H}_R$ with respect to the norm $\|\phi\|_\xi = \xi(\phi, \phi)^{1/2}$. Then $\mathcal{H}_R \subseteqq \mathcal{H}$.

Now, if $\psi \in \mathcal{H}$, then the form on $\mathcal{H}_R$ defined by

$$F_\psi(\phi) = \langle \phi, \psi \rangle_\xi, \qquad \phi \in \mathcal{H}_R$$

is continuous with respect to $\| \,.\, \|_R$ since

$$|F_\psi(\phi)| = |\langle \phi, \psi\rangle_\xi| \leqq \|\phi\|_\xi \|\psi\|_\xi \leqq C^{1/2} \|\phi\|_R \|\psi\|_\xi$$

Then the map

$$\psi \in \mathcal{H} \to F_\psi \in \mathcal{H}_{\bar{R}}$$

is well-defined, one-to-one and continuous. In fact, by the previous estimates it follows that

$$\|F_\psi\| \leqq C^{1/2} \|\psi\|_\xi \quad \square$$

Let now $(\mathcal{A}, *, R\mathcal{A}, \ )$ be a CQ*-algebra and ω_R a state (in usual sense) on the C*-algebra $(R\mathcal{A}, \ )$. Set

$$\mathcal{I} = \{ B \in R\mathcal{A} : \omega_R(B^\flat B) = 0 \}$$

and $\mathcal{D} = R\mathcal{A}/\mathcal{I}$, then we can construct, as is known, a Hilbert space $\mathcal{H}_R$ just taking the completion of $\mathcal{D}$ with respect to the norm $\|[B]\|_\omega^2 = \omega_R(B^\flat B)$ where $[B]$ denotes a rest class in $\mathcal{D}$. We will now exploit Lemma 4.2 to construct a representation of the whole CQ*-algebra $(\mathcal{A}, *, R\mathcal{A}, \ )$.

Let Ω be a positive sesquilinear form on $\mathcal{A}$ for which there exists $C>0$ such that

$$\Omega(B, B) \leqq C\,\omega_R(B^\flat B), \qquad \forall B \in R\mathcal{A} \tag{14}$$

We can now define a sesquilinear form $\langle \cdot, \cdot \rangle_\Omega$ on $\mathcal{H}_R$ putting

$$\langle \phi, \psi \rangle_\Omega = \lim_{n \to \infty} \Omega(B_n, C_n)$$

if $\phi = \lim_{n \to \infty} [B_n]$ and $\psi = \lim_{n \to \infty} [C_n]$ in $\mathcal{H}_R$.

Making use of (14), it is easy to see that $\langle \cdot, \cdot \rangle_\Omega$ is well-defined on $\mathcal{H}_R$. Now, we assume that

$$\text{if for } \phi \in \mathcal{H}_R,\ \langle \phi, \phi \rangle_\Omega = 0 \quad \text{then } \|\phi\|_\omega = 0. \tag{15}$$

In these conditions $\langle \cdot, \cdot \rangle_\Omega$ satisfies the assumptions of Lemma 4.2 and therefore if $\mathcal{H}$ is the completion of $\mathcal{H}_R$ with respect to the scalar product $\langle \cdot, \cdot \rangle_\Omega$, we get

$$\mathcal{H}_R \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{\bar{R}}$$

where $\hookrightarrow$ denotes, as before, a continuous imbedding with dense range.

Now we should define the representation. As a matter of fact the conditions (14) and (15) are not enough to ensure the continuity of the representation.

DEFINITION 4.3. – *Let* $(\mathcal{A}, *, R\mathcal{A}, \ )$ *be a* CQ*-*algebra and* ω_R *a state on the* C*-*algebra* $(R\mathcal{A}, \ )$,

Let Ω *be a positive sesquilinear form on* $\mathcal{A} \times \mathcal{A}$ *and* $\mathcal{H}_R$ *the carrier Hilbert space of the usual* GNS-*representation* π_R *of* $R\mathcal{A}$ *defined by* ω_R.

We say that Ω *is* admissible, *with respect to* ω_R (*or, simply,* ω_R-*admissible*) *if*

(i) $\phi \in \mathcal{H}_R$, *and* $\langle \phi, \phi \rangle_\Omega = 0$ *implies* $\|\phi\|_\omega = 0$

(ii) $\forall A \in \mathcal{A}$ *there exists* $K_A > 0$ *such that*

$$|\Omega(AB, C)|^2 \leqq K_A\, \omega_R(B^\flat B)\, \omega_R(C^\flat C), \qquad \forall B, C \in R\mathcal{A}$$

(iii) $\Omega(AB, C) = \Omega(B, A^* C)\ \forall A \in \mathcal{A}, \forall B, C \in R\mathcal{A}$

In the sequel we will say, simply, admissible instead of ω_R-admissible, whenever this will not create confusion.

We are now ready to build up our variant of the GNS-construction.

PROPOSITION 4.1. – *Let* ω_R *be a state on the* C*-*algebra* $R\mathcal{A}, \ )$ *and* Ω *an* ω_R-*admissible form on the* CQ*-*algebra, with unit* $\mathbb{I}$, $(\mathcal{A}, *, R\mathcal{A}, \ )$. *There exists a scale*

$$\mathcal{H}_R \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{\bar{R}}$$

of Hilbert spaces and a cyclic $*$-*representation* (π, π_R, ζ) *of* $(\mathcal{A}, *, R\mathcal{A}, \ )$ *into the* CQ*-*algebra* $(\mathcal{B}(\mathcal{H}_R, \mathcal{H}_{\bar{R}}), *, \mathcal{B}(\mathcal{H}_R), \ )$ *such that*

$$\Omega(A, \mathbb{I}) = \langle \pi(A)\zeta, \zeta \rangle_\Omega, \qquad \forall A \in \mathcal{A}$$

$$\omega_R(X) = \langle \pi_R(X)\zeta, \zeta \rangle_R, \qquad \forall X \in R\mathcal{A}$$

where $\langle .,. \rangle_\xi$ *denotes the form which puts* $\mathcal{H}_R$ *and* $\mathcal{H}_{\bar{R}}$ *in duality (and which extends the scalar product of* $\mathcal{H}$*) and* $\langle .,. \rangle_R$ *the scalar product of* $\mathcal{H}_R$.

Proof. — We only need to define the representation. If $A \in \mathcal{A}$ we first define $\pi(A)$ on $\mathcal{D} = R\mathcal{A}/\mathcal{I}$ as follows (we will take into account the last statement of Lemma 4.2). $\pi(A)[B]$ is the conjugate linear form acting on an arbitrary element [C] as

$$\pi(A)[B]([C]) = \langle \pi(A)[B], [C] \rangle_\Omega \equiv \Omega(AB, C)$$

Condition (ii) of Definition 4.3 implies that both the form $\pi(A)$ [B] and its value on [C] do not depend on the particular choice of the representatives of the rest classes [B] and [C].
Moreover, always by (ii), $\pi(A)$ is a bounded operator from $\mathcal{D}$ into $\mathcal{H}_{\bar{R}}$. Then it can be extended to the whole space $\mathcal{H}_R$.
From (iii) of Definition 4.3 it follows that π is a *-representation. Finally, we have to show that $\pi(AB) = \pi(A)\pi_R(B)\ \forall A \in \mathcal{A},\ B \in R\mathcal{A}$. We have

$$\pi(A)\pi_R(B)[C]([X]) = (\pi(A)[BC], [X]) = \Omega(ABC, X) = \pi(AB)[C]([X])$$

It is, now, easy to check that $\zeta = [\mathbb{I}]$ is a cyclic vector for π, in the sense that $\pi(\mathcal{A})$ is dense in $\mathcal{H}_{\bar{R}}$. This concludes the proof. □

Remark 1. — Given two cyclic representations (π, π_R, ζ) and $(\hat{\pi}, \hat{\pi}_R, \hat{\zeta})$ of $(\mathcal{A}, *, R\mathcal{A}, \ )$, constructed as above with respect to the same pair (Ω, ω_R), it is known that π_R and $\hat{\pi}_R$ are unitarily equivalent. Let V_0 be the corresponding unitary operator and let us define $V \equiv U' V_0 U^{-1}$. Here U and U′ are the two unitary operators defined by the Riesz Lemma respectively for $\mathcal{H}_R$ and for $\hat{\mathcal{H}}_R$. One can show that the map $V: \mathcal{H}_R \to \hat{\mathcal{H}}_R$ is unitary and that $\hat{\pi}(A)\hat{\xi} = V\pi(A)\xi\ \forall A \in \mathcal{A}$. In this sense we may say that the GNS-representation is determined up to unitary equivalence.

Remark 2. — In [22], the GNS-construction has been proved to be possible for an arbitrary partial *-algebra $\mathcal{A}$ with respect to any *invariant state*: a rigged quasi *-algebra is, by definition, a partial *-algebra but a positive sesquilinear form on a rigged quasi *-algebra is not necessarily invariant in the sense of [22]. The two notions at a first sight seem to be not directly comparable.

Since the main example of CQ*-algebra is, for us, that described in Example 3.3, we would expect that, *vector forms* over the CQ*-algebra $(\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar{\lambda}}), *, \mathcal{B}(\mathcal{H}_\lambda), \ )$ in the class of states for which the GNS-representation is possible. This is true if vector forms are defined in a suitable way.

If $\xi \in \mathcal{H}_\lambda$, we say that ξ is a *regular* vector if for any sequence $\{X_n\} \subset \mathcal{B}(\mathcal{H}_\lambda)$ such that $\|X_n \xi\|_{\bar{\lambda}} \to 0$ then $\|X_n \xi\|_\lambda \to 0$.
It is easy to check that if ξ is regular then $A\xi \in \mathcal{H}_\lambda\ \forall A \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_{\bar{\lambda}})$

For a regular ξ, we can now define a sesquilinear positive form Ω_ξ by

$$\Omega_\xi(\mathrm{A}, \mathrm{B}) = \langle \mathrm{A}\xi, \mathrm{B}\xi \rangle, \qquad \mathrm{A}, \mathrm{B} \in \mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar{\lambda}})$$

where $\langle ., . \rangle$ is the scalar product of $\mathscr{H}$.

We can now prove the following

PROPOSITION 4.2. – *For every regular $\xi \in \mathscr{H}_\lambda$, the vector form Ω_ξ is admissible with respect to the positive linear form ω_R on $\mathscr{B}(\mathscr{H}_\lambda)$ defined by*

$$\omega_\mathrm{R}(\mathrm{X}) = \langle \mathrm{X}\xi, \xi \rangle_\lambda, \qquad \mathrm{X} \in \mathscr{B}(\mathscr{H}_\lambda)$$

Proof. – It is readily checked that Ω_ξ fulfills the following condition:

$$|\Omega_\xi(\mathrm{AB}, \mathrm{C})| \leq \| \mathrm{A} \|_{\lambda, \bar{\lambda}} \| \mathrm{B}\xi \|_\lambda \| \mathrm{C}\xi \|_\lambda$$
$$\forall \mathrm{A} \in \mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar{\lambda}}), \qquad \forall \mathrm{B}, \mathrm{C} \in \mathscr{B}(\mathscr{H}_\lambda)$$

where $\| . \|_{\lambda, \bar{\lambda}}$ denotes the usual norm in $\mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar{\lambda}})$. This shows (ii) of Definition 4.3. Now if $\phi \in \mathscr{H}_\mathrm{R}$, with $\phi = \lim_{n \to \infty} [\mathrm{B}_n]$, $\mathrm{B}_n \in \mathscr{B}(\mathscr{H}_\lambda)$ and $\langle \phi, \phi \rangle_\Omega = 0$ we have

$$0 = \langle \phi, \phi \rangle_\Omega = \lim_{n \to \infty} \Omega_\xi(\mathrm{B}_n, \mathrm{B}_n) = \lim_{n \to \infty} \| \mathrm{B}_n \xi \| \geq \lim_{n \to \infty} \| \mathrm{B}_n \xi \|_{\bar{\lambda}}$$

and then, by the assumption, $\lim_{n \to \infty} \| \mathrm{B}_n \xi \|_\lambda = 0$.

This shows (i) of Definition 4.3. The proof of (iii) of the same definition is straightforward. □

Remark. – The condition for the regularity of ξ is similar to the condition for a sesquilinear form to be closable. In fact, if the sesquilinear form $\langle \mathrm{A}\xi, \mathrm{B}\xi \rangle_\lambda$, $\mathrm{A}, \mathrm{B} \in \mathscr{B}(\mathscr{H}_\lambda)$ is closable in $\mathscr{B}(\mathscr{H}_\lambda, \mathscr{H}_{\bar{\lambda}})$, then the vector ξ is regular. The converse is, however, not true.

Let us now give an example of admissible positive sesquilinear form on an abelian CQ*-algebra of functions.

Example 4.1. – Let $(\mathrm{L}^p(\mathrm{X}, d\mu), \mathrm{C}(\mathrm{X}))$, $p \geq 2$, be the CQ*-algebra considered in Example 3.4. Let us assume, in addition, that $\mu(\mathrm{X}) = 1$. Let $w \in \mathrm{L}^\infty(\mathrm{X}, d\mu)$ with $w > 0$ and define

$$\Omega(f, g) = \int_\mathrm{X} (fg^*)(x)\, d\mu(x)$$

By a simple application of Hölder inequality, it is readily checked that Ω is a continuous positive sesquilinear form on $\mathrm{L}^p(\mathrm{X}, d\mu)$.

Now, we assume that $w^{-1} \in \mathrm{L}^1(\mathrm{X}, d\mu)$ and that $\| w^{-1} \|_1 = 1$; then the linear form on $\mathrm{C}(\mathrm{X})$

$$\omega_\mathrm{R}(\phi) = \int_\mathrm{X} \phi(x) w^{-1}(x)\, d\mu(x), \qquad \phi \in \mathrm{C}(\mathrm{X})$$

is a state on $\mathrm{C}(\mathrm{X})$.

The conditions on w given above are, however, not enough to ensure that Ω is ω_R-admissible. To get this result we will consider the following set

$$\mathscr{D}^p_w(X, d\mu) = \{ f \in L^p(X, d\mu) : fw \in L^\infty(X, d\mu) \}$$

Since $C(X) \subset \mathscr{D}^p(X, d\mu)$ $\mathscr{D}^p(X, d\mu)$ is dense in $L^p(X, d\mu)$. If we endow $\mathscr{D}^p_w(X, d\mu)$ with the following norm

$$\| f \|_{w, p} = (\| f \|_p^p + \| fw \|_\infty^p)^{(1/p)}$$

then $\mathscr{D}^p_w(X, d\mu)$ is a Banach space and $C(X)$ is dense in it.
The norm $\| . \|_{w, p}$ satisfies, as is readily checked, the conditions of Proposition 3.2 and, therefore, $(\mathscr{D}^p_w(X, d\mu), C(X))$ is also a CQ*-algebra.
Now, if $f \in \mathscr{D}^p_w(X, d\mu)$ and $\phi, \psi \in C(X)$, we get

$$\begin{aligned} |\Omega(f, \phi, \psi)|^2 &= \left| \int_X fw \phi \psi^* w^{-1} d\mu \right|^2 \\ &\leqq \| fw \|_\infty^2 \int_X \phi^* \phi w^{-1} d\mu \int_X \psi^* \omega w^{-1} d\mu \\ &= \| fw \|_\infty^2 \omega_R(\phi^* \phi) \omega_R(\psi^* \psi) \end{aligned} \tag{16}$$

where we have used the Cauchy-Schwarz inequality in $L^2(X, w^{-1} d\mu)$.

Clearly, there is no matter of taking quotients, because either $\Omega(f, f) = 0$ or $\omega_R(f) = 0$ imply $f = 0$ almost everywhere. This Ω fulfills the conditions of Definition 4.3, *i.e.* Ω is admissible [(iii) of Definition 4.3 is actually trivially satisfied].

Let us see how the GNS-construction works in this case.
The completion of $C(X)$ with respect to the inner product defined by ω_R is the Hilbert space $L^2(X, w^{-1} d\mu)$ and the completion of $\mathscr{D}^p_w(X, d\mu)$ with respect to the scalar product defined by Ω is $L^2(X, d\mu)$. The fact that $w \in L^\infty(X, d\mu)$ implies that

$$L^2(X, w^{-1} d\mu) \subseteqq L^2(X, d\mu) \subseteqq L^2(X, w d\mu)$$

and $L^2(X, w^{-1} d\mu)$ and $L^2(X, w d\mu)$ are dual to each other with respect to the scalar product of $L^2(X, d\mu)$.
The representation π is then defined, for $f \in \mathscr{D}^p_w(X, d\mu)$, by $\pi(f) g = fg \; \forall g \in L^2(X, w^{-1} d\mu)$. It is easy to check that $fg \in L^2(X, w d\mu)$.

In this case we have, finally, $\pi_R = \pi |_{C(X)}$.

As expected, the inequality 16 implies that, for $f \in L^p(X, d\mu)$, $\pi(f)$ is a bounded operator from $L^2(X, w^{-1} d\mu)$ into $L^2(X, w d\mu)$.

Furthermore π is a regular representation in the sense of 4.2.

As we already remarked, the state Ω in Example 4.1 is continuous as a sesquilinear form on $L^p(X, d\mu)$. So, it is natural to ask ourselves whether any continuous state on a CQ*-algebra is admissible. Example 4.1 suggests that it is not so: in spite of the continuity, we had to choose a different CQ*-algebra, namely $\mathscr{D}^p_w(X, d\mu)$, where the conditions for admissibility were fulfilled.

4.2. Abelian CQ*-algebras. In this Section we will describe the structure of abelian CQ*-algebras.

For C*-algebras, as is known, the situation is completely clear: an abelian C*-algebra with unit is isometrically isomorphic to the space C(X) of continuous functions on the compact space X of characters. The correspondence stated in this way is the so-called *Gel'fand transform*.

We do not expect CQ*-algebras to behave so regularly: the first reason is that Proposition 3.2 allows the existence of non isomorphic CQ*-algebras over C(X) (think also of Example 3.4); the second reason is that, as is known [29], [37] already for Banach *-algebras the Gel'fand transform is not, in general, an isometric isomorphism.

As shown before, every abelian CQ*-algebra is necessarily proper (thus $R\mathscr{A} = L\mathscr{A} = \mathscr{A}_0$) but the involution   of $\mathscr{A}_0$ may be different from the involution $*$ of $\mathscr{A}$.

PROPOSITION 4.3. – *Let $(\mathscr{A}, \mathscr{A}_0)$ be an abelian CQ*-algebra, X the space of characters of $\mathscr{A}_0$. Then the following statements hold true.*

(i) *If $\mathscr{A}_0$ admits a faithful state ω which is continuous with respect to the norm $\|.\|$ of $\mathscr{A}$, then there exists a homomorphism ϕ from $\mathscr{A}$ into $C'(X)$ (the dual of the Banach space of continuous functions on X); the restriction of ϕ to $\mathscr{A}_0$ is an isomorphism of $\mathscr{A}_0$ onto C(X).*

(ii) *If, in addition, for some positive constant $K > 0$*

$$|\omega(C^* B)| \leq K \|B\| \|C\|$$

then there exists a regular Baire measure μ on X and a homomorphism ϕ from $\mathscr{A}$ into $L^2(X, d\mu)$; the restriction of ϕ to $\mathscr{A}_0$ is an isomorphism of $\mathscr{A}_0$ onto C(X).
In both cases, if $= *$, then ϕ preserves the involution.

Proof. – (i) Let $\mathscr{A}'_0$ be the dual space of $\mathscr{A}_0$ (with respect to its own norm). The Gel'fand transform extends to the whole $\mathscr{A}'_0$ in the following way: to $F \in \mathscr{A}'_0$ we associate the continuous linear functional $\hat{F}$ on C(X) defined by $\hat{F}\hat{B} = F(B)$ where $\hat{B}$ denotes the Gel'fand transform of $B \in \mathscr{A}_0$. Now, let ω be a faithful state ω which is continuous with respect to the norm $\|.\|$ of $\mathscr{A}$; then ω has a continuous extension (denote with the same symbol) to the whole $\mathscr{A}$; for each $A \in \mathscr{A}$, the linear functional ω_A defined by $\omega_A(B) = \omega(AB)$ $B \in \mathscr{A}_0$ is, therefore, bounded on $\mathscr{A}_0$ and then it has, as shown before, a Gel'fand transform $\hat{\omega}_A$. Let us define the map

$$\phi: \quad A \in \mathscr{A} \rightarrow \phi(A) = \hat{\omega}_A \in C'(X)$$

It is easily shown that ϕ is a homomorphism of $\mathscr{A}$ into $C'(X)$.

The fact that $\phi|_{\mathscr{A}_0}$ is an isomorphism can be seen as follows: if $B \in \mathscr{A}_0$ and $\omega_B(\hat{C})\ \forall C \in \mathscr{A}_0$ then $\omega_B(\hat{B}^\flat) = \omega(BB^\flat) = 0$ and then, by the faithfulness of ω, $B = 0$.

(ii) Set

$$\Omega_0(B, C) = \omega(B^* C)$$

By the assumption, Ω_0 can be extended, by continuity, to a positive sesquilinear form Ω on $\mathscr{A} \times \mathscr{A}$.
Since ω is continuous on $\mathscr{A}_0$, $\hat{\omega}$ is continuous on $C(X)$ and then by the Riesz representation theorem there exists a unique Borel measure μ on X such that

$$\omega(B) = \hat{\omega}(\hat{B}) = \int_X \hat{B}(\omega)\, d\mu(\omega), \qquad \forall B \in \mathscr{A}_0$$

where the operator $\hat{}$ denotes the Gel'fand transform. □
For $A \in \mathscr{A}$ the anti linear form F_A on $C(X)$ defined by $F_A(\hat{B}) = \Omega(A, B)$ is bounded in $L^2(X, d\mu)$, due to the Schwarz inequality. Therefore there exists a function $\hat{A}(\omega) \in L^2(X, d\mu)$ such that

$$F_A(\hat{B}) = \int_X \hat{A}(\omega)\, \overline{\hat{B}(\omega)}\, d\mu(\omega)$$

The map $\phi : A \in \mathscr{A} \to \phi(A) = \hat{A}(\phi) \in L^2(X, d\mu)$ is, as is readily checked a homomorphism of $\mathscr{A}$ into $L^2(X, d\mu)$ preserving the involution of $\mathscr{A}_0$. It is easy to see that ϕ satisfies the requirements of our proposition. The fact that $\phi|_{\mathscr{A}_0}$ is an isomorphism of $\mathscr{A}_0$ onto $C(X)$ follows, as before, from the faithfulness of ω.

Remark. – It is clear that the assumption $|\omega(C^* B)| \leqq K \|B\|\, \|C\|$ made in (ii), implies the continuity of ω required in (i). The converse is, however, not true in general. This could appear in contradiction with the fact that any separately continuous sesquilinear form on a Banach space, as $\mathscr{A}$ is, is necessarily jointly continuous. But, as matter of fact, the continuity of ω does not imply the separate continuity of Ω. There is, however, one relevant exception: if the state ω of (i) can be taken to be *pure*, then it is multiplicative and, as is easy to check,

$$\Omega_0(A, B) = \omega(A)\, \overline{\omega(B)}, \qquad \forall A, B \in \mathscr{A}_0$$

is then separately continuous with respect to the norm $\|.\|$ of $\mathscr{A}$; the same holds true, clearly, for its extension Ω to the whole $\mathscr{A}$.

A complete study of representation of CQ*-algebras should include, as is clear, several other topics: irreducibility, properties of the commutant, decomposition theory etc. These questions will be investigated in a further publication. Two natural questions are, in our opinion, of particular interest.

First, given a representation π_R of the C*-algebra (R $\mathcal{A}$, $\sharp$) is there a representation π of the whole CQ*-algebra $\mathcal{A}$ "extending" π_R to $\mathcal{A}$ (*i.e.*, satisfying the conditions of Definition 4.2)?

Second, is a CQ*-algebra ($\mathcal{A}$, $*$, R $\mathcal{A}$, $\sharp$) isomorphic to a CQ*-algebra of bounded operators in a scale of Hilbert spaces?

We do not have a definite answer to these questions which we then leave, at present, open.

5. LCQ*-ALGEBRAS

In this section we will consider quasi *-algebras which are algebraic inductive limits of CQ*-algebras and discuss the example of the quasi*-algebra of operators in Nested Hilbert spaces.

We need first some notation and definition.

Let P be a partially ordered set. An *order reversing involution* is a bijective map $r \to \bar{r}$ in P such that $\bar{p} \leqq \bar{q}$ if, and only if, $p \geqq q$ and $\bar{\bar{r}} = r$. P is said to be *directed to the right* if $\forall r, s \in$ P there exists $q \in$ P such that $r, s \leqq q$.

By I we will denote, from now on, a partially ordered set satisfying the following conditions

(A1) I is directed to the right

(A2) In I an order-reversing involution $r \to \bar{r}$ is defined (thus I is also directed to the left)

(A3) there exists a unique element $\varepsilon \in$ I such that $\bar{\varepsilon} = \varepsilon$.

DEFINITION 5.1. — *Let* ($\mathcal{A}$, $\mathcal{A}_0$) *be a quasi *-algebra and* I *a set of indices satisfying* (A1), (A2) *and* (A3). *We say that* ($\mathcal{A}$, $\mathcal{A}_0$) *is a* LCQ**-algebra if* $\forall r \in$ I *there exists a* CQ**-algebra* ($\mathcal{A}_r$, $*$, R $\mathcal{A}_r$, $\flat_r$) *with the properties*

(i) $\mathcal{A}_r \subseteq \mathcal{A} \, \forall r \in$ I

(ii) R $\mathcal{A}_{\bar{r}}$ = L $\mathcal{A}_r \, \forall r \in$ I

(iii) $\mathcal{A}_\varepsilon \equiv$ R $\mathcal{A}_\varepsilon$ (= L $\mathcal{A}_\varepsilon$) *is a* C**-algebra*

(iv) *For* $r \leqq s$ $\mathcal{A}_r \subseteq \mathcal{A}_s$ *and the identity is continuous and with dense range*

(v) $\forall r, s \in$ I $\exists q \in$ I *with* $r, s \geqq q$ *such that* $\mathcal{A}_q = \mathcal{A}_r \cap \mathcal{A}_s$

(vi) $\mathcal{A} = \bigcup \{\mathcal{A}_r, r \in$ I$\}$

(vii) $\forall r \in$ I *there exists a bilinear map* $m_r : \mathcal{A}_r \times \mathcal{A}_{\bar{r}} \to \mathcal{A}$ *such that*

(vii.*a*) m_r(A, B) = AB *whenever* A $\in \mathcal{A}_0$ *or* B $\in \mathcal{A}_0$;

(vii.*b*) *if* (A, B) $\in (\mathcal{A}_r \times \mathcal{A}_{\bar{r}}) \cap (\mathcal{A}_s \times \mathcal{A}_{\bar{s}})$ *then* m_r(A, B) = m_s(A, B)

(viii) $\mathcal{A}$ *is endowed with a locally convex topology* ξ *which is equivalent to the inductive limit topology defined by* $\{\mathcal{A}_r, r \in$ I$\}$ *and such that* ($\mathcal{A}$[ξ], $\mathcal{A}_0$) *is a complete topological quasi *-algebra*;

(ix) $\mathscr{A}_0$ *is the completion of* $\bigcap \{ R\mathscr{A}_r, r \in I \}$ *with respect to the reduced topology* ξ_0 *of* ξ.

If the set I *is countable we will say that* $(\mathscr{A}, \mathscr{A}_0)$ *is a* strict LCQ*-*algebra.*

Remark. – As is clear from the above definition, only (viii) and (ix) involve the topological structure of $(\mathscr{A}, \mathscr{A}_0)$; we shall see below a non-trivial but quite general example where the algebraic conditions (i)-(vii) are all fulfilled; however the topological ones hold true only under additional assumptions.

Condition (vii) of the previous definition allows us to refine the multiplicative structure that $(\mathscr{A}, \mathscr{A}_0)$ carries as quasi *-algebra.

PROPOSITION 5.1. – *Let* $(\mathscr{A}, \mathscr{A}_0)$ *be a quasi *-algebra fulfilling Conditions* (i)-(vii) *of the Definition* 5.1.

Let us define

$$\Gamma = \{ (A, B) \in \mathscr{A} \times \mathscr{A} \mid \exists r \in I : A \in \mathscr{A}_r \quad \textit{and} \quad B \in A_{\bar{r}} \}$$

then $\mathscr{A}$ *is a partial *-algebra when for* $(A, B) \in \Gamma$ *we set* $AB = m_r(A, B)$ *where* $r \in I$ *is such that* $A \in \mathscr{A}_r$ *and* $B \in A_{\bar{r}}$.

Proof. – First of all we remark that because of (vii.*a*) and (vii.*b*) the product AB, when it is defined, does not depend on the pair $(\mathscr{A}_r, \mathscr{A}_{\bar{r}})$ which we have used to define it. We check here only the distributive property, the other points of the definition of partial *-algebra being fulfilled trivially.

Let $(A, B) \in \Gamma$ and $(A, C) \in \Gamma$; then there exist $r, s \in I$ such that $(A, B) \in \mathscr{A}_r \times \mathscr{A}_{\bar{r}}$ and $(A, C) \in \mathscr{A}_s \times \mathscr{A}_{\bar{s}}$; thus by (v) of Definition 5.1, $A \in \mathscr{A}_r \cap \mathscr{A}_s = \mathscr{A}_q$ for $q \leqq r, s$; therefore $B + \lambda C \in \mathscr{A}_{\bar{q}}$ since $\mathscr{A}_{\bar{r}} \cup \mathscr{A}_{\bar{s}} \subseteqq \mathscr{A}_{\bar{q}}$. Now the equality

$$A(B + \lambda C) = AB + \lambda (AC)$$

follows immediately from the bilinearity of m_q. □

We do not go further in developing a theory of LCQ*-algebras, leaving it to future papers. Here we are more interested in showing that certain spaces of operators that frequently occurs in physical applications fit into our framework. This is the case of some families of operators acting in *nested Hilbert spaces* [30]; for reader's convenience we recall the basic definitions. A nested Hilbert space (NHS) is a vector space $\mathscr{D}'$ together with

(1) a family $\{ \mathscr{H}_r, r \in I \}$ of vector subspaces which covers $\mathscr{D}'$; when ordered by inclusion the family admits an order reversing involution $\mathscr{H}_r \leftrightarrow \mathscr{H}_{\bar{r}}$.

(2) a hermitean, positive definite form $\langle ., . \rangle$ is defined on $\bigcup (\mathscr{H}_r \times \mathscr{H}_{\bar{r}})$ and the subspace $\mathscr{D} = \bigcap \mathscr{H}_r$ separates points of $\mathscr{D}'$.

The family $\{ \mathscr{H}_r \}$ fulfills the following requirements

(3) for each $r \in I$, $\mathcal{H}_r$ is a Hilbert space with respect to a scalar product $\langle \cdot, \cdot \rangle_r$;

(4) there exists a unique element $\varepsilon \in I$ such that $\mathcal{H}_\varepsilon = \mathcal{H}_{\bar{\varepsilon}}$;

(5) the family $\{\mathcal{H}_r\}$ is stable under intersection.

If the set I is countable, following the terminology of Gel'fand and Vilenkin [38] we say that $\mathcal{D}'$ is a *countably Hilbert space*. A detailed study of this structure has been carried out in [39].

If the family $\{\mathcal{H}_r, r \in I\}$ is ordered by inclusion, then the set I can also be ordered in natural way: $r \leqq s$ if, and only if, $\mathcal{H}_r \subseteq \mathcal{H}_s$. If $r \leqq s$, we denote with E_{sr} the bounded linear injective map which embeds $\mathcal{H}_r$ into $\mathcal{H}_s$. The map E_{sr} has dense range.

Because of (2), $\forall r \in I$, $\mathcal{H}_r$ and $\mathcal{H}_{\bar{r}}$ are dual to each other. Let us denote with $U_{\bar{r}r}$ the unitary operator from $\mathcal{H}_r$ into $\mathcal{H}_{\bar{r}}$ defined by the Riesz lemma.

The definition of operator in a NHS is rather tricky and we will omit it, referring, for details, to [17], [37] and [38]. We will consider below a family of operators in a NHS which provides an example of the structures defined above.

Remarks:

(1) The definition of NHS given here differs from the original one given by Grossmann [30]; here we will adopt that given in [40] which is a little more restrictive but more convenient for our purposes.

(2) Nested Hilbert spaces are a particular case of PIP-space as defined by Antoine and Grossmann ([41], [42]).

(3) It is clear that the extreme spaces $\mathcal{D}$ and $\mathcal{D}'$ together with the central Hilbert space $\mathcal{H}_\varepsilon$ can be considered as a Rigged Hilbert space: in fact if $\mathcal{D}$ carries the projective limit topology defined by the family $\{\mathcal{H}_r\}$, then $\mathcal{D}'$ is exactly its topological dual whose strong dual topology is just the inductive limit topology defined by the family $\{\mathcal{H}_r\}$ (*see*, e.g., [43] Ch. 4, § 22, n° 6). So it turns out that $\mathcal{D}$ is a semi reflexive and weakly quasi-complete space and $\mathcal{D}'$ is barreled. Moreover $\mathcal{D}$ is dense in $\mathcal{D}'$.

(4) In the case of a countably Hilbert space D is Fréchet and reflexive, being a projective limit of countably many Hilbert spaces.

Let $\{\mathcal{D}', \{\mathcal{H}_r\}, I\}$ be a nested Hilbert space and $\mathcal{D} = \bigcap \mathcal{H}_r$. The *-invariant Banach space $\mathcal{B}(\mathcal{H}_r, \mathcal{H}_{\bar{r}})$ of all bounded linear operators from $\mathcal{H}_r$ into $\mathcal{H}_{\bar{r}}$ is, as is easily seen, a subspace of the set $\mathcal{L}(\mathcal{D}, \mathcal{D}\pi')$ of all continuous linear maps from $\mathcal{D}$ (with the projective topology) into $\mathcal{D}'$ (with the strong dual topology).

Let $O(\mathcal{D}') \subseteq \mathcal{L}(\mathcal{D}, \mathcal{D}')$ be the linear span of all the spaces $\mathcal{B}(\mathcal{H}_r, \mathcal{H}_{\bar{r}})$, $r \in I$ and $\mathcal{B}(\mathcal{D})$ the intersection of all C*-algebras $\mathcal{B}(\mathcal{H}_r)$, $r \in I$. As we have seen in Example 3.3, $\mathcal{B}(\mathcal{H}_r, \mathcal{H}_{\bar{r}})$ is a CQ*-algebra over $\mathcal{B}(\mathcal{H}_r)$.

Finally let $O(\mathscr{D})$ be the *-algebra of all elements of $O(\mathscr{D}')$ which (together with their adjoints) leave $\mathscr{D}$ invariant. Then as is easy to see $(O(\mathscr{D}'), O(\mathscr{D}))$ is a quasi *-algebra. Moreover we have

LEMMA 5.1. – *Let* $\{\mathscr{D}', \{\mathscr{H}_r\}, I\}$ *be a nested Hilbert space. Then the quasi *-algebra* $(O(\mathscr{D}'), O(\mathscr{D}))$ *satisfies Conditions* (i)-(vii) *of Definition* 5.1.

The proof is very simple and we omit it.

In order to get that also Conditions (viii) and (ix) of Definition 5.1 be fulfilled we have to add restrictions to the above general set up. In fact, if the nested Hilbert space we are dealing with reduces to the chain of Hilbert spaces generated by an unbounded self-adjoint operator then the quasi *-algebra $(O(\mathscr{D}'), O(\mathscr{D}))$ coincides with $(\mathscr{L}(\mathscr{D}, \mathscr{D}'), \mathscr{L}^+(\mathscr{D}))$ and it gives really a LCQ*-algebra.

Let T be an unbounded self-adjoint operator, with $T \geqq \mathbb{I}$. Then for each $r \in \mathbb{N}$, $D(T^r)$ is a Hilbert space with respect to the scalar product $\langle f, g \rangle_r = \langle T^r f, T^r g \rangle$, where $\langle ., . \rangle$ denotes the scalar product of $\mathscr{H}$. Set $\mathscr{H}_r = D(T^r)$ and let $\mathscr{H}_{\bar{r}}$ its dual, which is also a Hilbert space where the scalar product is defined as $\langle f, g \rangle_{\bar{r}} = \langle T^{-r} f, T^{-r} g \rangle$. We have so obtained a totally ordered (chain) family of Hilbert spaces $\{\mathscr{H}_r, r \in \mathbb{Z}\}$ with $\mathscr{H}_0 = \mathscr{H}$. Let $\mathscr{D} = \lim\operatorname{proj} \mathscr{H}_r$ and $\mathscr{D}' = \lim\operatorname{ind} \mathscr{H}_r$. As already mentioned, $\mathscr{D}$ is Fréchet and reflexive and $\mathscr{D}'$ is its dual. For this domain, $(\mathscr{L}(\mathscr{D}, \mathscr{D}'), \mathscr{L}^+(\mathscr{D}))$ is a quasi *-algebra and the quasi uniform topology τ of $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ ([11]-[13]) can be described by the following seminorms

$$\|A\|_f = \|f(T)\, A\, f(T)\| \tag{17}$$

where $\|.\|$ denotes the C*-norm of $\mathscr{B}(\mathscr{H})$ and f runs over the set $\mathscr{F}$ of all positive, bounded and continuous functions on $[0, \infty)$ such that $\sup_{x \in [0, \infty)} x^k f(x) < \infty \ \forall k \in \mathbb{N}$.

Analogously a topology τ_* on $\mathscr{L}^+(\mathscr{D})$ can be defined by the seminorms

$$\|A\|^{f, k} = \max\{\|T^k A f(T)\|, \|f(T) A T^k\|\} \qquad f \in \mathscr{F},\ k > 0 \tag{18}$$

The following known results are here stated as a lemma for convenience

LEMMA 5.2. – *Let* $\{\mathscr{D}', \{\mathscr{H}_r\}, \mathbb{Z}\}$ *be the chain of Hilbert spaces generated by an unbounded self-adjoint operator* T. *Then* $\mathscr{D} = \mathscr{D}^\infty(T)$ *and*

(i) $\mathscr{L}(\mathscr{D}, \mathscr{D}') = O(\mathscr{D}\pi')$ *and* $\mathscr{L}^+(\mathscr{D}) = O(\mathscr{D})$;

(ii) $(\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau], \mathscr{L}^+(\mathscr{D}))$ *is a complete topological quasi *-algebra*;

(iii) *the topology* τ_* *of* $\mathscr{L}^+(\mathscr{D})$ *coincides with the reduced topology* τ_0 *of* τ.

Proof. – (i) *See* [43] Sect. 19, n° 5(4) and [23] Cor. 3.1.14; (ii) has been stated in [12], Lemma 2.3 and (iii) in [13], Lemma 3.11. □

Remark. — Statement (ii) of the above lemma does not hold for general domains. In [44], for instance, an example of a domain $\mathscr{D}$ for which $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is not τ-complete has been given. The density of $\mathscr{L}^+(\mathscr{D})$ in $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ for Fréchet domains has been proved in [15]; other results in this direction can be found in [23] (Ch. 3). Statement (iii) holds also for the so-called *basic* domains [13].

We can now prove the main result of this Section

PROPOSITION 5.2. — *Let $\mathscr{D}$ be the projective limit of the chain of Hilbert spaces generated by an unbounded self-adjoint operator* T; *then* $(\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau], \mathscr{L}^+(\mathscr{D}))$, *where τ is the quasi-uniform topology, is a* LCQ*-*algebra.*

Proof. — Taking into account the previous lemmas, it remains only to show that $\mathscr{L}(\mathscr{D}, \mathscr{D}') = \lim\operatorname{ind} \mathscr{B}(\mathscr{H}_r, \mathscr{H}_{\bar{r}})$ and that $\mathscr{L}^+(\mathscr{D})$ is the τ_*-completion of $\mathscr{B}(\mathscr{D}) = \bigcap \mathscr{B}(\mathscr{H}_r)$. The second part is easier and we prove it first.

Let $\{E(\lambda), \lambda \geqq 1\}$ be the spectral family of T. Then if $A \in \mathscr{L}^+(\mathscr{D})$ and $\Delta_n = [1, n)$ then it is easily seen that $E(\Delta_n) A E(\Delta_n) \in \mathscr{B}(\mathscr{H}_r)\ \forall r \in \mathbb{Z}$ and that $E(\Delta_n) A E(\Delta_n)$ converges to A with respect to τ_*.

Let us finally prove that τ coincides with the inductive limit topology τ_{ind} defined by $\mathscr{B}(\mathscr{H}_r, \mathscr{H}_{\bar{r}})$, $r \in \mathbb{Z}$.

Let $\mathscr{S}$ denote the space of nuclear operators in $\mathscr{H}$ and for $\rho \in \mathscr{S}$ set $\|\rho\|_1 = \operatorname{tr}((\rho^+ \rho)^{1/2})$.

Let

$$\mathscr{S}_r = \{\rho \in \mathscr{B}(\mathscr{H}) : T^r \rho T^r \in \mathscr{S}\}$$

Then $\mathscr{S}_r$ is a Banach space under the norm $\|\rho\|^{(r)} = \|T^r \rho T^r\|_1$.

With techniques completely analogous to the case of $\mathscr{S}$ (*see*, for instance [34] Sect. 2.4.1) it can be proved that the strong dual $\mathscr{S}'_r$ of $\mathscr{S}_r$ is $\mathscr{B}(\mathscr{H}_r, \mathscr{H}_{\bar{r}})$ with its natural norm.

Let, now,

$$\mathscr{S}(\mathscr{D}) = \{\rho \in \mathscr{L}^+(\mathscr{D}) : T^r \rho T^r \in \mathscr{S},\ \forall r \in \mathbb{Z}\}$$

be endowed with the topology β^* ([13], [45]) defined by the seminorms

$$\|\rho\|_1^{(r)} = \|T^r \rho T^r\|_1, \qquad r \in \mathbb{Z}$$

Then as is clear $\mathscr{S}(\mathscr{D}) = \bigcap \{\mathscr{S}_r, r \in \mathbb{Z}\}$ and $\mathscr{S}(\mathscr{D})$ is the projective limit of the spaces $\mathscr{S}_r$. Then for the duals we get

$$\mathscr{S}(\mathscr{D})[\beta^*]' = \lim_r \operatorname{ind} \mathscr{B}(\mathscr{H}_r, \mathscr{H}_{\bar{r}})$$

But, as stated in [13], $\mathscr{S}(\mathscr{D})[\beta^*]' = \mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau]$. This completes the proof. □

6. SOME PHYSICAL APPLICATION

In this final section we will discuss in some details two physical models (already mentioned in the Introduction) whose mathematical formulation seems to fit quite well into our framework. In both cases, the C*-algebras formulation is known to be insufficient for a complete and satisfactory mathematical description. Some of the problems that arise in that formulation seem to be solvable in the present framework.

6.1. (Almost) mean field Ising model. The basic mathematical ingredient for the study of this model is the usual spin C*-algebra $\mathcal{A}_S$. We will adopt here the same notations as in [4], but we will give the main statement in a simpler formulation which is more adherent to this paper.

The finite volume Hamiltonian which describes this model is

$$H_V = \frac{1}{V^\gamma} \sum_{i,j \in |V|} \sigma_3^i \sigma_3^j \tag{19}$$

For $\gamma = 1$ this is a typical mean field model, extensively discussed in the literature [2, 3, 11]. The case $0 < \gamma < 1$ has been considered in [4].

The Heisenberg equations of motion are given by

(20) $$\alpha_V^t(\sigma_\beta^k) = i[H_V, \alpha_V^t(\sigma_\beta^k)] \qquad \beta = 1, 2, 3$$

whose solution can be easily computed as in [4].

As is known, the thermodynamical limit of this local Heisenberg dynamics does not belong to $\mathcal{A}_S$.

To any sequence $\{\underline{n}\}$ of three-vectors it corresponds a state $|\{\underline{n}\}\rangle$ of the system. Such a state defines, via the GNS-construction, a realization $\pi_{\{\underline{n}\}}(\mathcal{A}_S)$ in the Hilbert space $\mathcal{H}_{\{\underline{n}\}}$. This representation is faithful, since $\mathcal{A}_S$ is a simple C*-algebra. In the space $\mathcal{H}_{\{\underline{n}\}}$ one can choose an orthonormal basis obtained from $|\{\underline{n}\}\rangle$ by flipping a finite number of spins. A basis vector is, now, labeled by a sequence $\{m\}$ of 0's and 1's with exactly m ones.

In $\mathcal{H}_{\{\underline{n}\}}$ we can define an unbounded self-adjoint operator M by

(21) $$M|\{m\}, \{\underline{n}\}\rangle = (1 + \sum_p m_p)|\{m\}, \{\underline{n}\}\rangle$$

The operator $M-1$ counts the number of flipped spins with respect to the ground state $|\{0\}, \{\underline{n}\}\rangle$. Then M is, by definition, a number operator. It turns out that the operator e^M is a densely defined self-adjoint operator. Let $\mathcal{D}$ denote its domain. Then $\mathcal{D}$ can be made into a Hilbert space, denoted as $\mathcal{H}_M$, in canonical way. The norm in $\mathcal{H}_M$ is given by

$$\|f\|_M = \|e^M f\|.$$

Taking the conjugate dual $\mathcal{H}_{\bar{M}}$ of $\mathcal{H}_M$, with respect to the scalar product of $\mathcal{H}_{\{\underline{n}\}}$, we get the scale

$$\mathcal{H}_M \subset \mathcal{H}_{\{\underline{n}\}} \subset \mathcal{H}_{\bar{M}}$$

Once this triplet is obtained, it is natural to consider the CQ*-algebra $(\mathcal{B}(\mathcal{H}_M, \mathcal{H}_{\bar{M}}), *, \mathcal{B}(\mathcal{H}_M), \sharp)$ [*see* Example 3.3] as a framework where to discuss the existence of the thermodynamical limit of α_V^t.

Indeed, by simple modifications of the calculations of [4], one can prove the existence of the thermodynamical limit of the local Heisenberg dynamics in $\mathcal{B}(\mathcal{H}_M, \mathcal{H}_{\bar{M}})$, provided that the state $\{\underline{n}\}$ satisfies the condition

$$\lim_{|V|, \infty} \frac{1}{|V|^\gamma} \sum_p \underline{n}_p = \eta\, \underline{n}, \quad |\eta| = 1; \qquad \underline{n} = (0, 0 \pm 1).$$

It is worth mentioning that the natural norm of $\mathcal{B}(\mathcal{H}_M, \mathcal{H}_{\bar{M}})$ is given by

$$\|X\|_{M, \bar{M}} = \|e^{-M} X e^{-M}\|$$

and that this norm automatically fulfills the conditions given in Section 3.

6.2. Wightman fields. In this subsection we discuss some facts of axiomatic Quantum Field Theory which seem to be more conveniently described in the language of LCQ*-algebras.

As is known, the algebraic formulation of Quantum Field Theory exists in two variants: the Haag-Araki theory, where the algebras of local observables are von Neumann algebras and the Haag-Kastler theory where the local algebras are abstract C*-algebras. Clearly the link between the two theories is provided by representations.

In a concrete realization, the point-like field $A(x)$ is mathematically represented as a sesquilinear form on a certain domain $\mathcal{D}$ in Hilbert space $\mathcal{H}$ ([46], [47]) and one of the basic Wightman axioms is that

$$A(f) = \int_{\mathbb{R}^4} A(x) f(x)\, dx, \qquad f \in C_0^\infty(\mathbb{R}^4)$$

is a well defined operator in $\mathcal{D}$. The local von Neumann algebra $\mathcal{A}(\mathcal{O})$ is then generated by the Wightman fields $A(f)$ such that $\operatorname{supp} f \subset \mathcal{O}$. It is quite clear that this construction cannot be performed starting from an arbitrary point-like field $A(x)$.

Specializing some idea of [9], in [8] Epifanio and one of us proposed to consider a point-like field A as a linear map

$$x \in \mathbb{R}^4 \mapsto A(x) \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$$

with $\mathcal{D} = \mathcal{D}^\infty(H)$ where H is the energy operator. It turns out that for such a field there exists a number $k > 0$ (independent of x) such that $R^k A(x) R^k$ is a bounded operator, where $R = (\mathbb{I} + H)^{-1}$. The boundedness of $R^k A(x) R^k$ is exactly the regularity condition (high energy bound) required by Fredenhagen and Hertel in [10] to get that $A(x)$ satisfies both the Wightman axioms and those of a local theory in the Haag-Araki setup.

All these facts can be, more conveniently, reformulated in the language of the present paper.

Indeed, let us consider the chain of Hilbert spaces $\{\mathscr{H}_r, r\in\mathbb{Z}\}$ generated by the unbounded self-adjoint operator $R^{-1}=\mathbb{I}+H$ and $\mathscr{D}=\mathscr{D}^\infty(R^{-1})$. As shown in Proposition 5.2, in this case $(\mathscr{L}(\mathscr{D}, \mathscr{D}'), \mathscr{L}^+(\mathscr{D}))$ is a LCQ*-algebra with respect to the quasi-uniform topology τ. In is easy to see that the boundedness of $R^k A(x) R^k$ is equivalent to the fact that $A(x)\in\mathscr{B}(\mathscr{H}_k, \mathscr{H}_{-k})$. Taking this fact into account, one can show, as in [8], that if $A(x)$ satisfies the usual physical requirements (translation invariance, existence of a translation invariant vacuum, spectrum condition) then the *smeared* field $A(f)$, $f\in C_0^\infty(\mathbb{R}^4)$ satisfies all Wightman axioms; and in particular $A(f)\in\mathscr{L}^+(\mathscr{D})$, $\forall f\in C_0^\infty(\mathbb{R}^4)$.

Two features seem to us very relevant in this mathematical formulation (whose main point is the fact that the point-like field takes its values in an LCQ*-algebra).

First, by the definition itself, elements of an LCQ*-algebra enjoy very nice approximation properties in terms of very regular elements (Proposition 5.2) and this can be very useful in many applications.

Second, an LCQ*-algebra carries a natural structure of partial *-algebra and this can give a sense to many formal computations with point-like fields which are often performed as they really were operators (think, for instance, of the Lagrangian itself in a lagrangian formulation of a field theory [48] or even of the commutators $[A(x), B(y)]$ usually evaluated to check the local commutativity of two fields). In fact a product of the form $A(x)B(y)$ is not always meaningless. As shown in Section 5, this will be well-defined if $A(x)\in\mathscr{B}(\mathscr{H}_k, \mathscr{H}_{-k})$ and $B(y)\in\mathscr{B}(\mathscr{H}_{-k}, \mathscr{H}_k)$ for a certain $k\in\mathbb{Z}$.

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