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H. SAZDJIAN

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The Foldy-Wouthuysen transformation in the two-particle case

by

H. SAZDJIAN
Division de Physique Théorique (*), Institut de Physique Nucléaire,
Université Paris XI, 91406 Orsay Cedex, France

ABSTRACT. — We construct the Foldy-Wouthuysen transformation operator for wave equations describing two interacting particle systems, and possessing Poincaré invariance and manifest covariance. We consider systems composed of one spin-$\frac{1}{2}$ fermion and one spin-0 boson, and also of one spin-$\frac{1}{2}$ fermion and one spin-$\frac{1}{2}$ antifermion. We first construct the above operator in the free case. Then, we show that for several classes of interaction potential it can also be constructed in a compact form. As compared to the one-particle case, manifest covariance facilitates here the construction of the Foldy-Wouthuysen transformation operator, although the number of particles is larger.

RÉSUMÉ. — On construit l'opérateur de la transformation de Foldy-Wouthuysen pour des équations d'onde décrivant des systèmes de deux particules en interaction, et possédant l'invariance de Poincaré et la covariance manifeste. On considère des systèmes composés d'un fermion de spin $\frac{1}{2}$ et d'un boson de spin 0, et aussi d'un fermion de spin $\frac{1}{2}$ et d'un antifermion de spin $\frac{1}{2}$. On construit d'abord cet opérateur dans le cas libre. Puis on montre que pour plusieurs classes de potentiels d'interaction, il peut aussi

(*) Laboratory associated with C. N. R. S.

être construit sous une forme compacte. En comparaison avec le cas d'une particule, la covariance manifeste facilite ici la construction de l'opérateur de la transformation de Foldy-Wouthuysen, bien que le nombre des particules soit plus grand.

1. INTRODUCTION

The Foldy-Wouthuysen (FW) transformation [*1*] [*2*] [*3*] of the Dirac equation provides several decisive advantages for the understanding and interpretation of the physical properties of this equation. It permits the search for the solutions of the Dirac equation by means of two- component spinors. But its main achievement consists in classifying these solutions in two different subspaces, each of them corresponding to a definite sign of the energy eigenvalues.

This feature leads in turn to other positive consequences. First, one is led in a natural way to identify the physical Hilbert space of the theory with the subspace of (normalizable) solutions having positive energies. Second, it is in the Foldy-Wouthuysen representation that a consistent definition of observables emerges. For instance, the usual operators of position, spin and orbital angular momentum must be defined in the Foldy-Wouthuysen representation. The reason for this is that operators defined in this representation leave invariant the subspaces of solutions having a definite energy sign. Thus, in particular, they leave invariant the physical Hilbert space and can be considered, if they are hermitian, as observables. In the Dirac representation, the expressions of observables are more complicated and without the Foldy-Wouthuysen transformation one probably would have difficulties to find them.

The Foldy-Wouthuysen transformation can also be used for the Klein-Gordon equation, with a two-component formalism [*4*] [*5*].

In the presence of interaction, the FW transformation has not, in general, a compact form and one usually uses series expansion methods, either in $1/c^2$ or in the coupling constant, to find its expression in an approximate way. However, there are classes of interaction, represented for instance by the static magnetic potentials, for which the FW transformation still has a compact form [*6*].

The purpose of the present paper is to construct the FW transformation operator in the case of two-particle relativistic quantum mechanics involving spin-$\frac{1}{2}$ particles. In recent years, the use of the manifestly covariant formalism with constraints led to a consistent formulation and construction of two-particle relativistic quantum mechanics [7] [8] [9]. Therefore, in problems where spin-$\frac{1}{2}$ particles are present, the FW transformation may again be in order.

Although two-particle problems are in general more complicated than one-particle problems, the former display, however, a particular advantage with the fact that two-particle relativistic quantum mechanics, with no external potentials, is Poincaré invariant. This has the consequence that wave equations, as well as many transformation operators, can be constructed in a manifestly covariant form. Contrary to the one-particle case, where manifest covariance is broken, we shall see that the FW transformation operator in the two-particle case has a manifestly covariant expression (This is also the reason why we shall not consider the FW transformation for spin-0 particles, in which case manifest covariance is again broken.)

The plan of the paper is as follows. In Section 2 we present a brief review of two-particle relativistic quantum mechanics, mainly concerning the expressions of the wave equations and the scalar product of states.

Section 3 is devoted to a discussion of the realization problem of the unitarity property of the FW transformation. This question is not usually paid sufficient attention in one-particle quantum mechanics. The FW transformation does not, in a rigorous sense, define an operator in a given Hilbert space $\mathscr{H}$, for it does not transform the states of $\mathscr{H}$ into one another. Rather, it transforms the Hilbert space $\mathscr{H}$ into another Hilbert space $\tilde{\mathscr{H}}$, by establishing a one-to-one connection between the states ψ of $\mathscr{H}$ and the states $\tilde{\psi}$ of $\tilde{\mathscr{H}}$. For this reason, the unitarity property of this transformation is to be established in a more precise way than usually. To this end, it is sufficient to demand that the scalar product in $\mathscr{H}$ is left invariant when passing to $\tilde{\mathscr{H}}$, that is, $(\psi, \phi) = (\tilde{\psi}, \tilde{\phi})$. However, the definition of the scalar product in $\tilde{\mathscr{H}}$ is a matter of convention. Provided the transformation operator from $\mathscr{H}$ to $\tilde{\mathscr{H}}$ is nonsingular, then it is always possible to define a scalar product in $\tilde{\mathscr{H}}$ in such a way that the above property, and hence the unitarity of the transformation, be ensured. In the on-particle case, the kernel of the scalar product in the Dirac representation (i. e., in $\mathscr{H}$) is simply 1; in the FW representation (i. e., in $\tilde{\mathscr{H}}$) one can also choose it to be 1; then the unitarity property of the transformation is verified formally in either of the scalar products and the above subtleties do not occur. The problem would certainly be more complicated and less clear if the kernel of the scalar product in $\mathscr{H}$ was some nontrivial operator K. It is precisely this kind of situation which is met in the two-particle case, and for this reason, this question is analyzed in some detail in Section 3.

In Section 4, we construct the FW transformation operator in the free case, for systems composed of one fermion and one boson, and one fermion and one antifermion, respectively (The treatment of two fermion systems follows similar lines as those of fermion-antifermion systems and will not be dealt with in this article.)

In Section 5, we construct the FW transformations for the above systems, in the presence of special classes of interaction, in compact form. The construction of the FW transformation for arbitrary types of interaction, necessitates, as usual, the use of a series expansion method.

Conclusion follows in Section 6.

2. TWO-PARTICLE RELATIVISTIC QUANTUM MECHANICS

In the manifestly covariant formalism, the two-particle wave function satisfies two independent wave equations [9].

For a system of two spin-0 particles these equations read

$$H_1\Psi(x_1, x_2) \equiv (p_1^2 - m_1^2 - V)\Psi(x_1, x_2) = 0\,, \tag{2.1 a}$$

$$H_2\Psi(x_1, x_2) \equiv (p_2^2 - m_2^2 - V)\Psi(x_1, x_2) = 0\,, \tag{2.1 b}$$

where V is a manifestly covariant Poincaré invariant interaction potential.

For a system composed of one spin-$\frac{1}{2}$ (particle 1) and one spin-0 (particle 2) particles, the wave equations are

$$H_1\Psi \equiv (\gamma\cdot p_1 - m_1 - V)\Psi(x_1, x_2) = 0\,, \tag{2.2 a}$$

$$H_2\Psi \equiv \left[p_2^2 - m_2^2 - (\gamma\cdot p_1 + m_1)V\right]\Psi(x_1, x_2) = 0\,. \tag{2.2 b}$$

Here Ψ is a four-component spinor function:

$$\Psi = \Psi_\alpha(x_1, x_2)\,, \tag{2.3}$$

and the potential V may also depend on the Dirac matrices.

For a system composed of one fermion (particle 1) and one antifermion (particle 2) of spin-$\frac{1}{2}$, the wave equations are

$$H_1\Psi \equiv [\gamma\cdot p_1 - m_1 - (-\eta\cdot p_2 + m_2)V]\Psi(x_1, x_2) = 0\,, \tag{2.4 a}$$

$$H_2\Psi \equiv [\eta\cdot p_2 + m_2 + (\gamma\cdot p_1 + m_1)V]\Psi(x_1, x_2) = 0\,, \tag{2.4 b}$$

where Ψ is a spinor function of rank two:

$$\Psi = \Psi_{\alpha_1\alpha_2}(x_1, x_2)\,, \tag{2.5}$$

and the Dirac matrices γ and η are defined as those acting on the spinor indices of particles 1 and 2, respectively:

$$\gamma_\mu \Psi \equiv \gamma_{1\mu} \Psi = (\gamma_\mu)_{\alpha_1 \beta_1} \Psi_{\beta_1 \alpha_2},$$
$$\eta_\mu \Psi \equiv \Psi \gamma_{2\mu} = \Psi_{\alpha_1 \beta_2} (\gamma_\mu)_{\beta_2 \alpha_2},$$
$$\eta_\mu \eta_\nu \Psi = \Psi \gamma_{2\nu} \gamma_{2\mu}, \qquad \eta_5 \Psi = \Psi \gamma_{25}. \tag{2.6}$$

We also define

$$\sigma_{\mu\nu} = \frac{1}{2i} [\gamma_\mu, \gamma_\nu], \qquad \xi_{\mu\nu} = \frac{1}{2i} [\eta_\mu, \eta_\nu],$$
$$\tilde{\gamma}_\mu = \gamma_\mu \gamma_5, \qquad \tilde{\eta}_\mu = \eta_\mu \eta_5. \tag{2.7}$$

Here, the potential V may also depend on the matrices γ and η.

Because the wave function Ψ satisfies two independent equations, the latter must be compatible among themselves. The compatibility condition is

$$[H_1, H_2]\Psi = 0. \tag{2.8}$$

When applied to the above wave equations, this equation yeilds an equation for V which is

$$[p_1^2 - p_2^2, V] = 0. \tag{2.9}$$

The potential V must be a solution of this equation. By introducing the notations

$$p = p_1 + p_2, \qquad v = \frac{1}{2}(p_1 - p_2), \qquad X = \frac{1}{2}(x_1 + x_2),$$
$$x = x_1 - x_2, \qquad x_\mu^T = x_\mu - (\hat{p}\cdot x)\hat{p}_\mu, \qquad x_L = \hat{p}\cdot x,$$
$$\hat{p}_\mu = p_\mu/(p^2)^{1/2}, \qquad p^2 > 0, \qquad v_\mu^T = v_\mu - (\hat{p}\cdot v)\hat{p}_\mu,$$
$$\nabla_\mu = \partial_{1\mu} + \partial_{2\mu}, \qquad \partial_\mu = \frac{1}{2}(\partial_{1\mu} - \partial_{2\mu}), \tag{2.10}$$

equation (2.9) means that V depends on x through its transverse components x^T alone:

$$V = V(x^T, \ldots), \tag{2.11}$$

where the dots stand for the momenta and eventually for the Dirac matrice (We also recall that V must be a Poincaré invariant function of its arguments.)

Equations (2.1), (2.2) and (2.4) lead also to the following equation for Ψ (which is a consequence of them)

$$(p_1^2 - p_2^2)\Psi = (m_1^2 - m_2^2)\Psi. \tag{2.12}$$

This equation determines the evolution law of Ψ with respect to the covariant relative time variable x_L. For eigenfunctions of the total momentum p, the solution of Eq. (2.12) is

$$\Psi(X, x) = e^{-ip.X} e^{-i(m_1^2 - m_1^2)p.x/(2p^2)} \psi(x^T), \tag{2.13}$$

where $\psi(x^T)$ now represents the internal wave function and since it depends on x^T alone, the internal dynamics (or the eigenvalue equation) will be three-dimensional, besides the spin degrees of freedom.

We notice that, because of the structure of the potential V [Eq. (2.11)], the wave equation operators H_1 and H_2 commute with the longitudinal components $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$ of p_1 and p_2. Therefore the wave equations may be considered as yielding eigenvalues for $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$. It turns out that they rather yield eigenvalues for $(\hat{p}\cdot p_1)^2$ and $(\hat{p}\cdot p_2)^2$, which are also related by Eq. (2.12). We admit that the physically acceptable potentials are those which yield positive eigenvalues for $(\hat{p}\cdot p_1)^2$ and $(\hat{p}\cdot p_2)^2$. Then both $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$ have two real eigenvalues with opposite signs. Thus the space of normalizable solutions of the wave equations splits into four subspaces according to the signs of the eigenvalues of $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$. The physical Hilbert space is identified with that subspace where both $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$ have a positive sign. In this case the expressions of the latter are, in terms of p^2 and the masses,

$$\hat{p}\cdot p_{1,2} = \frac{(p^2)^{1/2}}{2}\left(1 \pm \frac{m_1^2 - m_2^2}{p^2}\right), \qquad p^2 > |\, m_1^2 - m_2^2 \,| . \tag{2.14}$$

This also ensures the positivity of $(p^2)^{\frac{1}{2}}$.

This definition of the physical Hilbert space is a direct generalization of a similar definition in one-particle relativistic quantum mechanics, where the physical Hilbert space is identified with the positive energy solutions [5].

In order to construct the scalar product of the theory one searches for a tensor current of rank two, $j_{\mu\nu}$, satisfying two independent conservation laws:

$$\begin{aligned} \partial_1^\mu j_{\mu\nu}(x_1, x_2) &= 0 , \\ \partial_2^\nu j_{\mu\nu}(x_1, x_2) &= 0 . \end{aligned} \tag{2.15}$$

The current $j_{\mu\nu}$ must be a bilinear function of two wave functions Ψ and Φ, say, with eigenvalues p' and p, respectively, and must have a kernel which is translation invariant and covariant. Thus it should have the structure

$$j_{\mu\nu} = \Psi * K_{\mu\nu} * \Phi , \tag{2.16}$$

where the stars on both sides of $K_{\mu\nu}$ indicate that the latter may represent an integral operator. The kernel $K_{\mu\nu}$ is translation invariant and a Lorentz tensor of rank two.

The scalar product is then constructed according to the formula

$$(\Psi, \Phi) = \int_{\Sigma_1, \Sigma_2} j_{\mu\nu}(x_1, x_2) d\sigma_1^\mu(x_1) d\sigma_2^\nu(x_2) , \tag{2.17}$$

where the surfaces Σ_1 and Σ_2 are three-dimensional and spacelike.

The conservation laws (2.15), the structure (2.16) of $j_{\mu\nu}$ and formula (2.17) then guarantee the hermiticity property of the Poincaré group generators and the unitary realization of the group.

The expressions of the currents $j_{\mu\nu}$, satisfying the conservation laws (2.15) and having a structure of the form (2.16), have been presented in Ref. [*10*]. The scalar products and norms were calculated on parallel constant hyperplanes Σ_1 and Σ_2, perpendicular to a timelike vector n:

$$n \cdot x_1 = t_1, \qquad n \cdot x_2 = t_2, \qquad n = (1, \vec{0}). \tag{2.18}$$

For two spin-0 particle systems, the scalar product is

$$\begin{aligned}(\Psi_{p'}, \Phi_p)_{p'^2 \neq p^2} = & \int d^3\vec{X} d^3\vec{x} \Psi^*(X, x) \Big\{ i^2 \overleftrightarrow{\partial}_{10} \overleftrightarrow{\partial}_{20} \\ & - (p'_0 + p_0) \frac{[V(x, p' + i\varepsilon n, \ldots) - V(x, p - i\varepsilon n, \ldots)]}{(p'_0 - p_0 + 2i\varepsilon)} \Big\} \Phi(X, x) \\ & = (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \int d^3\vec{x} e^{ip'_0 X^0} \psi^*(x) \Big\{ i^2 \overleftrightarrow{\partial}_{10} \overleftrightarrow{\partial}_{20} \\ & - (p'_0 + p_0) \frac{[V(x, p' + i\varepsilon n, \ldots) - V(x, p - i\varepsilon n, \ldots)]}{(p'_0 - p_0 + 2i\varepsilon)} \Big\} \phi(x) e^{-ip_0 X^0}, \end{aligned} \tag{2.19}$$

where the limit $\varepsilon = 0$ is understood and the potential V is assumed to be superficially hermitian, that is, hermitian in the usual L^2 norm, when p_μ are replaced by real eigenvalues (In the expressions (2.19) the transverse variables x^T in V are calculated with respect to $p' + i\varepsilon n$ and $p - i\varepsilon n$, respectively.)

The expression of the norm is

$$\begin{aligned}(\Psi_{p',a} \Psi_{p,b})_{p'^2 = p^2} = & (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \int d^3\vec{x} \\ & e^{ip'_0 X^0} \psi_a^*(x) \left[i^2 \overleftrightarrow{\partial}_{10} \overleftrightarrow{\partial}_{20} - 4p_0^2 \frac{\partial V}{\partial p^2} \right] \psi_b(x) e^{-ip_0 X^0} = (2\pi)^3 2p_0 \delta^3(\vec{p}' - \vec{p}) \delta_{ab} f_a(p^2), \end{aligned} \tag{2.20}$$

where the labels a, b distinguish different eigenfunctions with the same mass squared p^2. The normalization factor f_a has a field theoretic origin and reflects the fact that one uses here the normalization condition of physical states $| p >$. It is determined by means of the relationship of two-particle relativistic quantuum mechanics with the Bethe-Salpeter equation [*11*] and has the expression

$$f = \left[\frac{\hat{p} \cdot p_1}{2(m_1^2 - \langle v^{T^2} \rangle)^{1/2}} + \frac{\hat{p} \cdot p_2}{2(m_2^2 - \langle v^{T^2} \rangle)^{1/2}} \right]^{-1}, \tag{2.21}$$

where $\langle v^{T^2} \rangle$ represents the mean value of v^{T^2}, calculated in the L^2 norm in the C. M. frame (v^T is defined in (2.10).)

The expressions of the scalar product (2.19) and of the norm (2.20) simplify in the C. M. frame, where the operators $i\partial_{10}$ and $i\partial_{20}$ become identical to the longitudinal components $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$, respectively, which have well-defined eigenvalues, (2.14).

Furthermore, in the C. M. frame, the vector x^T has the components $(0, \vec{x})$ and is independent of p^2. Therefore the only p^2-dependence of V comes through its explicit dependence on p^2 (for instance, through coupling constants). If V is explicitly independent of p^2, then in the C. M. frame $\frac{\partial V}{\partial p^2} = 0$ and the kernel of the norm reduces to the product $\hat{p}\cdot p_1 \ \hat{p}\cdot p_2$ which is positive, because, in the physical Hilbert space, $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$ are separately positive. Hence, the norm is positive. If $\frac{\partial V}{\partial p^2} \neq 0$ in the C. M. frame, then the norm is no longer manifestly positive. However, one can still show [10], with arguments based on the properties of the domain of positivity of p^2, that it is actually positive for the physical states, characterized by the eigenvalues $\hat{p}\cdot p_1 > 0$, $\hat{p}\cdot p_2 > 0$.

For a fermion-boson system, the expressions of the scalar product and of the norm are the following:

$$(\Psi_{p'}, \Phi_p)_{p'^2 \neq p^2} = \int d^3\vec{X} d^3\vec{x}\, \overline{\Psi}(X, x) \Big\{ i\gamma_0 \overleftrightarrow{\partial}_{20} - (p'_0 + p_0) \frac{[V(x, p' + i\varepsilon n, \ldots) - V(x, p - i\varepsilon n, \ldots)]}{(p'_0 - p_0 + 2i\varepsilon)} \Big\} \Phi(X, x)$$

$$= (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \int d^3\vec{x}\, e^{ip'_0 X^0} \overline{\psi}(x) \Big\{ i\gamma_0 \overleftrightarrow{\partial}_{20} - (p'_0 + p_0) \frac{[V(x, p' + i\varepsilon n, \ldots) - V(x, p - i\varepsilon n, \ldots]}{(p'_0 - p_0 + 2i\varepsilon)} \Big\} \phi(x) e^{-ip_0 X^0}, \quad (2.22)$$

$$(\Psi_{p',a}, \Psi_{p,b})_{p'^2 = p^2} = (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \int d^3\vec{x}\, e^{ip'_0 X^0} \overline{\psi}_a(x) \left[i\gamma_0 \overleftrightarrow{\partial}_{20} - 4p_0^2 \frac{\partial V}{\partial p^2} \right] \psi_b(x) e^{-ip_0 X^0} = (2\pi)^3 2p_0 \delta^3(\vec{p}' - \vec{p}) \delta_{ab} f_a(p^2), \quad (2.23)$$

where $\overline{\Psi} = \Psi^+ \gamma_0$ and V is again assumed to be superficially hermitian ($\gamma_0 V^+ \gamma_0 = V$ in the usual L^2 norm, when p_μ are replaced by real eigenvalues); f_a has the same expression as in Eq. (2.21). The comments made about the positivity of the norm in the two-boson case remain also valid here.

For fermion-antifermion systems, the expressions of the scalar product and of the norm are:

$$
\begin{aligned}
&(\Psi_{p'}, \Phi_p)_{p'^2 \neq p^2} \\
&= \int d^3\vec{X} d^3\vec{x} \,\mathrm{Tr}\,\Big\{ \overline{\Psi}(X, x)\Big[\gamma_0\eta_0 - V(x, p'+i\varepsilon n, \ldots)\gamma_0\eta_0 V(x, p-i\varepsilon n, \ldots) \\
&+ (p'_0+p_0)\frac{[V(x, p'+i\varepsilon n, \ldots) - V(x, p-i\varepsilon n, \ldots)]}{(p'_0-p_0+2i\varepsilon)}\Big]\Phi(X, x)\Big\} \\
&= (2\pi)^3\delta^3(\vec{p}\,'-\vec{p})e^{i(p'_0-p_0)X^0} \\
&\int d^3\vec{x}\,\mathrm{Tr}\,\Big\{ \overline{\psi}(x)\Big[\gamma_0\eta_0 - V(x, p'+i\varepsilon n, \ldots)\gamma_0\eta_0 V(x, p-i\varepsilon n, \ldots) \\
&+ (p'_0+p_0)\frac{[V(x, p'+i\varepsilon n, \ldots) - V(x, p-i\varepsilon n, \ldots)]}{(p'_0-p_0+2i\varepsilon)}\Big]\phi(x)\Big\},
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
(\Psi_{p',a}, \Psi_{p,b})_{p'^2=p^2} &= (2\pi)^3\delta^3(\vec{p}\,' - \vec{p}) \\
&\int d^3\vec{x}\,\mathrm{Tr}\left\{ \overline{\psi}_a(x)\left[\gamma_0\eta_0 - V\gamma_0\eta_0V + 4p_0^2\frac{\partial V}{\partial p^2}\right]\psi_b(x)\right\} \\
&= (2\pi)^3 2p_0\delta^3(\vec{p}\,' - \vec{p})\delta_{ab}f_a(p^2),
\end{aligned}
\tag{2.25}
$$

where $\overline{\Psi} = [\gamma_0\eta_0\Psi]^+$ and V is assumed to be superficially hermitian ($\gamma_0\eta_0V^+\gamma_0\eta_0 = V$ in the usual L^2 norm, when p_μ are replaced by real eigenvalues); f_a has the same expression as in (2.21).

If V is independent of p^2 in the C. M. frame, the scalar product and the norm do not reduce here to their « free » expressions, as in the two preceding cases, because of the presence of the term $V\gamma_0\eta_0V$. However if V satisfies the inequality

$$\frac{1}{4}\,\mathrm{Tr}\,(\gamma\cdot\hat{p}\eta\cdot\hat{p}V)^2 < 1\,, \tag{2.26}$$

one can apply the transformations

$$\Psi = [1 - (\gamma\cdot\hat{p}\eta\cdot\hat{p}V)^2]^{-1/2}\Psi' \tag{2.27}$$

on the wave functions and bring the scalar product and the norm to their free expressions in the C. M. frame [9]. Then, the norm becomes manifestly positive. If $\dfrac{\partial V}{\partial p^2} \neq 0$ in the C. M. frame, the norm still remains positive for physical states, although its manifest positivity is lost.

3. ON THE UNITARITY PROPERTY OF THE FW TRANSFORMATION

In order to discuss the unitarity property of the FW transformation in the two-particle case, we shall first consider the general case of a transformation in one-particle quantum mechanics, which illustrates this question in some detail.

Let us assume that we dispose, in one-particle quantum mechanics, of a wave equation satisfied by wave functions ψ belonging to a Hilbert space $\mathcal{H}$:

$$H\psi = 0. \tag{3.1}$$

(H is not the Hamiltonian).

The scalar product for the corresponding wave functions is assumed to be defined by means of a kernel K in the three-dimensional space:

$$(\psi, \phi) = \int d^3\vec{x}\psi^+ K\phi, \tag{3.2}$$

where K may also act as an integral operator. ψ^+ is the adjoint of ψ, i. e., the complex conjugate of ψ in the spin-0 case and the adjoint spinor in the spin-$\frac{1}{2}$ case.

Let us now assume that there exists a nonsingular transformation operator S which modifies the wave equation operator H (3.1), as well as the wave functions ψ:

$$SHS^{-1} = \tilde{H}, \tag{3.3 a}$$

$$S\psi = \tilde{\psi}. \tag{3.3 b}$$

In this case, the Hilbert space $\mathcal{H}$ is transformed into another Hilbert space $\tilde{\mathcal{H}}$.

Strictly speaking, the transformation S does not define an operator in the usual sense, for, it does not transform the states of the Hilbert space $\mathcal{H}$ into one another. Instead, it establishes a one-to-one connection between two different Hilbert spaces. For this reason the study of the unitarity property of the above transformation demands some care.

We admit that the transformation S is unitary if it preserves the scalar product of the Hilbert space $\mathcal{H}$. In this case, we must have

$$(\psi, \phi) = (\tilde{\psi}, \tilde{\phi}). \tag{3.4}$$

However, the scalar product in $\tilde{\mathcal{H}}$ is not defined *a priori*. We can use Eq. (3.4) for the definition of the scalar product $(\tilde{\psi}, \tilde{\phi})$; then the transformation S will automatically be unitary.

Equations (3.2), (3.3 *b*) and (3.4) imply that the scalar product in $\tilde{\mathcal{H}}$ must be defined as

$$(\tilde{\psi}, \tilde{\phi}) = \int d^3\vec{x}\tilde{\psi}^+(S^+)^{-1}KS^{-1}\tilde{\phi}, \tag{3.5}$$

where S^+ is the hermitic conjugate of the operator S, calculated in the usual L^2 norm (with kernel 1). This shows that the kernel of the scalar product in $\tilde{\mathcal{H}}$ is now

$$\tilde{K} = (S^+)^{-1}KS^{-1}. \tag{3.6}$$

Once the scalar product in $\tilde{\mathcal{H}}$ is defined by Eq. (3.5), the transformation S will have a unitary realization. We emphasize here the fact in the above considerations, it is the nonsingular nature of the operator S which is crucial. We could multiply S by arbitrary constants and still have a unitary realization for it by means of a corresponding modification of the right-hand side of Eq. (3.5).

It can also be seen that the definition (3.4) of the unitarity of S preserves the hermiticity property of the operators in their transformation. Let $\tilde{A}$ be a hermitian operator in $\tilde{\mathcal{H}}$. Then, it satisfies the equation

$$\int d^3\vec{x}\tilde{\psi}^+(S^+)^{-1}KS^{-1}\tilde{A}\tilde{\phi} = \int d^3\vec{x}\tilde{\psi}^+\tilde{A}^+(S^+)^{-1}KS^{-1}\tilde{\phi}, \tag{3.7}$$

where again $\tilde{A}^+$ is the hermitic conjugate of $\tilde{A}$ in the usual L^2 norm (with kernel 1). The transform of $\tilde{A}$ in $\mathcal{H}$ is

$$A = S^{-1}\tilde{A}S. \tag{3.8}$$

Then Eq. (3.8), together with (3.3 *b*) yields

$$\int d^3\vec{x}\psi^+KA\phi = \int d^3\vec{x}\psi^+A^+K\phi, \tag{3.9}$$

which shows that A is hermitian in $\mathcal{H}$.

Coming now back to the particular case of the FW transformation in one-particle quantum mechanics, we notice that the kernel K here is simply equal to unity; this feature simplifies the preceding formulas. It is customary to choose the arbitrary multiplicative factor in S in such a way that the kernel $\tilde{K}$ (3.6), is also to unity, that is, now, S is unitary in the ordinary L^2 norm.

The problem is more complicated in the two-particle case. The expressions of the scalar products (2.19)-(2.25) show that their kernels are not equal to unity and therefore the general analysis of this section must be applied. What is actually important, is the nonsingular nature of the FW transformation. Its unitarity is realized by an appropriate transformation of the kernel of the scalar product of the initial Hilbert space $\mathcal{H}$.

Let $\Psi_{p'}$ and Φ_p be two eigenfunctions of the total momentum operator $i\nabla_\mu$ with eigenvalues p'_μ and p_μ respectively. We write the scalar products (2.19)-(2.25) in the condensed form

$$(\Psi_{p'}, \Phi_p) = \int d^3\vec{X}d^3\vec{x}\Psi_{p'}K_{p'p}\Phi_p. \tag{3.10}$$

We assume that the FW transformation S in the two-particle case has been constructed, as a nonsingular operator, up to an arbitrary multiplicative constant:

$$\begin{aligned} S\Phi_p &= \tilde{\Phi}_p, \\ S'\Psi_{p'} &= \tilde{\Psi}_{p'}. \end{aligned} \tag{3.11}$$

The notation S′ indicates that the eigenvalue of the operator $i\nabla_\mu$ in S has been replaced by p'_μ instead of p_μ.

Then the unitarity of S will be ensured by the equality of the scalar products in $\mathscr{H}$ and $\tilde{\mathscr{H}}$:

$$(\Psi_{p'}, \Phi_p) = (\tilde{\Psi}_{p'}, \tilde{\Phi}_p), \tag{3.12}$$

which shows that the expression of the scalar product in $\tilde{\mathscr{H}}$ must be

$$(\tilde{\Psi}_{p'}, \tilde{\Phi}_p) = \int d^3\vec{X}d^3\vec{x}\tilde{\Psi}^+_{p'}(S'^+)^{-1}K_{p'p}S^{-1}\tilde{\Phi}_p, \tag{3.13}$$

where the kernel is now

$$\tilde{K}_{p'p} = (S'^+)^{-1}K_{p'p}S^{-1}. \tag{3.14}$$

In order to fix the arbitrary multiplicative constant in S we make the following convention. We noticed in Section 2, that when the potential V is explicitly independent of p^2 (that is, independent of p^2 in the C. M. frame), then the kernels of the various scalar products reduce in the C. M. frame to their « free » expressions (In the fermion-antifermion case, one must also make the transformation (2.27).) We shall fix the arbitrary multiplicative constant in S in such a way that when V is explicitly independent of p^2, then S becomes unitary in the C. M. frame in the « free » expressions of the scalar products in $\mathscr{H}$. In other words, the kernel $\tilde{K}$ (3.14), takes also in this case its « free » expression.

4. THE FW TRANSFORMATION IN THE FREE CASE

We consider in this section the case of two free particle systems, the wave function of which is an eigenfunction of the total momentum $i\nabla_\mu$ with eigenvalue p_μ. We shall concentrate, for the sake of manifest covariance, on systems containing at least one fermion. It is worthwhile to recall that in the two-particle case, the role of the energy eigenvalue of the one-particle case is played by the two Lorentz invariant eigenvalues of the longitudinal component operators $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$ (2.14).

We first consider the fermion-boson system. Eqs. (2.2) become

$$\mathrm{H}_1\Psi = (\gamma\cdot p_1 - m_1)\Psi = 0\,, \tag{4.1 a}$$

$$\mathrm{H}_2\Psi = (p_2^2 - m_2^2)\Psi = 0\,, \tag{4.1 b}$$

where Ψ is a four component spinor function (2.3).

We define the FW transformation as the transformation which brings the four-component spinor Ψ into a two-component spinor $\tilde{\Psi}$, the latter being defined as an eigenfunction of the matrix $\gamma\cdot\hat{p}$. This transformation will mainly concern here the fermionic part of the system (particle 1), that is, Eq. (4.1 *a*). The transformation $\mathrm{S}_1 \equiv \mathrm{S}$ is then defined by the equations

$$\mathrm{S}_1\gamma\cdot\hat{p}(\gamma\cdot p_1 - m_1)\mathrm{S}_1^{-1} = \hat{p}\cdot p_1 - \gamma\cdot\hat{p}[m_1^2 - (\gamma\cdot p_1)^2 + (\hat{p}\cdot p_1)^2]^{1/2}, \tag{4.2}$$

$$\mathrm{S}_1\Psi = \tilde{\Psi}\,, \tag{4.3}$$

$$\tilde{\mathrm{H}}_1\tilde{\Psi} = \{\, \hat{p}\cdot p_1 - \gamma\cdot\hat{p}[m_1^2 - (\gamma\cdot p_1)^2 + (\hat{p}\cdot p_1)^2]^{1/2} \,\}\,\tilde{\Psi} = 0\,, \tag{4.4}$$

$$\tilde{\mathrm{H}}_1 \equiv \mathrm{S}_1\gamma\cdot\hat{p}\mathrm{H}_1\mathrm{S}_1^{-1}\,. \tag{4.5}$$

Since the wave equation (4.4) commutes with $\gamma\cdot\hat{p}$, then $\tilde{\Psi}$ can be classified according to the eigenvalues of the matrix $\gamma\cdot\hat{p}$ and will have two non-zero components only:

$$\gamma\cdot\hat{p}\tilde{\Psi} = \pm\,\tilde{\Psi}\,. \tag{4.6}$$

It is at the same time a solution of the equation

$$(p_1^2 - m_1^2)\tilde{\Psi} = 0\,, \tag{4.7}$$

and of Eq. (4.1 *b*), which is not affected by the operator S.

For positive eigenvalues of $\gamma\cdot\hat{p}$, $\hat{p}\cdot p_1$ will have, through Eq. (4.4), positive eigenvalues, given by (2.14).

The expression of the operator S (4.2)-(4.3), is

$$\mathrm{S}_1 = \mathrm{A}_1\left(1 + a_1\gamma\cdot\hat{p} - a_1\,\frac{\gamma\cdot p_1}{\hat{p}\cdot p_1}\right), \tag{4.8}$$

where the quantities a_1 and A_1 have the expressions:

$$a_1 = \frac{1}{\hat{p}\cdot p_1}\{\,[m_1^2 - (\gamma\cdot p_1)^2 + (\hat{p}\cdot p_1)^2]^{1/2} - m_1\,\}\left(1 - \frac{(\gamma\cdot p_1)^2}{(\hat{p}\cdot p_1)^2}\right)^{-1}, \tag{4.9}$$

$$\mathrm{A}_1 = \left\{\frac{1}{2}\cdot\frac{[m_1^2 - (\gamma\cdot p_1)^2 + (\hat{p}\cdot p_1)^2]^{1/2} + m_1}{[m_1^2 - (\gamma\cdot p_1)^2 + (\hat{p}\cdot p_1)^2]^{1/2}}\right\}^{1/2}, \tag{4.10}$$

and the multiplicative factor A has been fixed according to the convention made at the end of Section 3.

The inverse of the operator S is

$$S_1^{-1} = A_1\left(1 - a_1\gamma\cdot\hat{p} + a_1\frac{\gamma\cdot p_1}{\hat{p}\cdot p_1}\right). \tag{4.11}$$

In practical calculations one solves Eqs. (4.1 *b*) and (4.7) for $\tilde{\Psi}$ and then searches for the expression of Ψ in terms of $\tilde{\Psi}$. Using Eqs. (4.1 *b*) and (4.7), one finds for Ψ, for the positive eigenvalues of $\hat{p}\cdot p_1$:

$$\Psi = S_1^{-1}\tilde{\Psi} = \left[\frac{1}{2\hat{p}\cdot p_1(m_1 + \hat{p}\cdot p_1)}\right]^{1/2}(m_1 + \gamma\cdot p_1)\tilde{\Psi}. \tag{4.12}$$

It is to be noticed that if one takes for the mass of the boson the limit $m_2 \to \infty$, the vector $\hat{p}$ will then have components $(1, \vec{0})$ and the expression of the operator S_1 becomes identical to that of one-particle mechanics.

We next consider the fermion-antifermion case. Eqs. (2.4) become

$$H_1\Psi = (\gamma\cdot p_1 - m_1)\Psi = 0, \tag{4.13 a}$$

$$H_2\Psi = (\eta\cdot p_2 + m_2)\Psi = 0, \tag{4.13 b}$$

where Ψ is a 16-component spinor function of rank two (2.5), or a 4×4 matrix function.

The FW transformation transforms Ψ into a 2×2 matrix function $\tilde{\Psi}$, which is simultaneoulsy an eigenfunction of $\gamma\cdot\hat{p}$ and $\eta\cdot\hat{p}$:

$$\gamma\cdot\hat{p}\tilde{\Psi} = \pm\,\tilde{\Psi},$$
$$\eta\cdot\hat{p}\tilde{\Psi} = \pm\,\tilde{\Psi}. \tag{4.14}$$

The FW transformation operator S can be constructed as a product of two independent operators S_1 and S_2, acting on the variables of particles 1 and 2, respectively:

$$S = S_1S_2, \tag{4.15}$$

$$S\Psi = \tilde{\Psi}. \tag{4.16}$$

The operator S_1 has the same properties and expression as the one considered in the fermion-boson case [Eqs. (4.2), (4.4), (4.5) and (4.8)-(4.11)]. The operator S_2 is analogously defined:

$$S_2\eta\cdot\hat{p}(\eta\cdot p_2 + m_2)S_2^{-1} = \hat{p}\cdot p_2 + \eta\cdot\hat{p}[m_2^2 - (\eta\cdot p_2)^2 + (\hat{p}\cdot p_2)^2]^{1/2}, \tag{4.17}$$

$$\tilde{H}_2\tilde{\Psi} = \{\,\hat{p}\cdot p_2 + \eta\cdot\hat{p}[m_2^2 - (\eta\cdot p_2)^2 + (\hat{p}\cdot p_2]^{1/2}\,\}\,\tilde{\Psi} = 0, \tag{4.18}$$

$$\tilde{H}_2 \equiv S_2\eta\cdot\hat{p}_2H_2S_2^{-1}. \tag{4.19}$$

The expression of S_2 is given by:

$$S_2 = A_2\left(1 - a_2\eta\cdot\hat{p} + a_2\,\frac{\eta\cdot p_2}{\hat{p}\cdot p_2}\right), \tag{4.20}$$

with

$$a_2 = \frac{1}{\hat{p}\cdot p_2}\left\{\left[m_2^2-(\eta\cdot p_2)^2 + (\hat{p}\cdot p_2)^2\right]^{1/2} - m_2\right\}\left(1 - \frac{(\eta\cdot p_2)^2}{(\hat{p}\cdot p_2)^2}\right)^{-1}, \tag{4.21}$$

$$A_2 = \left\{\frac{1}{2}\cdot\frac{\left[m_2^2 - (\eta\cdot p_2)^2 + (\hat{p}\cdot p_2)^2\right]^{1/2} + m_2}{\left[m_2^2 - (\eta\cdot p_2)^2 + (\hat{p}\cdot p_2)^2\right]^{1/2}}\right\}^{1/2}, \tag{4.22}$$

$$S_2^{-1} = A_2\left(1 + a_2\eta\cdot\hat{p} - a_2\,\frac{\eta\cdot p_2}{\hat{p}\cdot p_2}\right). \tag{4.23}$$

The wave function $\tilde{\Psi}$ satisfies the Klein-Gordon equation (4.7) and

$$(p_2^2 - m_2^2)\tilde{\Psi} = 0. \tag{4.24}$$

Furthermore the positive eigenvalue solution in $\hat{p}\cdot p_2$ corresponds to the negative eigenvalue of $\eta\cdot\hat{p}$ (The antifermion has a negative parity.)

Finally, the expression of Ψ, in terms of $\tilde{\Psi}$, for positive eigenvalues of $\hat{p}\ p_1$ and $\hat{p}.p_2$, is given by:

$$\Psi = S^{-1}\tilde{\Psi}$$

$$= \left[\frac{1}{2\hat{p}\cdot p_1(m_1 + \hat{p}\cdot p_1)}\right]^{1/2}\left[\frac{1}{2\hat{p}\cdot p_2(m_2 + \hat{p}\cdot p_2)}\right]^{1/2}(m_1 + \gamma\cdot p_1)(m_2 - \eta\cdot p_2)\tilde{\Psi}. \tag{4.25}$$

5. THE FW TRANSFORMATION IN THE INTERACTING CASE

This section is devoted to the construction of the FW transformation in the case of special classes of interaction where it takes a compact form. A similar phenomenon also occurs in the one-particle case. For instance, if the fermion interacts with a static magnetic potential, then the corresponding FW transformation has a compact form and need not be calculated by series expansion methods [*6*]. In effect, in this case, the Dirac Hamiltonian is

$$p_0 = \gamma_5\vec{\sigma}\cdot(\vec{p} - \vec{A}) + m\gamma_0, \qquad \vec{A} = \vec{A}(\vec{x}), \tag{5.1}$$

and its square is

$$E^2 \equiv p_0^2 = (\vec{\sigma}\cdot(\vec{p} - \vec{A}))^2 + m^2. \tag{5.2}$$

It commutes with γ_0 and therefore the FW transformation operator has a form similar to that of the free case:

$$S = \frac{1}{2}\left(\frac{2E}{m + E}\right)^{1/2}\left(1 + \gamma_0\,\frac{p_0}{E}\right). \tag{5.3}$$

It transforms p_0 into $\tilde{p}_0$:

$$\tilde{p}_0 = \gamma_0 \mathrm{E} . \tag{5.4}$$

It turns out that also in the two-particle case there are classes of interaction where the FW transformation takes a form similar to that of the free case, the only modification consisting in changing the definition of the variables used in the latter case. We shall consider again the two systems considered in Section 4, the fermion-boson and the fermion-antifermion systems.

5.1. **Fermion-boson systems.**

The wave equations of the systems are given by Eqs. (2.2). However, Eq. (2.2 *b*) can still be modified. We can bring the operator $\gamma \cdot p_1$ on the right of V and use Eq. (2.2 *a*) to eliminate the operators $(\gamma \cdot p_1 - m_1)$ from Eq. (2.2 *b*). We then get the equation

$$\mathrm{H}_2 \Psi \equiv (p_2^2 - m_2^2 - [\gamma \cdot p_1, \mathrm{V}]_+ + \mathrm{V}^2)\Psi = 0 . \tag{5.5}$$

($[\, , \,]_+$ is the anticommutator.)

We now consider the classes of interaction potential satisfying the following condition:

$$[\gamma \cdot \hat{p}, \mathrm{V}]_+ = 0 . \tag{5.6}$$

This means that for parity conserving interactions, the potential V essentially represents a vector interaction of the transverse type:

$$\mathrm{V} = \gamma^{\mathrm{T}} \cdot \mathrm{W}^{\mathrm{T}} \tag{5.7}$$

where the function $\mathrm{W}_\mu^{\mathrm{T}}$ does no longer contain the Dirac matrices (except, eventually, the longitudinal matrix $\gamma \cdot \hat{p}$), and the transverse vectors γ^{T} and W^{T} are defined like x^{T} in Eqs. (2.10).

Condition (5.6) has several consequences. We define two vectors $p'_{1\mu}$ and $p'_{2\mu}$ such that

$$p'_{1\mu} = p_{1\mu} - \mathrm{W}_\mu^{\mathrm{T}} , \tag{5.8 a}$$

$$p'_{2\mu} = p_{2\mu} + \mathrm{W}_\mu^{\mathrm{T}} . \tag{5.8 b}$$

They satisfy the obvious properties

$$\hat{p} \cdot p'_1 = \hat{p} \cdot p_1 , \tag{5.9 a}$$

$$\hat{p} \cdot p'_2 = \hat{p} \cdot p_2 . \tag{5.9 b}$$

Equations (2.2 *a*) and (5.5) can then be written as

$$\mathrm{H}_1 \Psi \equiv (\gamma \cdot p'_1 - m_1)\Psi = 0 , \tag{5.10 a}$$

$$\mathrm{H}_2 \Psi \equiv [(\gamma \cdot p'_2)^2 - m_2^2]\Psi = 0 . \tag{5.10 b}$$

The « square » of Eq. (5.10 *a*) is also equal to

$$\mathrm{H}_1(\mathrm{H}_1 + 2m_1)\Psi = [(\gamma \cdot p'_1)^2 - m_1^2]\Psi = 0 , \tag{5.11}$$

from which we deduce that

$$(\gamma\cdot p_1')^2 - (\gamma\cdot p_2')^2 = p_1^2 - p_2^2\,, \tag{5.12}$$

which is nothing but Eq. (2.12) in its strong form. Furthermore, because of Eqs. (5.12) and (2.9), the operator $\gamma\cdot p_1'$ commutes with $(\gamma\cdot p_2')^2$:

$$[\gamma\cdot p_1', (\gamma\cdot p_2')^2] = 0\,. \tag{5.13}$$

As a last consequence of condition (5.6), by using the mixed type Jacobi identity

$$[\mathrm{A}, [\mathrm{B}, \mathrm{C}]_+] = [[\mathrm{A}, \mathrm{B}]_+, \mathrm{C}] - [\mathrm{B}, [\mathrm{A}, \mathrm{C}]_+]\,, \tag{5.14}$$

we deduce from Eqs. (5.5), (5.10 *b*), (2.9) and (5.12) that the matrix $\gamma\cdot\hat{p}$ commutes with the operators $(\gamma\cdot\hat{p}_2')^2$ and $(\gamma\cdot\hat{p}_1')^2$:

$$[\gamma\cdot\hat{p}, (\gamma\cdot p_2')^2] = [\gamma\cdot\hat{p}, (\gamma\cdot p_1')^2] = 0\,. \tag{5.15}$$

This means that the solutions of the « square » operator equation (5.10 *b*) or (5.11) can be classified according to the eigenvalues of the matrix $\gamma\cdot\hat{p}$.

Properties (5.15), (5.13), (5.9) and (5.6) are satisfied in the free case, with p_1' and p_2' replaced by p_1 and p_2, respectively [Section 4]. This has the consequence that the FW transformation operator can be obtained from its expression of the free case by the formal replacements of p_1 and p_2 by p_1' and p_2', respectively. We get:

$$\mathrm{S}_1 = \mathrm{A}_1\left(1 + a_1\gamma\cdot\hat{p} - a_1\frac{\gamma\cdot p_1'}{\hat{p}\cdot p_1}\right),$$

$$\mathrm{S}_1^{-1} = \mathrm{A}_1\left(1 - a_1\gamma\cdot\hat{p} + a_1\frac{\gamma\cdot p_1'}{\hat{p}\cdot p_1}\right),$$

$$a_1 = \frac{1}{\hat{p}\cdot p_1}\{[m_1^2-(\gamma\cdot p_1')^2+(\hat{p}\cdot p_1)^2]^{1/2}-m_1\}\left(1-\frac{(\gamma\cdot p_1')^2}{(\hat{p}\cdot p_1)^2}\right)^{-1},$$

$$\mathrm{A}_1 = \left\{\frac{1}{2}\cdot\frac{[m_1^2-(\gamma\cdot p_1')^2+(\hat{p}\cdot p_1)^2]^{1/2}+m_1}{[m_1^2-(\gamma\cdot p_1')^2+(\hat{p}\cdot p_1)^2]^{1/2}}\right\}^{1/2},$$

$$\mathrm{S}_1\gamma\cdot\hat{p}(\gamma\cdot p_1'-m_1)\mathrm{S}_1^{-1} = \hat{p}\cdot p_1-\gamma\cdot\hat{p}[m_1^2-(\gamma\cdot p_1')^2+(\hat{p}\cdot p_1)^2]^{1/2},$$

$$\mathrm{S}_1\Psi = \tilde{\Psi},$$

$$\tilde{\mathrm{H}}_1 \equiv \mathrm{S}_1\gamma\cdot\hat{p}\mathrm{H}_1\mathrm{S}_1^{-1},$$

$$\tilde{\mathrm{H}}_1\tilde{\Psi} = \{\hat{p}\cdot p_1-\gamma\cdot\hat{p}[m_1^2-(\gamma\cdot p_1')^2+(\hat{p}\cdot p_1)^2]^{1/2}\}\tilde{\Psi} = 0\,,$$

$$\gamma\cdot\hat{p}\tilde{\Psi} = \pm\tilde{\Psi},$$

$$[(\gamma\cdot p_1')^2 - m_1^2]\tilde{\Psi} = [(\gamma\cdot p_2')^2 - m_2^2]\tilde{\Psi} = 0\,. \tag{5.16}$$

For the positive eigenvalues of $\hat{p}\cdot p_1$, corresponding to the positive eigenvalues of $\gamma\cdot\hat{p}$ for $\tilde{\Psi}$, we also get:

$$\Psi = \mathrm{S}_1^{-1}\tilde{\Psi} = \left[\frac{1}{2\hat{p}\cdot p_1(m_1 + \hat{p}\cdot p_1)}\right]^{1/2}(m_1 + \gamma\cdot p_1')\tilde{\Psi}\,. \tag{5.17}$$

Examples of potentials V satisfying condition (5.6) were presented in Ref. [*9*]:

a)
$$\mathrm{V} = \gamma_\mu^{\mathrm{T}}(\mathrm{C}v^{\mathrm{T}\mu} + i\hbar\dot{\mathrm{C}}x^{\mathrm{T}\mu}),$$
$$\mathrm{C} = \mathrm{C}(x^{\mathrm{T}^2}, p^2), \qquad \dot{\mathrm{C}} = \frac{\partial \mathrm{C}}{\partial x^{\mathrm{T}^2}},$$
$$\begin{aligned}\mathrm{H}_2 = {} & p_2^2 - m_2^2 + \mathrm{C}(2 + \mathrm{C})v^{\mathrm{T}^2} + 4i\hbar\dot{\mathrm{C}}(1 + \mathrm{C})x^{\mathrm{T}}\cdot v^{\mathrm{T}} \\ & - \frac{\hbar^2}{2}(1 + \mathrm{C})(\partial^2\mathrm{C}) - \hbar^2 x^{\mathrm{T}^2}\dot{\mathrm{C}}^2 \\ & - \frac{4}{p^2}\mathrm{W_L}\cdot\mathrm{W_{1S}}\dot{\mathrm{C}}(1 + \mathrm{C})\,,\end{aligned} \tag{5.18}$$

where $\mathrm{W_L}$ and $\mathrm{W_{1S}}$ are the orbital angular momentum and spin operators:

$$\mathrm{W}_{\mathrm{L}\mu} = \varepsilon_{\mu\nu\alpha\beta}p^\nu x^{\mathrm{T}\alpha}v^\beta, \qquad \varepsilon_{0123} = +1\,, \tag{5.19}$$

$$\mathrm{W}_{1\mathrm{S}\mu} = -\frac{\hbar}{4}\varepsilon_{\mu\nu\alpha\beta}p^\nu\sigma^{\alpha\beta}\,. \tag{5.20}$$

b)
$$\mathrm{V} = \gamma_\mu^{\mathrm{T}}\mathrm{C}(x^{\mathrm{T}^2}v^{\mathrm{T}\mu} - x^{\mathrm{T}\mu}x^{\mathrm{T}}\cdot v^{\mathrm{T}} - i\hbar x^{\mathrm{T}\mu}),$$
$$\mathrm{C} = \mathrm{C}(x^{\mathrm{T}^2}, p^2), \qquad \dot{\mathrm{C}} = \frac{\partial \mathrm{C}}{\partial x^{\mathrm{T}^2}},$$
$$\begin{aligned}\mathrm{H}_2 = {} & p_2^2 - m_2^2 + \hbar^2(3\mathrm{C} + 2x^{\mathrm{T}^2}\dot{\mathrm{C}} + x^{\mathrm{T}^2}\mathrm{C}^2) \\ & - \frac{2}{p^2}\left(1 + x^{\mathrm{T}^2}\frac{\mathrm{C}}{2}\right)\mathrm{W_L^2} \\ & - \frac{2}{p^2}\mathrm{W_L}\cdot\mathrm{W_{1S}}(3\mathrm{C} + 2x^{\mathrm{T}^2}\dot{\mathrm{C}} + x^{\mathrm{T}^2}\mathrm{C}^2)\,.\end{aligned} \tag{5.21}$$

Application of this potential to confining interactions and fermion bound states was presented in Ref. [*12*].

5.2. **Fermion-antifermion systems.**

The wave equations of the system are given by Eqs. (2.4). The structure of these equations can be modified by « diagonalizing » them with respect to the operators $(\gamma\cdot p_1 - m_1)$ and $(\eta\cdot p_2 + m_2)$. To this end we can bring in Eq. (2.4 *a*) the operator $\eta\cdot p_2$ on the right of V and then eliminate it by using Eq. (2.4 *b*). A similar procedure can be applied for the operator $\gamma \cdot p_1$ in Eq. (2.4 *b*). Equations (2.4) then become

$$\mathrm{H}_1\Psi \equiv \{ \gamma \cdot p_1 - m_1 + (1-\mathrm{V}^2)^{-1}([\eta \cdot p_2, \mathrm{V}]_+ + \mathrm{V}[\gamma \cdot p_1, \mathrm{V}]_+) \} \Psi = 0, \qquad (5.22\,a)$$

$$\mathrm{H}_2\Psi \equiv \{ \eta \cdot p_2 + m_2 + (1-\mathrm{V}^2)^{-1}([\gamma \cdot p_1, \mathrm{V}]_+ + \mathrm{V}[\eta \cdot p_2, \mathrm{V}]_+) \} \Psi = 0. \qquad (5.22\,b)$$

Furthermore, transformation (2.27) can be applied, in order to have the « free » norm in the C. M. frame for p^2-independent potentials there (in order to eliminate the quadratic term in V in the kernel of the scalar product (2.24)-(2.25)).

We shall write the resulting equations in the more condensed form (by dropping the prime from the new wave function Ψ')

$$\mathrm{H}_1\Psi \equiv (\gamma \cdot p_1 - m_1 - \mathrm{V}'_1)\Psi = 0, \qquad (5.23\,a)$$

$$\mathrm{H}_2\Psi \equiv (\eta \cdot p_2 + m_2 + \mathrm{V}'_2)\Psi = 0, \qquad (5.23\,b)$$

where the new potentials V'_1 and V'_2 are functions of the initial potential V and of its derivatives, of the momentum operators and of the Dirac matrices. They satisfy Eq. (2.9):

$$[p_1^2 - p_2^2, \mathrm{V}'_1] = [p_1^2 - p_2^2, \mathrm{V}'_2] = 0, \qquad (5.24)$$

and are related to each other by charge conjugation invariance:

$$\mathrm{V}'_1 = \mathrm{V}'_1(1, 2; \gamma, \eta),$$
$$\mathrm{V}'_2 = \mathrm{V}'_1(2, 1; -\eta, -\gamma), \qquad (5.25)$$

the indices 1 and 2 inside the parentheses representing particle indices in the coordinates, momenta and masses.

The wave equations (5.23) can be rewritten in a still more condensed form, by introducing, as in the fermion-boson case, generalized momentum operators p'_1 and p'_2:

$$\mathrm{H}_1\Psi \equiv (\gamma \cdot p'_1 - m_1)\Psi = 0, \qquad (5.26\,a)$$

$$\mathrm{H}_2\Psi \equiv (\eta \cdot p'_2 + m_2)\Psi = 0. \qquad (5.26\,b)$$

This notation is always possible. If the potential V'_1, say, contains the γ matrix, the latter can be factorized in the form $\mathrm{V}'_1 = \gamma_\mu \mathrm{W}_1^{\mu}$; otherwise we can write $\mathrm{V}'_1 = \frac{1}{4} \gamma_\mu \gamma^\mu \mathrm{V}'_1$.

We now come to the specification of the classes of potential which are of interest for the present purpose.

We assume that the potentials V'_1 and V'_2 satisfy the following anti-commutativity or commutativity relations with the matrices $\gamma \cdot \hat{p}$ and $\eta \cdot \hat{p}$:

$$[\gamma \cdot \hat{p}, \mathrm{V}'_1]_+ = 0, \qquad [\eta \cdot \hat{p}, \mathrm{V}'_1] = 0,$$
$$[\gamma \cdot \hat{p}, \mathrm{V}'_2] = 0, \qquad [\eta \cdot \hat{p}, \mathrm{V}'_2]_+ = 0. \qquad (5.27)$$

These conditions are generalizations of the condition (5.6) imposed in the fermion-boson case. They can be rewritten in terms of the operators $\gamma \cdot p'_1$ and $\eta \cdot p'_2$ [Eqs. (5.26)]:

$$
\begin{aligned}
&[\gamma \cdot \hat{p}, \gamma \cdot p'_1]_+ = 2\hat{p} \cdot p_1, \qquad &&[\eta \cdot \hat{p}, \gamma \cdot p'_1] = 0, \\
&[\gamma \cdot \hat{p}, \eta \cdot p'_2] = 0, \qquad &&[\eta \cdot \hat{p}, \eta \cdot p'_2]_+ = 2\hat{p} \cdot p_2 .
\end{aligned} \tag{5.28}
$$

Furthermore, we assume that the compatibility condition (2.8) is satisfied in the strong sense, which means that the operators $\gamma \cdot p'_1$ and $\eta \cdot p'_2$ commute strongly:

$$[\gamma \cdot p'_1, \eta \cdot p'_2] = 0 . \tag{5.29}$$

This equation also implies, of course, the strong commutativity of the squared operators $(\gamma \cdot p'_1)^2$ and $(\eta \cdot p'_2)^2$.

The « squares » of equations (5.26) are:

$$H_1(H_1 + 2m_1)\Psi = [(\gamma \cdot p'_1)^2 - m_1^2]\Psi = 0 , \tag{5.30 a}$$

$$H_2(H_2 - 2m_2)\Psi = [(\eta \cdot p'_2)^2 - m_2^2]\Psi = 0 . \tag{5.30 b}$$

Because of Eqs. (5.24) and (2.12), we also have:

$$(\gamma \cdot p'_1)^2 - (\eta \cdot p'_2)^2 = p_1^2 - p_2^2 , \tag{5.31}$$

which is the analog of Eq. (5.12) of the fermion-boson case [The strong equality results from the fact that $(\gamma \cdot p'_1)^2$ and $(\eta \cdot p'_1)^2$ commute strongly, Eq. (5.29). Then $[(\gamma \cdot p'_1)^2 - (\eta \cdot p'_2)^2, (\gamma \cdot p'_1)^2 + (\eta \cdot p'_2)^2] = 0$. The second term in the commutator contains the potentials V'_1 and V'_2; since the latter satisfy Eqs. (5.24), this equality can in general be satisfied only if Eq. (5.31) holds in the strong sense.]

As a lest result, we deduce from Eqs. (5.28), (5.24) and the mixed type Jacobi identity (5.14), that the matrices $\gamma \cdot \hat{p}$ and $\eta \cdot \hat{p}$ commute with the operators $(\gamma \cdot p'_1)^2$ and $(\eta \cdot p'_2)^2$:

$$
\begin{aligned}
&[\gamma \cdot \hat{p}, (\gamma \cdot p'_1)^2] = [\gamma \cdot \hat{p}, (\eta \cdot p'_2)^2] = 0 , \\
&[\eta \cdot \hat{p}, (\gamma \cdot p'_1)^2] = [\eta \cdot \hat{p}, (\eta \cdot p'_2)^2] = 0 .
\end{aligned} \tag{5.32}
$$

This means that the solutions of the « square » equations (5.30) can be classified according to the eigenvalues of the matrices $\gamma \cdot \hat{p}$ and $\eta \cdot \hat{p}$.

Properties (5.27)-(5.32) are satisfied in the free case, with $\gamma \cdot p'_1$ and $\eta \cdot p'_2$ replaced by $\gamma \cdot p_1$ and $\eta \cdot p_2$, respectively [Section 4]. This implies that the FW transformation operator can be obtained from its expression of the free case by the formal replacements of $\gamma \cdot p_1$ and $\eta \cdot p_2$ by $\gamma \cdot p'_1$ and $\eta \cdot p'_2$, respectively ($\hat{p} \cdot p_1$ and $\hat{p} \cdot p_2$ remain unchanged because of properties (5.28).)

The FW transformation operator S is constructed as a product of two independent and commuting operators S_1 and S_2, acting on the variables of particles 1 and 2, respectively:

$$S = S_1 S_2, \tag{5.33}$$

$$S\Psi = \tilde{\Psi}, \tag{5.34}$$

$$\gamma\cdot\hat{p}\tilde{\Psi} = \pm\tilde{\Psi}, \qquad \eta\cdot\hat{p}\tilde{\Psi} = \pm\,\tilde{\Psi}. \tag{5.35}$$

The operator S_1 has the same properties and expression as the corresponding one considered in the fermion-boson case [Eqs. (5.16)-(5.17)]. Furthermore, because of Eqs. (5.28)-(5.29) is commutes with $\eta\cdot p_2'$:

$$[S_1, \eta\cdot p_2'] = 0. \tag{5.36}$$

The operator S_2 is obtained from formulae (4.17)-(4.23) by the replacement of $\eta\cdot p_2$ by $\eta\cdot p_2'$:

$$S_2 = A_2\left(1 - a_2\eta\cdot\hat{p} + a_2\frac{\eta\cdot p_2'}{\hat{p}\cdot p_2}\right),$$

$$S_2^{-1} = A_2\left(1 + a_2\eta\cdot\hat{p} - a_2\frac{\eta\cdot p_2'}{\hat{p}\cdot p_2}\right),$$

$$a_2 = \frac{1}{\hat{p}\cdot p_2}\{[m_2^2 - (\eta\cdot p_2')^2 + (\hat{p}\cdot p_2)^2]^{1/2} - m_2\}\left(1 - \frac{(\eta\cdot p_2')^2}{(\hat{p}\cdot p_2)^2}\right)^{-1},$$

$$A_2 = \left\{\frac{1}{2}\cdot\frac{[m_2^2 - (\eta\cdot p_2')^2 + (\hat{p}\cdot p_2)^2]^{1/2} + m_2}{[m_2^2 - (\eta\cdot p_2')^2 + (\hat{p}\cdot p_2)^2]^{1/2}}\right\}^{1/2},$$

$$S_2\eta\cdot\hat{p}(\eta\cdot p_2' + m_2)S_2^{-1} = \hat{p}\cdot p_2 + \eta\cdot\hat{p}[m_2^2 - (\eta\cdot p_2')^2 + (\hat{p}\cdot p_2]^{1/2},$$

$$\tilde{H}_2 \equiv S_2\eta\cdot\hat{p}H_2S_2^{-1},$$

$$\tilde{H}_2\tilde{\Psi} = \{\hat{p}\cdot p_2 + \eta\cdot\hat{p}[m_2^2 - (\eta\cdot p_2')^2 + (\hat{p}\cdot p_2)^2]^{1/2}\}\tilde{\Psi} = 0,$$

$$[S_2, \gamma\cdot p_1'] = 0. \tag{5.37}$$

The positive eigenvalue solution in $\hat{p}\cdot p_2$ corresponds to the negative eigenvalue of $\eta\cdot\hat{p}$ (The antifermion has a negative parity.)

The physical solutions satisfy therefore the eigenvalue equations

$$\gamma\cdot\hat{p}\tilde{\Psi} = -\,\eta\cdot\hat{p}\tilde{\Psi} = \tilde{\Psi}, \tag{5.38}$$

$$[(\gamma\cdot p_1')^2 - m_1^2]\tilde{\Psi} = [(\eta\cdot p_2')^2 - m_2^2]\tilde{\Psi} = 0, \tag{5.39}$$

with the choice of positive eigenvalues of $\hat{p}\cdot p_1$ and $\hat{p}\cdot p_2$ from Eq. (5.39) (The two equations (5.39) are actually the same one, once Eqs. (2.12) and (5.31) are used.) For these solutions the expression of the wave function Ψ in terms of $\tilde{\Psi}$ is:

$$\Psi = S^{-1}\tilde{\Psi}$$

$$= \left[\frac{1}{2\hat{p}\cdot p_1(m_1 + \hat{p}\cdot p_1)}\right]^{1/2}\left[\frac{1}{2\hat{p}\cdot p_2(m_2 + \hat{p}\cdot p_2)}\right]^{1/2}(m_1 + \gamma\cdot p_1')(m_2 - \eta\cdot p_2')\tilde{\Psi}. \tag{5.40}$$

A first example, of a class of potentials satisfying the properties mentioned above is provided by the pseudoscalar type potential [9]

$$V = \gamma_5\eta_5 W, \qquad W = W(x^{T^2}, p^2), \tag{5.41}$$

leading to the wave equations [(5.23) or (5.26)]

$$H_1\Psi \equiv (\gamma\cdot p_1 - m_1 + i\hbar A\gamma_5\tilde{\eta}\cdot x^T)\Psi = 0, \tag{5.42 a}$$

$$H_2\Psi \equiv (\eta\cdot p_2 + m_2 - i\hbar A\tilde{\gamma}\cdot x^T\eta_5)\Psi = 0, \tag{5.42 b}$$

where A is defined as

$$A = \frac{-2\dot{W}}{1 - W^2}, \qquad \dot{W} = \frac{\partial W}{\partial x^{T^2}}. \tag{5.43}$$

(The matrices $\tilde{\gamma}$ and $\tilde{\eta}$ are defined by (2.7).)

One can check on Eqs. (5.42) that all the properties (5.24)-(5.32) are explicitly satisfied. The « square » equations (5.30) and consequently (5.39) become (after using Eq. (2.12))

$$\tilde{H}\tilde{\Psi} \equiv \left\{\frac{1}{4}p^2 - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4p^2} + v^{T^2} + \hbar^2A^2x^{T^2}\right.$$

$$\left. - \frac{4}{p^2}\dot{A}\gamma\cdot\hat{p}\eta\cdot\hat{p}W_{1S}\cdot W_{2S} - \frac{8}{p^2}\dot{A}\gamma\cdot\hat{p}\eta\cdot\hat{p}(W_{1S}\cdot x^T)(W_{2S}\cdot x^T)\right\}\tilde{\Psi} = 0, \tag{5.44}$$

where $\dot{A} = \partial A/(\partial x^{T^2})$. The spin operator W_{1S} was defined in Eq. (5.20); W_{2S} is the spin operator of the antifermion:

$$W_{2S\mu} = -\frac{\hbar}{4}\varepsilon_{\mu\nu\alpha\beta}p^\nu\xi^{\alpha\beta}, \tag{5.45}$$

where the matrix ξ was defined in (2.7).

This class of potentials is relevant for the study of confining interactions with spontaneous breakdown of chiral symmetry [13].

Other types of potentials satisfying the properties mentioned in this Subsection are provided by vector interactions of the transverse kind (the analogs of those considered at the end of Subsection 5.1) [9].

a)

$$H_1 = \gamma\cdot p_1 - m_1 - \gamma_\mu(Cv^{T\mu} + i\hbar\dot{C}x^{T\mu} + \hbar\dot{C}\xi^{TT\mu\alpha}x_\alpha^T), \tag{5.46 a}$$

$$H_2 = \eta\cdot p_2 + m_2 + \eta_\mu(Cv^{T\mu} + i\hbar\dot{C}x^{T\mu} + \hbar\dot{C}\sigma^{TT\mu\alpha}x_\alpha^T). \tag{5.46 b}$$

b)

$$H_1 = \gamma\cdot p_1 - m_1 - \gamma_\mu[C(x^{T^2}v^{T\mu} - x^{T\mu}x^T\cdot v^T - i\hbar x^{T\mu}) + \hbar B\xi^{TT\mu\nu}x_\nu^T], \tag{5.47 a}$$

$$H_2 = \eta\cdot p_2 + m_2 + \eta_\mu[C(x^{T^2}v^{T\mu} - x^{T\mu}x^T\cdot v^T - i\hbar x^{T\mu}) + \hbar B\sigma^{TT\mu\nu}x_\nu^T], \tag{5.47 b}$$

where

$$B = \frac{3C + 2\dot{C}x^{T^2} - x^{T^2}C^2}{2(1 - Cx^{T^2})}, \qquad C = C(x^{T^2}, p^2), \qquad \dot{C} = \frac{\partial C}{\partial x^{T^2}}. \tag{5.49}$$

The expressions of the « squares » of these operators can be found in Ref. [9], Section VII D.

6. CONCLUSION

We considered the Foldy-Wouthuysen transformation in the two-particle case. For Poincaré invariant and manifestly covariant two-particle systems, the problem is not essentially more complicated than in the one-particle case. On the contrary, one gains here manifest covariance which facilitates the construction of the FW transformation operator. This construction can be used in particular in the interacting case, where for several classes of potential, the transformation operator keeps a compact form. Here, one transforms the wave equations into equations of the Klein-Gordon type, but having much the structure of nonrelativistic quantum mechanics, with analogous quantum numbers.

It is also possible to introduce generalized FW transformations which would classify the transformed states with respect to the eigenvalues of the matrices γ_5 or $\gamma.\hat{p}\gamma_5$ [14]. However, in these cases the parity operator is also transformed, and in the new basis the classification of the solutions as parity eigenstates becomes more difficult.

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