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Moments and Huygens' principle for conformally invariant field equations in curved space-times

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ABSTRACT. – By means of a certain conformal covariant differentiation process we define an infinite sequence of conformally invariant tensors (moments) for Weyl's neutrino equation in a curved space-time. In the cases of the conformally invariant-scalar wave equation and Maxwell's equations such moments were introduced by Günther. We prove some properties of the moments and study the relationship between the moments and the validity of Huygens' principle for these conformally invariant field equations. Using suitable generating systems of conformally invariant tensors we derive the first moment equations and obtain from them results on Huygens' principle.

Key words : Conformally invariant field equations, moments, Huygens' principle.

RÉSUMÉ. – Grâce à un certain processus de différentiation de covariant conformes, nous définissons une suite infinie de tenseurs invariants conformes (moments) pour l'équation de Weyl dans un espace-temps courbe. Ces moments sont introduits par Günther pour l'équation des ondes scalaires et les équations de Maxwell. Nous prouvons quelques propriétés des moments et étudions le lien entre ces moments et la validité du principe de Huygens pour les équations de champ invariantes conformes. En utilisant des tenseurs invariants conformes adaptés, nous déduisons les premières équations de moments et obtenons des résultats à propos du principe de Huygens.

1. INTRODUCTION

In a four-dimensional pseudo-Riemannian manifold (M, g) with a smooth metric of Lorentzian signature the following conformally invariant field equations are considered:

Scalar wave equation $g^{ab}\nabla_a\nabla_b u - \frac{1}{6}\mathrm{Ru} = O$ $\quad E_1$

Maxwell's equations $d\omega = O$, $\delta\omega = O$ $\quad E_2$

Weyl's neutrino equation $\nabla^A_{\dot{X}}\varphi_A = O$ $\quad E_3$

P. Günther [G4] defined for the equations E_1 and E_2 an infinite sequence of symmetric, trace-free, conformally invariant tensors

$$I^{\sigma}_{i_1\ldots i_\nu}(x),\ \sigma = 1,\ 2,\ \nu = 0,\ 1,\ 2,\ \ldots$$

which he called moments of the equation E_σ of order ν. He derived these moments by means of a certain conformal covariant differentation process. The moments are of particular importance in the theory of Huygens' principle for E_1 and E_2. For one of the equations E_σ Huygens' principle (in the sense of Hadamard's "minor premise") is valid if and only if the corresponding tail term vanishes [Ha; G2, 4; W4]. Consequently, if $(M,\ g)$ is analytic we have the following relationship between the moments and the validity of Huygens' principle: The equation E_1 or E_2 satisfies Huygens' principle if and only if all corresponding moments vanish on M. These moment equations $I^{\sigma}_{i_1}\ldots i_\nu(x) = O$, $(\sigma = 1,\ 2)$ are determined explicitly up to now for $O \leqq \nu \leqq 4$ (*see* [G1, 2; W1, 2, 4; McL 2]). For example, we have $I^{\sigma}_{i_1 i_2} = \alpha^{(\sigma)} B_{i_1 i_2}$, where $\alpha^{(\sigma)} \in \mathbb{R}\backslash\{0\}$ and $B_{i_1 i_2}$ denotes the Bach tensor [G1, 2; W4].

In this paper we define such a sequence of conformally invariant tensors also for Weyl's neutrino equation E_3. Using some results on the theory of conformally invariant tensors [GW 2, 3], in particular, the notion "conformal covariant derivative" and suitable linear independent systems of conformally invariant tensors, we give new information about the general structure of the moment equations $I^{\sigma}_{i_1\ldots i_v} = O$ for $\sigma \in \{1,\ 2,\ 3\}$ and $\nu \in \{0,\ 1,\ \ldots,\ 6\}$, which imply some results on Huygens' principle for E_σ.

The paper is organized as follows. After some preliminaries we give in Section 3 necessary and sufficient conditions for the validity of Huygens' principle for the equations E_σ and, further, the transformation law for the corresponding tail terms under conformal transformations of the metric. In Section 4, Günther's and the author's contributions to the theory of polynomial conformally invariant tensors [GW2, 3] are generalized by including the Levi-Civita pseudo-tensor and conformally invariant spinors. Further, some classes of conformally invariant tensors, which are important for the moments and necessary conditions for (M, g) to be conformal to an Einstein space-time are given. In Section 5, we introduce moments also for Weyl's neutrino equation and derive some properties of the moments. Using the results of Section 4 on generating systems of conformally invariant tensors we obtain information about the algebraic structure of the first moments. Finally, in Section 6, we show the importance of the moments for the validity of Huygens' principle, especially in the case of a Petrov type N space-time.

2. PRELIMINARIES

Let (M, g) be a space-time, i.e. a 4-manifold together with a smooth metric of Lorentzian signature, and g_{ab}, g^{ab}, ∇_a, R_{abcd}, R_{ab}, R, C_{abcd} the local coordinates of the covariant and contravariant metric tensor, the Levi-Civita connection, the curvatur tensor, the Ricci tensor, the scalar curvature and the Weyl curvature tensor, respectively. $\mathcal{J}$ and Λ^p denote the space of the C^∞ scalar fields and the p-forms of class C^∞, respectively. On Λ^p the exterior derivative d, the co-derivative δ and $\Delta := -(d\delta + \delta d)$ are defined. Assuming that (M, g) can be equipped with a spin structure we denote the complex spinor bundles of covariant and contravariant 1-spinors and their conjugates by S, S^*, $\bar{S}$, $\bar{S}^*$, the set of all cross sections of S, S^*, $\bar{S}$ $\bar{S}^*$ by $\mathcal{S}$, $\mathcal{S}^*$, $\bar{\mathcal{S}}$, $\bar{\mathcal{S}}^*$, respectively, the coordinates of $\varphi \in \mathcal{S}$, $\psi \in \bar{\mathcal{S}}$, the connection quantities (generalized Pauli-matrices), the Levi-Civita spinor, the connection coefficients and the spinor covariant derivative by [PR]

$$\varphi_A,\ \psi_{\dot{X}},\ \sigma^a_{A\dot{X}},\ \varepsilon_{AB},\ \Gamma^B_{aA},\ \nabla_{A\dot{X}} := \sigma^a_{A\dot{X}}\nabla_a, \qquad \begin{array}{l} A \in \{1, 2\} \\ \dot{X} \in \{\dot{1}, \dot{2}\}. \end{array} \tag{2.1}$$

If we define for $\varphi \in \mathcal{S}$, $\psi \in \bar{\mathcal{S}}$

$$(\mathcal{M}\,\varphi)_{\dot{X}} := \nabla^A_{\dot{X}}\,\varphi_A, \qquad (\mathcal{N}\,\psi)_A := \nabla^{\dot{X}}_A\,\psi_{\dot{X}}, \tag{2.2}$$

we have [W4; PR]

$$-2\,(\mathcal{N}\mathcal{M}\,\varphi)_A = g^{ab}\,\nabla_a\,\nabla_b\,\varphi_A - \frac{1}{4}\,R\,\varphi_A =: (\mathcal{L}^{(1/2)}\,\varphi)_A. \tag{2.3}$$

In the following we consider the conformally invariant wave equation

$$\mathcal{L}^{(0)}\,u \equiv g^{ab}\,\nabla_a\,\nabla_b\,u - \frac{1}{6}\,R\,u = O, \qquad u \in \mathcal{J} \tag{E$_1$}$$

the (source-free) Maxwell equations

$$du = O, \qquad \delta u = O, \qquad u \in \Lambda^2 \tag{E_2}$$

and Weyl's neutrino equation

$$\mathcal{M}\, u = O, \qquad u \in \mathcal{S}. \tag{E_3}$$

Let M be a causal domain [F; $G4$] and $\Gamma(x, y)$ the square of the geodesic distance of $x, y \in M$. For any fixed $y \in M$ the set $\{x \in M \mid \Gamma(x, y) > O\}$ decomposes naturally into two open subsets of M; one of them is called the future $D_+(y)$ and the other one the past $D_-(y)$ of y. The characteristic semi null cones $C_\pm(y)$ are defined as the boundary sets of $D_\pm(y)$, respectively.

Let $G_\pm^{(0)}(y)$, $G_\pm^{(1)}(y)$ and $G_\pm^{(1/2)}(y)$ be the fundamental solutions of the linear operators $\mathcal{L}^{(0)}$, Δ, $\mathcal{L}^{(1/2)}$ and $T^{(\alpha)}(\cdot, y)$, $\alpha = 0, 1, 1/2$ the tail terms of $G_\pm^{(\alpha)}(y)$ with respect to y, respectively. The tail term is just the factor of the regular part of the corresponding fundamental solution, which is a distribution supported inside the future of y [F; $G4$]. For $T^{(\alpha)}$ there is an asymptotic expansion in Γ

$$T^{(\alpha)} \sim \sum_{k=0}^{\infty} \frac{1}{2^k . k!} U_{(k+1)}^{(\alpha)} \Gamma^k, \tag{2.4}$$

where the Hadamard coefficients $U_{(k)}^{(\alpha)}$ are determined recursively by the transport equations [F, G4, W4].

3. HUYGENS' PRINCIPLE

Let F be a space-like 3 dimensional submanifold of M, $D_F(y)$ that part of the interior of the past semi null cone $C_-(y)$, which is bounded by the hypersurface F and, finally, $F(y) := F \cap D_F(y)$. Cauchy's problem for one of the equations E_σ is the problem of determining a solution which assumes given values u (and their normal derivative for E_1) on the given submanifold F. These values are called Cauchy data. Local existence and uniqueness of the solution of Cauchy's problem has been proved by Hadamard [Ha] for E_1, by Günther [G2] for E_2 and by the author [W4] for E_3

DEFINITION 3.1. – *One of the equations E_σ, $\sigma \in \{1, 2, 3\}$, is said to satisfy Huygens' principle (in the sense of Hadamard's "minor premise") if and only if for every Cauchy problem and for every $y \in M$ the solution*

u depends only on the Cauchy data in an arbitrarily small neighbourhood of $F(y)$.

Only if Huygens' principle is valid the wave propagation is free of tails [F; G4; McL2; W4], that is the solution depends on the sources distributions on the past null cone of the field point only and not on the sources inside the cone. Huygens' principle plays an important role also in quantum field theory in curved space-times. According to Lichnerowicz [L; B] the support of the commutator – or the anticommutator – distribution, respectively, lies on the null cone if and only if Huygens' principle holds for the corresponding field equation.

In [Ha; F; G2; W4] it was proved:

PROPOSITION 3.1. – *The equation E_σ, $\sigma \in \{1, 2, 3\}$ satisfies Huygens' principle iff*

$$\forall x, y \in M : T^{(0)}(x, y) = O \qquad \textit{in the case } \sigma = 1 \tag{3.1}$$

$$\forall x, y \in M : K(x, y) : d^{(1)} d^{(2)} T^{(1)}(x, y) = O \textit{ in the case } \sigma = 2 \tag{3.2}$$

$$\forall x, y \in M : N(x, y) := \mathcal{M}^{(1)} T^{(1/2)}(x, y) = O \textit{ in the case } \sigma = 3. \tag{3.3}$$

Here the superscripts $^{(1)}$, $^{(2)}$ indicate whether the derivative is meant with respect to x or y.

DEFINITION 3.2. – *The terms*

$$T^{(0)}(x, y), \qquad K(x, y), \qquad N(x, y) \tag{3.4}$$

are called tail terms of the equation E_1, E_2, E_3, respectively.

If the metric g undergoes a conformal transformation

$$\tilde{g}_{ab} = e^{2\Phi} g_{ab}, \qquad \phi \in C^\infty(M), \tag{3.5}$$

the tail terms (3.4) transform according to [GW1; McL2; W4, 5]

$$\tilde{T}^{(0)}(x, y) = e^{-[\phi(x)+\phi(y)]} T^{(0)}(x, y) \tag{3.6}$$

$$\tilde{K}(x, y) = K(x, y) \tag{3.7}$$

$$\tilde{N}(x, y) = e^{-3/2[\phi(x)+\phi(y)]} N(x, y). \tag{3.8}$$

Consequently, a conformal transformation (3.5) preserve the Huygens' character of the equation E_σ, $\sigma \in \{1, 2, 3\}$ [G4; W4; ∅]. In particular, the conditions (3.1)-(3.3) are fulfilled for flat metrics [G4; W4], which implies that if g is conformally flat for the equations E_σ Huygens' principle is (trivially) valid.

Because the functional relationship between the tail terms (3.4) and the metric is not clear, the problem of determination of all metrics, for which any equation E_σ satisfies Huygens' principle, is not yet completely solved (*see* [G4; W2, 4; McL3; CM; I]). A step forward is the derivation of suitable infinite sequences of conformally invariant tensors, the so called moments of E_σ.

4. CONFORMALLY INVARIANT TENSORS

We consider polynomial tensors, i.e. tensors, whose coordinates are polynomial in g^{ab} and the partial derivatives of g_{ab}. These tensors are just the elements of the tensor algebra $\mathcal{R}_0$ generated by the tensors.

$$g^{ab},\ g_{ab},\ \nabla_{(i_1} \ldots \nabla_{i_r} R_{i_{r+1}|ab|i_{r+2})}, \qquad r = 0, 1, 2, \ldots \tag{4.1}$$

by means of the usual tensor operations. Furthermore, let $\mathcal{R}$ be the algebra generated by the tensors (4.1) and the Levi-Civita tensor e_{abcd}.[1)]

DEFINITION 4.1. – *A tensor $T(g) \in \mathcal{R}$ is said to be conformally invariant of weight ω, if under the conformal transformation* (3.5) *$T(g)$ has the transformation law*

$$T(\tilde{g}) = e^{2\omega\phi}\, T(g). \tag{4.2}$$

It is an important problem to give a survey of all conformally invariant tensors or to give methods for constructing special classes of such tensors [GW2, 3; W3].

LEMMA 4.1. – *$T(g) \in \mathcal{R}$ is conformally invariant iff it is invariant under all infinitesimal conformal transformations, i.e. iff*

$$P(g, \Phi) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[T(e^{2\varepsilon\Phi} g) - e^{2\varepsilon\omega\Phi}\, T(g) \right] = O. \tag{4.3}$$

Let τ be the subalgebra of those elements of $\mathcal{R}$ which contain only *first* derivatives of Φ in their transformation law [GW 2]. From Lemma 4.1 it follows that $T(g) \in \tau$ iff $P(g, \Phi)$ has the form

$$P(g, \Phi) = X^k(T)\, \nabla_k \Phi. \tag{4.4}$$

For the linear operator X^k, defined on τ by (4.4), holds Leibniz's rule [GW2].

COROLLARY 4.1. – *$T(g) \in \mathcal{R}$ is conformally invariant iff $T(g) \in \tau$ and $X^k(T) = 0$.*

Examples. – $X^k(g_{ab}) = 0$, $X^k(C_{abcd}) = 0$, $X^k(\nabla_u C^u{}_{.abc}) = C^k{}_{.abc}$. If $T(g) \in \tau$ then in general we have $\nabla_a T \notin \tau$. Let be

$$L_{ab} := -R_{ab} + \frac{1}{6} R g_{ab}, \quad {}^*C_{abd} := \frac{1}{2} e_{abkl} C^{kl}{}_{..cd}.$$

DEFINITION 4.2. – *For $T \in \tau$ the tensor*

$$\overset{c}{\nabla}_a T := \nabla_a T - \frac{1}{2} X^k(T) L_{ka} \tag{4.5}$$

is called the conformal covariant derivative of T [GW2].

In [GW 2; GeW 1] it was proved:

PROPOSITION 4.1. – (i) *The conformal covariant derivative $\overset{c}{\nabla}_a$ is linear, obeys Leibniz's rule and commutes with contractions.*

(ii) $\overset{c}{\nabla}_a : \tau \to \tau$

(iii) *τ is generated by the tensors*

$$g^{ab}, \; g_{ab}, \; e_{abcd}, \; \overset{c}{\nabla}_{(i_1} \ldots \overset{c}{\nabla}_{i_r} C_{i_{r+1}|ab|i_{r+2})}, \qquad r = 0, 1, 2, \ldots \tag{4.6}$$

(iv) *If $T \in \tau$ has the weight ω, then*

$$X^k(\overset{c}{\nabla}_a T) - \overset{c}{\nabla}_a (X^k(T)) = 2\omega \delta^k_a T + \Delta^k_a T, \tag{4.7}$$

where

$$\begin{aligned}\Delta^k_a (T_{ij\ldots}{}^{lm}\ldots) := \; & \Delta^{kl}_{as} T_{ij\ldots}{}^{sm\ldots} + \Delta^{km}_{as} T_{ij\ldots}{}^{ls\ldots} + \ldots \\ & - \Delta^{ks}_{ai} T_{sj\ldots}{}^{lm\ldots} - \Delta^{ks}_{aj} T_{is\ldots}{}^{lm\ldots} - \ldots\end{aligned} \tag{4.8}$$

and

$$\Delta^{ks}_{ai} := \delta^k_a \delta^s_i + \delta^k_i \delta^s_a - g_{ai} g^{ks}.$$

Examples. – (i) $\overset{c}{\nabla}_k C_{abcd} = \nabla_k C_{abcd}$, $\overset{c}{\nabla}_k {}^* C_{abcd} = \nabla_k {}^* C_{abcd}$

(ii) $B_{i_1, i_2} := \nabla_a \nabla_b C^a{}_{.i_1 i_2 .}{}^b + \frac{1}{2} C^a{}_{\cdot i_1 i_2 \cdot}{}^b R_{ab} = \overset{c}{\nabla}_a \overset{c}{\nabla}_b C^a{}_{\cdot i_1 i_a \cdot}{}^b$ (Bach tensor) $X^k(B_{i_1 i_2}) = 0$. Therefore, the Bach tensor is a conformally invariant tensor of weight -1.

Whenever in the following latin indices with subindices arise (e.g. $i_1 \ldots i_r$) we assume that symetrization has been carried out over the indices. If T is any tensor with covariant rank r $(r \geqq 2)$, then we denote by $TS\,(T)$ the trace-free part of T. For a symmetric tensor $T_{i_1 \ldots i_r}$ with $r \geqq 2$ we write

$$T_{i_1 \ldots i_r} \equiv O \,(\mathrm{mod}\, g) \qquad \text{iff} \quad TS\,(T_{i_1 \ldots i_r}) = O.$$

LEMMA 4.2. – *If $T_{i_1 \ldots i_{k-l}}$ is a symmetric, conformally invariant tensor with covariant rank $(k-l)$ and weight ω then*

$$\begin{aligned} &X^\gamma \,(\overset{c}{\nabla}_{i_1} \ldots \overset{c}{\nabla}_{i_l} T_{i_{l+1} \ldots i_k}) \\ &\equiv l\,(2\,\omega + l - 2\,k + 1)\, \delta^\gamma_{i_1}\, \overset{c}{\nabla}_{i_2} \ldots \overset{c}{\nabla}_{i_l} T_{i_{l+1} \ldots i_k} \,(\mathrm{mod}\, g) \end{aligned} \tag{4.9}$$

Proof. – By (4.7) one gets (*see* [G4], p. 510)

$$\begin{aligned} X^\gamma \,&(\overset{c}{\nabla}_{i_1} \ldots \overset{c}{\nabla}_{i_l} T_{i_{l+1} \ldots i_k}) \\ &\equiv l\,(2\,\omega - l + 1)\, \delta^\gamma_{i_1}\, \overset{c}{\nabla}_{i_2} \ldots \overset{c}{\nabla}_{i_l} T_{i_{l+1} \ldots i_k} \\ &\quad + l\, \overset{c}{\nabla}_{i_1} \ldots \overset{c}{\nabla}_{i_{l-1}} \Delta^\gamma_{i_l} \,(T_{i_{l+1} \ldots i_k})\,(\mathrm{mod}\, g). \end{aligned}$$

On account of (4.8) it follows $\Delta^{\gamma r}_{i_1\, i_2} \equiv 2\, \delta^\gamma_{i_1}\, \delta^r_{i_2}\,(\mathrm{mod}\, g)$ and, therefore,

$$\Delta^\gamma_{i_l} \,(T_{i_{l+1} \ldots i_k}) = -2\,(k - l)\; \delta^\gamma_{i_l}\, T_{i_{l+1} \ldots i_k}$$

and the assertion (4.9).

A conformally invariant tensor T is called trivial if T is generated by

$$\{g^{ab},\, g_{ab},\, e_{abcd},\, C_{abcd}\}.$$

Lemma 4.1, Corollary 4.1, Definition 4.2 and Proposition 4.1 are very useful for the construction of nontrivial conformally invariant tensors.

Let $\mathcal{S}_{2r}$, $\mathcal{S}_{2r+1}$ $(r = 0,\, 1,\, 2,\, \ldots)$ be the set of all symmetric, trace-free, conformally invariant tensors contained in $\mathcal{R}_0$ and $\mathcal{R}$, respectively, with weight –1 and covariant rank $2\,r$ and $2\,r + 1$, respectively. Using the above results, in [GW3; GeW1, 2] linear independent generating systems of $\mathcal{S}_r$ for $0 \leqq r \leqq 6$ were derived. [2)]

PROPOSITION 4.2. – *If* $S_r = \{S_{i_1 \ldots i_r}\} \in \mathcal{S}_r$ *then one has*

$$S_r = 0 \qquad \text{for} \quad r = 0,\, 1,\, 3, \qquad S_2 = \alpha\, B$$

$$S_4 = \sum_{m=1}^{3} \beta_m\, W^{(m)}, \qquad S_5 = \gamma^{(2)}\, S_5^{(2)} + \sum_{k=1}^{4} \gamma_k^{(3)}\, S_5^{(3,\,k)}$$

$$S_6 = \sum_{k=1}^{2} \delta_k^{(2)}\, S_6^{(2,\,k)} + \sum_{l=1}^{6} \delta_l^{(3)}\, S_6^{(3,\,l)},$$

where B is the Bach tensor,

$$\begin{aligned} W^{(1)}_{i_1 \ldots i_4} := TS\, [&\nabla^a\, C^{b}_{\cdot i_1 i_2 \cdot}{}^{c}\, \nabla_a\, C_{b i_3 i_4 c} + 16\, \nabla_u\, C^{u}_{\cdot i_1 i_2 a}\, \nabla_k\, C^{k}_{\cdot i_3 i_4 \cdot}{}^{a} \\ &+ 4\, C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\, \{2\, \nabla_a\, \nabla_u\, C^{u}_{\cdot i_3 i_4 b} - C_{a i_3 i_4 \cdot}{}^{c}\, L_{bc}\}], \end{aligned}$$

$$\begin{aligned} W^{(2)}_{i_1 \ldots i_4} := TS\, [&2\, \nabla_{i_1}\, C^{b}_{\cdot i_2 i_3 \cdot}{}^{c}\, \nabla_u\, C^{u}_{\cdot abi_4} + 2\, \nabla_u\, C^{u}_{\cdot i_1 i_2 a}\, \nabla_k\, C^{k}_{\cdot i_3 i_4 \cdot}{}^{a} \\ &- C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\, \{2\, \nabla_{i_3}\, \nabla_u\, C^{u}_{\cdot abi_4} - C^{c}_{\cdot abi_3} L_{ci_4}\}], \end{aligned}$$

$$W^{(3)}_{i_1 \ldots i_4} := TS\, [C^{abcd}\;\; C_{ai_1 i_2 d}\, C_{bi_3\, i_4 c}],$$

$$\begin{aligned} S^{(2)}_{i_1 \ldots i_5} := TS\, [&\nabla^{c*} C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\, \nabla_c\, C_{a i_3 i_4 b} + 4^{*} C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\, \nabla_{i_3}\, \nabla_a\, \nabla_u\, C^{u}_{\cdot i_4 i_5 b} \\ &- 6\, \nabla^{*}_{i_1} C^{a}_{\cdot i_2 i_3 \cdot}{}^{b}\, \nabla_a\, \nabla_u\, C^{u}_{\cdot i_4 i_5 b} \\ &+ 26\, \nabla^{*}_{u} C^{u}_{\cdot i_1 i_2 \cdot}{}^{a}\, \nabla_{i_3}\, \nabla_k\, C^{k}_{\cdot i_4 i_5 a} \\ &- 5^{*} C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\, \nabla_{i_3}\, C^{c}_{\cdot i_4 i_5 b}\, L_{ac} - 4^{*} C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\, \nabla_a\, C^{c}_{\cdot i_3 i_4 b}\, L_{i_5 c} \\ &- 4^{*} C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\, \nabla_u\, C^{u}_{\cdot abi_3}\, L_{i_4 i_5} + 21^{*} C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\, \nabla_u C^{u}_{\cdot i_3 i_4 a} L_{i_5 b}], \end{aligned}$$

$$S^{(3,\,1)}_{i_1 \ldots i_5} := TS\, [12^{*} C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\;\; C_{a i_3 i_4 \cdot}{}^{c}\, \nabla_u\;\, C^{u}_{\cdot bci_5} + C_{a \cdot \cdot i_1}^{\;\; cd}\;\; C_{bcdi_2}\, \nabla_{i_3}\; {}^{*} C^{a}_{\cdot i_4 i_5 \cdot}{}^{b}],$$

$$S^{(3,\,2)}_{i_1 \ldots i_5} := TS\, [8^{*} C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\;\; C_{a i_3 i_4 \cdot}{}^{c}\, \nabla_u\;\, C^{u}_{\cdot bci_5} - {}^{*} C_{a \cdot \cdot i_1}^{\;\; cd}\;\; C_{bi_2 cd}\, \nabla_{i_3}\, C^{a}_{\cdot i_4 i_5 \cdot}{}^{b}],$$

$$S^{(3,\,3)}_{i_1 \ldots i_5} := TS\, [12\, C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\;\; C_{a i_3 i_4 \cdot}{}^{c}\, \nabla_u\;\, C^{u}_{\cdot bci_5} + C_{a \cdot \cdot i_1}^{\;\; cd}\;\; C_{bcdi_2}\, \nabla_{i_3}\, C^{a}_{\cdot i_4 i_5 \cdot}{}^{b}],$$

$$S^{(3,\,4)}_{i_1 \ldots i_5} := TS\, [8\; C^{a}_{\cdot i_1 i_2 \cdot}{}^{b}\;\; C_{a i_3 i_4 \cdot}{}^{c}\, \nabla_u\;\, C^{u}_{\cdot bci_5} - C_{a \cdot \cdot i_1}^{\;\; cd}\;\; C_{bi_2 cd}\, \nabla_{i_3}\, C^{a}_{\cdot i_4 i_5 \cdot}{}^{b}]$$

and α, β_m, $\gamma^{(2)}$, $\gamma_k^{(3)}$, $\delta_k^{(2)}$, $\delta_l^{(3)}$ $\in \mathbb{R}$.

For the definition of the tensors $S_6^{(2,k)}$, $S_6^{(3,l)}$, which contain many monomials see [GeW 2, pp. 127-136]. [3)]

Remark 4.1. – In the case of $R_{ab}=0$ the tensors B, $W^{(2)}$ and $\mathcal{S}_6{}^{(2,2)}$ vanish. Consequently, if $(M,\ g)$ is conformally related to an Einstein space-time with $R_{ab}=0$ then [W8]

$$B=0,\ W^{(2)}=0,\ S_6{}^{(2,2)}=0.$$

Remark 4.2. – McLenaghan and Leroy investigate in [McL, L] the class of generalized plane wave metrics

$$ds^2 = 2\,dx^1[dx^2 + \{a\,(z+\bar{z})\,x^2 + Dz^2 + \bar{D}\bar{z}^2 + ez\bar{z} + Fz + \bar{F}\bar{z}\}\,dx^1$$

$$-2\,[dz + az^2\,dx^1]\,[d\bar{z} + a\bar{z}^2\,dx^1], \tag{4.10}$$

where $z=x^3+ix^4$ and $a=\bar{a}, D, e, F$ are arbitrary functions of x^1 only. For the special case $a=0$ we obtain the important subclass of plane wave metrics [S2, G3, 4; DC; W4]. In [AW; Ge W2] it was shown:

LEMMA 4.3. – *For a metric* (4.10) *it holds*

$$B=0;\ W^{(m)}=0,\ S_5{}^{(2)}=0,\ S_5{}^{(3,k)}=0;\ S_6{}^{(3,l)}=0$$
$$(m=1,\ 2,\ 3;\ k=1,\ \ldots,\ 4;\ l=1,\ \ldots,\ 6)$$

$$S_6{}^{(2,1)} = 11\,S_6{}^{(2,2)} = 363\,Z,$$

where

$$Z_{i_1\ldots i_6} := TS\,[C^{a}{}_{\cdot i_1 i_2 \cdot}{}^{b}\,C_{a i_3 i_4 b}\,R_{i_5}\,{}^{c}_{\cdot}\,R_{i_6 c}].$$

LEMMA 4.4. – *A generalized plane wave metric is a plane wave metric iff* $Z=0$.

Consequently, in virtue of Remark 4.1 a generalized plane wave metric g is conformally related to an Einstein metric iff g is a plane wave metric.

It is well known that the equations E_1-E_3 are conformally invariant [PR; W4]. In particular, for Weyl's neutrino operator under the conformal transformation [PR; W4]

$$\tilde{g}_{ab} = e^{2\Phi}\,g_{ab},\ \tilde{\sigma}_a{}^{A\dot{X}} = e^{\Phi}\cdot\sigma_a{}^{A\dot{X}},\ \tilde{\varepsilon}_{AB} = \varepsilon_{AB}$$

we get

$$\tilde{\mathcal{M}}\,\tilde{\varphi} = e^{-(5/2)\,\Phi}\,\mathcal{M}\,\varphi,$$

where $\tilde{\varphi} := e^{-(3/2)}\, \varphi$. In Section 5 we need a conformal covariant derivative of a conformal invariant spinor $S_{\dot{X}}$ with the weight ω, i.e. a spinor with the transformation law

$$\tilde{S}_{\dot{X}} = e^{2\omega\phi}\, S_{\dot{X}}, \qquad \omega \in \mathbb{R}.$$

Similarly to (4.5) one can define inductively for $r > 0$:

$$\begin{aligned} &\overset{c}{\nabla}_{i_1} S_{\dot{X}} = \nabla_{i_1} S_{\dot{X}}, \qquad \overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_r} S_{\dot{X}} \\ &= \nabla_{i_1} (\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_r} S_{\dot{X}}) - \frac{1}{2} X^k (\overset{c}{\nabla}_{i_2} \dots \overset{c}{\nabla}_{i_r} S_{\dot{X}})\, L_{k i_1}. \end{aligned} \tag{4.11}$$

Here $X^k (\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_r} S_{\dot{X}})$ is again given by (*see* (4.3), (4.4))

$$\begin{aligned} \overset{\tilde{c}}{\nabla}_{i_1} \dots \overset{\tilde{c}}{\nabla}_{i_r} \tilde{S}_{\dot{X}} = e^{2\omega\phi} [&\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_r} S_{\dot{X}} \\ &+ X^k (\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_r} S_{\dot{X}})\, \nabla_k \phi + \dots], \end{aligned}$$

where we omit terms which are nonlinear in the derivatives of ϕ. The transformation law of $\overset{c}{\nabla}_{i_1} \dots \overset{c}{\nabla}_{i_r} S_{\dot{X}}$ contains only first derivatives of ϕ.

LEMMA 4.5. – *A conformally invariant spinor $S_{\dot{X}}$ with weight ω satisfies*

$$\begin{aligned} X^\gamma (\overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_k} S_{\dot{X}}) \equiv{}& k\,(2\,\omega - k + 1)\, \delta^\gamma_{i_1}\; \overset{c}{\nabla}_{i_2} \cdots \overset{c}{\nabla}_{i_k} S_{\dot{X}} \\ &- k\, \sigma_{[i_1}{}^{Y\dot{Z}}\, \sigma_{k]\,\dot{X}Y}\, g^{\gamma k}\; \overset{c}{\nabla}_{i_2} \cdots \overset{c}{\nabla}_{i_k} S_{\dot{Z}} \pmod{g}. \end{aligned}$$

Proof. – Induction with respect to k: by $X^\gamma (S_{\dot{X}}) = 0$ the formula (4.11) is correct if $k = 0$. If (4.11) is fulfilled for any k, then on account of (*see* [W 4])

$$\nabla_a S_{\dot{X}} = \partial_a S_{\dot{X}} - \tilde{\Gamma}^{\dot{Y}}_{a\dot{X}}\, S_{\dot{Y}}, \quad \tilde{\Gamma}^B_{aA} = \Gamma^B_{aA} + g^{bc}\, \sigma_{[a}{}^{B\dot{Y}}\, \sigma_{b]\,A\dot{Y}}\, \nabla_c \phi$$

we have analogues to (4.7)

$$\begin{aligned} X^\gamma (\overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_{k+1}} S_{\dot{X}}) \equiv{}& \overset{c}{\nabla}_{i_1} X^\gamma (\overset{c}{\nabla}_{i_2} \cdots \overset{c}{\nabla}_{i_{k+1}} S_{\dot{X}}) \\ &+ 2\,(\omega - k)\, \delta^\gamma_{i_1}\; \overset{c}{\nabla}_{i_2} \cdots \overset{c}{\nabla}_{i_{k+1}} S_{\dot{X}} \\ &- \sigma_{[i_1}{}^{A\dot{Y}}\, \sigma_{m]\,A\dot{X}}\, g^{\gamma m}\; \overset{c}{\nabla}_{i_2} \cdots \overset{c}{\nabla}_{i_{k+1}} S_{\dot{Z}} \pmod{g}. \end{aligned}$$

In order to calculate $\overset{c}{\nabla}_{i_1} X^\gamma (\overset{c}{\nabla}_{i_2} \cdots \overset{c}{\nabla}_{i_{k+1}} S_{\dot{X}})$ we use the induction hypothesis and obtain under consideration of $\overset{c}{\nabla}_a\, g_{bd} = 0$, $\overset{c}{\nabla}_a\, \sigma_b^{A\dot{X}} = O$ the assertion.

5. MOMENTS OF THE FIELD EQUATIONS

1. *The moments of the scalar wave equation*

From (2.4) and the transport equations for the Hadamard coefficients it follows, that the coincidence values $\nabla^{(1)}_{i_1} \cdots \nabla^{(1)}_{i_r} T^{(0)}(x, x)$ are elements of $\mathcal{R}_0$ [G4]. Here $\nabla^{(1)}$ denotes the covariant derivative of $T^{(0)}$ with respect to the first variables. Because of (3.6) and Proposition 4.1 the conformal covariant derivatives $\overset{c}{\nabla}{}^{(1)}_{i_1} \cdots \overset{c}{\nabla}{}^{(1)}_{i_r} T^{(0)}(x, x)$, $r = 0, 1, 2 \dots$ are contained in τ. In [G4] it was defined and proved:

DEFINITION 5.1. – *The covariant r-tensors given recursively by*

$$I^1_{i_1 \cdots i_r}(x) := TS\left[\overset{c}{\nabla}{}^{(1)}_{i_1} \cdots \overset{c}{\nabla}{}^{(1)}_{i_r} T^{(0)}(x, x) - \sum_{l=1}^{r} d^1_{l, r}\, \overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_l} I^1_{i_{l+1} \cdots i_r}(x)\right]$$

with

$$d^1_{l, r} := \binom{r}{l}^2 \Big/ \binom{2r - l + 1}{l}$$

are called moments of the scalar wave equation E_1 of order r.

PROPOSITION 5.1. – *It holds $I^1_{i_1 \cdots i_r} \in \mathcal{S}_r$, $r \geqq 0$ and $I^1_{i_1 \cdots i_r} \equiv O$ if r odd.* [4)]

2. *The moments of Maxwell's equations E_2*

The tail term $K(x, y)$ of Maxwell's equations E_2 (*see* (3.2)) is a double differential form of degree 2:

$$K(x, y) = K_{ij, \alpha\beta}(x, y)\, dx^i \wedge dx^j\, dy^\alpha \wedge dy^\beta.$$

One defines for every $x \in M$ and $r \geqq 2$ the covariant r tensors [G4; W2]

$$M_{i_1 \cdots i_r}(x) := g^{j\beta}(x)\ \overset{c}{\nabla}{}^{(1)}_{i_1} \cdots \overset{c}{\nabla}{}^{(1)}_{i_{r-2}} K_{i_{r-1} j, i_r \beta}(x, x)$$

and for r even recursively

$$I'_{i_1 \cdots i_r} := TS\left[M_{i_1 \cdots i_r} - \sum_{l=1}^{r-2} d^2_{l, r}\ \overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_l} I'_{i_{l+1} \cdots i_r}\right]$$

with

$$d^2_{l, r} := \binom{r-2}{l}\binom{r}{l} \Big/ \binom{2r - l + 1}{l}$$

On the other hand let be

$$\begin{aligned} {}^*K(x, y) &= {}^*K_{ij,\,\alpha\beta}(x, y)\, dx^i \wedge dx^j\, dy^\alpha \wedge dy^\beta \\ &:= \frac{1}{2}\, e_{ijlk}\, K^{lk}_{\cdot\cdot\alpha\beta}(x, y)\, dx^i \wedge dx^j \wedge dy^\alpha \wedge dy^\beta. \end{aligned}$$

Then ${}^*K(x, y)$ is the left dual of $K(x, y)$ which is analogues to $K(x, y)$ (*see* (3.7)) conformally invariant with weight zero. Now one defines for $x \in M$ and $r \geqq 2$ [G4; W2]

$${}^*M_{i_1\cdots i_r} := g^{j\beta}(x)\ \overset{c}{\nabla}{}^{(1)}_{i_1} \cdots \overset{c}{\nabla}{}^{(1)}_{i_{r-2}}\ {}^*K_{i_{r-1}\,j,\,i_r\,\beta}(x, x)$$

and for r odd recursively

$$I''_{i_1\cdots i_r} := TS\left[{}^*M_{i_1\cdots i_r} - \sum_{l=1}^{r-2} d^2_{l,\,r}\ \overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_l}\ I''_{i_{l+1}\cdots i_r}\right] \tag{5.1}$$

DEFINITION 5.2. – *The covariant* r*-tensors given by*

$$I^2_{i_1\cdots i_r} := \begin{cases} I'_{i_1\cdots i_r} & \text{if} \quad r \geqq 2, \text{ even} \\ I''_{i_1\cdots i_r} & \text{if} \quad r \geqq 3, \text{ odd} \end{cases} \tag{5.2}$$

are called moments of Maxwell's equations of order r. [5)]

In [G4] the following was shown, using in particular (3.2):

PROPOSITION 5.2. – *The moments (5.2) are elements of* $\mathcal{S}_r$ $(r \geqq 2)$.

3. *The moments of Weyl's neutrino equation*

The tail term $N(x, y)$ of the equation E_3 (*see* (3.3)) is a spinor of $\bar{\mathcal{S}}_{(x)} \otimes \mathcal{S}_{(y)}$ [W4]. We denote by $N_{\dot{X}\underline{A}}(x,y)$ its coordinates. Here the underlined indices refer to y. In [W4, p. 69] it was proved:

LEMMA 5.1. – *It is* $N(x, y) = -N(y, x)$ *for* $x, y \in M$.

Now we define for every $x \in M$ and $r \geqq 1$ the covariant (complex) r-tensors [W4]

$$N_{i_1\cdots i_r}(x) := \sigma_{i_1}{}^{\underline{A}\dot{X}}(x)\ \overset{c}{\nabla}{}^{(1)}_{i_2} \cdots \overset{c}{\nabla}{}^{(1)}_{i_r}\ N_{\dot{X}\underline{A}}(x, x) \tag{5.3}$$

and for $r \geqq 1$ recursively

$$I^*_{i_1\cdots i_r} := N_{i_1\cdots i_r} - \sum_{l=1}^{r-1} d^3_{l,\,r}\ \overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_l}\ I^*_{i_{l+1}\cdots i_r} \tag{5.4}$$

with

$$d^3_{l,\,r} := \binom{r-1}{l}\binom{r}{l} \Big/ \binom{2r-l+1}{l} \tag{5.5}$$

PROPOSITION 5.3. – *The tensor* $i^r\, I^*_{i_1\cdots i_r}$ *is real for* $r \geqq 1$.

Proof. – Let us write in this proof instead of $I^*_{i_1\cdots i_r}$, $N_{i_1\cdots i_r}$, $\overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_r}$, $\overset{c}{\nabla}{}^{(1)}_{i_1} \cdots \overset{c}{\nabla}{}^{(1)}_{i_r}$, $\overset{c}{\nabla}{}^{(2)}_{i_1} \cdots \overset{c}{\nabla}{}^{(2)}_{i_r}$ shortly I_r, N_r, ∇_r, ∇^1_r, ∇^2_r, respectively.

Then it may be shown that for $r \geqq 1$

$$I_r + (-1)^{r+1}\, \bar{I}_r = 0. \tag{5.6}$$

Using the differentiation rule $\overset{c}{\nabla} = \overset{c}{\nabla}{}^{(1)} + \overset{c}{\nabla}{}^{(2)}$ along the diagonal of $M \times M$ and Lemma 5.1 we obtain

$$\begin{aligned}
\nabla^1_l\, N_{\dot{X}A}(x,\, x) &= -\nabla^2_l\, \bar{N}_{\dot{X}A}(x,\, x) = (\nabla_l - \nabla^2_l)\, N_{\dot{X}A}(x,\, x) \\
&= \sum_{n=0}^{l} (-1)^{l-n} \binom{l}{n} \nabla_n\, \nabla^2_{l-n}\, N_{\dot{X}A}(x,\, x) \\
&= \sum_{n=0}^{l} (-1)^{l-n+1} \binom{l}{n} \nabla_n\, \nabla^1_{l-n}\, \bar{N}_{\dot{X}A}(x,\, x),
\end{aligned}$$

hence

$$N_{l+1} = \sum_{n=0}^{l} (-1)^{l-n+1} \binom{l}{n} \nabla_n\, \bar{N}_{l-n+1}. \tag{5.7}$$

We prove (5.6) by induction with respect to r. By (5.7) we have for $r = 1$:

$$I_1 = N_1 = -\bar{N}_1 = -\bar{I}_1.$$

Now let (5.6) be fulfilled for $1 \leqq r \leqq (k-1)$. Then from (5.4) and (5.7) it follows for $r = k$

$$\begin{aligned}
\bar{I}_k + (-1)^{k+1} I_k &= \bar{N}_k + (-1)^{k+1} N_k \\
&- \sum_{l=1}^{k-1} d^3_{l,\,k} [\nabla_l \bar{I}_{k-l} + (-1)^{k+1} \nabla_l I_{k-l}] \\
&= \sum_{n=1}^{k-1} (-1)^{k-n} \binom{k-1}{n} \nabla_n \left[I_{k-n} + \sum_{l=1}^{k-n-1} d^3_{l,\,k-n} \nabla_l I_{k-n-l} \right] \\
&+ \sum_{n=1}^{k-1} d^3_{n,\,k} (-1)^k (1 - (-1)^n) \nabla_n I_{k-n} \\
&= \sum_{n=2}^{k-1} (-1)^k \nabla_n I_{k-n} \left[\sum_{l=0}^{n} (-1)^l \binom{k-1}{l} d^3_{n-l,\,k-l} - (-1)^n d^3_{n,\,k} \right] \\
&= \sum_{n=2}^{k-1} (-1)^k d^3_{n,\,k} \nabla_n I_{k-n} [F(-n,\, n-2k-1,\, -k,\, 1) - (-1)^n] = O,
\end{aligned}$$

where F is the hypergeometric polynomial at the point 1.

DEFINITION 5.3. – *The covariant real r-tensors given by*

$$I^3_{i_1 \cdots i_r} = \begin{cases} TS\,[I^*_{i_1 \cdots i_r}] & \text{if } r \geqq 2, \text{ even} \\ TS\,[i\, I^*_{i_1 \cdots i_r}] & \text{if } r \geqq 1, \text{ odd} \end{cases}$$

are called moments of Weyl's neutrino equation of order r.

LEMMA 5.2. – *If* $I^3_{i_1 \cdots i_l}(x) = 0$ *for every* $x \in M$ *and* $1 \leqq l \leqq r-1$ *then*

$$I^3_{i_1 \cdots i_r} = \begin{cases} I_{i_1 \cdots i_r} & \text{if } r \geqq 2, \text{ even} \\ i\, I_{i_1 \cdots i_r} & \text{if } r \geqq 1, \text{ odd,} \end{cases}$$

where

$$I_{i_1 \cdots i_r}(x) := \sigma_{i_1}{}^{A\dot{X}}(x)\, \partial^{(1)}_{i_2} \cdots \partial^{(1)}_{i_r} N_{\dot{X}A}(x,\, x).$$

Proof. – Induction with respect to r. Obviously, the assertion is true for $r = 1$. Now we assume, that the assertion is valid for $r - 1$. Then we have

$$\left.\begin{aligned} I_{i_1 \cdots i_l}(x) = \sigma_{i_1}{}^{A\dot{X}}(x)\, \partial^{(1)}_{i_2} \cdots \partial^{(1)}_{i_l} N_{\dot{X}A}(x,\, x) = O, \\ 0 \leqq l \leqq r-1, \end{aligned}\right\} \tag{5.8}$$

from which it follows

$$\sigma_{i_1}{}^{A\dot{X}} \nabla^{(1)}_{i_2} \cdots \nabla^{(1)}_{i_l} N_{\dot{X}A}(x,\, x) = O, \quad 0 \leqq l \leqq r-1. \tag{5.9}$$

Lemma 4.5 and (4.11) imply that

$$\overset{c}{\nabla}{}^{(1)}_{i_1} \cdots \overset{c}{\nabla}{}^{(1)}_{i_{r-1}} N_{\dot{X}\cdot} - \nabla^{(1)}_{i_1} \cdots \nabla^{(1)}_{i_{r-1}} N_{\dot{X}\cdot}$$

can be represented as linear form in the symmetric covariant derivatives of $N_{\dot{X}}$ up to the order $r-2$. Consequently, on account of (5.8) and (5.9) we get

$$\overset{c}{\nabla}{}^{(1)}_{i_1} \cdots \overset{c}{\nabla}{}^{(1)}_{i_{r-1}} N_{\dot{X}A}(x, x) = \partial^{(1)}_{i_1} \cdots \partial^{(1)}_{i_{r-1}} N_{\dot{X}A}(x, x)$$

(*see* also [G4], Lemma 4.10), hence

$$I_{i_1 \cdots i_r} \equiv \sigma_{i_1}{}^{A\dot{X}}\, \partial^{(1)}_{i_2} \cdots \partial^{(1)}_{i_r} N_{\dot{X}A} \pmod{\mathrm{g}}. \tag{5.10}$$

In [W4, p. 154] it was proved, that (5.10) is symmetric and trace-free. Hence, the assertion is valid also for r.

PROPOSITION 5.4. – *The moments $I^3_{i_1 \cdots i_r}$ are elements of $\mathcal{S}_r$, $r \geqq 1$.*

Proof. – The spinor equivalents of the moments are polynomial with respect to the covariant derivatives of the curvature spinors with real coefficients [W4].

Therefore

$$I^3_{i_1 \cdots i_{2r}} \in \mathcal{R}_0, \quad I^3_{i_1 \cdots i_{2r+1}} \in \mathcal{R}, \quad r = 0, 1, 2, \ldots, \tag{5.11}$$

where the moments contain the Levi-Civita tensor linearly if r odd (*see* [W4]). Now we have to show, that the moments are conformally invariant of weight (-1). Obviously, it is sufficient to prove this for the tensors (5.4).

Induction with respect to r. From (5.3), (5.4), (3.8) and $\tilde{\sigma}_a{}^{A\dot{X}} = e^{\Phi}\, \sigma_a{}^{A\dot{X}}$ it follows for $r = 1$:

$$\tilde{I}^*_{i_1}(x) = \tilde{N}_{i_1}(x, x) = e^{-2\Phi(x)} N_{i_1}(x, x) = e^{-2\phi(x)} I^*_{i_1}(x).$$

Lemma 4.2 implies for $l \geqq 1$

$$\begin{aligned} & X^{\gamma}(\overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_l} I^*_{i_{l+1} \cdots i_r}) \\ & \equiv -l\,(2r - l + 1)\, \delta^{\gamma}_{i_1}\, \overset{c}{\nabla}_{i_2} \cdots \overset{c}{\nabla}_{i_l} I^*_{i_{l+1} \cdots i_r} \pmod{g}. \end{aligned} \tag{5.12}$$

Further, we get from (5.3), Lemma 4.5, (3.8) and

$$\sigma_{i_1}{}^{A\dot{X}}\, \sigma_{[i_2}{}^{Y\dot{Z}}\, \sigma_{k]Y\dot{X}} \equiv \frac{1}{2}\, \sigma_{i_1}{}^{A\dot{Z}}\, g_{ki_2} \pmod{g}$$

(*see* [Sch]) the formula

$$
\begin{aligned}
X^\gamma (N_{i_1 \cdots i_r}) &\equiv \sigma_{i_1}{}^{A\dot{X}}\, X^\gamma (\overset{c}{\nabla}{}^{(1)}_{i_2} \cdots \overset{c}{\nabla}{}^{(1)}_{i_r} N_{\dot{X}A} \\
&\equiv \sigma_{i_1}{}^{A\dot{X}} \Bigg[(r-1) \left(\frac{1}{2} - r \right) \delta^\gamma_{i_2}\ \overset{c}{\nabla}{}^{(1)}_{i_3} \cdots \overset{c}{\nabla}{}^{(1)}_{i_r} N_{\dot{X}A} \\
&\qquad + (1-r)\, \sigma_{[i_2}{}^{Y\dot{Z}}\, \sigma_{k]\dot{X}Y}\, g^{\gamma k}\ \overset{c}{\nabla}{}^{(1)}_{i_3} \cdots \overset{c}{\nabla}{}^{(1)}_{i_r} N_{\dot{Z}A} \Bigg] \\
&\equiv r(1-r)\, \delta^\gamma_{i_1} N_{i_2 \cdots i_r} \pmod{g}.
\end{aligned}
\tag{5.13}
$$

Finally, from (5.4), (5.5), (5.12), (5.13) it follows

$$
\begin{aligned}
X^\gamma (I^*_{i_1 \cdots i_r}) &= X^\gamma (N_{i_1 \cdots i_r}) - \sum_{l=1}^{r-1} d^3_{l,r}\, X^\gamma (\overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_l} I^*_{i_{l+1} \cdots i_r}) \\
&\equiv -r(r-1)\, \delta^\gamma_{i_1} N_{i_2 \cdots i_r} + \sum_{l=1}^{r-1} d^3_{l,r}\, l\,(2r-l+1)\, \delta^\gamma_{i_r}\ \overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_{l-1}} I^*_{i_l \cdots i_{r-1}} \\
&\equiv r(1-r)\, \delta^\gamma_{i_1} \left[I^*_{i_1 \cdots i_{r-1}} + \sum_{l=1}^{r-2} d^3_{l,r-1}\ \overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_l} I^*_{i_{l+1} \cdots i_{r-1}} \right] \\
&\qquad + \sum_{l=1}^{r-1} d^3_{l,r}\, l\,(2r-l+1)\, d^\gamma_{i_r}\ \overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_{l-1}} I^*_{i_l \cdots i_{r-1}} \\
&\equiv \delta^\gamma_{i_r} \sum_{l=1}^{r-2} [r(1-r)\, d^3_{l,r-1} + (l+1)(2r-l)\, d^3_{l+1,r}]\ \overset{c}{\nabla}_{i_1} \cdots \overset{c}{\nabla}_{i_l} I^*_{i_{l+1} \cdots i_{r-1}} \\
&\equiv O \pmod{\mathrm{g}}.
\end{aligned}
$$

Consequently $I^3_{i_1 \cdots i_r}$ is conformally invariant. The proof of the weight is easily done.

We summarize the results of this Section:

THEOREM 5.1. – *Every field equation* E_σ, $\sigma \in \{1, 2, 3\}$ *implies an infinite sequence* $\{I^\sigma_{i_1 \cdots i_r}\}$ *of symmetric, trace-free, conformally invariant* r*-tensors of weight* (-1).

4. *The first moments*

The propositions 5.1, 5.2, 5.4 and 4.2 imply

PROPOSITION 5.5. – *There are real coefficients* $\alpha^{(\sigma)}$, $\beta^{(\sigma)}_k$, $\gamma^{(2,\sigma)}$, $\gamma^{(3,\sigma)}_l$, $\delta^{(2,\sigma)}_m$, $\delta^{(3,\sigma)}_p$ *such that for the moments* $I^\sigma_{i_1 \cdots i_r}$ *it holds*

$$I^1 = O,\ I^1_{i_1} = I^3_{i_1} = O,\ I^\sigma_{i_1 i_2} = \alpha^{(\sigma)} B_{i_1 i_2},\ I^\sigma_{i_1 i_2 i_3} = O,$$
$$I^\sigma_{i_1 \cdots i_4} = \sum_{k=1}^{3} \beta^{(\sigma)}_k W^{(k)}_{i_1 \cdots i_4},\ I^\sigma_{i_1 \cdots i_5} = \gamma^{(2,\sigma)} S^{(2)}_{i_1 \cdots i_5} + \sum_{l=1}^{2} \gamma_l{}^{(3,\sigma)} S^{(3,l)}_{i_1 \cdots i_5},$$
$$I^\sigma_{i_1 \cdots i_6} = \sum_{m=1}^{2} \delta^{(2,\sigma)}_m S^{(2,m)}_{i_1 \cdots i_6} + \sum_{p=1}^{6} \delta_p{}^{(3,\sigma)} S^{(3,p)}_{i_1 \cdots i_6} \quad (\sigma = 1,\ 2,\ 3). \tag{5.14}$$

6. HUYGENS' PRINCIPLE AND MOMENT EQUATIONS

The following theorem shows the importance of the moments for the validity of Huygens' principle:

THEOREM 6.1. – *Let $(M,\ g)$ be an analytic space-time. The field equation E_σ, $\sigma \in \{1,\ 2,\ 3\}$ satisfies Huygens' principle if and only if all the corresponding moments $I^\sigma_{i_1 \cdots i_r}$ or order r vanish on M.*

Proof. – For the cases $\sigma = 1,\ 2$ *see* [G4, W2]. If for E_3 Huygens' principle is valid, then because of (3.3) we have for all $x \in M$ and $r \geqq 0$

$$\partial^{(1)}_{i_1} \cdots \partial^{(1)}_{i_r} N_{\dot{X}A}(x,\ x) = O, \tag{6.1}$$

hence, by Lemma 5.2

$$I^3_{i_1 \cdots i_r} = 0, \qquad r \geqq 1\ ^{6)} \tag{6.2}$$

Conversely, from (6.2) and Lemma 5.2 it follows (6.1). Since $N_{\dot{X}A}(x,\ y)$ is analytic the spinor $N_{\dot{X}A}(x,\ y)$ can be determined with the help of the Taylor expansion of $N_{\dot{X}A}(x,\ y)$ at the point $y = x$, which by (6.1) has only zero coefficients. Hence, $N(x,\ y) = 0$ and E_3 satisfies Huygen's principle.

DEFINITION 6.1. – *The equations*

$$I^\sigma_{i_1 \cdots i_r} = 0, \quad \sigma \in \{1,\ 2,\ 3\} \tag*{$(\mathrm{ME})^\sigma_r$}$$

are called moment equations for E_σ of order r.

During the last forty years in the investigations of Huygens' principle for E_σ, $\sigma \in \{1,\ 2,\ 3\}$, the first moments were determined, using the transport equations for Hadamard's coefficients, suitable test metrics and often normal coordinates [A; G1, 2, 4; GW1; McL1, 2; S3; W1, 2, 4]. By complicate calculations one obtained in this way more detailed information about the coefficients of Proposition 5.5.

LEMMA 6.1. – *It holds* $\alpha^{(1)} = -1/60$ [G1], $\alpha^{(2)} = 1/20$ [G2], $\alpha^{(3)} = -1/80$ [W4],

$$\beta_1^{(1)} = -1/2^3 \cdot 5 \cdot 7, \quad \beta_2^{(1)} = 1/2 \cdot 3 \cdot 5 \cdot 7, \quad \beta_3^{(1)} = O \quad \text{[W1]},$$
$$\beta_1^{(2)} = 1/2^3 \cdot 3 \cdot 7, \quad \beta_2^{(2)} = -2/3 \cdot 5 \cdot 7, \quad \beta_3^{(2)} = O \quad \text{[W2]},$$
$$\beta_1^{(3)} = -1/2^2 \cdot 3 \cdot 5 \cdot 7, \quad \beta_2^{(3)} = 13/2^5 \cdot 3 \cdot 5 \cdot 7, \quad \beta_3^{(3)} = O \quad \text{[W4]},$$
$$\gamma^{(2,\sigma)} \neq 0 \ for \ \sigma = 2,\, 3 \,\text{[AW, A]}$$

and [RW; GeW2]

$$I^1_{i_1 \cdots i_6} = 1/2^3 \cdot 3^2 \cdot 5 \cdot 7 \left[\frac{1}{33} S^{(2,1)}_{i_1 \cdots i_6} + \frac{7}{66} S^{(2,2)}_{i_1 \cdots i_6} + \frac{5}{2} S^{(3,1)}_{i_1 \cdots i_6} + 2\, S^{(3,2)}_{i_1 \cdots i_6} - \frac{5}{2} S^{(3,3)}_{i_1 \cdots i_6} - 5\, S^{(3,4)}_{i_1 \cdots i_6} \right].$$

COROLLARY 6.1. – *If the equation* E_σ, $\sigma \in \{1, 2, 3\}$ *satisfies Huygens' principle, then, in particular, the following moment equations* [7)] *must hold*

$$B_{i_1 i_2} = O \qquad (ME)_2^\sigma$$
$$W^{(1)}_{i_1 \cdots i_4} - k_\sigma\, W^{(2)}_{i_1 \cdots i_4} = O \qquad (ME)_4^\sigma$$
$$I^\sigma_{i_1 \cdots i_5} = O \qquad (ME)_5^\sigma$$
$$I^\sigma_{i_1 \cdots i_6} = O \qquad (ME)_6^\sigma$$

where

$$k_1 = 4/3,\ k_2 = 16/5,\ k_3 = 13/8.$$

PROPOSITION 6.1. – (i) *If* g *is a plane wave metric, then for all* r *and* σ *it holds* $I^\sigma_{i_1 \cdots i_r} = 0$.

(ii) *If* g *is an Einstein metric, a central symmetric metric a (2,2)-decomposable metric or a conformally recurrent metric and* $\sigma \in \{1, 2, 3\}$, *then from* $\{(ME)_k^\sigma \mid k = 2, 4\}$ *it follows, that* g *is a conformally flat or a plane wave metric.*

(iii) *Let* g *be conformally equivalent to a metric with* $\nabla_a C_{abcd} = 0$ *and* $\sigma \in \{1, 2, 3\}$. *Then the equations* $\{(ME)_k^\sigma \mid k = 2, 4\}$ *imply, that* g *is of Petrov type* N.

(iv) *Let g be of Petrov type N and $\sigma \in \{2, 3\}$. The moment equations $\{(ME)_k^\sigma \mid k = 2, 4, 5\}$ are satisfied if and only if g is conformally equivalent to a generalized plane wave metric.*

(v) *Let g be of Petrov type N, $\sigma \in \{1, 2, 3\}$ and*

$$11\,\delta_1^{(2,\sigma)} + \delta_2^{(2,\sigma)} \neq O. \tag{6.3}$$

Then the equations $\{(ME)_k^\sigma \mid k = 2, 4, 5, 6\}$ are fulfilled if and only if g is conformally equivalent to a plane wave metric. [8)]

(vi) *Let g be of Petrov type D and $\sigma \in \{1, 2, 3\}$. There are no metrics, for which the equations $\{(ME)_k^\sigma \mid k = 2, 4\}$ are valid.*

Proof. – If g is a plane wave metric then every equation E_σ satisfies Huygens' principle [G3; S1; W4]. Hence, Theorem 6.1 implies the assertion (i). The assertion (ii) was proved in [W4]. For the case $R_{ab} = 0$ see [McL 1]. For the proof of (iii) see [W6]. If g is of Petrov type N and $(ME)_k^\sigma$ hold for $k = 2$ and 4, then g is conformally equivalent to special cases of complex recurrent metrics [CM]. If in addition $(ME)_5^\sigma$, $\sigma \in \{2, 3\}$, is satisfied, then g is conformally equivalent to a generalized plane wave metric (*see* [AW], p. 81). Conversely, if g is conformally equivalent to a generalized plane wave metric, then by Lemma 4.3 the equations $(ME)_k^\sigma$ are satisfied for $k = 2, 4, 5$. In order to prove (v) we remark, that g is conformally equivalent to a plane wave metric, iff g is of Petrov type N and the equations $(ME)_k^1$ for $k = 2, 4, 6$ are satisfied [CM]. For $\sigma \in \{2, 3\}$ by (iv) from $\{(ME)_k^\sigma \mid k = 2, 4, 5\}$ it follows, that g is conformally equivalent to a generalized plane wave metric. Now (5.14) and Lemma 4.3 imply

$$I^\sigma_{i_1 \cdots i_6} = 33\,(11\,\delta_1^{(2,\sigma)} + \delta_2^{(2,\sigma)})\,Z_{i_1 \cdots i_6} = O,$$

from which by Lemma 4.4 and (6.3) it follows, that g is conformal equivalent to a plane wave metric. The conversion is clear because of (i). Finally, in [W7, CM] it was proved the assertion (vi). [9)]

COROLLARY 6.2. – *Let g be of Petrov type N and the condition* (6.3) [8)] *fulfilled. Every field equation E_σ satisfies Huygens' principle if and only if g is conformally equivalent to a plane wave metric.*

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NOTES

[1] The Levi-Civita tensor e_{abcd} is strictly speaking a pseudo-tensor.

[2] In [GW3] generating systems were derived also for higher dimensions.

[3] It is easy to prove, that there are no trivial tensors in S_5 and S_6.

[4] In [G4] moments for a general hyperbolic equation for higher dimensions were defined, too.

[5] If r odd then because of (4.12) the moments contain the Levi-Civita pseudo-tensor lienar and the moments are strictly speaking pseudo-tensors.

[6] For this assertion it is sufficient for g to require to be of class C^∞.

[7] A conjecture is, that the moment equations $(ME)_r^\sigma$ $(r = 2, 4, 5, 6)$ are also sufficient for the validity of Huygens' principle for E_σ, $\sigma = 1, 2, 3$, and that these equations are fulfilled if and only if g is conformally flat or a plane wave metric [W4, 6; CM].

[8] For $\sigma = 1$ by Lemma 6.1 the condition (6.3) is satisfied. For $\sigma = 2, 3$ we show this in a subsequent paper.

[9] In order to prove (ii)-(vi) we mainly use the two-component spinor formalism of Penrose and the spin coefficient formalism of Newman and Penrose [PR]

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