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Behaviour of the Wilson parameter in U(1) lattice gauge theory with long range gauge invariant interactions

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ABSTRACT. — The U(1) lattice gauge model with fermions can be expressed after integration over the fermionic variables as a « long range gauge model »: the effective action is a sum over all possible gauge field loops with corresponding weight factors. Different behaviors of the Wilson parameter are shown according to the hypothesis on the weight factors.

RÉSUMÉ. — Le modèle de théorie de jauge sur réseau avec fermions peut être exprimé après intégration sur les variables fermioniques comme un modèle de jauge avec interactions à longue portée : l'action effective est une somme sur chaque boucle, du produit des variables de champ de jauge associées aux liens de la boucle, chaque terme étant affecté d'un facteur de poids. Différents comportements du paramètre de Wilson sont exhibés suivant les hypothèses sur les facteurs de poids.

1. INTRODUCTION

The purpose of this paper is to study the behavior of the Wilson parameter [*1*] in U(1) lattice gauge theory with long range gauge invariant

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interactions occuring in particular in lattice gauge theories with fermions [2]. These theories have been intensively studied analytically and recently also numerically by the Monte-Carlo method. The usual groups considered in a gauge invariant field theory are U(1), SU(N). One way to study the models consists in doing the « integration out » over the fermionic variables proposed by Matthews and Salam [3] [4] [5]; this « integration out » leads to an effective action which can be expressed as a sum over all possible gauge field loops affected with weight factors [2]. In the U(1) case the result is simple. For example in two-space-time dimension and for Susskind fermions [6], the lattice fermionic action coupled to a gauge field is given by (see [7]:

$$S = S_F + S_G$$

(1) $$\begin{aligned} S_F &\equiv (\bar\psi, G(u)\psi) \\ &\equiv \sum_{i,j} \{ \bar\psi_{ij} U^X_{ij,i+1j}\psi_{i+1j} - \bar\psi_{ij} U^X_{ij,i-1j}\psi_{i-1j} \\ &\qquad - i(-1)^{i+j}(\bar\psi_{ij}U^Y_{ij,ij+1}\psi_{ij+1} - \bar\psi_{ij}U^Y_{ij,ij-1}) + m\bar\psi_{ij}\psi_{ij} \} \end{aligned}$$

ψ and $\bar\psi$ are Grassman variables representing the fermion field. The couple (i, j) of integers represents the sites of the lattice. The one component variable ψ_{ij} with $i + j$ even or odd can be taken to represent respectively the field ψ^+ or ψ^-. $U^X_{ij,i+1j}$, $U^Y_{ij,ij+1}$ are the gauge field variables belonging to U(1) and indexed by links. They verify $U_{a,b} = \bar U_{b,a}$. S_G is the usual Wilson's lattice action.

(2) $$S_G = \beta \sum_p Re\,[\mathrm{tr}\, U(p)]$$

p represents an elementary square (plaquette) of the lattice and $U(p)$ is the product of the link variables associated to the plaquette p. To integrate out over the Grassman variables one uses the well known formulae (see [5])

$$\int d\psi d\bar\psi \exp(\bar\psi, Q\psi) = \det Q$$

Expanding $\det G(U) = \exp \mathrm{tr} \log G(U)$ by random walk techniques [8] [9], one obtains an effective action of the form

(3) $$S_{eff} = \sum_\Gamma J_\Gamma(m)\, Re\,[\mathrm{tr}\, U_\Gamma)$$

where U_Γ is the product of the link variables associated to the closed path Γ. The corresponding weight factors $J_\Gamma(m)$ depend on m and on Γ:

$$J_\Gamma(m) = \varepsilon(\Gamma) m^{-|\Gamma|},$$

$|\Gamma|$ representing the length of the path and $\varepsilon(\Gamma) = \pm 1$ according to the geometry of Γ. For « naive » fermions the result is similar.

The purpose of this paper is to study the behavior of the Wilson parameter for this kind of action according to different hypothesis on the interaction $J_\Gamma(m)$ in particular the interaction obtained from the Matthews-Salam expansion. The pure lattice gauge theory with action given by (2) is known to have a linear confinement in two dimension [*10*] a logarithmic confinement if three dimension [*11*] and is not confining at low temperature in four dimension [*12*]. We shall show that if the interaction does not decrease sufficiently with $|\Gamma|$ the model can have a non confining behavior at all temperature: this occurs for ferromagnetic interactions, where $J_\Gamma \geqslant 0$ for all Γ. In the converse case we show that if the interaction decreases rapidly enough with $|\Gamma|$ then the model has a confining behavior at all temperature in dimensions two and three. These results are stated precisely in Section II, the proofs are given in Section III.

II. DEFINITIONS AND RESULTS

We consider an infinite d-dimensional hypercubic lattice of unit spacing $\Lambda \equiv \mathbb{Z}^d \, (d \geqslant 2)$. The basic objects on the lattice are the sites

$$x \equiv \{ x^1, x^2, \dots, x^d \} \in \mathbb{Z}^d,$$

the links $\langle x, x' \rangle$ where x and x' are nearest neighbours and the plaquettes p (elementary squares).

A walk on the lattice is an ordered set of oriented links

$$\omega \equiv \{ \langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \dots, \langle x_{k-1}, x_k \rangle \}$$

A closed walk is a walk such that $x_k = x_1$. We divide the set of closed walks into equivalent classes by letting ω_1, ω_2 be equivalent whenever ω_1, ω_2 have the same links and the order of the links in ω_1 is a cyclic permutation of the order of the links in ω_2. We call the equivalent classes « loops » and denotes by $\Lambda(\Gamma)$ the set of the loops.

To a loop Γ we associate a loop $\gamma(\Gamma)$ obtained from Γ by eliminating two by two the terms $\langle x_m, x_{m+1} \rangle, \langle x_n, x_{n+1} \rangle$ such that: $x_n = x_{m+1}$ and $x_{n+1} = x_m$. We denote by $\Lambda(\gamma)$ the set of these loops. $|\Gamma|$ (resp. $|\gamma|$) denotes the number of links of Γ (resp. γ).

A connected surface S is a connected set of plaquettes. $|S|$ denotes the number of plaquettes of S and $\Lambda(S)$ the set of connected surfaces.

Let $\mathscr{L}$ be the set of links of Λ. To each link $l = \langle x, x' \rangle$ of Γ we associate a random variable $A(l)$ with value in $[-\pi, \pi]$ and such that

$$A(x, x') = -A(x', x).$$

We denote by A_Γ the sum of the link variables of the loop Γ and by B_S

the sum over the plaquettes p of S of $B_{(p)}$ where $B_{(p)} = A_{\partial_p}$, ∂ being the boundary operator.

We now consider the following actions

$$H^1_\Lambda = - \sum_{\Gamma \in \Lambda(\Gamma)} J_\Gamma \cos A_\Gamma \tag{4}$$

$$H^2_\Lambda = - \sum_{S \in \Lambda(S)} K_S \cos B_S \tag{5}$$

where J_Γ and B_S are real parameters.

Remark. – H^1 and H^2 can be rewritten as

$$H_\Lambda = - \sum_{\gamma \in \Lambda(\gamma)} I_\gamma \cos A_\gamma \tag{6}$$

with

$$I_\gamma = \sum_{\substack{\Gamma \in \Lambda(\Gamma) \\ \gamma \text{ is a loop} \\ \text{associated to } \Gamma}} J_\Gamma \quad \text{for} \quad H^1$$

$$I_\gamma = \sum_{\substack{S \in \Lambda(S) \\ :\partial S = \gamma}} K_S \quad \text{for} \quad H^2$$

The Wilson parameter is given by

$$W_\beta(C) = \langle e^{iA_c} \rangle (\beta) = Z^{-1}(\beta) \prod_{l \in \mathscr{L}} \int_{-\pi}^{\pi} \frac{dA(l)}{2\pi} e^{iA_c} e^{-\beta H_\Lambda} \tag{7}$$

$$Z(\beta) = \prod_{l \in \mathscr{L}} \int_{-\pi}^{\pi} \frac{dA(l)}{2\pi} e^{-\beta H}$$

where $dA/2\pi$ is the invariant measure on S(1). The formulae (7) are to be interpreted as the thermodynamic limit $\Lambda' \to \mathbb{Z}^d$ of the corresponding finite volume quantities $\langle e^{iA_c} \rangle_{\Lambda'}(\beta)$ defined by the same expressions but with links restricted to a finite box Λ'. Let C be a rectangular loop of sides of length L and T, for pure gauge model given by (2) we consider $E(L) = \lim_{T \to \infty} - \frac{1}{T} \operatorname{Log} W_\beta(C)$ as the energy between static quarks separated by a distance L.

We denote by $n(l)$ the number of loops of length l containing a given link. It is known that $n(l) \leqslant (2d)^l$. If $N(s)$ denotes the number of connected surfaces of area s containing a given plaquettes then $N(s) \leqslant \nu_d^s$, where ν_d is a positive number depending on the dimension d of the lattice. This follows by drawing the graphs whose edges connect the centers of the plaquettes containing a same link and by using the following fact: on every connected graph there is a path that passes through every edge at most twice [*13*].

We will now consider the following conditions.

CONDITION 1. — At large $|\Gamma|$, $|J_\Gamma| \sim |\Gamma|^r e^{-\mu_1|\Gamma|}$ with $\mu_1 > \text{Log}\, 2d$, $r < +\infty$.

CONDITION 2. — At large $|\Gamma|$, $|J_\Gamma| \sim |\Gamma|^r e^{-\mu_2|\Gamma|\log|\Gamma|}$ with $\mu_2 > 0$, $r < +\infty$.

CONDITION 3. — At large $|S|$, $|K_S| \sim e^{-\mu_3|S|}$ with $\mu_3 > \text{Log}\, v_d$.

The condition 3 implies that I_γ decreases as exp { − cste minimal area with boundary γ }.

The conditions 1, 2, 3 imply the existence of the thermodynamic limit and give sufficient conditions of the Matthews-Salam expansion. The condition $n > 2d$ is a sufficient condition for the existence of the Matthews-Salam expansion.

THEOREM 1. — Let C be any loop. Consider the action given by (4) and assume that J_Γ verifies the condition 1, then:

a) $\langle e^{iA_c} \rangle (\beta) \leqslant e^{-k_1|\gamma(C)|}$ for any positive β
k_1 is a positive constant and at large β, $k_1 \sim k'_1/\beta$ (k'_1 being a positive constant).

b) If moreover: $J_\Gamma \geqslant 0$ for all Γ then

$$\beta_0/2\,|\gamma(C)|^r e^{-\mu_1|\gamma(C)|} \leqslant \langle e^{iA_c} \rangle (\beta)$$

for any positive β_0 sufficiently small and any β such that $\beta \geqslant \beta_0$.

THEOREM 2. — Let C be a rectangular loop of sides of length L and T. Consider the action given by (4) and assume that J_Γ verifies the condition 2, then for any positive β

a) if $d = 2$ $\quad \langle e^{iA_c} \rangle (\beta) \leqslant e^{-k_2 T(\log L + \text{cste})}$

b) if $d = 3$ $\quad \langle e^{iA_c} \rangle (\beta) \leqslant e^{-k_3 T(\log L + \text{cste})}$

c) if $d \geqslant 4$ $\quad \langle e^{iA_c} \rangle (\beta) \leqslant e^{-k_4(T+L)}$

k_2, k_3 and k_4 are positive constants and at large β $k_i \sim k'_i/\beta$, k'_i being positive constants.

d) If moreover: $J_\Gamma \geqslant 0$ for all Γ, then

$$\beta_0|T + L|^r \exp\{ -2\mu_2|T+L|\,\text{Log}\,|T+L| \} \leqslant \langle e^{iA_c} \rangle (\beta)$$

for any positive β_0 sufficiently small and any β such that $\beta \geqslant \beta_0$.

THEOREM 3. — Let C be a rectangular loop of sides of length L and T. Consider the action given by (5) and assume that K_S verifies the condition 3. Then for any positive β,

a) if $d = 2$ $\langle e^{iA_c} \rangle (\beta) \leqslant e^{-k_5 TL}$

b) if $d = 3$ $\langle e^{iA_c} \rangle (\beta) \leqslant e^{-k_6 T (\log L + \text{cste})}$

c) if $d \geqslant 4$ $\langle e^{iA_c} \rangle (\beta) \leqslant e^{-k_7 (T + L)}$

k_5, k_6 and k_7 are positive constants and at large β, $k_i \sim k_i'/\beta$, k_i' being positive constants

d) if moreover: $K_S \geqslant 0$ for all S then

$$\beta_0/2 e^{-\mu_3 T.L} \leqslant \langle e^{iA_c} \rangle (\beta)$$

for any positive β_0 sufficiently small and any β such that $\beta \geqslant \beta_0$.

Remarks. — We can see that the upper bounds obtained in Theorem 1 for $d = 4$, in part *b* and *c* of Theorem 2 and in part *a*, *b*, *c* of Theorem 3 are of the same kind than those obtained for the U(1) pure lattice gauge theory with action given by (2).

If the interaction is ferromagnetic and in the 4-dimensional case one can obtain better lower bounds (exp $\{ -\text{cste} (T + L) \}$) than those obtained under the conditions 2 and 3 by using Ginibre inequality [*14*] and Guth's lower bound [*12*].

The inequality a of Theorem 1 can be applied to the lattice gauge theory with fermions since the weight factors are given by $\varepsilon(\Gamma) m^{-|\Gamma|}$. Nevertheless the lower bounds are only obtained in the ferromagnetic case and cannot be applied to this theory.

III. PROOF OF THEOREMS

In the proof of upper bounds the idea consists in a comparison with Gaussian process. So we first use the method of complex translation of Mac Bryan and Spencer [*15*]. Our starting point is the following estimate due to Mac Bryan and Spencer (see also Glimm and Jaffe [*11*] for Gauge model).

LEMMA 1. — Let $\{ a(l) \}_{l \in \mathscr{L}}$ be some configuration of links. Then

$$a) \quad \langle e^{iA_c} \rangle (\beta) \leqslant \exp \{ -a_C \} \exp \left\{ \beta \sum_{\Gamma \in \Lambda(\Gamma)} J_\Gamma (\operatorname{ch} a_\Gamma - 1) \right\}$$

$$b) \quad \langle e^{iA_c} \rangle (\beta) \leqslant \exp \{ -a_C \} \exp \left\{ \beta \sum_{S \in \Lambda(S)} K_S (\operatorname{ch} b_S - 1 \right\}$$

where $$b_S = \sum_{p \in S} b(p), \qquad b(p) = a_{\partial p}.$$

We refer the reader to [15], [11] for the proof of this lemma. For the proof of the lower bounds one uses Ginibre's inequality [14], [16]. In terms of gauge model it can be rewritten as follows:

$$\langle \cos A_\Gamma \rangle_{J'} \leqslant \langle \cos A_\Gamma \rangle_J \quad \text{if} \quad | J'_\Gamma | \leqslant J_\Gamma \quad \text{for all} \quad \Gamma \tag{8}$$

III.1. Proofs of the Lower Bounds in Theorems 1, 2, 3.

In formula (7), let $J_\Gamma = 0$ for all Γ excepted for $\Gamma = \gamma(C)$. Then by using inequality (8), we obtain if the interaction is ferromagnetic

$$\langle e^{iA_c} \rangle (\beta) \geqslant \frac{\int_{-\pi}^{\pi} \prod \frac{dA(l)}{2\pi} e^{iA_c} e^{\beta J_{\gamma(c)} \cos A_{\gamma(c)}}}{\int_{-\pi}^{\pi} \prod_l \frac{dA(l)}{2\pi} e^{\beta J_{\gamma(c)} \cos A_{\gamma(c)}}} \tag{9}$$

The right hand side of inequality (9) is equal to $\frac{I_1(\beta J_{\gamma(C)})}{I_0(\beta J_{\gamma(C)})}$ where $I_k(x)$ is the modified Bessel function.

Then one can show that

$$\frac{I_1(\beta J_{\gamma(C)})}{I_0(\beta J_{\gamma(C)})} \geqslant \beta/2 J_{\gamma(C)}$$

According to the different hypothesis on J_Γ we obtain the statement *b* of Theorem 1 and the statement *d* of Theorem 2. The statement *d* of Theorem 3 is obtained in the same way.

III.2. Proof part *a*) of Theorem 1 and part *c*) of Theorem 2.

Let C be an oriented loop. We consider a configuration $\{ a(l) \}_{l \in \mathscr{L}}$ verifying the following condition.

$$\begin{cases} a(l) = \dfrac{1}{\beta k} & \text{for all } l \text{ in } C, \quad l \text{ is oriented in the sense of } C \\ a(l) = 0 & \text{if} \quad l \notin C \end{cases} \tag{10}$$

k is a positive constant chosen later.

Let *l* be some link such that $\gamma(C)$ contains *l*. By using part *a*) of Lemma 1 we obtain

$$\langle e^{iA_c} \rangle (\beta) \leqslant \exp\left\{ -\frac{|\gamma(C)|}{\beta k} \right\} \exp\left\{ \beta |\gamma(C)| \sum_{\substack{\Gamma \in \Lambda(\Gamma) \\ \Gamma \supset l}} J_\Gamma (\text{ch } a_\Gamma - 1) \right\}$$

Let $P = \sum_{\substack{\Gamma\in\Lambda(\Gamma)\\ \Gamma\supset l}} J_\Gamma(\mathrm{ch}\, a_\Gamma - 1)$.

For βk large enough $\left(\text{we take } \beta > \beta_0 \text{ with } \beta_0 \gg \frac{1}{k}\right)$ we can write

$$P = \sum_{\substack{\Gamma:|\Gamma|<\beta k\\ \Gamma\supset l}} J_\Gamma(\mathrm{ch}\, a_\Gamma - 1) + \sum_{\substack{\Gamma:|\Gamma|\geqslant\beta k\\ \Gamma\supset l}} J_\Gamma(\mathrm{ch}\, a_\Gamma - 1)$$

Since $|A_\Gamma| \leqslant |\Gamma|/\beta k$; we can use for $|\Gamma| < \beta k$ the estimate

$$\mathrm{ch}\, a_\Gamma - 1 \leqslant (|\Gamma|/\beta k)^2 .$$

For $|\Gamma| \geqslant \beta k$ we use the estimate

$$\mathrm{ch}\, a_\Gamma - 1 \leqslant \exp\left\{\frac{|\Gamma|}{\beta k}\right\}$$

Then under condition 1 we have

$$P \leqslant \sum_{\substack{4\leqslant l<\beta k\\ l\in\mathbb{N}}} n(l)e^{-\mu_1 l}\frac{l^{r+2}}{\beta^2 k^2} + \sum_{\substack{l\geqslant\beta k\\ l\in\mathbb{N}}} n(l)l^r e^{-\mu_1 l}e^{l/\beta k}$$

where $\mu_1 \geqslant \mathrm{Log}\, 2d+\alpha$, with $\alpha > 0$. Since $n(l) \leqslant (2d)^l$ we have

$$P \leqslant \sum_{l<\beta k} e^{-\alpha l}l^{r+2}\beta^{-2}k^{-2} + \sum_{l\geqslant\beta k} l^r e^{-\alpha l}e^{\beta^{-1}k^{-1}}$$

Let β_1 such that $\beta_1 k > \frac{1}{\alpha}$. Then for $\beta \geqslant \sup\{\beta_0, \beta_1\}$ we obtain

$$P \leqslant A\beta^{-2}k^{-2} + A' e^{-\alpha\beta k}$$

where A and A′ are positive constants. Therefore

$$\langle e^{iA_c}\rangle(\beta) \leqslant \exp\{-|\gamma(C)|\beta^{-1}k^{-1}(1-Ak^{-1}-\beta^2 kA'e^{-\alpha\beta k})\}$$

we choose $k > 2A$. Let β_2 such that $\beta_2^2 kAe^{-\alpha\beta_2 k} < 1/2$. Then for $\beta \geqslant \sup\{\beta_0, \beta_1, \beta_2\}$ we obtain statement A of Theorem 2 for large β. By using inequality (8) one extends the proof to any positive β. The same method is applied to prove statement c of Theorem 2.

III.3. Proof of part a) of Theorem 3.

Let $d = 2$, and S_1 be the rectangle of vertices $O \equiv \{0, 0\}$ $x_1 \equiv \{T, 0\}$, $x_2 \equiv \{T, L\}$, $x_3 \equiv \{0, L\}$. Let S_2 be the symmetric of S_1 with respect to Ox^1 axis and $S_0 = S_1 \cup S_2$

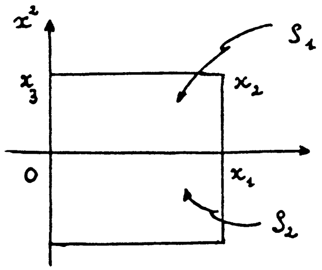

FIG. 1.

We now choose a configuration $\{ a(l) \}_{l \in \mathscr{L}}$ verifying the following conditions.

$$
(11)\quad \begin{cases}
\text{for the links } l \text{ such that } l \in \Lambda/S_0 \text{ we take } a(l) = 0 \\
\text{for the links } l \text{ such that } l \in \partial S_0 \text{ we take } a(l) = 0 \\
\text{for the links } l \text{ parallel to the direction } Ox^2 \text{ we take } a(l) = 0 \\
\text{for the links } l \text{ parallel to the direction } Ox^1 \text{ we take} \\
\text{if } x^2 \geqslant 0 \; a[\{ x^1, x^2 \}, \{ x^1 + 1, x^2 \}] \\
\qquad - a[\{ x^1, x^2 + 1 \}, \{ x^1 + 1, x^2 + 1 \}] = \dfrac{1}{\beta k} \\
\text{if } x^2 < 0 \; a[\{ x^1, x^2 \}, \{ x^1 + 1, x^2 \}] \\
\qquad = a[\{ x^1, - x^2 \}, \{ x^1 + 1, - x^2 \}]
\end{cases}
$$

k is a positive constant chosen later.

Under these conditions, for the $b(p)$ variables we have $| b(p) | = \beta^{-1}k^{-1}$ if $p \in S_0$, $b(p) = 0$ otherwise.

Let p be some plaquettes of S_0. By using part b) of Lemma 1 we obtain

$$\langle e^{iA\partial S_1} \rangle (\beta) \leqslant \exp \{ - L.T\beta^{-1}k^{-1} \} \exp \left\{ 2\beta LT \sum_{S \supset p} K_S (\mathrm{ch}\, b_S - 1) \right\}$$

If βk is large enough $\left(\beta > \beta_0 \text{ with } \beta_0 \gg \dfrac{1}{k} \right)$ we can write

$$Q = \sum_{S \supset p} K_S (\mathrm{ch}\, b_S - 1) = \sum_{\substack{S \supset p \\ |S| < \beta k}} K_S (\mathrm{ch}\, b_S - 1) + \sum_{\substack{S \supset p \\ |S| \geqslant \beta k}} K_S (\mathrm{ch}\, b_S - 1)$$

For $| S | < \beta k$ we use the estimate

$$\mathrm{ch}\, b_S - 1 \leqslant (| S | \beta^{-1}k^{-1})^2$$

For $| S | \geqslant \beta k$ we use $\mathrm{ch}\, b_S - 1 \leqslant e^{|S| \beta^{-1}k^{-1}}$

Then under condition 3 we have:

$$Q \leqslant \sum_{\substack{s<\beta k \\ s\in\mathbb{N}}} v_d^s e^{-\mu_3 s} s^2 \beta^{-2} k^{-2} + \sum_{\substack{s\geqslant\beta k \\ s\in\mathbb{N}}} v_d^s e^{-\mu_3 s} e^{s\beta^{-1}k^{-1}}$$

where $\mu_3 \geqslant \operatorname{Log} v_d + \alpha$, $\alpha > 0$. Let β_1 be such that $\beta_1 k > \frac{1}{\alpha}$.

For $\beta \geqslant \sup(\beta_0, \beta_1)$ we obtain:

$$Q \leqslant A\beta^{-2}k^{-2} + A' e^{-\alpha\beta k}$$

A and A′ are positive constants. The proof of inequality *a*) of Theorem 3 ends analogously to III.2. To prove statement *c* of Theorem 2, we use the same method but in choosing the configuration given by (10).

We now consider the 3-dimensional case. The idea of the proof consists in choosing a configuration $\{ a(l) \}_{l\in\mathscr{L}}$ to reduce it to a bidimensional problem. We first introduce some notations.

III.4. Notations.

Let $x \equiv \{ x^1, x^2, x^3 \}$ be a site of Λ. We denote by $d(x)$ the distance of x to the Ox^1 axis

$$d(x) = \operatorname{dist}(x, Ox^1) = \sup \{ | x^2 |, x^3 | \}$$

We define the projection of x on the half-plane $\{ x^3 = 0, x^2 \geqslant 0 \}$

$$\operatorname{Proj} [\{ x^1, x^2, x^3 \}] = \{ y^1, y^2, y^3 \}$$

where $y^1 = x^1$, $y^2 = d(x)$, $y^3 = 0$.

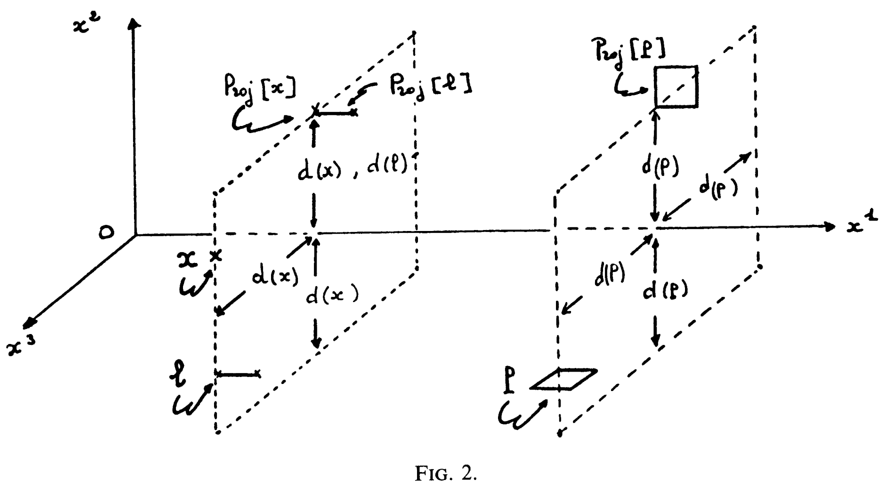

FIG. 2.

Let $l = \langle x, y \rangle$ be a link. We define the projection of the link l on the half-plane $\{ x^3 = 0, x^2 \geqslant 0 \}$

$$\text{Proj}\ [l] = \langle\ \text{Proj}\ [x], \text{Proj}\ [y\}\ \rangle$$

We consider the links $l = \langle x, y \rangle$ parallel to Ox^1 and introduce the distance of l to Ox^1

$$d(l) = d(x) = d(y)$$

Let $p = (x_1, x_2, x_3, x_4)$ be some plaquettes such that

$$\text{Proj}\ [x_i] \neq \text{Proj}\ [x_j] \qquad \forall i, \ \forall j \qquad i \neq j$$

We define the projection of the plaquette p as

$$\text{Proj}\ [p] = (\text{Proj}\ [x_1], \text{Proj}\ [x_2], \text{Proj}\ [x_3], \text{Proj}\ [x_4])$$

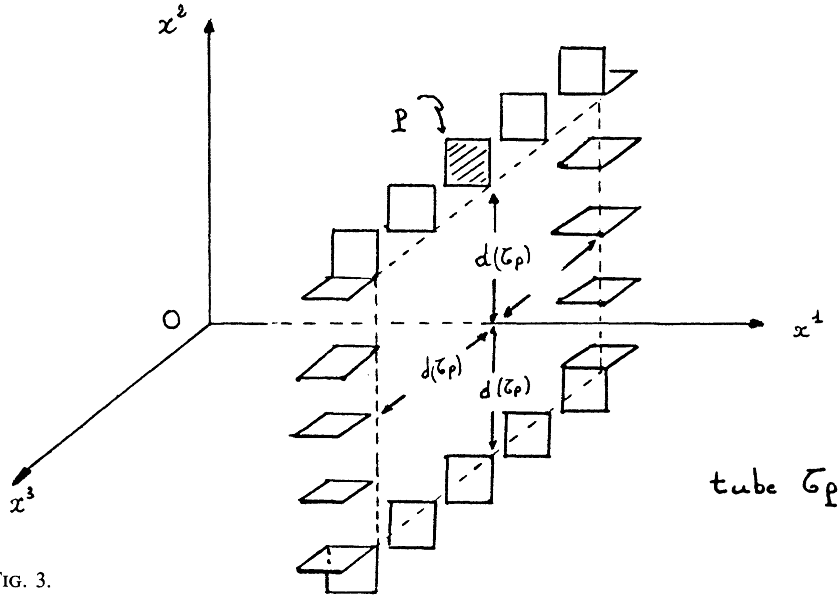

FIG. 3.

Let p be a plaquette on the half-plane $\{ x^3 = 0, x^2 \geqslant 0 \}$.
We define the « tube » τ_p associated to the plaquette p by

$$\tau_p = \{\ \text{set of plaquettes } q \text{ such that } \{\ \text{Proj}\ [q] = p\ \}$$

We define the distance of the plaquette $p = (x_1, x_2, x_3, x_4)$ to Ox^1

$$d(p) = \min_{x_i \in p} d(x)$$

The distances of the tube τ_p to Ox^1 are given by

$$d(\tau_p) = d(p)$$

III.5. **Proof of statement** *b*) **of Theorem 3.**

We consider the rectangle S_1 of vertices $O \equiv \{0,0,0\}$, $x_1 \equiv \{T,0,0\}$, $x_2 \equiv \{T,L,0\}$, $x_3 \equiv \{0,L,0\}$, and the box

$$\Lambda_{LT} : \{0 \leqslant x^1 \leqslant T, -L \leqslant x^2 \leqslant L, -L \leqslant x^3 \leqslant L\}$$

We choose a configuration $\{a(l)\}_{l\in\mathscr{L}}$ verifying the following conditions:

(12)
- for all links l perpendicular to Ox^1 direction we take $a(l) = 0$
- for all links of $\partial\Lambda_{LT}$ and Λ/Λ_{LT} we take $a(l) = 0$
- for the links in Λ_{LT} parallel to Ox^1 and oriented in the Ox^1 direction we take

$$a(l) = \frac{1}{\beta k} \sum_{m=d(l)}^{L-1} \frac{1}{m+1}$$

k is a positive constant chosen later.

With this choice, for the $b(p)$ variables we have

$$\forall p \in S_1, \quad \forall q \in \tau_p \quad |b(q)| = \frac{1}{\beta k(d(p)+1)}$$
$$b(p) = 0 \quad \text{otherwise}$$

Using part *b* of Lemma 1 and assuming that the configuration verifies the condition (12) we obtain

$$(13) \quad \langle e^{iA\partial S_1} \rangle (\beta) \leqslant \exp\{-a_{\partial S_1}\} \exp\left\{\beta \sum_{p\in S_1} \sum_{q\in\tau_p} \sum_{S\supset q} K_S(\operatorname{ch} b_S - 1)\right\}$$

with

$$(14) \qquad \exp\{-a_{\partial S_1}\} = \exp\left\{-T\beta^{-1}k^{-1}\sum_{j=1}^{L}\frac{1}{j}\right\}$$

We can write

$$Q' = \beta \sum_{p\in S_1} \sum_{q\in\tau_p} \sum_{S\supset q} K_S(\operatorname{ch} b_S - 1) \leqslant \beta T \sum_{j=1}^{L} 4(2j-1) \sum_{\substack{S\supset p\\ d(p)=j}} K_S(\operatorname{ch} b_S - 1)$$

We can decompose the sum Q' as follows

$$Q' \leqslant \beta T \sum_{j=1}^{L} 4(2j-1) \sum_{\substack{S\supset p : d(p)=j\\ |S|<j/2}} K_S(\operatorname{ch} b_S - 1)$$
$$+ \beta T \sum_{j=1}^{L} 4(2j+-1) \sum_{\substack{S\supset p : d(p)=j,\\ |S|>j/2}} K_S(\operatorname{ch} b_S - 1)$$

In the first term of the R. H. S. of (15) we use the estimate

$$\operatorname{ch} b_S - 1 \leqslant \left(\frac{2|S|}{(j+1)\beta k}\right)^2$$

In the second term of R. H. S. of (15) we use the estimate

$$\operatorname{ch} b_S - 1 \leqslant e^{|S|\beta^{-1}k^{-1}}$$

Then under condition 3 on K_S

$$Q' \leqslant \beta T \sum_{j=1}^{L} 4(2j-1) \sum_{\substack{s<j/2 \\ s\in\mathbb{N}}} \frac{4s^2 e^{-\mu_3 s} v_d^s}{\beta^2 k^2 (j+1)^2} + \beta T \sum_{j=1}^{L} 4(2j-1) \sum_{\substack{s\geqslant j/2 \\ s\in\mathbb{N}}} e^{-\mu_3 s} v_d^s e^{s\beta^{-1}k^{-1}}$$

where $\mu_3 \geqslant \operatorname{Log} v_d + \alpha$, with $\alpha > 0$. For $\beta > \alpha^{-1}k^{-1}$ we obtain

$$Q' \leqslant \beta T \left\{ A\beta^{-2}k^{-2} \sum_{j=1}^{L} \frac{1}{j} + A' \right\} \tag{16}$$

where A, A′ are positive constants. By choosing $k > A$ statement *b* of Theorem 3 follows from (13), (14) and (16).

III.6. **Proof of statement *a*) of Theorem 2.**

We keep the notation of Sections III.3 and III.4. We consider a configuration $\{a(l)\}_{l\in\mathscr{L}}$ verifying the following conditions

(17)
- for all links of ∂S_0 and Λ/S_0 we take $a(l) = 0$
- for all links parallel to Ox^2 we take $a(l) = 0$
- for all links l in S_0 parallel to Ox^1 and oriented in the Ox^1 direction we take

$$a(l) = \frac{1}{\beta k} \sum_{m=d(l)}^{L-1} \frac{1}{m+1}$$

We shall assume $k = 1$. Under these conditions for the $b(p)$ variables we have

$$|b(p)| = \beta^{-1}k^{-1}(d(p)+1)^{-1} \quad \text{if} \quad p\in S_0, \qquad b(p) = 0 \quad \text{otherwise}$$

Using part *a*) of Lemma 1 for a configuration verifying the conditions (11) we obtain

$$\langle e^{iA\partial S_1} \rangle (\beta) \leqslant \exp\left\{ -\frac{T}{\beta k} \sum_{j=1}^{L} \frac{1}{j} \right\} \exp\left\{ \beta \sum_{\substack{\Gamma\in\Lambda(\Gamma) \\ a_\Gamma \neq 0}} J_\Gamma (\operatorname{ch} a_\Gamma - 1) \right\} \tag{18}$$

Let

$$R = \sum_{\Gamma : a_\Gamma \neq 0} J_\Gamma (\operatorname{ch} a_\Gamma - 1) \tag{19}$$

we can write

$$R \leqslant \sum_{p \in S_0} \sum_{\substack{\Gamma \in \Lambda(\Gamma): \\ \left\{ \substack{\gamma(\Gamma) \text{ contains} \\ \text{a link of } p} \right\}}} J_\Gamma (\operatorname{ch} a_\Gamma - 1)$$

It is clear that

$$R \leqslant 2T \sum_{j=1}^{L} \sum_{\substack{\Gamma: \\ \left\{ \substack{\gamma(\Gamma) \text{ contains} \\ \text{a link of } \rho;\, d(p)=j} \right\}}} J_\Gamma (\operatorname{ch} a_\Gamma - 1)$$

Let c a some positive constant larger than 3. For βk large enough we make the following decomposition of R.

$$\begin{aligned}
R \leqslant\ & 2cT \sum_{\substack{\Gamma: \\ \left\{ \substack{\gamma(\Gamma) \text{ contains} \\ \text{a given link} \\ |\Gamma| < \sqrt{\beta k}} \right\}}} J_\Gamma (\operatorname{ch} a_\Gamma - 1) + 2cT \sum_{\substack{\Gamma: \\ \left\{ \substack{\gamma(\Gamma) \text{ contains} \\ \text{a given link} \\ |\Gamma| \geqslant \sqrt{\beta k}} \right\}}} J_\Gamma (\operatorname{ch} a_\Gamma - 1) \\
& + 2T \sum_{j=c}^{L} \sum_{\substack{\Gamma: \\ \left\{ \substack{\gamma(\Gamma) \text{ contains} \\ \text{a link of } p \\ d(p)=j,\, |\Gamma| < 4\sqrt{j}} \right\}}} J_\Gamma (\operatorname{ch} a_\Gamma - 1) + 2T \sum_{j=c}^{L} \sum_{\substack{\Gamma: \\ \left\{ \substack{\gamma(\Gamma) \text{ contains} \\ \text{a link of } p \\ d(p)=j,\, |\Gamma| \geqslant 4\sqrt{j}} \right\}}} J_\Gamma (\operatorname{ch} a_\Gamma - 1)
\end{aligned} \tag{20}$$

Let R_1, R_2, R_3, R_4 the first second third and fourth terms of the R. H. S. of the inequality (20). We now use the estimates:

$$\operatorname{ch} a_\Gamma - 1 \leqslant \left(\frac{|\Gamma|}{\beta k} \right)^2 \quad \text{in} \quad R_1$$

$$\operatorname{ch} a_\Gamma - 1 \leqslant \exp \left\{ \frac{|\Gamma|}{2\beta k} \operatorname{Log}(|\Gamma|/2) \right\} \quad \text{in} \quad R_2 \quad \text{and} \quad R_4$$

$$\operatorname{ch} a_\Gamma - 1 \leqslant (|\Gamma|/j\beta k)^2 \quad \text{in} \quad R_3$$

Under the condition 2 on J_Γ we obtain for large β

$$R_1 \leqslant 2cT \sum_{l/2=2}^{\sqrt{\beta k}-1} e^{-\mu_2 l \log l^{r+4}} \beta^{-2} k^{-2} e^{l \log 2d} \leqslant A_1 T \beta^{-2}$$

$$R_2 \leqslant 2cT \sum_{l/2 \geqslant \sqrt{\beta k}} l^{r+2} e^{-\mu_2 l \log l} e^{2l\beta^{-1}k^{-1} \log l/2} e^{l \log 2d} \leqslant A_2 T$$

$$R_3 \leqslant 2T \sum_{j=c}^{L} \sum_{\frac{l}{2}=2}^{2\sqrt{j}-1} e^{-\mu\, l \log l} l^{r+2} (j\beta k)^{-2} e^{l \log 2d} \leqslant A_3 T \beta^{-2}$$

$$R_4 \leqslant 2T \sum_{j=c}^{L} \sum_{l \geqslant 4\sqrt{j}} l^r e^{-\mu_2 l \log l} e^{2l\beta^{-1}k^{-1} \log l/2} e^{l \log 2d} \leqslant A_4 T$$

A_1, A_2, A_3 and A_4 are positive constants. From these four inequalities and from (18), (19), (20) follows the proof of statement *a*) of Theorem 2 at large β. Ginibre inequality extends the proof to any positive β.

III.7. **Proof of Part *b*) of Theorem 2.**

In this case we choose a configuration $\{ a(l) \}_{l \in \mathscr{L}}$ verifying the condition (12) as in III.5. Using part *a*) of Lemma 1 for this configuration we obtain

$$(21) \quad \langle e^{iA\partial S_1} \rangle (\beta) \leqslant \exp\left\{ -\beta^{-1}k^{-1} \sum_{j=1}^{L} \frac{1}{j} \right\} \exp\left\{ \beta \sum_{\substack{\Gamma \in \Lambda(\Gamma) \\ a_\Gamma \neq 0}} J_\Gamma (\text{ch}\, a_\Gamma - 1) \right\}$$

Let
$$R' = \sum_{\Gamma : a_\Gamma \neq 0} J_\Gamma (\text{ch}\, a_\Gamma - 1)$$

We can write
$$R' \leqslant \sum_{p \in S_0} \sum_{q \in \tau_p} \sum_{\substack{\Gamma: \\ \{\gamma(\Gamma) \text{ contains a link of } q\}}} J_\Gamma (\text{ch}\, a_\Gamma - 1)$$

It is clear that
$$R' \leqslant T \sum_{j=1}^{L} 4(2j-1) \sum_{\substack{\Gamma: \\ \{\gamma(\Gamma) \text{ contains a link of } p,\ d(p)=j\}}} J_\Gamma (\text{ch}\, a_\Gamma - 1) \cdot$$

We remark that R′ differs from R only by the factor $2(2j-1)$. By using the same decomposition and estimates as in Section III.6 we obtain

$$R' = R'_1 + R'_2 + R'_3 + R'_4$$

with

$$R'_1 \leqslant TA'_1 \beta^{-1} ; \quad R'_2 \leqslant TA'_2 ; \quad R'_3 \leqslant TA'_3(\beta k)^{-2} \sum_{j=c}^{L} \frac{1}{j} ; \quad R'_4 \leqslant A'_4 T$$

where A'_1, A'_2, A'_3 and A'_4 are positive constants. By choosing $k > A'_3$ we obtain part *b*) of Theorem 2.

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