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Section A :
Physique théorique.

On $-\dfrac{d^2}{dx^2} + V$ where V has infinitely many « bumps »

by

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ABSTRACT. — We characterize the *negative part* of the essential spectrum of $H = -\dfrac{d^2}{dx^2} + V$ where V is made up of « bumps », their separation increasing with distance. We show that $\sigma_{ess}(H) \cap (-\infty, 0]$ consists precisely of those points which are accumulation points of eigenvalues of Hamiltonians having one bump.

RÉSUMÉ. — On caractérise la partie négative du spectre essentiel du Hamiltonien $H = -\dfrac{d^2}{dx^2} + V$, où V est formé de « bosses » dont la séparation croît avec la distance. On montre que $\sigma_{ess}(H) \cap (-\infty, 0]$ est l'ensemble des points d'accumulation de valeurs propres des Hamiltoniens à une bosse.

1. INTRODUCTION

To fix our ideas let us begin with some notation. Let $\{ a_i \}$ and $\{ b_i \}$ $(i \in \mathbb{Z})$ be two sequences of real numbers, satisfying $b_i < a_{i+1}$, $b_i - a_i \equiv 2d > 0$, $a_i \to \pm \infty$ and $a_{i+1} - b_i \to \infty$ as $i \to \pm \infty$. Let $c_i = \frac{1}{2}(a_i + b_i)$. In the

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sequel $\| \ \|_p$ denotes L_p norm, $\| \ \|_{p,q}$ the norm of an operator from L_p to L_q; simply $\| \ \|$ means $\| \ \|_2$ or $\| \ \|_{2,2}$. Suppose that

$$W_i \in L_1(\mathbb{R}) \qquad (i \in \mathbb{Z}) \tag{1.1}$$

obeys

$$\text{supp } W_i \in [-d, d] \tag{1.2}$$

and

$$\sup_i \| W_i \|_1 < \infty \tag{1.3}$$

Define

$$V(x) = \sum_{i=-\infty}^{\infty} W_i(x - c_i) \qquad (x \in \mathbb{R}) \tag{1.4}$$

so that V consists of « bumps » (W_i) whose separation $(a_{i+1} - b_i)$ increases as $i \to \pm \infty$. Let $H_0 = -d^2/dx^2$. Then W_i is infinitesimally form-bounded with respect to H_0, i. e.

$$|(f, W_i f)| \leqslant \varepsilon(f, H_0 f) + b_\varepsilon \| f \|^2, \tag{1.5}$$

where $\varepsilon > 0$ can be chosen arbitrarily small. Moreover, condition (1.3) allows us to choose b_ε independently of i. Similarly,

$$|(f, Vf)| \leqslant \varepsilon(f, H_0 f) + b_\varepsilon \| f \|^2 \tag{1.6}$$

follows from (1.5) by using « localisation » [*1*]. Then the KLMN theorem [*2*] gives meaning to $H = H_0 + V$. We recall that $Q(H) = Q(H_0)$ (i. e. the form domains agree) and

$$D(H) = \{ f \in D(H_0^{1/2}) \mid H_0 f + Vf \in L_2 \} \subset D(H_0^{1/2}) \subset D(|V|^{1/2}).$$

For any further properties of H we refer the reader to [*3*]. Finally, let $S = \{ E \leqslant 0 \mid \exists$ sequence $\{ i_n \} (i_n \in \mathbb{Z})$ and negative eigenvalues E_{i_n} of $H_0 + W_{i_n}$ such that $|i_n| \to \infty$ and $E_{i_n} \to E$ as $n \to \infty \}$. The purpose of this note is to prove:

THEOREM (1.1). — $\sigma_{ess}(H) = [0, \infty] \cup S$.

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2. For potentials of this type, provided the separation of the bumps increases *sufficiently rapidly*, Pearson [*4*] proved the remarkable result that $[0, \infty) \subset \sigma_{s.c.}(H)$ (the singular continuous spectrum). For the potentials considered in [*4*] it follows from our result that $\sigma_{s.c.}(H) = [0, \infty)$.

3. Any closed subset of $(-\infty, 0]$ can occur as a set S by suitably selec-

ting the potentials W_i. However, if $W_i = W \leqslant 0$ for all $i \in \mathbb{Z}$, then S coincides with the set of negative eigenvalues of $H_0 + W$. These points must be cluster points of eigenvalues of H (since they cannot be eigenvalues of infinite multiplicity). It was this structure of the negative spectrum which aroused our interest. In fact, we do not know of any previous example that would show such a behavior.

4. Of course, the idea behind Theorem (1.1) is that the bumps become more and more « independent » as $|i| \to \infty$. This easily translates into the result $[0, \infty) \cup S \subset \sigma_{ess}(H)$ by constructing suitable Weyl sequences. Since this method is so well-known we skip this part of the proof. For the opposite inclusion one has to find a sequence of potentials W_{i_n} whose eigenvalues E_{i_n} converge to any given (negative) $E \subset S$. This will follow indirectly, by exploiting the properties of the Birman-Schwinger kernel which allows us to decouple the bumps. That we can do without any specific growth condition on the spacing of the bumps is due to our using of the right compactness criterion for a certain integral kernel (Propositions (2.1) (2.2)).

2. PROOF OF THEOREM (1.1) AND SOME AUXILIARY RESULTS

We introduce

$$K_V(E) = |V|^{1/2}(H_0 - E)^{-1}V^{1/2} \qquad (V^{1/2} = |V|^{1/2}\text{sgn } V) \tag{2.1}$$

$$= \sum_i |\tilde{W}_i|^{1/2}(H_0 - E)^{-1}\tilde{W}_i^{1/2} + \sum_{i \neq j} |\tilde{W}_i|^{1/2}(H_0 - E)^{-1}\tilde{W}_j^{1/2} \tag{2.2}$$

$$\equiv \bigoplus_i K_i(E) + \sum_{i \neq j} K_{ij}(E) \tag{2.3}$$

where $\tilde{W}_i(x) \equiv W_i(x - c_i)$ and (2.3) should indicate that the first sum in (2.2) can be viewed as a direct sum of operators. By virtue of (1.5) and (1.6) all operators in (2.1)-(2.3) are bounded. Let now $E \in \sigma_{ess}(H)$ and pick a corresponding Weyl sequence $\{f_n\}$. Set $g_n = |V|^{1/2}f_n / \| |V|^{1/2}f_n \|$, noting that $\underline{\lim} \| |V|^{1/2}f_n \| > 0$, for otherwise $((H_0 - E)f_n, f_n) \to 0$ for a subsequence. Set $h_n = (H_0 - E)f_n$. Then a little calculation gives

$$g_n = -|V|^{1/2}(H_0 - E)^{-1}V^{1/2}g_n + (\| |V|^{1/2}f_n \|)^{-1}.(H_0 - E)^{-1}h_n,$$

and we see that $\|(K_V(E)+1)g_n\| \to 0$. Thus $-1 \in \sigma(K_V(E))$ has a Weyl sequence. Then Lemma (2.4) along with Prop. (2.2) provide us with a sequence $\{\lambda_{i_n}\}$, $\lambda_{i_n} \to -1$, each λ_{i_n} being eigenvalue of $K_{i_n}(E)$.

By the Birman-Schwinger principle [5], E is eigenvalue of $H_0-(1/\lambda_{i_n})W_{i_n}$ for all n. Then $H_0 + W_{i_n}$ has an eigenvalue E_{i_n} approaching E as $n \to \infty$, for, if not, we could find a $\delta > 0$ such that $(E-\delta, E+\delta) \in \rho(H_0+W_{i_n})$ if n is sufficiently large. But since W_{i_n} obeys (1.5) uniformly in n and $\lambda_{i_n} \to -1$, $H_0 + W_{i_n}$ is a small perturbation of $H_0 - (1/\lambda_{i_n})W_{i_n}$. Hence by perturbation theory [6]

$$(E - \delta, E + \delta) \cap \sigma(H_0 + W_{i_n}) \neq \emptyset$$

for n large enough. This contradiction implies $E \in S$. Theorem (1.1) is proved. ■

The following Propositions will lead us to Lemma (2.4) which has been used in the above proof. Our first goal is to prove that $\sum_{i \neq j} K_{ij}(E)$ is compact.

Let $\chi_R(x) = 1$ if $|x| < R$, $\chi_R(x) = 0$ if $|x| > R$ $(R \geqslant 0)$. If $K(x, y)$ is an integral kernel then K denotes the corresponding operator.

PROPOSITION (2.1). — Suppose that $K(x, y)$ is a $\mathbb{R} \times R$-measurable function. Let

$$h_1(R) = \sup_{|x| \geqslant R} \| K(x,.) \|_1, \qquad h_2(R) = \sup_{|y| \geqslant R} \| K(., y) \|_1$$

and suppose that $h_i(R) \to 0$ as $R \to \infty$ and $h_i(0) < \infty$ $(i = 1, 2)$. In addition suppose that $\chi_R K \chi_R : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is compact. Then $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is compact.

Proof. — It is well known that by interpolation one gets the estimate $\| K \|_{2,2} \leqslant h_1(0)^{1/2} \cdot h_2(0)^{1/2}$ (note that $\| K \|_{\infty,\infty} \leqslant h_1(0)$, $\| K \|_{1,1} \leqslant h_2(0)$). Estimating $\chi_R K$, $K\chi_R$ and $(1 - \chi_R)K(1 - \chi_R)$ by means of this bound yields $\lim \chi_R K \chi_R = K$ in norm as $R \to \infty$. Hence K is compact. ■

If $g : \mathbb{R} \to \mathbb{R}$ is any given function, let

$$g^{(N)}(x) = \begin{cases} N & \text{if} \quad g(x) > N \\ g(x) & \text{if} \quad |g(x)| \leqslant N \\ -N & \text{if} \quad g(x) < -N \end{cases} \qquad N > 0$$

Let $K_{ij}(E) = |\tilde{W}_i^{(N)}|^{1/2}(H_0 - E)^{-1}(\tilde{W}_i^{(N)})^{1/2}$.

PROPOSITION (2.2). — $\sum_{i \neq j} K_{ij}^{(N)}(E)$ is compact in $L_2(\mathbb{R})$ and converges in norm to $\sum_{i \neq j} K_{ij}(E)$ as $N \to \infty$.

Proof. — Given $n > 0$ pick $i_n > 0$ such that $a_{i+1} - b_i \geqslant n$ for $|i| \geqslant i_n$.

Let $\beta_n = \max\{|a_{i_n+1}|, |b_{-i_n}|\}$. Recall that $(H_0 - E)^{-1}$ has kernel $(2\alpha)^{-1}\exp(-\alpha|x-y|)$ where $E = -\alpha^2$, $\alpha > 0$. Then

$$\sup_{|x|\geqslant\beta_n} \| K_{ij}^{(N)}(x,.)\|_1 \leqslant \frac{1}{2\alpha}\sup_{|i|\geqslant i_n}\left\| \sum_{\substack{j\in Z\\ i\neq j}} |\tilde{W}_i^{(N)}(x)|^{1/2}\int \overline{e}^{\alpha|x-y|}|\tilde{W}_j^{(N)}(y)|^{1/2}dy\right\|_\infty \tag{2.4}$$

Assuming that $i \geqslant i_n > 0$ we see that the contribution from the sum over $j > i$ is bounded by

$$\frac{(2d)N}{2\alpha}(e^{-n} + e^{-2n-2d} + \dots) = \frac{dN}{\alpha}\frac{e^{-n}}{1-e^{-(n+d)}} \tag{2.5}$$

where we have used the simple estimate

$$\int e^{-\alpha|x-y|}|\tilde{W}_j^{(N)}(y)|^{1/2}dy \leqslant 2dN^{1/2}e^{-(j-i)(n+2d)}e^{2d} \tag{2.6}$$

Similarly, the contribution from summing over $j < i$ is dominated by

$$\frac{(2d)N}{2\alpha}(e^{-n} + e^{-n-d} + \dots) = \frac{dN}{\alpha}\frac{e^{-n}}{1-e^{-d}} \tag{2.7}$$

Thus

$$\sup_{|x|\geqslant\beta_n} \| K_{ij}^{(N)}(x,.)\|_1 \leqslant \frac{2dN}{\alpha}\frac{e^{-n}}{1-e^{-d}} \tag{2.8}$$

A similar bound holds if $i \leqslant -i_n$. Since $K_{ij}^{(N)}(x,y)$ is « symmetric » (apart from a unimportant factor sgn $(W_i^{(N)})$) and the right side of (2.8) can be made arbitrarily small by choosing n large, we can appeal to Proposition (2.1) and conclude that $\sum_{i,j} K_{ij}^{(N)}(E)$ is compact.

The norm convergence as $N \to \infty$ follows from that of $K_V^{(N)}$ and $\bigoplus_i K_i^{(N)}$. And this follows from estimates (1.5) (1.6) which can also be used to estimate $(f,(V - V^{(N)})f)$ to the effect that

$$|(f,(V - V^{(N)})f)| \leqslant g(N)(f,(H_0 + 1)f) \tag{2.9}$$

where $g(N) \to 0$ as $N \to \infty$ (Note that the assumption $\sup_i \| W_i\|_1 < \infty$ assures us that $\| W_i^{(N)} - W_i\|_1 \to 0$ uniformly in $i \in \mathbb{Z}$ as $N \to \infty$), Proposition (2.2) is proved. ■

Our next goal is to describe the spectrum of $\bigoplus_i K_i(E)$. In general $\sigma\left(\bigoplus_n A_n\right)$ can be much larger than $\overline{\bigoplus_n \sigma(A_n)}$ if A_n are arbitrary bounded operators (e. g. let $A_n = T\left(\frac{1}{n}\right)$ where T is defined in [6, p. 210, Ex.: 3.8]: $\overline{\bigcup_n \sigma(A_n)}$ is the unit circle while $\sigma\left(\bigoplus_n A_n\right)$ is the closed unit disk). In our case, however, we have

PROPOSITION (2.3). — $\sigma\left(\bigoplus_i K_i(E)\right) = \overline{\bigcup_i \sigma(K_i(E))}$.

Proof. — Only the $\subset$ inclusion is nontrivial. $K_i(E)$ is of the form A_iB_i with $A_i = |\tilde{W}_i|^{1/2}(H_0 - E)^{-1/2}$, $B_i = (H_0 - E)^{-1/2}\tilde{W}_i^{1/2}$ and A_i, B_i bounded. Since $\sigma(A_iB_i)\backslash\{0\} = \sigma(B_iA_i)\backslash\{0\}$ [7] and B_iA_i is self-adjoint, we see that $\overline{\bigcup_i \sigma(K_i(E))} \subset \mathbb{R}$. Pick $\lambda \in \mathbb{R}$, $\lambda \notin \overline{\bigcup_i \sigma(K_i(E))}$. Then there exists $\delta > 0$ such that $(\lambda - \delta, \lambda + \delta) \notin \sigma(K_i(E))$ for any $i \in \mathbb{Z}$. By a commutation formula [7] if $Z \in \rho(A_iB_i)$ then $(A_i \equiv A, B_i \equiv B)$

$$(AB - Z)^{-1} = \frac{1}{Z}(A(BA - Z)^{-1}B - 1) \tag{2.10}$$

so that

$$\|(AB - Z)^{-1}\| \leqslant \frac{1}{|Z|}\left(\frac{\|A\|\,\|B\|}{\rho(Z)} + 1\right) \tag{2.11}$$

where $\rho(Z) = \text{dist}\,[Z, \sigma(BA)]$. If $Z = \lambda$ we may replace $\rho(Z)$ by δ for any $i \in \mathbb{Z}$. Letting $K_{i,M} = \bigoplus_{-M}^{M} K_i(E)$; it is clear that $(K_{i,M} - Z)^{-1} \to \bigoplus_i (K_i - Z)^{-1}$ strongly as $M \to \infty$, first for $|Z|$ large enough and then by continuation [6, p. 427] also for $Z = \lambda$. Hence $\lambda \notin \sigma\left(\bigoplus_i K_i\right)$ and we are done. ■

LEMMA (2.4). — $\lambda \in \sigma\left(\bigoplus_i K_i\right)$ has a singular sequence $\Leftrightarrow$ $\exists$ sequences $\{i_n\}$ and $\{\lambda_{i_n}\}$ such that $\lambda_{i_n} \in \sigma(K_{i_n})$, $\lambda_{i_n} \to \lambda$ and $|i_n| \to \infty$ as $n \to \infty$.

Proof. — Only the $\Rightarrow$ direction has to be proved. By Proposition (2.3) if $\lambda \in \sigma\left(\bigoplus_i K_i\right)$ then $\lambda \in \overline{\bigcup_i \sigma(K_i)}$ and if no sequence with the above properties exists, $\lambda \in \sigma(K_i)$ for only finitely many i and λ is an isolated spectral point, i. e. λ is an eigenvalue of finite multiplicity and thus would not admit a singular sequence. ■

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Note added in proof. B. SIMON (private communication) has found another proof of Theorem (1.1).