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Classical limit of a quantum particle in an external Yang-Mills field

by

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ABSTRACT. — It is studied the classical limit of a quantum particle in an external non-abelian gauge field. It is shown that the unitary group describing the quantum fluctuations around any classic phase orbit has a classical limit when h tends to zero under very general conditions on the potentials. It is also proved the self-adjointness of the Hamilton's operator of the quantum theory for a large class of potentials. Some applications of the theory are finally exposed.

RÉSUMÉ. — On étudie la limite classique d'une particule quantique dans un champ extérieur non abelien. On démontre que l'existence de la limite classique du groupe unitaire qui décrit les fluctuations quantiques autour d'une orbite classique quelconque est assurée par des conditions peu restrictives sur les potentiels. On démontre aussi que l'opérateur hamiltonien de la théorie quantique est auto-adjoint pour une large classe de potentiels. On présente enfin quelques applications de cette théorie.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

Wong's equations are known to be the classical evolution equations of a particle moving in an external Yang-Mills field ([1], [2]); they were

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obtained as the formal limit for h tending to zero of the corresponding quantum equations of motion, where h is the Planck's constant.

The aim of this paper is to give a rigorous proof of this correspondence under very general hypotheses.

In this introduction the formulations of both the classical and quantum theories are stated and the ideas underlying the classical limit procedure are explained.

Let $\mathfrak{G}$ be a semisimple and compact Lie group and $\mathfrak{g}$ the associated Lie algebra, with commutation rules

$$[e_\alpha, e_\beta] = f^\gamma_{\alpha\beta} e_\gamma \tag{1.1}$$

where $\{e_\alpha\}$ is a basis for $\mathfrak{g}$ (summation over repeated indices is assumed hereafter unless otherwise specified).

In several previous papers on this and related subjects ([3], [4] and references therein) the main idea to obtain the classical limit of the quantum theory consists in letting the dimension m of the representation of the group tend to infinity in such a way that $m\,h = \text{const}$, when h tends to zero.

More precisely one chooses an irreducible representation π_l parametrized by its highest weight l and then to each natural number associates the irreducible representation π_{nl}, whose maximal weight is nl. The classical limit may be obtained in this way by letting h tend to zero as n^{-1}.

The classical Wong equations then include as a dynamical variable a classical isospin, which is a point of Γ_l the coadjoint orbit through l. This approach has however the disadvantage that it is necessary to change at each step of the limit procedure the Hilbert space of the quantum theory.

A different strategy is used here: the representation of $\mathfrak{g}$ is chosen infinite dimensional from the beginning; it is constructed in the most convenient way for to perform the classical limit in the spirit of Hepp [6], *i. e.* by making use of the bosonic creation and annihilation operators a^+ and a.

Consider to this end the Jordan-Schwinger transformation [5]: let X be an $m \times m$ matrix and let $\mathfrak{H}_m$ be the Hilbert space corresponding to m cinematically independent bosons; a bosonic operator $\mathrm{L}(\mathrm{X})$ representing X on $\mathfrak{H}_m$ is defined by

$$\mathrm{L}(\mathrm{X}) = a_i^+ \mathrm{X}_{ij} a_j. \tag{1.2}$$

Let now $\{\mathrm{T}\}$ be a faithful representation of the algebra $\mathfrak{g}$ on a m-dimensional vector space V_m.

$\{\mathrm{L}(\mathrm{T})\}$ constitutes a representation of $\mathfrak{g}$ on $\mathfrak{H}_m$ because

$$[\mathrm{L}(\mathrm{X}), \mathrm{L}(\mathrm{Y})] = \mathrm{L}([\mathrm{X}, \mathrm{Y}]) \tag{1.3}$$
$$\mathrm{L}(\lambda \mathrm{X} + \mu \mathrm{Y}) = \lambda \mathrm{L}(\mathrm{X}) + \mu \mathrm{L}(\mathrm{Y}) \tag{1.4}$$

It is obvious that $\{L(T)\}$ is a reducible representation of $\mathfrak{g}$ and that it contains a lot of the finite dimensional irreducible representations of $\mathfrak{g}$. To see this it suffices to note that $\mathfrak{H}_m$ may be rewritten as the direct sum of the eigenspaces of a quantum m-dimensional oscillator [*see* (4.3)]; by construction $\{L(T)\}$ leaves each of these spaces invariant and therefore the restriction of $\{L(T)\}$ to any of them defines a finite dimensional representation which is either irreducible or completely reducible. To say precisely which irreducible representations of $\mathfrak{g}$ are contained in $\{L(T)\}$ further information about $\{T\}$ is needed. For example consider the following realization of a basis of (complex) su (2):

$$J_+=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad J=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad J_3=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

the previous construction yields an infinite dimensional representation which contains each finite dimensional irreducible representation exactly once [5].

Now it is possible to write the quantum Hamiltonian of a particle moving in an external Yang-Mills field; to this end consider the following potentials:

$$A_j(x, t)\colon\ R^n \times R \to \mathfrak{g} \qquad \text{with} \quad 1 \leqq j \leqq n,$$
$$A_0(x, t)\colon\ R^n \times R \to \mathfrak{g} \qquad \text{and} \qquad V(x, t)\colon R^n \times R \to R.$$

The minimal coupling prescription leads to the following Hamilton operator (the mass of the particle is taken equal to one):

$$H=\frac{1}{2}(hp_j - ih \langle a, A_j(q, t)\, a \rangle)^2 + ih < a, A_0(q, t)\, a \rangle + V(q, t) \quad (1.5)$$

where

$$\langle a, A_j a \rangle = A_j^\mu (T_\mu)_{kl}\, a_k^+\, a_l \quad (1.6)$$

means the representation of the potentials A on the space $\mathfrak{H}_m$.

The operators q and p are the usual position and momentum operators. The potentials A_j account for the "magnetic" part of the Yang-Mills interaction while A_0 is responsible for the "electric" part. V describes an interaction that does not involve the internal degrees of freedom of the particle.

H operates on a Hilbert space of the form $\mathfrak{H}_s \otimes \mathfrak{H}_m$. $\mathfrak{H}_s$ corresponds to the spatial degrees of freedom of the particle while $\mathfrak{H}_m$ to the internal ones.

Clearly q and p operate on $\mathfrak{H}_s$ while a and a^+ operate on $\mathfrak{H}_m$. The last two operators should not be confused with

$$b_j = 2^{-1/2}(q_j + ip_j), \qquad b_j^+ = 2^{-1/2}(q_j - ip_j) \quad (1.7)$$

The only non zero commutators are

$$[q_j, p_k] = i\,\delta_{jk} \tag{1.8}$$
$$[a_j, a_k^+] = \delta_{jk} \tag{1.9}$$

Let z represent any of the operators q, p, a, a^+. In order to perform the classical limit one needs a representation of the canonical commutation rules symmetric with respect to h; to this end define [6]

$$z_h = h^{1/2} z \tag{1.10}$$

The following hamiltonian is obtained from (1.5) by a unitary scale transformation:

$$\mathrm{H}_h(z_h) = \frac{1}{2}(p_{hj} - i\langle a_h, \mathrm{A}_j(q_h, t)\, a_h\rangle)^2 + i\langle a_h, \mathrm{A}_0(q_h, t)\, a_h\rangle + \mathrm{V}(q_h, t) \tag{1.11}$$

Let now ξ_j, π_j, ϑ_k and ϑ_k^* be the classical variables corresponding to the operators q_{jh}, p_{jh}, a_{kh}, and a_{kh}^+, where the star means complex conjugation; the symbol ζ will be adopted to denote any of these classical variables. Bohr's correspondence principle implies that $\mathrm{H}_h(z_h)$ has the same form of the classical Hamiltonian $\mathrm{H}(\zeta)$ and so it follows that:

$$\mathrm{H}(\zeta) = \frac{1}{2}(\pi_j - i\langle \vartheta, \mathrm{A}_j(\xi, t)\,\vartheta\rangle)^2 + i\langle \vartheta, \mathrm{A}_0(\xi, t)\,\vartheta\rangle + \mathrm{V}(\xi, t) \tag{1.12}$$

(here the correspondence principle has been so to say inverted).

The only nonzero Poisson's brackets are:

$$\{\xi_j, \pi_k\} = \delta_{jk} \tag{1.13}$$
$$\{\vartheta_k, \vartheta_l^*\} = -i\,\delta_{kl} \tag{1.14}$$

The hamilton's equations easily follow:

$$\dot{\xi}_j = \pi_j - i\langle \vartheta, \mathrm{A}_j(\xi, t)\,\vartheta\rangle \tag{1.15}$$
$$\dot{\pi}_j = i\,\dot{\xi}_k\left\langle \vartheta, \frac{\partial \mathrm{A}_k}{\partial \xi_j}(\xi, t)\,\vartheta\right\rangle - i\left\langle \vartheta, \frac{\partial \mathrm{A}_0}{\partial \xi_j}(\xi, t)\,\vartheta\right\rangle - \frac{\partial \mathrm{V}}{\partial \xi_j}(\xi, t) \tag{1.16}$$
$$\dot{\vartheta}_a = -\dot{\xi}_k(\mathrm{T}_\mu)_{ab}\,\vartheta_b\,\mathrm{A}_k^\mu(\xi, t) + (\mathrm{T}_\mu)_{ab}\,\vartheta_b\,\mathrm{A}_0^\mu(\xi, t) \tag{1.17}$$

Equations analogous to these may be found in [7].

The definition of the classical isospin variable is given by

$$\mathrm{I}_\mu = i\,\vartheta_a^*(\mathrm{T}_\mu)_{ab}\,\vartheta_b \tag{1.18}$$

The equation of motion that one gets for the isospin variable are exactly those given by Wong:

$$\dot{\mathrm{I}}_\alpha + \dot{\xi}_k f_{\alpha\beta}^\gamma\,\mathrm{A}_k^\beta(\xi, t)\,\mathrm{I}_\gamma - f_{\alpha\beta}^\gamma\,\mathrm{A}_0^\beta(\xi, t)\,\mathrm{I}_\gamma = 0 \tag{1.19}$$

It has been displayed a richer structure underlying the Yang-Mills theory (the variables ϑ); this structure will be explicitly used to perform the classical limit in a way that avoids the use of the group-theoretical properties that are so crucial in the previous treatments [3]. Clearly more degrees of freedom have been used than those that are necessary to describe the classical isospin; however it is possible to recover in a simple way the minimal phase spaces $R^{2n}\times\Gamma$, where Γ is a certain coadjoint orbit of the group G: indeed one may construct a map $l:\ R^{2n}\times R^{2m}\to R^{2n}\times\mathfrak{g}^*$

($\mathfrak{g}^*$ is the dual of the algebra $\mathfrak{g}$) whose definition is the following:

$$l(\xi,\pi,\vartheta)=(\xi,\pi,-i\langle\vartheta,T_\mu\vartheta\rangle e'^{\mu}) \tag{1.20}$$

$\{e'^{\mu}\}$ is the dual of the previously chosen basis of $\mathfrak{g}$. It is easy to check that the Poisson brackets defined by (1.14) are nothing but the pull-back through l of the well-known Lie-Poisson brackets on $\mathfrak{g}^*$ [8]. The explicit expression of the Hamiltonian then implies that it is possible to limit oneself to consider as dynamical variables ξ, π and $l(\vartheta)$. The construction of Kirillov [9] finally gives $R^{2n}\times\Gamma_{l_0}$ as the phase space in which one can describe the classical system. In particular one derives directly equations (1.19) without having to pass through (1.17), which however remains hidden in this way; this does not seem to be an advantage because the use of the redundant degrees of freedom described by the ϑ variables greatly simplifies the classical limit procedure and may be useful in other problems related to this.

It is time to explain briefly what is the meaning of the words "classical limit" ([3], [4], [6], [10], [11], [12]). It is commonly said that classical mechanics is the limit of quantum mechanics when h tends to zero but a concrete mathematical definition of the limit procedure is needed. The most significant one is the following: let A_h a h-dependent quantum operator representing a certain observable whose classical expression is the phase space function $a(\zeta)$. What one has to do is to select certain quantum states $|h,\zeta\rangle$ depending on h such that

$$\lim_{h\to 0}(A_h)_{|h,\zeta\rangle}\to a(\zeta) \tag{1.21}$$

$$\lim_{h\to 0}(\Delta A_h)_{|h,\zeta\rangle}\to 0 \tag{1.22}$$

where the following definitions have been adopted:

$$(B)_{|\psi\rangle}=\langle\psi|B\psi\rangle \tag{1.23}$$

$$(\Delta B)_{|\psi\rangle}=\langle\psi|[B-(B)_{|\psi\rangle}]^2\psi\rangle^{1/2} \tag{1.24}$$

The symbol $\langle\ |\ \rangle$ denotes as usual the Hilbert space scalar product.

The only approrpiate quantum states for doing this are known as "coherent states", and are the states that minimize the uncertainity relations ([13], [14]).

Weyl's operators are crucial in the construction of the coherent states; their definition is the following:

$$C(\zeta)=\exp[i(\pi q-\xi p)+\vartheta a^+-\vartheta^* a] \tag{1.25}$$

with $\pi q = \pi_j q_j$ etc. It is well known that

$$C(\zeta)^+ z C(\zeta) = z + \zeta \tag{1.26}$$

The coherent states that will be used in the following are given by

$$|h, \zeta\rangle = C(h^{-1/2}\zeta)|\Psi\rangle = C_h(\zeta)|\psi\rangle \tag{1.27}$$

with ψ any normalized quantum state.

Let $s \to \zeta_s$ be a solution of the canonical equations with given initial conditions. The quantum time evolution operator is given by

$$U_h(t, s) = \exp i h^{-1} [(t-s) H_h(z_h)] \tag{1.28}$$

and the Heisenberg operators are

$$z_{jh}(t, s) = U_h^+(t, s) z_{jh} U_h(t, s) \tag{1.29}$$

The mean value of the operator $z_{jh}(t, s)$ on the state $|h, \zeta_s\rangle$ is then expected to converge to the solution of the canonical equations with the given initial conditions. Indeed one has that:

$$(z_{jh}(t . s))_{|h, \zeta_s\rangle} = \zeta_j(t) + (W_h(t, s)^+ z_{jh} W_h(t, s))_{|\psi\rangle} \tag{1.30}$$

$$(\Delta z_{jh}(t, s))_{|h, \zeta_s\rangle} = (W_h(t, s) [z_{jh} + \zeta_{jt} - (z_{jh}(t, s))_{|h, \zeta_s\rangle}]^2 W_h(t, s))_{|\psi\rangle} \tag{1.31}$$

where

$$W_h(t, s) = C_h(\zeta_t)^+ U_h(t, s) C_h(\zeta_s) \exp(-i\omega_h(t, s)) \tag{1.32}$$

$$\omega_h = h^{-1} \int_s^t \frac{1}{2} [\zeta_\tau \partial_\zeta H(\zeta_\tau) - H(\zeta_\tau)] d\tau \tag{1.33}$$

The operator $W_h(t, s)$ is the main mathematical object of this paper. It describes the evolution of the quantum fluctuations around the classical phase orbit. It will be shown that:

$$\frac{d}{ds} W_h(t, s) = i h^{-1} W_h(t, s) K_h(s) \tag{1.34}$$

$$\frac{d}{dt} W_h(t, s) = -i h^{-1} K_h(t) W_h(t, s) \tag{1.35}$$

where

$$K_h(r) = H(z_h + \zeta_r) - H(\zeta_r) - \partial_\zeta H(\zeta_r) z_h \tag{1.36}$$

The solution of the operator equations (1.33), (1.34) can be written

$$W_h(t, s) = T \exp\left[-i h^{-1} \int_s^t K_h(r) dr\right] \tag{1.37}$$

Define

$$h^{-1} K_h(r) = H_2(r) + R_h(r) \tag{1.38}$$

The operator $R_h(r)$ is a remainder of order $h^{1/2}$.

If the potentials are smooth enough it will happen that $R_h \to 0$ when $h \to 0$. Indeed if one defines

$$U_2(t,s)=T\exp\left(-i\int_s^t H_2(r)\,dr\right) \tag{1.39}$$

then it follows that

$$\lim_{h\to 0} W_h(t,s)=U_2(t,s) \tag{1.40}$$

and from (1.28) and (1.29) one obtains

$$\lim_{h\to 0} (z_{jh}(t,s))_{|h,\zeta_s\rangle} \to \zeta_{jt} \tag{1.41}$$

$$\lim_{h\to 0} (\Delta z_{jh}(t,s))_{|h,\zeta_s\rangle} \to 0 \tag{1.42}$$

The reader is referred to [6] and [10] for further material about the classical limit. The main results of this paper are concerning the selfadjointness of the operator (1.11) and the properties of the unitary group of operators (1.32). In particular the limit (1.40) will be rigorously proved while the proofs of (1.41) and (1.42) will not be reproduced here.

2. NOTATIONS AND GENERAL RESULTS

Let $\mathfrak{H}$ be a Hilbert space with inner product $\langle\ |\ \rangle_{\mathfrak{H}}$ and norm $\|\ \|_{\mathfrak{H}}$.

Let A be a linear operator in $\mathfrak{H}$ and denote its domain with D(A).

If A is closable $\bar{A}$ denotes its closure. If A is densely defined A^+ denotes the adjoint of A.

The set of bounded operators from $\mathfrak{H}$ to another Hilbert space $\mathfrak{H}'$ is denoted by the symbol $B(\mathfrak{H},\mathfrak{H}')$ and the corresponding operator norm is denoted by $\|\ \|_{\mathfrak{H},\mathfrak{H}'}$.

Let A be a self-adjoint and positive operator; it is possible to associate a scale of Hilbert spaces to A ([15], [16], [17]):

let τ be a real number; one defines with the help of the spectral theorem the operator

$$A_\tau=(1+A)^\tau \tag{2.1}$$

The Hilbert space

$$\mathfrak{H}_\tau=\text{completion of}\,(D(A_\tau),\langle\ |\ \rangle_{\mathfrak{H}_\tau}) \tag{2.2}$$

is defined by setting

$$\langle\varphi|\psi\rangle_{\mathfrak{H}_\tau}=\langle A_\tau\varphi|A_\tau\psi\rangle,\qquad \varphi,\psi\in D(A_\tau) \tag{2.3}$$

If $\tau_1 \geqq \tau_2$ then $\mathfrak{H}_{\tau_1} \subseteq \mathfrak{H}_{\tau_2}$ densely. (2.4)

Consider now the Hilbert space $L^2(\mathbb{R}^p)$ and define formally the operators

$$q_j \psi(x) = x_j \psi(x),$$
$$p_j \psi(x) = -i \frac{\partial}{\partial x_j} \psi(x), \qquad j=1, \ldots, p \tag{2.5}$$

These operators are self-adjoint on the domains

$$D(q_j) = \{ \psi \in L^2(\mathbb{R}^p) : q_j \psi \in L^2(\mathbb{R}^p) \} \tag{2.6}$$
$$D(p_j) = \{ \psi \in L^2(\mathbb{R}^p) : p_j \psi \in L^2(\mathbb{R}^p) \} \tag{2.7}$$

The operators b_j and b_j^+ defined in (1.7) are closed operators defined on the domain $D(b_j) = D(b_j^+) = D(q_j) \cap D(p_j)$.

Fix the index j and define

$$N_j = b_j^+ b_j = \frac{1}{2}(q_j^2 + p_j^2) - \frac{1}{2} \tag{2.8}$$

This operator is self-adjoint and positive on the domain

$$D(N_j) = \{ \psi \in L^2(\mathbb{R}^p) : q_j^2 \psi \text{ and } p_j^2 \psi \in L^2(\mathbb{R}^p) \} \tag{2.9}$$

Finally define

$$|q| = (\sum q_j^2)^{1/2}, \qquad |p| = (\sum p_j^2)^{1/2}, \qquad N = \sum N_j \tag{2.10}$$

These operators are self-adjoint and positive on the domains

$$D(|q|) = \{ \psi \in L^2(\mathbb{R}^p) : q_j \psi \in L^2(\mathbb{R}^p), j=1, \ldots, p \} \tag{2.11}$$
$$D(|p|) = \{ \psi \in L^2(\mathbb{R}^p) : p_j \psi \in L^2(\mathbb{R}^p), j=1, \ldots, p \} \tag{2.12}$$
$$D(N) = \{ \psi \in L^2(\mathbb{R}^p) : q_j^2 \psi \text{ and } p_j^2 \psi \in L^2(\mathbb{R}^p), j=1, \ldots, p \} \tag{2.13}$$

The scales of Hilbert spaces associated with $|q|$, $|p|$ and N are denoted respectively with $\{Q_\tau\}_{\tau \in \mathbb{R}}$, $\{P_\tau\}_{\tau \in \mathbb{R}}$, $\{X_\tau\}_{\tau \in \mathbb{R}}$.

Some remarkable properties of these scales are listed here:

$$\left(\prod_{j=1}^{m} z_j \right) \in B(X_{\tau+(m/2)} X_\tau) \quad \text{for each real } \tau \tag{2.14}$$

with z any of the q, p, b, b^+.

$$X_\tau \subset P_{2\tau}, \qquad X_\tau \subset Q_{2\tau} \tag{2.15}$$
$$X_{1/2} = Q_1 \cap P_1, \qquad X_1 = Q_2 \cap P_2 \tag{2.16}$$

In the introduction all the interesting quantities were written in a mixed form by making use of the operators position and momentum to describe the spatial degrees of freedom of the particle and of the operators creation and annihilation to describe the internal ones. Define now the $(n+m)$-vector operators $\tilde{q}$ and $\tilde{p}$ by

$$\tilde{q}_j=\left\{\begin{array}{ll} q_j, & 1\leqq j\leqq n \\ 2^{-1/2}(a_j+a_j^+), & n+1\leqq j\leqq n+m \end{array}\right\} \tag{2.17}$$

$$\tilde{p}_j=\left\{\begin{array}{ll} p_j, & 1\leqq j\leqq n \\ -i\,2^{-1/2}(a_j-a_j^+), & n+1\leqq j\leqq n+m \end{array}\right\} \tag{2.18}$$

with analogous definitions for their classical counterparts.

Everything can now be rephrased making use of these generalized position and momentum operators $\tilde{q}$ and $\tilde{p}$; analogously one could use generalized creation and annihilation operators $\tilde{b}$ and $\tilde{b}^+$. The scales of spaces that will be used in the following are those associated with $|\tilde{q}|$, $|\tilde{p}|$ and $\tilde{X}$; these operators are those of (2.11) (2.12) (2.13) with $p=n+m$.

Some well known facts about the Weyl operators are now listed; note that (1.23) is equivalent to the following expressions:

$$C(\tilde{\xi},\tilde{\pi})=\exp\left[i\sum_{j=1}^{n+m}(\tilde{\pi}_j\tilde{q}_j-\tilde{\xi}_j\tilde{p}_j)\right] \tag{2.19}$$

$$C(\tilde{\vartheta})=\exp\left[\sum_{j=1}^{n+m}(\tilde{\vartheta}_j\tilde{b}_j^+-\tilde{\vartheta}_j^*\tilde{b}_j)\right] \tag{2.20}$$

THEOREM 2.1:

$$C(\zeta)=C(\tilde{\xi},\tilde{\pi})=C(\tilde{\vartheta}) \tag{2.21}$$

$$C(\tilde{\xi},\tilde{\pi})\quad \textit{is a } L^2\textit{-unitary operator} \tag{2.22}$$

$$C(\tilde{\xi},\tilde{\pi})=\exp(i\tilde{\pi}\tilde{q})\exp(-i\tilde{\xi}\tilde{p})\exp(-i\tilde{\xi}\tilde{\pi}/2) \tag{2.23}$$

$$[C(\tilde{\xi},\tilde{\pi})\psi](\tilde{x})=[\exp(-i\tilde{\xi}\tilde{\pi}/2+i\tilde{\pi}\tilde{q})]\psi(\tilde{x}-\tilde{\xi}) \tag{2.24}$$

$C(\tilde{\xi},\tilde{\pi})$ *is* L^2*-strongly continuous in the* $2n+2m$ *arguments jointly* (2.25)

$C(\tilde{\xi},\tilde{\pi})$ *belongs to each of the sets* $B(X_\tau, X_\tau)$, $B(Q_\tau, Q_\tau)$, $B(P_\tau, P_\tau)$; *besides it is jointly strongly continuous*

on the spaces Q_τ, P_τ, X_τ *for each real* τ. (2.26)

If $\psi\in X_{1/2}$, *and the function* $t\to\zeta_t$ *is differentiable (in some real interval* I) *then*

$$\frac{d}{dt}C(\zeta_t)\psi=i\,C(\zeta_t)\,[(x+\xi_t/2)\,\dot{\pi}_t-(p+\pi_t/2)\,\dot{\xi}_t+i(a+\vartheta_t/2)\,\dot{\vartheta}_t^*-i(a^++\vartheta_t^*/2)\,\dot{\vartheta}_t]\,\psi \tag{2.27}$$

3. THE CLASSICAL LIMIT OF THE UNITARY GROUP W_h(t. s.)

Consider the canonical equations (1.15), (1.16), (1.17). There exists a local solution of these equations if the potentials fulfil the usual Lipschitz conditions. This solution is defined on a real interval I and is denoted by ζ_t (in the following the dependence of the solution of the canonical equations on the initial conditions is left implicit).

HYPOTHESES 3.1 (on the potentials).

For each $t \in \mathrm{I}$ it is given a positive number ρ_t; it is required that

(*a*) $$\rho_{\mathrm{K}} = \inf_{t \in \mathrm{K}} \rho_t > 0, \quad \text{where } \mathrm{K} \subset \mathrm{I} \text{ is a compact} \tag{3.1}$$

(*b*) The potentials $\mathrm{A}_{j\mu}(.,t)$ and $\mathrm{V}(.,t)$ are differentiable twice in the set $\mathfrak{U}_t = \mathrm{S}(\xi_t, \rho_t)$, (3.2)

with $\mathrm{S}(x, r) = \{ y \in \mathrm{R}^n, \| x - y \| < r \}$.

(*c*) The functions $\mathrm{A}_{j\mu}$, $\partial_{\xi_k} \mathrm{A}_{j\mu}$, $\partial^2_{\xi_k \xi_l} \mathrm{A}$, V, $\partial_{\xi_k} \mathrm{V}$, $\partial^2_{\xi_k \xi_l} \mathrm{V}$ are continuous on the set $\hat{\mathfrak{U}}_{\mathrm{I}}$ (3.3)

with $\hat{\mathfrak{U}}_{\mathrm{I}} = \text{closure of } \{ \bigcup_{t \in \mathrm{I}} \mathfrak{U}_t x \{ t \} \}$.

It is easy to verify that the hypotheses 3.1 are equivalent to the hypotheses 3.1′ in the important case in which the potentials do not depend explicitely on time:

HYPOTHESES 3.1′.

(*a*) For each $t \in \mathrm{I}$ it is given a positive number r_t such that the potentials $\mathrm{A}_{j\mu}(.)$ and $\mathrm{V}(.)$ belong to $\mathscr{C}^2(\mathrm{S}(\xi_t, r_t))$, the class of twice continuously differentiable functions of the set $\mathrm{S}(\xi_t, r_t)$ (3.4)

The reason for these hypotheses is that in order to prove the limit (1.38) it will be necessary to make a Taylor's expansion of the potentials up to the second order in a neighborhood of the classical ξ-orbit. Define now

$$\mathfrak{D}(\alpha)\, \psi(x) = \psi(\alpha x), \quad \alpha \text{ not zero} \tag{3.5}$$

$\mathfrak{D}(\alpha)$ is the operator of the dilations and is continuous.

$$\mathrm{D}_t = \left\{ \psi \in \tilde{\mathrm{X}}_2 : \psi = \sum_{j=1}^{k} f_j(x)\, g_j(y),\ k < \infty, \right.$$
$$\left. f_j \in \mathfrak{D}(h^{1/2})\, \mathrm{L}^2(\mathfrak{U}_t),\ g_j \in \mathrm{L}^2(\mathrm{R}^m),\ \operatorname{supp} g_j \text{ is a compact} \right\} \tag{3.6}$$

The following hypotheses will ensure the existence of the quantum theory:

HYPOTHESES 3.2 (on the operators).

It is given a family of self-adjoint operators depending on the parameter h denoted with $\{ \mathrm{H}_h(z_h, t),\ t \in \mathrm{I},\ h > 0 \}$. (3.7)

It is required that:

(*a*) $$\mathrm{D}(\mathrm{H}_h(z_h, t)) \supseteq \mathrm{D}_t \tag{3.8}$$

(*b*) For each $\psi \in D_t$ one has that

$$H_h(z_h, t)\psi = 1/2\{p_{jh}^2 \psi - i\langle a_h, A_j(q_h, t) a_h\rangle p_{jh}\psi - i p_{jh}\langle a_h, A_j(q_h, t) a_h\rangle \psi - \langle a_h, A_j(q_h, t) a_h\rangle^2 \psi\} + i\langle a_h, A_0(q_h, t) a_h\rangle \psi + V(q_h, t)\}\psi \quad (3.9)$$

(*c*) There exists a L^2-strongly continuous group of L^2-unitary operators denoted with $U_h(t, s)$ such that

$$\frac{d}{ds} U_h(t, s)\psi = \frac{i}{h} U_h(t, s) H_h(z_h, s)\psi, \qquad \psi \in D_s, \quad s \in I, \qquad (3.10)$$

Let $W_h(t, s)$ defined as in (1.32) with t and s belonging to I and $U_h(t, s)$ defined in (3.10) with $h>0$. It holds the following

THEOREM 3.3. — *$W_h(t, s)$ is a L^2-strongly continuous group of L^2-unitary operators.*

Proof. — This is easily obtained from theorem 2.1, the continuity of $\omega_h(t, s)$ and the hypothesis (3.10). ##

Consider now the operator $K_h(s)$ of (1.36). Define its domain as follows:

$$D(K_h(s)) = \left\{ \psi \in \bar{X}_2 : \psi = \sum_{j=1}^{k} f_j(x) g_j(y), k < \infty, \right. \\ \left. f_j \in \mathfrak{D}(h^{1/2}) L^2(\mathfrak{U}_s - \xi_s) g_j \in L^2(R^m), \operatorname{supp} g_j \text{ is compact} \right\} \quad (3.11)$$

On $D(K_h(s))$ one has:

$$K_h(s)\psi = \frac{1}{2}\{p_{jh}^2\psi - i\Delta_s(\langle \vartheta, A_j(\xi, t)\vartheta\rangle) p_{jh}\psi - i p_{jh}\Delta_s(\langle \vartheta, A_j(\xi, t)\vartheta\rangle)\psi - \Delta_s^1(\langle \vartheta, A_j(\xi, t)\vartheta\rangle^2)\psi\} - i\Delta_s^1(\langle \vartheta, (\pi_j A_j(\xi, t) - A_0(\xi, t))\vartheta\rangle)\psi + \Delta_s^1(V(\xi))\psi \quad (3.12)$$

The following definitions have been adopted:

$$\Delta_s(f(\xi, \vartheta, \vartheta^*, t)) = f(q_h + \xi_s, a_h + \vartheta_s, a_h^+ + \vartheta_s^*, s) - f(\xi_s, \vartheta_s, \vartheta_s^*, s) \quad (3.13)$$

$$\Delta_s^1(f(\xi, \vartheta, \vartheta^*, t)) = \Delta_s(f(\xi, \vartheta, \vartheta^*, t)) - \partial_{\xi_j} f(\xi_s, \vartheta_s, \vartheta_s^*, s) q_{jh} - \partial_{\vartheta_l} f(\xi_s, \vartheta_s, \vartheta_s^*, s) a_{lh} - \partial_{\vartheta_l^*} f(\xi_s, \vartheta_s, \vartheta_s^*, s) a_{lh}^+ \quad (3.14)$$

LEMMA 3.4. — *If $\psi \in D(K_h(s))$ with $s \in I$, then*

$$\frac{d}{ds} W_h(t, s)\psi = \frac{i}{h} W_h(t, s) K_h(s)\psi \qquad (3.15)$$

Proof. — From theorem 2.1 it follows that

$$C_h(\zeta_s)\psi \in D_s \subseteq D(H_h(z_h, s)) \qquad (3.16)$$

The identity

$$H_h(z_h, s) C_h(\zeta_s)\psi = C_h(\zeta_s) C_h(\zeta_s)^+ H_h(z_h, s) C_h(\zeta_s)\psi \qquad (3.17)$$

implies that

$$\mathrm{H}_h(z_h, s)\,\mathrm{C}_h(\zeta_s)\,\psi = \mathrm{C}_h(\zeta_s)\,\mathrm{H}_h(z_h+\zeta_s, s)\,\psi \tag{3.18}$$

$$\begin{aligned}\frac{1}{\delta}[\mathrm{U}_h(t, s+\delta)\,\mathrm{C}_h(\zeta_{s+\delta})\exp(-i\,\omega_h(t, s+\delta))\\ -\mathrm{U}_h(t, s)\,\mathrm{C}_h(\zeta_s)\exp(-i\,\omega_h(t, s))]\,\psi\\ = \frac{1}{\delta}[\mathrm{U}_h(t, s+\delta)-\mathrm{U}_h(t, s)]\,\mathrm{C}_h(\zeta_s)\exp(-i\,\omega_h(t, s)\,\psi\\ +\frac{1}{\delta}\mathrm{U}_h(t, s+\delta)\,[\mathrm{C}_h(\zeta_{s+\delta})-\mathrm{C}_h(\zeta_s)]\exp(-i\,\omega_h(t, s))\,\psi\\ +\frac{1}{\delta}\mathrm{U}_h(t, s+\delta)\,\mathrm{C}_h(\zeta_{s+\delta})\,[\exp(-i\,\omega_h(t, s+\delta))-\exp(-i\,\omega_h(t, s))]\,\psi\end{aligned} \tag{3.19}$$

From this one easily gets the result by exploiting the hypotheses 3.2 and (2.27). ##

Consider now the operator $\mathrm{H}_2(s)$ introduced in (1.38). Its explicit expression is the following:

$$\begin{aligned}\mathrm{H}_2(s) = \frac{1}{2}p_j^2 - i\langle\langle a, \mathrm{A}_j(\xi_s, s)\,\vartheta_s\rangle\rangle\, p_j\\ -\frac{i}{2}\langle \vartheta_s, \partial_{\xi_k}\mathrm{A}_j(\xi_s, s)\,\vartheta_s\rangle\,(q_k p_j + p_j q_k)\\ -\langle a, \mathrm{A}_j(\xi_s, s)\,a\rangle\langle \vartheta_s, \mathrm{A}_j(\xi, s)\,\vartheta_s\rangle - \frac{1}{2}\langle\langle a, \mathrm{A}_j(\xi, s)\,\vartheta_s\rangle\rangle^2\\ -\partial_\xi(\langle\langle a, \mathrm{A}_j(\xi_s, s)\,\vartheta_s\rangle\rangle\langle \vartheta_s, \mathrm{A}_j(\xi_s, s)\,\vartheta_s\rangle)\,q_k\\ -\frac{1}{4}\partial^2_{\xi_k\xi_l}\langle \vartheta_s, \mathrm{A}_j(\xi_s, s)\,\vartheta_s\rangle^2\, q_k q_l - i\langle a, (\pi_{js}\mathrm{A}_j-\mathrm{A}_0)\,(\xi_s, s)\,a\rangle\\ -i\langle\langle a, \partial_{\xi_k}(\pi_{js}\mathrm{A}_j-\mathrm{A}_0)\,(\xi_s, s)\,\vartheta_s\rangle\rangle\, q_k - \frac{i}{2}\langle \vartheta_s, \partial^2_{\xi_k\xi_l}(\pi_{js}\mathrm{A}_j-\mathrm{A}_0)\,(\xi_s, s)\,\vartheta_s\rangle\, q_k q_l\\ +\frac{1}{2}\partial^2_{\xi_k\xi_l}\mathrm{V}(\xi_s, s)\, q_k q_l\end{aligned} \tag{3.20}$$

with

$$\langle\langle a, \mathrm{A}\,\vartheta\rangle\rangle = \langle a, \mathrm{A}\,\vartheta\rangle + \langle \vartheta, \mathrm{A}\,a\rangle \tag{3.21}$$

This is not an unworkable object; indeed it is simply a quadratic operator in q, p, a and a^+. Apart from details the situation is exactly the same as in [12]; so it holds the following

THEOREM 3.5:

(*a*) *For each real* τ, $t \in \mathrm{I}$, $\mathrm{H}_2(t) \in \mathrm{B}(\tilde{\mathrm{X}}_{\tau+1}, \tilde{\mathrm{X}}_\tau)$ (3.22)

(*b*) $\mathrm{H}_2(t)$ *is essentialy self-adjoint on* $\tilde{\mathrm{X}}_\tau$, $\forall\, \tau \in [1, \infty]$ (3.23) (*c*) *It exists a* L^2*-strongly continuous group of* L^2*-unitary operators called* $U_2(t, s)$ *with* t, $s \in I$, *such that*

$$U_2(t, s) \in B(\tilde{X}_\tau, \tilde{X}_\tau), \qquad \forall \tau \in R \tag{3.24}$$

(*d*) *For each compact* $K \subset I$, *it exists a constant* $\alpha_{\tau, K}$ *such that*

$$\forall t, s \in K, \quad \| U_2(t, s) \|_{\tilde{X}_\tau, \tilde{X}_\tau} \leqq \exp(\alpha_{\tau, K} | t - s |) \tag{3.25}$$

(*e*) $U_2(t, s)$ *is jointly strongly continuous in* X_τ *with respect to* t *and* s, $\forall \tau \in R$. (3.26)

(*f*) *Let* ψ *belong to* $\tilde{X}_1$; *then*

$$\begin{aligned} &\frac{d}{dt} U_2(t, s) \psi = -i H_2(t) U_2(t, s) \psi \\ &\frac{d}{ds} U_2(t, s) \psi = i U_2(t, s) H_2(s) \psi \end{aligned} \tag{3.27}$$

Proof. — *See* [10], theorems 4.2, 4.3. ##

The following is the fundamental theorem of this section:

THEOREM 3.6. — *Let* $K \subset I$ *be a compact; if the hypotheses* 3.1 *and* 3.2 *are verified then it follows that*

$$\underset{h \to 0}{s\text{-}\lim}\, W_h(t, s) = U_2(t, s) \tag{3.28}$$

uniformly for t *and* s *belonging to* K.

Proof. — (3.1) implies that $\rho_K > 0$. Choose ρ such that $0 < \rho < \rho_K$; define

$$D_1 = \left\{ \psi \in \tilde{X}_2 : \psi = \sum_{j=1}^{k} f_j(x) g_j(y), k < \infty, f_j \in L^2(R^n), \right. \\ \left. g_j \in L^2(R^m), \operatorname{supp} g_j \text{ is compact} \right\} \tag{3.29}$$

Let χ a cut-off function whose definition and properties are listed after the end of the proof. If $\psi \in D_1$ one has:

$$\begin{aligned} [W_h(t, s) - U_2(t, s)] \psi &= W_h(t, s)(1 - \chi_{\rho/2}(q_h)) \psi \\ &- (1 - \chi_{\rho/2}(q_h)) U_2(t, s) \psi + [W_h(t, s) \chi_{\rho/2}(q_h) - \chi_{\rho/2}(q_h) U_2(t, s)] \psi \end{aligned} \tag{3.30}$$

The estimation of the first addendum gives:

$$\begin{aligned} \| W_h(t, s)[1 - \chi_{\rho/2}(q_h)] \psi \|_{\tilde{X}_0} &= \| [1 - \chi_{\rho/2}(q_h)] \psi \|_{\tilde{X}_0} \\ &\leqq 16 h^2 \rho^{-4} \| r^{-4}(1 - \chi(r)) \|_{L^\infty} \| \tilde{q}^4 \|_{\tilde{X}_2, \tilde{X}_0} \| \psi \|_{\tilde{X}_2} \end{aligned} \tag{3.31}$$

because of the unitariety of $W_h(t, s)$, the property (2.14) and the definition of the cut-off χ.

The second addendum is majorized thanks to (3.25):

$$\|[1-\chi_{\rho/2}(q_h)]\,\mathrm{U}_2(t,s)\psi\|_{\tilde{\mathrm{X}}_0} \leqq 16\,h^2\rho^{-4}\|r^{-4}(1-\chi(r))\|_{\mathrm{L}^\infty}\|\tilde{q}^4\,\mathrm{U}_2(t,s)\psi\|_{\tilde{\mathrm{X}}_0}$$
$$\leqq 16\,h^2\rho^{-4}\exp(\alpha_{2,\mathrm{K}}|t-s|)\,\|\tilde{q}^4\|_{\tilde{\mathrm{X}}_2,\tilde{\mathrm{X}}_0}\|\psi\|_{\tilde{\mathrm{X}}_2} \quad (3.32)$$

Now the estimation of the third term in (3.30) has to be faced; observe that

$$\left.\begin{array}{lll} \varphi=\chi_{\rho/2}(q_h)\,\mathrm{U}_2(t,s)\psi\in\mathrm{X}_2 & \text{with} & \varphi=\sum f_j(x)\,g_j(y) \\ f_j\in\mathfrak{D}(h^{1/2})\,\mathrm{L}^2(\mathfrak{U}_t-\xi_t), & g_j\in\mathrm{L}^2(\mathrm{R}^m), & \operatorname{supp} g_j \text{ is a compact.}\end{array}\right\} \quad (3.33)$$

It is possible to use the lemma 3.4:

$$\frac{d}{dr}\mathrm{W}_h(t,r)\,\chi_{\rho/2}(q_h)\,\mathrm{U}_2(r,s)\psi$$
$$=i\,\mathrm{W}_h(t,r)\,[h^{-1}\mathrm{K}_h(r)\,\chi_{\rho/2}(q_h)-\chi_{\rho/2}(q_h)\,\mathrm{H}_2(r)]\,\mathrm{U}_2(r,s)\psi \quad (3.34)$$

From (3.12) and (3.20) one gets:

$$[h^{-1}\mathrm{K}_h(r)\,\chi_{\rho/2}(q_h)-\chi_{\rho/2}(q_h)\,\mathrm{H}_2(r)]\,\mathrm{U}_2(r,s)\psi$$
$$=\Big\{-\frac{1}{2}h\,(\partial^2_{x_j x_j}\chi_{\rho/2})(q_h)-(\partial_{x_j}\chi_{\rho/2})(q_h)\,(i\,h^{1/2}p_j+h\langle a,\mathrm{A}_j(q_h+\xi_r,r)\,a\rangle$$
$$+h^{1/2}\langle\langle a,\mathrm{A}_j(q_h+\xi_r,r)\,\vartheta_r\rangle\rangle+\delta_r(\langle\vartheta,\mathrm{A}_j(\xi,t)\,\vartheta\rangle))$$
$$-i\,\chi_{\rho/2}(q_h)\,(h^{1/2}\langle a,\mathrm{A}_j(q_h+\xi_r,r)\,a\rangle+\delta_r(\langle\langle a,\mathrm{A}_j(\xi,t)\,\vartheta\rangle\rangle)$$
$$+h^{-1/2}\delta^1_r(\langle\vartheta,\mathrm{A}_j(\xi,t)\,\vartheta\rangle))\,p_j-\frac{1}{2}\chi_{\rho/2}(q_h)\,(h\langle a,\operatorname{div}\mathrm{A}(q_h+\xi_r,r)\,a\rangle$$
$$+h^{1/2}\langle\langle a,\operatorname{div}\mathrm{A}(q_h+\xi_r,r)\,\vartheta_r\rangle\rangle+\delta_r(\langle\vartheta,\operatorname{div}\mathrm{A}(\xi)\,\vartheta\rangle)$$
$$-\chi_{\rho/2}(q_h)\,((2h)^{-1}\delta^2_r(\langle\vartheta,\mathrm{A}_j(\xi,t)\,\vartheta\rangle^2))+\frac{1}{2}h\langle a,\mathrm{A}_j(q_h+\xi_r,r)\,a\rangle^2$$
$$+h^{-1/2}\delta^1_r(\langle\vartheta,\mathrm{A}_j(\xi,t)\,\vartheta\rangle\langle\langle a,\mathrm{A}_j(\xi,t)\,\vartheta\rangle\rangle)+\delta_r\Big(\frac{1}{2}\langle\langle a,\mathrm{A}_j(\xi,t)\,\vartheta\rangle\rangle^2$$
$$+\langle\vartheta,\mathrm{A}_j(\xi,t)\,\vartheta\rangle\langle a,\mathrm{A}_j(\xi,t)\,a\rangle\Big)+h^{1/2}\langle\langle a,\mathrm{A}_j(q_h+\xi_r,r)\,\vartheta_r\rangle\rangle\langle a,\mathrm{A}_j(q_h+\xi_r,r)\,a\rangle$$
$$+i\,h^{-1}\delta^2_r(\langle\vartheta,(\pi_j\mathrm{A}_j-\mathrm{A}_0)(\xi,t)\,\vartheta\rangle)+i\,h^{-1/2}\delta^1_r(\langle\langle a,(\pi_j\mathrm{A}_j-\mathrm{A}_0)(\xi,t)\,\vartheta\rangle\rangle)$$
$$+i\,\delta_r\langle a,(\pi_j\mathrm{A}_j-\mathrm{A}_0)(\xi,t)\,a\rangle-h^{-1}\delta^2_r(\mathrm{V}(\xi,t))\Big\}\,\mathrm{U}_2(r,s)\psi \quad (3.35)$$

The following definitions have been adopted:

$$\delta_r(f)=f(q_h+\xi_r,\vartheta_r,\vartheta^*_r,r)-f(\xi_r,\vartheta_r,\vartheta^*_r,r) \quad (3.36)$$
$$\delta^1_r(f)=\delta_r(f)-\partial_{\xi_j}f(\xi_r,\vartheta_r,\vartheta_r,r)\,q_{jh} \quad (3.37)$$
$$\delta^2_r(f)=\delta^1_r(f)-\frac{1}{2}\partial^2_{\xi_j\xi_l}f(\xi_r,\vartheta_r,\vartheta_r,r)\,q_{jh}\,q_{lh} \quad (3.38)$$

The strong continuity of the group $U_2(t,s)$ and the expression (3.35) allow one to write

$$W_h(t,s)\chi_{\rho/2}(q_h)\psi - \chi_{\rho/2}(q_h)U_2(t,s)\psi = \int_s^t \frac{d}{dr}[W_h(t,r)\chi_{\rho/2}(q_h)U_2(r,s)]\psi\,dr \quad (3.39)$$

and so it follows that:

$$\|W_h(t,s)\chi_{\rho/2}(q_h)\psi - \chi_{\rho/2}(q_h)U_2(t,s)\psi\|_{\tilde{X}_0} \leqq \left|\int_s^t \|[h^{-1}K_h(r)\chi_{\rho/2}(q_h) - \chi_{\rho/2}(q_h)H_2(r)]U_2(t,s)\psi\|_{\tilde{X}_0}\,dr\right| \quad (3.40)$$

Now the expression (3.35) must be used to estimate the norm in the integral. The prototypes of the terms contained in this estimate are the following: from the first term one gets the majorization:

$$\frac{1}{2}h\|\partial^2_{x_j x_j}\chi_{\rho/2}(q_h)U_2(r,s)\psi\|_{\tilde{X}_0} \leqq 2nc_2 h\rho^{-2}\exp(\alpha_{2,K}|t-s|)\|\psi\|_{\tilde{X}_2} \quad (3.41)$$

from the second term one obtains:

$$\|-ih^{1/2}(\partial_{x_j}\chi_{\rho/2})(q_h)p_j U_2(r,s)\psi\|_{\tilde{X}_0} \leqq 2nc_1 h^{1/2}\rho^{-1}\||p|\|_{\tilde{X}_{1/2},\tilde{X}_0}\exp(\alpha_{2,K}|t-s|)\|\psi\|_{\tilde{X}_2} \quad (3.42)$$

The estimate of the last term gives

$$\|h^{-1}\delta^2_r(V)\chi_{\rho/2}(q_h)U_2(r,s)\psi\|_{\tilde{X}_0} \leqq \frac{1}{2}\{\sum_{ij}[\sup_{\tau\in K,\,|y|<\rho}|\partial^2_{\xi_i\xi_j}V(y+\xi_\tau,\tau) - \partial^2_{\xi_i\xi_j}V(\xi_\tau,\tau)|]^2)^{1/2} \times(\sum_{ij}\|q_i q_j\|_{\tilde{X}_1,\tilde{X}_0})^2\|\psi\|_{\tilde{X}_2}\exp(\alpha_{2,K}|t-s|) \quad (3.43)$$

where the hypotheses 3.2 have been used; from the compactness of $\mathfrak{U}_K$ it follows that it exists a function $f(\rho)$ such that $f(\rho)\to 0$ when $\rho\to 0^+$ and for which one has:

$$\|h^{-1}\delta^2_r(V)\chi_{\rho/2}(q_h)U_2(r,s)\psi\|_{\tilde{X}_0} \leqq f(\rho)\exp(\alpha_{2,K}|t-s|)\|\psi\|_{\tilde{X}_2} \quad (3.44)$$

The procedure for the estimate of the other terms is the same as in the previous three; note that it is necessary to use $\|\psi\|_{\tilde{X}_2}$ to control the terms of the expression (3.35) that are quartic in the operators z's. From these estimates, the density of the set D_1 in L^2 and the unitarity of $U_2(t,s)$ and $W_h(t,s)$, the result may be finally obtained.

Appendix
Properties of cut-off functions

The function χ used during the previous proof has the following definition:

$$\left.\begin{array}{l}\chi \in \mathscr{C}^{\infty}([0, \infty), [0, \infty)); \qquad \chi(r)=1 \quad \text{if } 0 \leqq r \leqq 1, \\ \qquad\qquad \chi(r)=0 \quad \text{if } r \geqq 2.\end{array}\right\} \tag{3.45}$$

Let $x \in \mathrm{R}^n$; define

$$\chi_\nu(x)=\chi\left(\frac{|x|}{\nu}\right) \tag{3.46}$$

It follows that

$$\begin{array}{lll}\chi_\nu \in \mathscr{C}_0^{\infty}(\mathrm{R}^n), & 0 \leqq \chi_\nu \leqq 1, & \operatorname{supp} \chi_\nu = \mathrm{S}(0, 2\nu) \\ \forall m \in \mathrm{N}, \quad \exists c_m > 0: & \max\limits_{|\alpha|=m} \left|\dfrac{\partial^{|\alpha|}}{\partial x^\alpha} \chi_\nu(x)\right| \leqq c_m \nu^{-m}. & \end{array} \tag{3.47}$$

4. SELF-ADJOINTNESS OF THE HAMILTONIAN OPERATOR

Consider the formal Hamiltonian

$$\mathrm{H}_1 = \frac{1}{2}[p_j - i\langle a, \mathrm{A}_j(q)\, a\rangle]^2 + \mathrm{V}(q) \tag{4.1}$$

defined on some dense set of

$$\begin{aligned} &\mathrm{L}^2(\mathrm{R}^n) \times \mathrm{L}^2(\mathrm{R}^m) \\ &= \text{completion of} \left\{ \sum_{j=1}^{s} f_j(x)\, g_j(y), f_j \in \mathrm{L}^2(\mathrm{R}^n), g_j \in \mathrm{L}^2(\mathrm{R}^m) \right\} \end{aligned} \tag{4.2}$$

[There is no difference in using this space instead of $\mathrm{L}^2(\mathrm{R}^{n+m})$ because of the natural isomorphism connecting these two spaces.]

There are some well known facts that allow a great simplification in the study of the problem of the self-adjointness of the operator (4.1) on the space (4.2). Indeed one has that:

$$(a) \qquad \mathrm{L}^2(\mathrm{R}^n) \times \mathrm{L}^2(\mathrm{R}^m) = \bigoplus_{p=0}^{\infty} \mathrm{L}^2(\mathrm{R}^n) \times \mathrm{K}_p = \bigoplus_{p=0}^{\infty} \mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p) \tag{4.3}$$

where K_p is the eigenspace of a quantum m-dimensional oscillator relative to the eigenvalue p.

(b) The operator $a_i^+ (\mathrm{T}_\mu)_{il}\, a_l$ leaves invariant the space K_p. (4.4)

(c) $\langle \varphi \,|\, a_i^+ (\mathrm{T}_\mu)_{il}\, a_l \varphi \rangle + \langle a_i^+ (\mathrm{T}_\mu)_{il}\, a_l \varphi \,|\, \varphi \rangle = 0$ (4.5) as follows from the antihermiticity of the matrices T.

(*d*) $$\dim \mathrm{K}_p = \frac{(p+m-1)!}{p!\,(m-1)!} = \nu \tag{4.6}$$

These observations allow one to decompose the present problem in two successive steps: the first one is the study of the self-adjointness of (4.1) on the space $\mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p)$.

This is essentialy the same as in the case of the Schrodinger operator with electromagnetic potential.

Call H_1^p the restriction of H_1 to some subset of $\mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p)$.

THEOREM 4.1. – *If* $\mathrm{A}_j^\mu \in \mathrm{L}^4_{\mathrm{loc}}(\mathrm{R}^n, \mathrm{R})$, $\mathrm{V} \in \mathrm{L}^2_{\mathrm{loc}}(\mathrm{R}^n, \mathrm{R})$ *with* $\mathrm{V} \geqq 0$ $\operatorname{div} \mathrm{A}^\mu \in \mathrm{L}^2_{\mathrm{loc}}(\mathrm{R}^n, \mathrm{R})$, *then* H_1^p *is essentialy self-adjoint on* $\mathscr{C}_0^\infty(\mathrm{R}^n, \mathrm{K}_p)$.

Proof. – The proof of this theorem follows the same procedures of [18]. The modifications that are necessary are based on the observations (4.3), (4.4), (4.5) and (4.6). ##

Consider now the operator

$$\mathrm{H}^p = \mathrm{H}_1^p + i \langle a, \mathrm{A}_0(q)\, a \rangle + \mathrm{V}_1(q) \tag{4.7}$$

where $\mathrm{V}_1 \leqq 0$.

THEOREM 4.2. – *Suppose that*

$$\begin{gathered} \| \mathrm{V}_1(q) f \|_{\mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p)} \leqq b \, \| \Delta f \|_{\mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p)} + c \, \| f \|_{\mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p)} \\ \text{with } b < 1 \\ \| \mathrm{A}_0^\mu(q) f \|_{\mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p)} \leqq b_1 \, \| \Delta f \|_{\mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p)} + c_1 \, \| f \|_{\mathrm{L}^2(\mathrm{R}^n, \mathrm{K}_p)} \end{gathered} \tag{4.8}$$

$\mu = 1, \ldots, \dim \mathfrak{g}$, c_1 *arbitrarily small.*

If the hypotheses of theorem 4.1 *are true then* H^p *is essentialy self-adjoint on* $\mathscr{C}_0^\infty(\mathrm{R}^n, \mathrm{K}_p)$.

Proof. – The proof is obtained from lemma 6 of [14] and from the Kato-Rellich theorem [16]. ##

In the second step the complete Hamilton's operator is reconstructed; consider indeed the following operator:

$$\mathrm{H} = \bigoplus_{p=0}^{\infty} \mathrm{H}^p \tag{4.9}$$

defined on the space

$$\begin{aligned} \mathrm{D}_0 = \{ u = (u_1, \ldots, u_n, 0, 0, \ldots), \quad & n < \infty, \\ u_p \in \mathscr{C}_0^\infty(\mathrm{R}^n, \mathrm{K}_p) \} & \end{aligned} \tag{4.10}$$

The action of (4.9) on (4.10) is given by

$$\mathrm{H} u = \bigoplus_{p=0}^{\infty} \mathrm{H}^p u_p \tag{4.11}$$

LEMMA 4.3. — H *is essentialy self-adjoint on* D_0.

Proof. — It suffices to show that $\operatorname{Ran}(H \pm i)$ are dense in $L^2(R^n) \times L^2(R^m)$ [13]. This is immediate; indeed

$$0 = \langle \psi | (H \pm i) u \rangle = \sum_p \langle \psi_p | (H^p \pm i) u_p \rangle, \qquad \forall u \in D_0 \qquad (4.12)$$

implies $\psi_p = 0$ and so $\psi = 0$. ##

Consider now the set

$$D_2 = \{ \varphi = \sum f_j g_j, f_j \in \mathscr{C}_0^\infty(R^n), g_j \in \mathfrak{S}(R^m) \} \qquad (4.13)$$

$\mathscr{S}(R^m)$ is the Schwarz space [13]. Call besides $D(\bar{H})$ the domain of the closure of the operator (4.9). The final result is the following:

THEOREM 4.4. — *If the hypotheses of theorem* 4.2 *are satisfied the operator*

$$H = \frac{1}{2} [p_j - i \langle a, A_j(q) a \rangle]^2 + i \langle a, A_0(q) a \rangle + V(q) \qquad (4.14)$$

essentially self-adjoint on the set

$$\mathscr{C}_0^\infty(R^n) \times \mathscr{C}_0^\infty(R^m). \qquad (4.15)$$

Proof. — It is enough to observe that

$$D_0 \subset D_1 \subset D(\bar{H}) \qquad (4.16)$$

to conclude the proof. ##

5. APPLICATIONS

In this first remark it is shown that there are families of operators satisfying the hypotheses (3.2).

Consider the operator (1.11). It has been shown in theorem 4.4 that this operator is essentialy self-adjoint on the set (4.15).

If the hypotheses of theorem 4.5 and (3.4) are assumed, the family of operators

$$\text{closure of } (H_h(z_h) \downarrow_{\mathscr{C}_0^\infty(R^n) \times \mathscr{C}_0^\infty(R^m)})_{h>0} \qquad (5.1)$$

satisfies the hypotheses 3.2. Indeed the operators of (5.1) are self-adjoint; besides for each $\psi \in D$ it is possible to find a succession ψ_j in (4.15) such that $\psi_j \to \psi$ in $\tilde{X}_2$ and

$$\| H_h(z_h)(\psi_j - \psi_k) \|_{L^2} \to 0 \qquad (5.2)$$

Having in mind the Stone's theorem one concludes that hypotheses (3.2) are effectively satisfied.

In this second and conclusive remark it is shown how to applicate the results that have been obtained.

LEMMA 5.1. — *Let* $f: \mathrm{R}^n \to \mathrm{C}$ *a measurable and bounded function which is continuous in* $\mathrm{S}(\xi, \rho)$, $\rho \in \mathrm{R}$, $\xi \in \mathrm{R}^n$. *Then*

$$s\text{-}\lim_{h \to 0+} f(q_h + \xi) = f(\xi) \tag{5.3}$$

Proof. — Write the identity

$$\begin{aligned} &[f(q_h+\xi)-f(\xi)]\,\psi \\ &\quad = [f(q_h+\xi)\,\chi_{\rho/2}(q_h)-f(\xi)]\,\psi-[1-\chi_{\rho/2}(q_h)]f(q_h+\xi)\,\psi, \\ &\qquad \psi \in \mathrm{L}^2(\mathrm{R}^n) \end{aligned} \tag{5.4}$$

Then one easily gets the proof by noting that $q_{jh} \to 0$ for $h \to 0$ strongly in the resolvent sense and that

$$\|f(q_h+\xi)\|_{\mathrm{L}^2,\,\mathrm{L}^2} \leqq \|f\|_{\mathrm{L}^\infty} \quad \#\# \tag{5.5}$$

THEOREM 5.2. — *Let* $f: \mathrm{R}^n \to \mathrm{C}$ *a measurable bounded function, which is continuous in a neighborhood of* $(\xi_t)_{t \in \mathrm{I}}$; *if the hypotheses* 3.1 *and* 3.2 *are true then it follows that*

$$\lim_{h \to 0} \langle \mathrm{U}_h(t,s)\,\mathrm{C}_h(\zeta_s)\,\varphi \,|\, f(q_h)\,\mathrm{U}_h(t,s)\,\mathrm{C}_h(\zeta_s)\,\varphi \rangle = f(\xi_t) \tag{5.6}$$

with $t, s \in \mathrm{I}$, $\varphi \in \mathrm{L}^2(\mathrm{R}^n) \times \mathrm{L}^2(\mathrm{R}^m)$ *is normalized to* 1.

Proof. — Observe that (5.6) is equal to

$$\langle \mathrm{W}_h(t,s)\,\varphi \,|\, \mathrm{C}_h^+(\zeta_t)\,f(q_h)\,\mathrm{C}_h(\zeta_t)\,\mathrm{W}_h(t,s)\,\varphi \rangle \tag{5.7}$$

and therefore equal to

$$\langle \mathrm{W}_h(t,s)\,\varphi \,|\, f(q_h+\xi_t)\,\mathrm{W}_h(t,s)\,\varphi \rangle \tag{5.8}$$

Having in mind (5.5) and the theorem 3.6 one can conclude that

$$\begin{aligned} &\lim_{h \to 0} \langle \mathrm{W}_h(t,s)\,\varphi \,|\, f(q_h+\xi_t)\,\mathrm{W}_h(t,s)\,\varphi \rangle \\ &\qquad = \langle \mathrm{U}_2(t,s)\,\varphi \,|\, f(\xi_t)\,\mathrm{U}_2(t,s)\,\varphi \rangle = f(\xi_t) \end{aligned} \tag{5.9}$$

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