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HENRIQUE VON DREIFUS

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Bounds on the critical exponents of disordered ferromagnetic models

by

Henrique VON DREIFUS (*)
Departamento de Matemática Aplicada,
Instituto de Matemática e Estatística,
Universidade de São Paulo, Caixa Postal 20570,
01498 São Paulo SP, Brazil

ABSTRACT. – We show that the critical exponent ν associated with the correlation length $\xi(\beta)$ of disordered ferromagnetic models satisfies the bound $\nu \geqq \frac{2}{d}$. Our proof makes use of ideas related to the Lieb-Simon inequality.

RÉSUMÉ. – Nous montrons que l'exposant critique ν de la longueur de corrélation $\xi(\beta)$ d'un système ferromagnétique désordonné satisfait l'inégalité $\nu \geqq \frac{2}{d}$. Notre démonstration fait appel à des idées en rapport avec l'inégalité de Lieb-Simon.

1. INTRODUCTION

Critical exponents of disordered systems have been the main subject of a number of papers recently published ([ACCN], [By], [CCFS], [Ng]). In

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[CCFS], J. T. Chayes, L. Chayes, D. S. Fisher and T. Spencer proposed a definition of a finite size correlation length for disordered models which was very useful in obtaining bounds on the critical exponent ν associated with the actual correlation length $\xi(\beta)$. They proved the bound $\nu \geqq \frac{2}{d}$, for percolation model and ferromagnetic model in $\mathbb{Z}^d$. This result is of special interest when the corresponding ordered model satisfies the bound $\nu \geqq \frac{2}{d}$ [Ma]. In those situations the Harris criterion [Ha] leads one to expect that disorder is relevant for the critical behaviour *i.e.* the fixed point of the renormalization group transformation differs from the fixed point of the pure system. However these assertions are far from any kind of mathematical proof even in the simple models.

In this article we prove the bound $\nu \geqq \frac{2}{d}$ for ferromagnetic rotators with two components. The proof of this bound for Ising models presented in [CCFS], makes a significant use of the Fortuin-Kastelyn representation [FK], which expresses the Ising model as a percolation system. Since in the case of rotators we do not have an equivalent representation, a new technique had to be developed to extend the results. We first generalize to disordered models results obtained by B. Simon [Si] and E. Lieb [Li], in the context of ordered models to obtain a "local" condition that implies in the exponential decay of the two point function of disordered systems. We then prove the bound $\nu \geqq \frac{2}{d}$ by combining this result with the ideas introduced in [CCFS].

This article is organized as follows. In section 2 we prove theorem 2.3 which consists in the generalization of the results by B. Simon and E. Lieb. In section 3 we give the proof of the bound $\nu \geqq \frac{2}{d}$ for rotators.

2. LIEB-SIMON INEQUALITY AND THE EXPONENTIAL DECAY OF THE TWO-POINT FUNCTIONS OF DISORDERED FERROMAGNETIC MODELS

We start with some notations and definitions.

For each point $x \in \mathbb{Z}^d$ we consider a random variable $\sigma(x)$ which takes values in S^{N-1}. Given, for each $x \in \Lambda \subset \mathbb{Z}^d$, a choice of $\sigma(x)$, we call $\tilde{\sigma} = \{\sigma(x)\}_{x \in \Lambda}$ and denote by Ω_Λ the set of configurations $\tilde{\sigma}$. On Ω_Λ we define a energy function H_Λ (Hamiltonian), which takes values in $\mathbb{R}$, by

$$H_\Lambda(\tilde{\sigma}) = \sum_{\substack{x,\, y \in \Lambda \\ |x-y|=1}} -J_{xy}\, \sigma(x) . \sigma(y) \tag{2.1}$$

where $J_{xy} \in \mathbb{R}$; $J_{xy} \geqq 0$ and $\sigma(x) . \sigma(y)$ denote scalar products in $\mathbb{R}^N$.

In general we will consider Dirichlet (free) boundary conditions *i.e.* $J_{xy}=0$ for all pairs x, y such that $x \in \partial\Lambda$ and $y \notin \Lambda$, with $\partial\Lambda$ defined as

$$\partial\Lambda = \{ x \in \Lambda \mid \exists y \notin \Lambda,\ |x-y|=1 \} \tag{2.2}$$

In this article we prove results for the cases with $N=1$ and $N=2$. For $N=1$, $S^0=\{+1, -1\}$, and the model is called the Ising model. For $N=2$, $S^1 \subset \mathbb{R}^2$ is the unit circle, and the model is called the plane rotator. For a function defined on Ω_Λ, the expectation of F, denoted $\langle F \rangle_\Lambda$ is given by

$$\left.\begin{aligned} \langle F \rangle_\Lambda &= Z_\Lambda^{-1} \int F(\tilde{\sigma})\, e^{-\beta H_\Lambda(\tilde{\sigma})} \prod_{x \in \Lambda} d\mu(\sigma(x)) \\ Z_\Lambda &= \int e^{-\beta H_\Lambda(\tilde{\sigma})} \prod_{x \in \Lambda} d\mu(\sigma(x)) \end{aligned}\right\} \tag{2.3}$$

where $d\mu(\sigma(x)) = \delta(\sigma^2(x)-1)\, d\sigma(x)$ for the Ising model and $d\mu(\sigma(x))$ is the Haar measure on the unit circle for the plane rotator model.

The existence of the thermodynamic limit ($\Lambda \to Z^d$) of certain correlation functions denoted by $\lim_{\Lambda \to Z^d} \langle\ .\ \rangle_\Lambda = \langle\ .\ \rangle$ can be obtained from the well known Griffith's Inequalities [Gi]. Our results are restricted to the cases with $N=1, 2$ because Griffith's Inequalities have not been proved for $N>2$.

The following result due to B. Simon [Si] and E. Lieb [Li], is a key ingredient for our proofs.

THEOREM 2.1 (Generalized Lieb-Simon Inequality). — *Let* X *be a set of ordered pairs* (u, t) *of sites in the lattice* Z^d *such that if* $(u, t) \in X$ *so is* tu. *Let* $\langle\ \rangle_X$ *denote the infinite volume expectation defined by setting* $J_{ut}=0$ *for* $ut \in X$. *Then for* $x, y \in Z^d$.

$$\begin{aligned} 0 \leqq \langle \sigma(x) . \sigma(y) \rangle - \langle \sigma(x) . \sigma(y) \rangle_X & \\ \leqq \sum_{ut \in X} \langle \sigma(x) . \sigma(u) \rangle_X\, J_{ut}\, \beta \langle \sigma(t) . \sigma(y) \rangle & \end{aligned} \tag{2.4}$$

COROLLARY 2.2. — *Let* $\Lambda \subset Z^d$ *be a finite cube. For* x *and* y *such that* $x \in \Lambda$ *and* $y \notin \Lambda$, *we have*

$$\langle \sigma(x) . \sigma(y) \rangle \leqq \sum_{ut \in \partial\Lambda^*} \langle \sigma(x) . \sigma(u) \rangle_{\partial\Lambda} . \beta J_{ut} \langle \sigma(t) . \sigma(y) \rangle \tag{2.5a}$$

$$\langle \sigma(x) . \sigma(y) \rangle \leqq \sum_{ut \in \partial\Lambda^*} \langle \sigma(x) . \sigma(u) \rangle . \beta J_{ut} \langle \sigma(t) . \sigma(y) \rangle_{\partial\Lambda^*} \tag{2.5b}$$

where

$\partial\Lambda^* = \{(x, y) \in (\mathbb{Z}^d)^2, |x-y| = 1 \mid$ *either* $x \in \Lambda$ *and* $y \notin \Lambda$ *or* $x \notin \Lambda$ *and* $y \in \Lambda\}$.

For a proof of those results *see* [Br].

We shall consider the nearest neighbor coupling J_{xy} as independent random variables with a *common* independent distribution density given by

$$d\mu(J_{xy}) = g(J_{xy})\, dJ_{xy} \tag{2.6}$$

where dJ_{xy} denotes the Lebesgue measure on $\mathbb{R}$. These models are studied to analyse the effects of impurities. We begin with very general conditions for g, requiring only that $g(J_{xy}) \geqq 0$, $\int g(J_{xy})\, dJ_{xy} = 1$ and $g(J_{xy}) = 0$ for $J_{xy} < 0$. The last condition is to assure the ferromagnetism. We denote by $\bar{J}$ the common mean value of the variables J_{xy} *i.e.* $\bar{J} = \int J_{xy}\, d\mu(J_{xy})$ and by $(Z^d)^2$ the set of pairs (x, y); $x, y \in \mathbb{Z}^d$ with $|x-y| = 1$. With this modification $\langle F \rangle$ is a function of the J_{xy} configuration $J\,(J = \{J_{xy}\}_{(x,y) \in (Z^d)^2})$, and we shall consider

$$\overline{\langle F \rangle} = \int \langle F \rangle (J)\, d\rho(J) \tag{2.7}$$

where

$$d\rho(J) = \prod_{(x,\, y) \in (\mathbb{Z}^d)^2} g(J_{xy})\, dJ_{xy}$$

Results obtained for ordered models can be very useful in the analysis of disordered models. A simple example is the existence of phase transition if inf [supp g] $\geqq J_0 > 0$. Since for the ordered model with $J_{xy} = J_0$ the existence of phase transition is known [GJ], the same conclusion follows for the disordered model from Griffith's Inequalities. The behaviour of correlations as $T \searrow T_c$ require new ideas as in [CCFS] where it was proved that for the correlation length $\xi(\beta)$, of a disordered Ising Model, defined as

$$\xi^{-1}(\beta) = -\lim_{|x| \to \infty} \log \frac{\{\overline{\langle \sigma(0) . \sigma(x) \rangle}\}}{|x|} \tag{2.8}$$

and β_c defined as

$$\beta_c = \sup \{ \beta \mid \overline{\langle \sigma(0) . \sigma(x) \rangle} \leqq e^{-|x|/\xi} \text{ as } |x| \to \infty, \text{ for some } \xi > 0 \} \tag{2.9}$$

the following bound for $\xi(\beta)$, as $\beta \to \beta_c$, holds

$$\xi(\beta) \leqq (\beta_c - \beta)^{-2/d} \tag{2.10}$$

To extend the above result on $\xi(\beta)$ to rotators we will need the following theorem.

THEOREM 2.3. – *Let* $\Lambda_L(x)$ *be a finite cube in* $\mathbb{Z}^d$ *with side* $L=2k$. ($k\in\mathbb{Z}$) *and center at* $x\in\mathbb{Z}^d$. *If*

$$\beta^2(\bar{J})^2 \sum_{u\in\partial\Lambda_L(x)} \overline{\langle\sigma(0).\sigma(x)\rangle}_{\partial\Lambda_L^*(x)} < \frac{1}{(L+4)^{d-1}n(d)} \qquad (2.11)$$

where $n(d)$ *is a geometric constant which gives the number of hyperfaces of a cube in d-dimensions.*

Then, for $|x-y|\gg L$

$$\overline{\langle\sigma(x).\sigma(y)\rangle} \leqq C e^{(-x-y)/\xi(\beta)}$$

where C *is a positive constant and*

$$\xi(\beta) \leqq -\frac{1}{2}\frac{L+4}{\log\delta(\beta)}$$

with

$$0<\delta(\beta)=(L+4)^{d-1}\beta^2(\bar{J}) \sum_{u\in\partial\Lambda_L(x)} \overline{\langle\sigma(0).\sigma(x)\rangle}_{\partial\Lambda_L^*(x)}$$

We also prove.

THEOREM 2.4. – *For* $\xi(\beta)$ *and* β_c *defined as in* (2.8) *and* (2.9)

$$\xi(\beta)\log\xi(\beta) \geqq \text{const.}(\beta_c-\beta)^{-2/d} \qquad (2.12)$$

A similar version of theorem 2.3, in the context of ordered models, was proved by E. Lieb and B. Simon. The difficulty in extending their proof to disordered models lies in the fact that the expectation in the coupling variables of a product of two point functions does not, in general, factorize into the product of expectations. The idea of our proof for the random case, which is very simple, makes a double use of the generalized Lieb-Simon inequality in order to bound the two point function by a sum of products of local two point functions which can be factored with respect to the expectation in the coupling variables.

Proof of theorem 2.3. – For a given configuration J and $|x-y|\gg L$, (2.5 *a*) implies in,

$$\langle\sigma(x).\sigma(y)\rangle(J) \leqq \sum_{(u,t)\in\partial\Lambda_L^*(x)} \langle\sigma(x).\sigma(u)\rangle_{\partial\Lambda_L^*(x)}(J) \times\beta J_{ut}\langle\sigma(t).\sigma(y)\rangle(J) \qquad (2.13)$$

Consider the cube $\Lambda_{L+2}(x)$ which has center at x and side $L+2$. Note that $t\in\partial\Lambda_{L+2}(x)$.

Applying (2.5 *b*) to $\langle\sigma(t).\sigma(y)\rangle$ we obtain

$$\langle\sigma(t).\sigma(y)\rangle(J) \leqq \sum_{sw\in\partial\Lambda_{L+2}^*(x)} \langle\sigma(t).\sigma(s)\rangle\beta J_{sw}\langle\sigma(w).\sigma(y)\rangle_{\partial\Lambda_{L+2}^*(x)} \qquad (2.14)$$

Thus from (2.13), (2.14) and the trivial bound $\langle \sigma(t).\sigma(s) \rangle \leqq 1$ we have

$$\langle \sigma(x).\sigma(y) \rangle (\mathrm{J}) \leqq \sum_{ut \in \partial\Lambda^*_{\mathrm{L}+2}(x)} \Big(\sum_{sw \in \partial\Lambda^*_{\mathrm{L}+2}(x)} \langle \sigma(x).\sigma(u) \rangle_{\partial\Lambda^*_{\mathrm{L}}(x)} (\mathrm{J}) \langle \sigma(w).\sigma(y) \rangle_{\partial\Lambda^*_{\mathrm{L}+2}(x)} (\mathrm{J})\, \beta^2 \mathrm{J}_{ut} \mathrm{J}_{sw} \Big) \quad (2.15)$$

observing that on the right hand side, $\langle \sigma(x).\sigma(u) \rangle_{\partial\Lambda^*_{\mathrm{L}}(x)}(\mathrm{J})$ is a function of the values of J_{xy} inside $\Lambda_{\mathrm{L}}(x)$ only and that $\langle \sigma(w).\sigma(y) \rangle_{\partial\Lambda^*_{\mathrm{L}+2}(x)}(\mathrm{J})$ is a function of the values of J_{xy} outside $\Lambda_{\mathrm{L}+4}(x)$ only, we see that the average over J_{xy} is factorized, *i.e.*

$$\overline{\langle \sigma(x).\sigma(y) \rangle} \leqq \beta^2 \sum_{u \in \partial\Lambda_{\mathrm{L}}(x)} \Big(\sum_{u \in \partial\Lambda_{\mathrm{L}}(x)} \overline{\langle \sigma(x).\sigma(u) \rangle}_{\partial\Lambda^*_{\mathrm{L}}(x)} \overline{\langle \sigma(w).\sigma(y) \rangle}_{\partial\Lambda^*_{\mathrm{L}+2}(x)} \Big) \quad (2.16)$$

Let w_0 be site $w \in \partial\Lambda_{\mathrm{L}+4}(x)$ such that $\overline{\langle \sigma(w_0).\sigma(y) \rangle}_{\partial\Lambda^*_{\mathrm{L}+2}(x)}$ is maximum. Then

$$\overline{\langle \sigma(x).\sigma(y) \rangle} \leqq \beta^2 (\bar{\mathrm{J}})^2 n(d) (\mathrm{L}+4)^{d-1} \times \sum_{u \in \partial\Lambda_{\mathrm{L}}(x)} \overline{\langle \sigma(x).\sigma(u) \rangle \langle \sigma(w_0).\sigma(y) \rangle} \leqq \delta(\beta) \overline{\langle \sigma(w_0).\sigma(y) \rangle} \quad (2.17)$$

where

$$\delta(\beta) = \beta^2 (\bar{\mathrm{J}})^2 n(d) (\mathrm{L}+4)^{d-1} \sum_{u \in \partial\Lambda_{\mathrm{L}}(x)} \overline{\langle \sigma(x).\sigma(u) \rangle}_{\partial\Lambda^*_{\mathrm{L}}(x)} \quad (2.18)$$

Note that we have used Griffith's inequality to disregard the subscript $\partial\Lambda^*_{\mathrm{L}+2}(x)$ in $\langle \sigma(w_0).\sigma(y) \rangle_{\partial\Lambda^*_{\mathrm{L}+2}(x)}$. We can iterate this estimate at least $2^{|x-y|/(\mathrm{L}+4)} - 1$ times. Also from the way that we defined the distribution of the J_{xy} variables we have translational invariance over the averaged expectations. Therefore from (2.17) we conclude that

$$\overline{\langle \sigma(x).\sigma(y) \rangle} \leqq [\delta(\beta)]^{(2|x-y|/(\mathrm{L}+4)) - 1} \quad (2.19)$$

By the hypothesis (2.11) in the Theorem 2.3 $\delta(\beta) < 1$. Then from (2.19) we have

$$\overline{\langle \sigma(x).\sigma(y) \rangle} \leqq \mathrm{C}\, e^{-|x-y|/\xi(\beta)}$$

where C is a positive constant and

$$\xi(\beta) \leqq -\frac{1}{2} \frac{\mathrm{L}+4}{\log \delta(\beta)} \quad \blacksquare$$

COROLLARY 2.5. — *For* $\beta = \beta_c$ *the two point function satisfies*

$$\overline{\langle \sigma(0).\sigma(x) \rangle} \geqq \mathrm{const.}\, x^{-d-2-\eta} \quad (2.20)$$

with $\eta \leqq d$.

Proof. — First observe that for $\beta=\beta_c$ we can not have exponential decay, because if that is so, there exists $\Lambda_L(0)$ large enough such that

$$\overline{\langle\sigma(0).\sigma(u)\rangle_{\partial\Lambda_L^*(0)}}(\beta_c)$$

satisfies the hypothesis of the Theorem 2.3 and we will have

$$\delta(\beta_c)=n(d)(L+4)^{d-1}\beta_c^2(\bar{J})\sum_{u\in\partial\Lambda_L(0)}\overline{\sigma(0).\sigma(u)\rangle_{\Lambda_L^*(u)}}(\beta_c)<1 \quad (2.21)$$

But in (2.21) $\delta(\beta_c)$ is a continuous function of β, so exist $\beta>\beta_c$ such that $\delta(\beta)$ still less than 1, and therefore we still have the exponential decay, which contradicts the definition of β_c. Finally it is easy to see that if $\eta\geqq d+\varepsilon$

$$\delta(\beta_c)\leqq\text{const.}\ \frac{n(d)\,\beta_c^2(\bar{J})^2}{L^\varepsilon}.$$

Thus the hypothesis in the Theorem 2.3 is satisfied for L big enough and therefore we have exponential decay, in contradiction to the definition of β_c. ■

Remark. — It would be interesting to strengthen the bound on η.

3. BOUND ON THE CRITICAL EXPONENT ν

In this section we prove Theorem 2.4. We shall consider in the remainder of this article the following

(i) $\text{supp}(g)\subset[0,\infty)$.

(ii) $g\in C^2(0,\infty)$, with bounded second derivative.

(iii) the model exhibits a phase transition for some finite value β_c of β.

The assumption (i) on g is to assure the ferromagnetism and (ii) is been taking for technical reasons (*see* proof of Lemma 3.5). With respect to the assumption about the existence of a finite β_c we call the attention of the reader to the case in which $\inf[\text{supp}(g)]\geqq J_0>0$. As noted in the section 2, using Griffith's inequality and well known results for uniform non random coupling constants J_{xy}, the existence of such β_c follows easily.

Given $l>0$ let $\mathscr{J}(\beta)$ be the set of configurations J of variables J_{xy} for which

$$\beta^2(\bar{J})^2\sum_{x\in\partial\Lambda_l(0)}\langle\sigma(0).\sigma(x)\rangle_{\partial\Lambda_l(0)^*}(\beta,J)\geqq\frac{1}{2(l+4)^{d-1}n(d)} \quad (3.1)$$

where

$n(d)$ is the geometric constant defined in the chapter I.

$\Lambda_l(0)$ denote the cube with center at $0\in\mathbb{Z}^d$ and site $l=2k$, $k\in\mathbb{Z}$.

$\partial\Lambda_l(0)$ and $\partial\Lambda_l^*(0)$ are as defined in section 2.

Let $\chi_l(J, \beta)$ denote the characteristic function of $\mathscr{I}(\beta)$. The measure of $\mathscr{I}(\beta)$ for $\beta \leqq \beta_c$ is

$$P_l(\beta) = \int \chi_l(J, \beta)\, d\mu(J) \tag{3.2}$$

and for $\beta < \beta_c$ we define

$$l(\beta) = \inf \left\{ l \,|\, P_l(\beta) < \frac{1}{n^2(d)\, \beta^2 (\bar{J})^2 (l+4)^{4d-4}} \right\} \tag{3.3}$$

Observe that $P_l(\beta)$ is the probability that (3.1) holds at inverse temperature β. In order to show that $l(\beta)$ is finite let us suppose the contrary, *i.e.*

$$P_l(\beta) > \frac{1}{n^2(d)\, \beta^2 (\bar{J})^2 (l+4)^{4d-4}}$$

holds for all l. Then, from (3.1)

$$\beta^2 (\bar{J})^2 \sum_{x \in \Lambda_l(0)} \overline{\langle \sigma(0) . \sigma(x) \rangle}_{\partial \Lambda_l^*(0)}(\beta) \geqq \text{const.} \frac{1}{(l+4)^{5d-5}}$$

holds for all l. Therefore for all l there exists at least one $x \in \partial \Lambda_l(0)$ such that

$$\overline{\langle \sigma(0) . \sigma(x) \rangle}(\beta) \geqq \overline{\langle \sigma(0) . \sigma(x) \rangle}_{\partial \Lambda_l^*(0)}(\beta) \geqq \text{const.} \frac{1}{\beta^2 (\bar{J})^2 |x|^{6d-6}} \tag{3.4}$$

in contradiction with the fact that for $\beta < \beta_c$, $\overline{\langle \sigma(0) . \sigma(x) \rangle}(\beta)$ decays exponentially. Note that the first inequality is a consequence of Griffith's inequality which implies that Dirichlet boundary conditions decrease correlations.

LEMMA 3.2. — *Let $\beta < \beta_c$. There exists constants c_1 and c_2 such that*

$$\frac{c_1\, l(\beta)}{\log[l(\beta)]} \leqq \xi(\beta) \leqq c_2\, l(\beta) \tag{3.5}$$

Proof. — The upper bound is an immediate consequence of Theorem 2.3 since there exists $l > c l(\beta)$ such that

$$P_l(\beta) < \frac{1}{n^2(d)\, \beta^2 (\bar{J})^2 (l+4)^{4d-4}}$$

and thus for $\Lambda_l(x)$, (3.1) and the above bound imply that the hypothesis (2.11) is satisfied.

To obtain the lower bound we use the FKG property of the measure $d\mu(J)$ *i.e.* Let F and G be non decreasing functions of J in the sense that for any two configurations J and J′ such that $J_{xy} \leqq J'_{xy}$ for all $(x, y) \in \mathbb{Z}^d$, we have $F(J) \leqq F(J')$ and $G(J) \leqq G(J')$. Then

$$\int F(J)\, d\mu(J) \int G(J)\, d\mu(J) \leqq \int F(J)\, G(J)\, d\mu(J) \tag{3.6}$$

Since for all $s, t \in \mathbb{Z}^d$, $\langle \sigma(s).\sigma(t) \rangle$ is a non decreasing function of J_{xy}, the FKG property of $d\mu(J)$ implies that

$$\begin{aligned}
&\overline{\langle \sigma(0).\sigma(z_1) \rangle_{\partial\Lambda_l^*(0)}}\, \overline{\langle \sigma(z_1).\sigma(z_2) \rangle_{\partial\Lambda_l^*(z_1)}} \\
&\qquad \times \overline{\langle \sigma(z_2).\sigma(z_3) \rangle_{\partial\Lambda_l^*(z_2)}} \ldots \overline{\langle \sigma(z_n).\sigma(x) \rangle_{\partial\Lambda_l^*(z_n)}} \\
&\leqq \overline{\langle \sigma(0).\sigma(z_1) \rangle_{\partial\Lambda_l^*(0)} \langle \sigma(z_1).\sigma(z_2) \rangle_{\partial\Lambda_l^*(z_1)} \langle \sigma(z_2).\sigma(z_3) \rangle_{\partial\Lambda_l^*(z_2)} \ldots \langle \sigma(z_n).\sigma(x) \rangle_{\partial\Lambda_l^*(z_n)}} \\
&\qquad \leqq \overline{\langle \sigma(0).\sigma(z_1) \rangle \langle \sigma(z_1).\sigma(z_2) \rangle \langle \sigma(z_2).\sigma(z_3) \rangle \ldots \langle \sigma(z_n).\sigma(x) \rangle}
\end{aligned} \tag{3.7}$$

where in the last inequality we have used Griffth's inequality to disregard the restriction on the expectation. By using the inequality $\langle \sigma(x).\sigma(y) \rangle \langle \sigma(y).\sigma(z) \rangle \leqq 2 \langle \sigma(x).\sigma(z) \rangle$ (*see* [Dr])

$$\begin{aligned}
&\overline{\langle \sigma(0).\sigma(z_1) \rangle_{\partial\Lambda_l^*(0)}}\, \overline{\langle \sigma(z_1).\sigma(z_2) \rangle_{\partial\Lambda_l^*(z_1)}}\, \overline{\langle \sigma(z_2).\sigma(z_3) \rangle_{\partial\Lambda_l^*(z_2)}} \\
&\qquad \times \ldots \times \overline{\langle \sigma(z_n).\sigma(x) \rangle_{\partial\Lambda_l^*(z_n)}} \leqq 2^n \overline{\langle \sigma(0).\sigma(x) \rangle}
\end{aligned} \tag{3.8}$$

Now from definition of $l(\beta)$, for $l = \frac{l(\beta)}{2}$ there exists $x \in \partial\Lambda_l(0)$ such that (3.4) holds. This together with (3.8) imply that for all $n \in \mathbb{Z}$, there exists $x \in \mathbb{Z}^d$ such that $|x| = nl$ and

$$\left(\frac{\text{const.}}{\beta^2 (\bar{J})^2 (l)^{6d-6}}\right)^{|x|/l} \leqq \overline{\langle \sigma(0).\sigma(x) \rangle} \tag{3.9}$$

From the definition of $\xi(\beta)$ we have

$$\left(\frac{\text{const.}}{\beta^2 (\bar{J})^2 (l(\beta)/2)^{6d-6}}\right)^{2|x|/l(\beta)} \leqq \exp\left(-\frac{|x|}{\xi(\beta)}\right) \tag{3.10}$$

Therefore there exist positive constants c and c' such that

$$(cl(\beta))^{-c'(|x|/l(\beta))} \leqq \exp\left(-\frac{|x|}{\xi(\beta)}\right)$$

The lower bound follows by taking the logarithm on both sides. ■

LEMMA 3.3. – *For all values of l the following holds*

$$P_l(\beta_c) \geqq \frac{1}{n^2(d)(\bar{J})^2 \beta_c^2\, 2(l+4)^{2d-2}} \tag{3.11}$$

Proof. – First note that

$$\beta_c^2 (\bar{J})^2 \sum_{x \in \partial\Lambda_l(0)} \overline{\langle \sigma(0).\sigma(x) \rangle}(\beta_c)$$

$$\leqq \beta_c^2 (\bar{J})^2 n(d) l^{d-1} P_l(\beta_c) + \frac{1}{2(l+4)^{d-1} n(d)} (1 - P_l(\beta_c)) \quad (3.12)$$

where we have used in the first term on the right hand side the bound $\overline{\langle \sigma(0).\sigma(x) \rangle} \leqq 1$ and for the second term we used the fact for $J \in \mathcal{J}(\beta)$,

$$\beta_c^2 (\bar{J})^2 \sum_{x \in \partial\Lambda_l(0)} \langle \sigma(0).\sigma(x) \rangle_{\partial\Lambda_l^*(0)} < \frac{1}{2(l+4)^{d-1} n(d)}$$

Now suppose that the contrary of (3.11) holds *i.e.* There exists l such that

$$P_l(\beta_c) < \frac{1}{n^2(d)(\bar{J})^2 \beta_c^2 2(l+4)^{2d-2}} \quad (3.13)$$

From (3.12) we obtain

$$\beta_c^2 (\bar{J})^2 \sum_{x \in \partial\Lambda_l(0)} \overline{\langle \sigma(0).\sigma(x) \rangle_{\partial\Lambda_l^*(0)}}(\beta_c) < \frac{1}{(l+4)^{d-1} n(d)}$$

which by the Theorem 2.3 implies in exponentiel decay. As we discussed in the proof of Corollary 2.5 this is in contradiction to the definition of β_c.

COROLLARY 3.4. – *There exists a positive constant c such that*

$$|P_{l(\beta)}(\beta_c) - P_{l(\beta)}(\beta)| \geqq \frac{c}{(l(\beta)+4)^{2d-2}}. \quad (3.14)$$

Proof. – From the definition of $l(\beta)$ and Lemma 3.3 we have

$$|P_{l(\beta)}(\beta_c) - P_{l(\beta)}(\beta)|$$

$$\geqq \frac{1}{n^2(d)(\bar{J})^2 \beta_c^2 2(l+4)^{2d-2}} - \frac{1}{n^2(d)(\bar{J})^2 \beta_c^2 (l+4)^{4d-4}} \geqq \frac{c}{(l+4)^{2d-2}} \quad \blacksquare$$

LEMMA 3.5. – *There exists positive constants c_1 and c_2 depending only on β_c and g such that*

$$|P_{l(\beta)}(\beta_c) - P_{l(\beta)}(\beta)| \leqq \frac{c_2}{(l(\beta)+4)^{2d-2}} (e^{c_1 l^d(\beta)(\beta_c-\beta)^2} - 1)^{1/2} \quad (3.15)$$

Proof. – We first observe the dependence of $\chi(J, \beta)$ comes from the Hamiltonian **H**, which is a function of the product βJ only. From this it is clear that $\chi_l(J, \beta)$ has the following property

$$\chi(J, \beta_c) = \chi\left(\frac{\beta_c J}{\beta}, \beta\right) \quad (3.16)$$

Hence

$$\int \chi(J, \beta_c)\, d\mu(J) = \int \chi(J, \beta_c) \prod_{(x, y) \in (Z^d)^2} g(J_{xy})\, dJ_{xy}$$
$$= \int \chi\left(\frac{\beta_c}{\beta} J, \beta\right) \prod_{(x, y) \in (Z^d)^2} g(J_{xy})\, dJ_{xy}$$
$$= \int \chi(J, \beta) \prod_{(x, y) \in (Z^d)^2} \frac{\beta}{\beta_c} g\left(\frac{\beta}{\beta_c} J_{xy}\right) dJ_{xy} \tag{3.17}$$

We recall that in the above expression $\chi(J, \beta)$ is a function of J_{xy} with $x, y \in \Lambda_l(0)$ only. Thus we can consider ourselves restricted to the cube $\Lambda_l(0)$ with Dirichlet boundary conditions. From (3.17) we obtain

$$P_l(\beta_c) - P_l(\beta)$$
$$= \int \chi(J, \beta) \left[\prod_{(x, y)} \frac{\beta}{\beta_c} g\left(\frac{\beta}{\beta_c} J_{xy}\right) - \prod_{(x, y)} g(J_{xy}) \right] \prod_{(x, y)} \tag{3.18}$$
$$= \int \prod_{(x, y)} \frac{(\beta/\beta_c)\, g((\beta/\beta_c) J_{xy}) - \prod_{(x, y)} g(J_{xy})}{\prod_{(x, y)} g(J_{xy})} \chi(J, \beta) \prod_{(x, y)} g(J_{xy})\, dJ_{xy} \tag{3.19}$$

By applying Schwartz inequality with respect to the measure

$$d\mu(J) = \prod_{(x, y) \in (Z^d)^2} g(J_{xy})\, dJ_{xy}$$

we obtain, for $l = l(\beta)$

$$|P_l(\beta_c) - P_l(\beta)|$$
$$\leq \left[\int \frac{\left(\prod_{(x, y)} (\beta/\beta_c)\, g(\beta/\beta_c) J_{xy}) - \prod_{(x, y)} g(J_{xy}) \right)^2}{\prod_{(x, y)} g(J_{xy})} \prod_{(x, y)} dJ_{xy} \right]^{1/2} \frac{1}{(l+3)^{2d-2}} \tag{3.20}$$

[Recall the definition of $l(\beta)$]. Using the fact that $\int g(J_{xy})\, dJ_{xy} = 1$, we obtain

$$|P_l(\beta_c) - P_t(\beta)| \leq \left[\int \frac{\prod_{(x, y)} (\beta^2/\beta_c^2)\, g^2(\beta/\beta_c) J_{xy})}{\prod_{(x, y)} g(J_{xy})} \prod_{(x, y)} d(J_{xy}) - 1 \right]^{1/2}$$
$$\times \frac{1}{(l(\beta) + 3)^{2d-2}} \tag{3.21}$$

Observe that the integrals on the right hand side of (3.21) can be factorized in the product of the integrals, resulting in

$$|P_l(\beta_c)-P_l(\beta)| \leqq \frac{1}{(l(\beta)+4)^{2d-2}} \times \left[\left(\int \frac{(\beta^2/\beta_c^2)\, g^2((\beta/\beta_c)\, J_{xy})}{g(J_{xy})}\, d(J_{xy}) \right)^{l^d(\beta)} - 1 \right]^{1/2} \quad (3.22)$$

Now (3.15) follows from (3.22) and the following bound

$$\left(\int \frac{\beta^2 g^2((\beta/\beta_c)\, J_{xy})}{\beta_c^2\, g(J_{xy})}\, dJ_{xy} \right)^{l_d} \leqq \exp(c_1 l^d (\beta_c-\beta)^2) \quad (3.23)$$

where c_1 depends on the bound for g'' [recall assumption (ii)].

To prove (3.23) we define

$$\exp(h(\beta)) = \left(\int \frac{\beta^2 g^2((\beta/\beta_c)\, J_{xy})}{\beta_c^2\, g(J_{xy})}\, dJ_{xy} \right) = f(\beta)$$

For $\beta=\beta_c$, $f(\beta_c)=1$ and thus $h(\beta_c)=0$. With the assumption (ii) on g, we can consider the Taylor development of $h(\beta)$ in powers of $(\beta_c-\beta)$ up to second order. Observing that

$$h'(\beta_c) = \frac{1}{f(\beta_c)} f'(\beta_c) = f'(\beta_c)$$

and that

$$f'(\beta) = \int \left(\frac{2\beta g^2((\beta/\beta_c)\, J_{xy}) + (2\beta^2 J_{xy}/\beta_c)\, g((\beta/\beta_c)\, J_{xy})\, g'((\beta/\beta_c)\, J_{xy})}{\beta_c^2\, g(J_{xy})} \right) dJ_{xy}$$

we obtain for $\beta=\beta_c$

$$f'(\beta_c) = \frac{2}{\beta_c} \int (g(J_{xy}) + J\, g'(J_{xy}))\, dJ_{xy} = 0.$$

This implies that

$$h(\beta) \leqq c_1 (\beta_c-\beta)^2$$

where the constant c_1 dependes upon the bound for g'' ($c_1 \leqq \max(g'')$).

The expression above implies in (3.23), which is the desired bound. ■

COROLLARY 3.6. – *Let* $l(\beta)$ *be defined as in* (3.3), *then*

$$l(\beta) \geqq \text{const.}\, (\beta_c-\beta)^{-(2/d)} \quad (3.24)$$

Proof. – From (3.14) and (3.15) we have

$$\frac{1}{(l(\beta)+4)^{2d-2}} \leqq \frac{(e^{c_1 l^d(\beta)(\beta_c-\beta)^2}-1)^{1/2}}{(l(\beta)+4)^{2d-2}}$$

or equivalently

$$2 \leqq e^{c_1 l^d(\beta)(\beta_c-\beta)^2}$$

This implies in

$$l^d(\beta)(\beta_c-\beta)^2 \geqq \text{const.} \quad \blacksquare$$

Remark. – We observe that the choice of the exponent $4d-4$ in the definition of $l(\beta)$ [*see* (3.12)] is rather arbitrary. Different choices will generate logarithmic corrections in (3.24).

Proof of Theorem 2.4. – From Lemma 3.2

$$\frac{l(\beta)}{\log l(\beta)} \leqq \text{const}\,\xi(\beta)$$

hence

$$\log l(\beta) - \log\log l(\beta) \leqq \log \text{const.} + \log \xi(\beta)$$

Therefore there exists a positive constant c such that

$$\log l(\beta) \leqq c \log \xi(\beta).$$

Thus, by using again the lower bound from Lemma 3.2 we obtain

$$c\,\xi(\beta)\log\xi(\beta) \geqq \xi(\beta)\log l(\beta) \geqq \text{const.}\, l(\beta)$$

Finally from the result of Corollary 3.6 we obtain

$$\xi(\beta)\,\log\,\xi(\beta) \geqq \text{const.}\,(\beta_c-\beta)^{-2/d} \quad \blacksquare$$

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