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Section A :
Physique théorique

Predictive relativistic mechanics of systems of N particles with spin. II. The electromagnetic interaction

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ABSTRACT. — In a preceding paper we have developed the Predictive Relativistic Mechanics of systems of particles with spin. In this paper we apply our general results to the case of systems of spinning, electric poles-magnetic dipoles. As a test of our formalism we derive the classical version of Breit's Hamiltonian.

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1. INTRODUCTION

In an earlier article *[1]*, which we shall hereafter refer to as BM, we have expounded the theoretical foundations of *Predictive Relativistic Mechanics* (P. R. M.) of isolated systems of particles with spin in interaction. The object of the present work is to apply the results of that article to electromagnetic interaction. For this purpose we shall assume that the particles in the system are electrically charged and possess a magnetic moment considered to be associated with a spin or intrinsic angular momentum in the standard form. Hereafter we shall use the term electric pole-magnetic dipole to designate this type of particles. Furthermore the term *motion*, with reference to such an object, will include both space-time motion and the evolution of the magnetic moment.

Section 2 discusses the results obtained from Classical Field Theory for the interaction of two EP-MD's. To this end we shall divide this Section into three Subsections, the first of which will be concerned with calculating the electromagnetic field created by a EP-MD undergoing a known but arbitrary motion. The second will consider the equations of motion of a EP-MD subject to the action of an external electromagnetic field. These equations involve two coupled systems of differential equations; the first, of the 2nd order, determines the space-time motion of the particle and is obtained by adding to the Lorentz force the force exerted by a field on a magnetic dipole; the second system, of the first order, determines the evolution of the magnetic moment and is a trivial generalization of the well-known equations of Bargmann, Michel and Telegdi [*2*]. Finally, in the third Subsection we assume that the external electromagnetic field in the previous Subsection is precisely the field calculated in the first one i. e., that which creates another EP-MD in arbitrary motion. We examine then the possibility of obtaining equations of motion for the system of two EP-MD's in interaction. As in the case of two simple electric poles, we conclude that the Lienard-Wiechert type of equations thus obtained are only useful as « boundary conditions ».

In Section 3 we study the interaction of two EP-MD's (assumed to be isolated from the rest of the objects in the Universe) from the point of view of the theory expounded in BM. For this purpose we first consider that this interaction is described by a Projectable System in the sense of the definition 3.2 in BM. We then suppose that both the 4-accelerations and the « 4-precessions » of the particles are developable in power series of the parameter which is the product of the electric charges. In this way, as in the case of particles without magnetic moment [*3*], it is possible, using the « boundary conditions » considered in Section 2 to obtain unique expressions for these quantities. In the present work we shall limit ourselves to the first-order calculation of these series for reasons of simplicity, and because, as we shall see, this is already a significant approximation. We believe, nevertheless, that the second-order calculation, even though exceedingly laborious, could lead to results of great interest.

Section 4 constitutes a direct application of the results obtained in the last part of BM. To this end, we make use of the first-order expressions for the 4-accelerations and « 4-precessions » calculated in the previous Section, to obtain, likewise in the first order of approximation, the expressions for the total Linear Momentum and Angular Momentum of the system of two EP-MD's. We also calculate in the same order of approximation the corresponding simplectic form.

Finally, in Section 5, we carry out a development in power series of $1/c$

(c = speed of light in vacuum) of some of the quantities relative to the Projection (see BM) of the approximate Projectable System of previous sections. This process leads in particular to the obtaining of the classic version of the well-known quantized hamiltonian of Breit [*4*, *5*]. We consider this result as a good test of reliability for the more general ones which are presented in this paper.

2. INTERACTION OF TWO EP-MD'S IN CLASSICAL FIELD THEORY

A. Field created by a EP-MD.

Let us consider Minkowski's space time $\mathscr{M}_4$ referred to a Galilean system of coordinates $\{ x^\alpha \}$ ($\alpha, \beta, \lambda, \mu, \ldots = 0, 1, 2, 3$), where $x^0 = $ ct represents the time coordinate and $\{ x^i \}$ ($i, j, k, \ldots = 1, 2, 3$) the space coordinates. Let there be then a EP-MD performing a known but arbitrary motion, whose space-time part is described by the following curve of $\mathscr{M}_4$:

$$x^\alpha_{a'} = \Phi^\alpha_{a'}(\tau) \tag{2.1}$$

where the subindex a' serves to individualize the object and where we choose τ in such a way that:

$$\eta_{\alpha\beta}\dot{\Phi}^\alpha_{a'}.\dot{\Phi}^\beta_{a'} \equiv - m^2_{a'}c^2 \quad , \quad \left(\cdot \equiv \frac{d}{d\tau}\right) \tag{2.2}$$

$\eta_{\alpha\beta}$ being the metric tensor of $\mathscr{M}_4$ and $m_{a'}$ the mass of the particle considered [*6*]. Let us assume also that the evolution of the magnetic moment is described by the following functions:

$$\mathrm{M}^{\lambda\mu}_{a'} = \mathrm{Q}^{\lambda\mu}_{a'}(\tau), \tag{2.3}$$

where $\mathrm{M}^{\lambda\mu}_{a'}$ is, for each value of τ, an antisymmetrical tensor orthogonal to the trajectory of the particle [*7*], i. e.:

$$\mathrm{Q}^{\lambda\mu}_{a'}.\dot{\Phi}_{a'\lambda} = 0 \quad , \quad \forall\tau. \tag{2.4}$$

Thus $\mathrm{M}^{\lambda\mu}_{a'}$ is in a one to one correspondence with the following 4-vector, which will play the role of magnetic moment,

$$\mathrm{M}^\mu_{a'} = \overset{*}{\mathrm{Q}}{}^{\lambda\mu}_{a'}.\dot{\Phi}_{a'\lambda} \equiv -\frac{1}{2}\eta^{\mu\lambda\alpha\beta}\dot{\Phi}_{a'\lambda}\mathrm{Q}_{a'\alpha\beta}, \tag{2.5}$$

$\eta^{\mu\lambda\alpha\beta}$ being the Levi-Civita *tensor* [*8*]. Note that, according to (2.5), the 4-vector $\mathrm{M}^\mu_{a'}$ is trivially orthogonal to the trajectory and so space-like.

Let us now see what is the electromagnetic field created by the previous

EP-MD. For this purpose we first consider the associated current, of which the expression in any point (y^ρ) of $\mathcal{M}_4$ is as follows:

$$J^\alpha_{a'}(y^\rho) = e_{a'}\int_{-\infty}^{+\infty} d\tau . \dot{\Phi}^\alpha_{a'}(\tau) . \delta[y^\rho - \Phi^\rho_{a'}(\tau)] - m^2_{a'} c \int_{-\infty}^{+\infty} d\tau . Q^{\lambda\alpha}_{a'}(\tau) \cdot \frac{\partial}{\partial y^\lambda} \delta[y^\rho - \Phi^\rho_{a'}(\tau)]. \quad (2.6)$$

Moreover, using the Lorentz gauge and taking into account Maxwell's equations, the electromagnetic potential $A^\alpha_{a'}$ is, as is well known, determined in each point (χ^ρ_a) of $\mathcal{M}_4$ by the following expression:

$$A^\alpha_{a'}(x^\rho_a; \varepsilon) = \int \eta_y . D_{(\varepsilon)}(x^\rho_a - y^\rho) . J^\alpha_{a'}(y^\rho), \quad (2.7)$$

where:

$$\eta_y \equiv dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \quad (2.8))$$

represents the volume element of $\mathcal{M}_4$, and where:

$$D_{(\varepsilon)}(x^\rho_a - y^\rho) \equiv 2 . \theta[- \varepsilon(x^0_a - y^0)] . \delta\{(x_{a\lambda} - y_\lambda)(x^\lambda_a - y^\lambda)\}. \quad (2.9)$$

represents the retarded ($\varepsilon = -1$) or advanced ($\varepsilon = +1$) Green's propagator, θ being Heaviside's step function. Let us also remark that the sub-index a serves to distinguish between the field point and the source points. So, substituting (2.6) in (2.7) and carrying out the integrations in the standard manner, we finally obtain [10]:

$$A^\alpha_{a'}(x^\rho_a; \varepsilon) = \frac{e_{a'}}{c} m^{-1}_{a'} \hat{r}^{-1}_{aa'} \hat{\pi}^\alpha_{aa'} - \frac{1}{c} \hat{r}^{-2}_{aa'} \hat{l}_{aa'\mu} \left\{ \varepsilon \hat{\dot{M}}^{\mu\alpha}_{aa'} - \frac{1}{c} m^{-1}_{a'} \hat{r}^{-1}_{aa'} \hat{M}^{\mu\alpha}_{aa'} (m^2_{a'} . c^2 + \hat{\theta}_{aa'\rho} \hat{l}^\rho_{aa'}) \right\}, \quad (2.10)$$

where the following notations have been used:

$$\hat{l}^\alpha_{aa'} \equiv x^\alpha_a - \Phi^\alpha_{a'}(\tau_\varepsilon) \quad (2.11\,a)$$

$$\hat{\pi}^\alpha_{aa'} \equiv \dot{\Phi}^\alpha_{a'}(\tau_\varepsilon) \quad , \quad \hat{\theta}^\alpha_{aa'} \equiv \ddot{\Phi}^\alpha_{a'}(\tau_\varepsilon) \quad (2.11\,b)$$

$$\hat{M}^{\mu\alpha}_{aa'} \equiv Q^{\mu\alpha}_{a'}(\tau_\varepsilon) \quad , \quad \dot{M}^{\mu\alpha}_{aa'} \equiv \dot{Q}^{\mu\alpha}_{a'}(\tau_\varepsilon) \quad (2.11\,c)$$

$$\hat{r}_{aa'} \equiv \frac{\varepsilon}{m_{a'} c} \hat{\pi}_{aa'\rho} \hat{l}^\rho_{aa'} > 0, \quad (2.11\,d)$$

τ_ε ($\varepsilon = \pm 1$) being two values of the parameter τ defined as:

$$\hat{l}_{aa'\rho} \hat{l}^\rho_{aa'} \equiv 0 \quad , \quad \varepsilon \hat{l}^0_{aa'} < 0, \quad (2.12)$$

that is, the values of the parameter corresponding to the points of intersection between the curve (2.1) and the half past ($\varepsilon = -1$) or future ($\varepsilon = +1$) light cone with vertex at the point (x^ρ_a).

Observe that if the particle describes a straight line of space-time and the spin remains constant, the expression (2.10) is reduced to the following in the *appropriate* referential:

$$\mathrm{A}^0_{aa'} = e_{a'} r^{-1}_{aa'} \tag{2.13}$$

$$\vec{\mathrm{A}}_{aa'} = -\frac{1}{c} r^{-3}_{aa'} \vec{l}_{aa'} \wedge \vec{\mathrm{M}}_{a'} \tag{2.14}$$

where $\vec{l}_{aa'}$ is the three-dimensional vector joining the source point and the field point and where the symbol Λ denotes here the *usual* vector product. Furthermore we have put $r_{aa'} \equiv \sqrt{\vec{l}^{\,2}_{aa'}}$. In formulae (2.13) and (2.14) we identify, respectively, the classical scalar potential and vector potential created by a EP-MD at rest.

From expression (2.10) we obtain the electromagnetic field tensor $\mathrm{F}^{\alpha\beta}_{a'}(x^\rho_a;\varepsilon)$ with the usual formula:

$$\mathrm{F}^{\alpha\beta}_{a'}(x^\rho_a;\varepsilon) = \frac{\partial \mathrm{A}^\beta_{a'}}{\partial x_{\alpha\beta}} - \frac{\partial \mathrm{A}^\alpha_{a'}}{\partial x_{\alpha\beta}}. \tag{2.15}$$

In calculating these derivatives we should bear in mind that from (2.12) and (2.11 *d*) it follows that:

$$\frac{\partial \tau_\varepsilon}{\partial x_{\alpha\beta}} = \frac{\varepsilon}{c} m^{-1}_{a'} \widehat{r}^{\,-1}_{aa'} \widehat{l}^{\,\beta}_{aa'}, \tag{2.16}$$

with which we obtain from (2.11 *a*) and (2.11 *d*):

$$\frac{\partial l^\alpha_{aa'}}{\partial x_{\alpha\beta}} = \eta^{\alpha\beta} - \frac{\varepsilon}{c} m^{-1}_{a'} \widehat{r}^{\,-1}_{aa'} \widehat{\pi}^\alpha_{aa'} . \widehat{l}^{\,\beta}_{aa'} \tag{2.17 a}$$

$$\frac{\partial \widehat{r}_{aa'}}{\partial x_{\alpha\beta}} = \frac{\varepsilon}{c} m^{-1}_{a'} \widehat{\pi}^\beta_{aa'} + \frac{1}{c^2} m^{-2}_{a'} \widehat{r}^{\,-1}_{aa'} (m^2_{a'} c^2 + \widehat{\theta}_{aa'\rho} \widehat{l}^{\,\rho}_{aa'}) \widehat{l}^{\,\beta}_{aa'}. \tag{2.17 b}$$

Thus, deriving (2.10) taking these results into account, we obtain [*10*]:

$$\begin{aligned}
\frac{\partial \mathrm{A}^\alpha_{a'}}{\partial x_{a\beta}} = -\frac{e_{a'}}{c^2} m^{-2}_{a'} \widehat{r}^{\,-2}_{aa'} \Big\{ & \varepsilon \widehat{\pi}^\alpha_{aa'} \widehat{\pi}^\beta_{aa'} - \left(\varepsilon \widehat{\theta}^\alpha_{aa'} - \frac{1}{c} m^{-1}_{a'} \widehat{r}^{\,-1}_{aa'} \widehat{\mathrm{L}}_{aa'} \widehat{\pi}^\alpha_{aa'} \right) \widehat{l}^{\,\beta}_{aa'} \Big\} \\
& + \frac{1}{c} \widehat{r}^{\,-2}_{aa'} \left(\varepsilon \widehat{\dot{\mathrm{M}}}{}^{\alpha\beta}_{aa'} - \frac{1}{c} m^{-1}_{a'} \widehat{r}^{\,-1}_{aa'} \widehat{\mathrm{L}}_{aa'} \widehat{\mathrm{M}}^{\alpha\beta}_{aa'} \right) + \frac{1}{c^2} m^{-1}_{a'} \widehat{r}^{\,-3}_{aa'} \widehat{\pi}_{aa'\mu} \widehat{\dot{\mathrm{M}}}{}^{\mu\alpha}_{aa'} \widehat{l}^{\,\beta}_{aa'} \\
& + \frac{1}{c^2} m^{-1}_{a'} \widehat{r}^{\,-3}_{aa'} \widehat{l}_{aa'\mu} \Big\{ \widehat{\mathrm{M}}^{\mu\alpha}_{aa'} \widehat{\theta}^\beta_{aa'} + \left(2\widehat{\dot{\mathrm{M}}}{}^{\mu\alpha}_{aa'} - \varepsilon \frac{3}{c} m^{-1}_{a'} \widehat{r}^{\,-1}_{aa'} \widehat{\mathrm{L}}_{aa'} \widehat{\mathrm{M}}^{\mu\alpha}_{aa'} \right) \widehat{\pi}^\beta_{aa'} \\
& \qquad - \left(\widehat{\ddot{\mathrm{M}}}{}^{\mu\alpha}_{aa'} - \varepsilon \frac{1}{c} m^{-1}_{a'} \widehat{r}^{\,-1}_{aa'} \widehat{\dot{\theta}}_{aa'\rho} \widehat{l}^{\,\rho}_{aa'} \widehat{\mathrm{M}}^{\mu\alpha}_{aa'} \right) \widehat{l}^{\,\beta}_{aa'} \\
& \qquad + \frac{3}{c} m^{-1}_{a'} \widehat{r}^{\,-1}_{aa'} \widehat{\mathrm{L}}_{aa'} \left(\varepsilon \widehat{\dot{\mathrm{M}}}{}^{\mu\alpha}_{aa'} - \frac{1}{c} m^{-1}_{a'} \widehat{r}^{\,-1}_{aa'} \widehat{\mathrm{L}}_{aa'} \widehat{\mathrm{M}}^{\mu\alpha}_{aa'} \right) \widehat{l}^{\,\beta}_{aa'} \Big\},
\end{aligned} \tag{2.18}$$

in which we have used the following notations:

$$\widehat{\mathrm{L}}_{aa'} \equiv m_{a'}^2 c^2 + \widehat{\theta}_{aa'\rho} \widehat{l}^{\rho}_{aa'} \tag{2.19}$$

$$\widehat{\dot{\theta}}^{\alpha}_{aa'} \equiv \dddot{\Phi}^{\alpha}_{a'}(\tau_\varepsilon) \quad , \quad \widehat{\ddot{\mathrm{M}}}^{\mu\alpha}_{aa'} \equiv \ddot{\mathrm{Q}}^{\mu\alpha}_{a'}(\tau_\varepsilon). \tag{2.20}$$

By substituting (2.18) in (2.15) we arrive at the desired expression of the electromagnetic field tensor at point (x_a^ρ). Observe that in this expression we encounter as new elements with respect to the potential (2.10), the super-acceleration $\widehat{\dot{\theta}}^{\alpha}_{aa'}$ and the « superprecession » $\widehat{\ddot{\mathrm{M}}}^{\mu\alpha}_{aa'}$ of the EP-MD in the retarded ($\varepsilon = -1$) or advanced ($\varepsilon = +1$) position.

B. Action of an external field on a EP-MD.

Let us now consider a EP-MD subjected to the action of an external electromagnetic field which we shall designate $\mathrm{F}^{\alpha\beta}$. The object of the present Subsection will be to expound and comment on the equations of motion of this object for such a situation.

Regarding *space-time* equations of motion, we shall use the following:

$$\left\{ \begin{aligned} & \frac{dx_a^\alpha}{d\tau} = \pi_a^\alpha & (2.21\,a) \\ & \frac{d\pi_a^\alpha}{d\tau} = \frac{e_a}{c}\mathrm{F}^{\alpha\rho}\pi_{a\rho} + \frac{1}{2c^2}(\eta^{\alpha\beta} + \pi_a^{-2}\pi_a^\alpha\pi_a^\beta)\eta_{\rho\sigma\mu\nu}\pi_a^\rho \mathrm{M}_a^\sigma \frac{\partial \mathrm{F}^{\mu\nu}}{\partial x_a^\beta} & (2.21\,b) \end{aligned} \right.$$

where x_a^α represents the position of the particle and e_a its electric charge, and also make use of the notation $\pi_a^2 \equiv -\pi_a^\rho \pi_{a\rho}$. Furthermore the vector M_a^σ represents the corresponding magnetic moment, which is assumed to be orthogonal to the object's trajectory, in other words,

$$\mathrm{M}_a^\sigma \pi_{a\sigma} = 0, \tag{2.22}$$

whereby it has no time component in its own instantaneous referential. Note that, in view of the structure of the second member of (2.21 *b*), the quantity π_a^2 is trivially a first integral of system (2.21), to which we shall always assign the value:

$$\pi_a^2 = m_a^2 c^2, \tag{2.23}$$

m_a being the mass of the particle. This choice of value for π_a^2 is equivalent to interpreting the parameter τ so that the product $m_a\tau$ is the proper time.

The justification of equations (2.21) offers no difficulty once the relations (2.22) and (2.23) have been imposed. In fact, writing (2.21 *b*) in the particle's proper referential, we obtain trivially:

$$m_a \frac{dv_a^i}{dt} = e_a \mathrm{E}^i + \frac{1}{c}\mathrm{M}_{aj}\frac{\partial \mathrm{B}^j}{\partial x_{ai}}, \tag{2.24}$$

where v_a^i represents the velocity of the particle and where (E^i, B^j) are the electric field vector and magnetic field vector respectively; as can be seen, the second member of (2.24) coincides with the classic force exerted by an electromagnetic field on a EP-MD at rest. Thus the second member of (2.21 *b*) is, except for one factor, the sum of the usual Lorentz force and another term representing the covariant expression of the force exerted by a magnetic field on a magnetic dipole.

Now let us deal with the equations of motion *relative to the magnetic moment.* Here let us assume that the EP-MD considered possesses an intrinsic angular momentum or spin described by a vector S_a^μ, to be related to the magnetic moment vector M_a^μ by the standard formula:

$$M_a^\mu = \lambda_a \frac{e_a}{2m_a} S_a^\mu, \tag{2.25}$$

λ_a being the *giromagnetic coefficient* of the particle. We shall use then as equations of motion the so-called equations of Bargmann, Michel and Telegdi [*2*, *11*], which can be written as follows:

$$\frac{dS_a^\alpha}{d\tau} = \frac{\lambda_a e_a}{2c}(\pi_a^{-2}\pi_{a\mu}S_{a\nu}F^{\mu\nu}\pi_a^\alpha - S_{a\mu}F^{\mu\alpha}) + \pi_a^{-2}S_{a\rho}\dot{\pi}_a^\rho\pi_a^\alpha, \tag{2.26}$$

where, naturally, $\dot{\pi}_a^\alpha$ is given by the second member of (2.21 *b*). Let us observe that (2.26) allows us to deduce immediately that the quantity $(\pi_a S_a) \equiv \pi_a^\rho S_{a\rho}$ is a first integral, and this result fits in with hypotheses (2.22) and (2.25). What is more, in consequence of this, it turns out that the quantity:

$$s_a^2 \equiv S_{a\rho}S_a^\rho \tag{2.27}$$

is also a first integral of (2.26), i. e., the modulus of spin is a constant of motion as was to be expected. For this reason from here on we shall not use the spin vector S_a^α but the corresponding unitary vector γ_a^α, so that:

$$S_a^\alpha \equiv s_a\gamma_a^\alpha. \tag{2.28}$$

Let us point out, finally, that by writing equation (2.26) in the particle's own instantaneous referential the following is obtained:

$$\frac{d\vec{S}_a}{dt} = \frac{\lambda_a e_a}{2m_a c}\vec{S}_a \wedge \vec{B}, \tag{2.29}$$

which is the well-known classical equation of spin evolution.

C. Lienard-Wiechert equations for two EP-MD's.

Let us now consider an isolated system of two EP-MD's in interaction and analyze the « equations of motion » given by Classical Field Theory.

To obtain these equations it is enough to assume that the electromagnetic field tensor $F^{\mu\nu}$, which appears in equations (2.21) and (2.26) in the previous Subsection, is precisely the tensor $F^{\mu\nu}_{a'}(x_a;\,\varepsilon)$ calculated in Subsection A, making subindices a and a' necessarily take the values (1, 2) with $a \neq a'$. The formal result thus obtained, taking into account expression (2.18) and equations (2.21) and (2.26), is the following:

$$\frac{d\pi^\alpha_a}{d\tau} = W^\alpha_a(\widehat{l}_{aa'}\,;\,\pi_a,\,\widehat{\pi}_{aa'}\,;\,\gamma_a,\,\widehat{\gamma}_{aa'}\,;\,\widehat{\theta}_{aa'},\,\widehat{\Delta}_{aa'}\,;\,\widehat{\dot{\theta}}_{aa'},\,\widehat{\dot{\Delta}}_{aa'}\,;\,\widehat{\ddot{\theta}}_{aa'},\,\widehat{\ddot{\Delta}}_{aa'}) \tag{2.30 a}$$

$$\frac{d\gamma^\alpha_a}{d\tau} = Z^\alpha_a(\widehat{l}_{aa'}\,;\,\pi_a,\,\widehat{\pi}_{aa'}\,;\,\gamma_a,\,\widehat{\gamma}_{aa'}\,;\,\widehat{\theta}_{aa'},\,\widehat{\Delta}_{aa'}\,;\,\widehat{\dot{\theta}}_{aa'},\,\widehat{\dot{\Delta}}_{aa'}\,;\,\widehat{\ddot{\theta}}_{aa'},\,\widehat{\ddot{\Delta}}_{aa'}), \tag{2.30 b}$$

where W^α_a and Z^α_a are known functions of their arguments, in which the 4-vectorial Greek index has been omitted to simplify writing. What is more, the following evident notations have been utilized:

$$\widehat{\gamma}^\alpha_{aa'} \equiv \Psi^\alpha_{a'}(\tau_\varepsilon) \quad , \quad \widehat{\Delta}^\alpha_{aa'} \equiv \dot{\Psi}^\alpha_{a'}(\tau_\varepsilon) \quad , \quad \widehat{\dot{\Delta}}^\alpha_{aa'} \equiv \ddot{\Psi}^\alpha_{a'}(\tau_\varepsilon) \quad , \quad \widehat{\ddot{\Delta}}^\alpha_{aa'} \equiv \dddot{\Psi}^\alpha_{a'}(\tau_\varepsilon) \tag{2.31 a}$$

$$\widehat{\ddot{\theta}}^\alpha_{aa'} \equiv \ddddot{\Phi}^\alpha_{a'}(\tau_\varepsilon), \tag{2.31 b}$$

where, in line with (2.28), (2.25), and (2.5) functions $\Psi^\alpha_{a'}(\tau)$ are such that:

$$\lambda_{a'}\frac{e_{a'}s_{a'}}{2m_{a'}}\Psi^\alpha_{a'} \equiv -\frac{1}{2}\eta^{\alpha\beta\mu\nu}\dot{\Phi}_{a'\beta}Q_{a'\mu\nu} \tag{2.32}$$

or inversely:

$$Q_{a'\mu\nu} = \lambda_{a'}\frac{e_{a'}s_{a'}}{2m^3_{a'}c^2}\eta_{\mu\nu\alpha\beta}\dot{\Phi}^\alpha_{a'}\Psi^\beta_{a'}. \tag{2.33}$$

It should be pointed out that the supersuperacceleration $\widehat{\ddot{\theta}}_{aa'}$ and the supersuperprecession $\widehat{\ddot{\Delta}}_{aa'}$ in equations (2.30) arise from the derivative of the tensor field, which appears in the right-hand member of (2.21 *b*) and consequently also in (2.26). This derivative was not explicitly calculated in Subsection A, since it gives rise to an expression of excessive length. Nevertheless it will be calculated in the next Section to a certain approximation.

Equations (2.30) are not equations of motion in the ordinary sense, but rather a differential system with retarded ($\varepsilon = -1$) or advanced ($\varepsilon = +1$) arguments and also of the so-called *neutral type*, for which the existence and uniqueness theorem is not verified. This situation is not new as it already appeared in the case of simple electric poles, but the complication is now greater due to the presence of first and second derivatives of $\widehat{\theta}_{aa'}$ and $\widehat{\Delta}_{aa'}$.

The interest of equations (2.30) arises, however, as in the case of spinless particles [*3*], from their utilization as « boundary conditions ». That is to say, for the determination of ordinary equations of motion whose solutions will satisfy these equations of motion. It is in this way that they will be used in the following Section.

3. FIRST ORDER PREDICTIVE INTERACTION OF TWO EP-MD'S

As before, let us consider an isolated system of two EP-MD's in interaction. We will now assume that this interaction is governed by a Projectable System in the sense of theorem 3.2 of BM, that is, by a differential system of the following type (with obvious notations):

$$\left\{\begin{aligned} &\frac{dx_a^\alpha}{d\tau} = \pi_a^\alpha \quad , \quad \frac{d\pi_a^\alpha}{d\tau} = \theta_a^\alpha(x_b^\beta, \pi_c^\mu, \gamma_d^\nu) && (3.1\,a) \\ & && (a, b, c, d = 1, 2) \\ &\frac{d\gamma_a^\alpha}{d\tau} = \Delta_a^\alpha(x_b^\beta, \pi_c^\mu, \gamma_d^\nu), && (3.1\,b) \end{aligned}\right.$$

where functions θ_a^α and Δ_a^α must satisfy the two following groups of equations:

$$\left\{\begin{aligned} \pi_{a\rho}\theta_a^\rho &= 0 && (3.2\,a) \\ \pi_{a\rho}\Delta_a^\rho + \gamma_{a\rho}\theta_a^\rho &= 0 && (3.2\,b) \\ \gamma_{a\rho}\Delta_a^\rho &= 0 && (3.2\,c) \end{aligned}\right.$$

$$\left\{\begin{aligned} &\pi_{a'}^\rho \frac{\partial\theta_a^\alpha}{\partial x^{a'\rho}} + \theta_{a'}^\rho \frac{\partial\theta_a^\alpha}{\partial \pi^{a'\rho}} + \Delta_{a'}^\rho \frac{\partial\theta_a^\alpha}{\partial \gamma^{a'\rho}} = 0 && (3.3\,a) \\ & && (a' \neq a) \\ &\pi_{a'}^\rho \frac{\partial\Delta_a^\alpha}{\partial x^{a'\rho}} + \theta_{a'}^\rho \frac{\partial\Delta_a^\alpha}{\partial \pi^{a'\rho}} + \Delta_{a'}^\rho \frac{\partial\Delta_a^\alpha}{\partial \gamma^{a'\rho}} = 0. && (3.3\,b) \end{aligned}\right.$$

As we know, equations (3.2) imply that the six quantities $\pi_a^2 \equiv -\pi_a^\rho\pi_{a\rho}$, $(\pi_a\gamma_a) \equiv \pi_{a\rho}\gamma_a^\rho$ and $\gamma_a^2 \equiv \gamma_a^\rho\gamma_{a\rho}$ are first integrals of system (3.1). Regarding the last four, we shall always assign to them the following values:

$$(\pi_a\gamma_a) = 0 \quad , \quad \gamma_a^2 = 1, \tag{3.4}$$

so that the number of independent initial conditions is reduced to 20. As for the first two integrals π_a^2, these will be interpreted *a posteriori* so that the quantities:

$$m_a \equiv c^{-1}\pi_a \tag{3.5}$$

will represent the masses of the particles.

We shall also assume that functions θ_a^α and Δ_a^α are developable in power series of the product $g \equiv e_a e_{a'}$ of the electric charges of the two EP-MD's, i. e.,

$$\theta_a^\alpha = \sum_{n=1}^{\infty} g^n \theta_a^{\alpha(n)} \tag{3.6 a}$$

$$\Delta_a^\alpha = \sum_{n=1}^{\infty} g^n \Delta_a^{\alpha(n)}, \tag{3.6 b}$$

where both functions have been made to be zero for $g = 0$.

We will limit ourselves to the first order calculation of series (3.6), making use for this purpose of the supplementary condition that the solutions of system (3.1) should verify the « boundary conditions » (2.30) of the previous Section. To this end we shall begin by writing out these equations (2.30) to the first order of approximation in g, that is, obtain functions $W_a^{\alpha(1)}$ and $Z_a^{\alpha(1)}$ (with obvious notations).

In accordance with (2.18), (2.19) and (2.23) we have:

$$\begin{aligned}\frac{\partial A^\alpha_{a'}}{\partial x_{\alpha\beta}} \sim &- e_{a'}\widehat{\pi}_{aa'}^{-2}\widehat{r}_{aa'}^{-2}(\varepsilon\widehat{\pi}^\beta_{aa'} + \widehat{\pi}_{aa'}\widehat{r}_{aa'}^{-1}\widehat{l}^\beta_{aa'})\widehat{\pi}^\alpha_{aa'} \\ &- \frac{1}{2}\lambda_{a'}e_{a'}s_{a'}\widehat{\pi}_{aa'}^{-2}\widehat{r}_{aa'}^{-3}\eta^{\alpha\beta\mu\nu}\widehat{\pi}_{aa'\mu}\widehat{\gamma}_{aa'\nu} \\ &+ \frac{3}{2}\lambda_{a'}e_{a'}s_{a'}\widehat{\pi}_{aa'}^{-3}\widehat{r}_{aa'}^{-4}(\varepsilon\widehat{\pi}^\beta_{aa'} + \widehat{\pi}_{aa'}\widehat{r}_{aa'}^{-1}\widehat{l}^\beta_{aa'})\eta^{\alpha\mu\nu\rho}\widehat{l}_{aa'\mu}\widehat{\pi}_{aa'\nu}\widehat{\gamma}_{aa'\rho},\end{aligned} \tag{3.7}$$

where the symbol $\sim$ denotes equality excepting higher order terms in the electric charges. Now, substituting (3.7) in (2.15) we get:

$$\begin{aligned}F^{\alpha\beta}_{a'} \sim &- e_{a'}\widehat{\pi}_{aa'}^{-1}\widehat{r}_{aa'}^{-3}(\widehat{l}^\alpha_{aa'}\widehat{\pi}^\beta_{aa'} - \widehat{l}^\beta_{aa'}\widehat{\pi}^\alpha_{aa'}) + \lambda_{a'}e_{a'}s_{a'}\widehat{\pi}_{aa'}^{-2}\widehat{r}_{aa'}^{-3}\eta^{\alpha\beta\mu\nu}\widehat{\pi}_{aa'\mu}\widehat{\gamma}_{aa'\nu} \\ &+ \frac{3}{2}\lambda_{a'}e_{a'}s_{a'}\widehat{\pi}_{aa'}^{-3}\widehat{r}_{aa'}^{-4}\{(\varepsilon\widehat{\pi}^\alpha_{aa'} + \widehat{\pi}_{aa'}\widehat{r}_{aa'}^{-1}\widehat{l}^\alpha_{aa'})\eta^{\beta\mu\nu\rho} \\ &- (\varepsilon\widehat{\pi}^\beta_{aa'} + \widehat{\pi}_{aa'}\widehat{r}_{aa'}^{-1}\widehat{l}^\beta_{aa'})\eta^{\alpha\mu\nu\rho}\}\widehat{l}_{aa'\mu}\widehat{\pi}_{aa'\nu}\widehat{\gamma}_{aa'\rho}.\end{aligned} \tag{3.8}$$

Let us now calculate the derivative of (3.7) with the same order of approximation. Taking (2.16) and (2.17) into account, we find:

$$\begin{aligned}\frac{\partial^2 A^\alpha_{a'}}{\partial x_{a\sigma}\partial x_{a\beta}} \sim &\, 2e_{a'}\widehat{\pi}_{aa'}^{-3}\widehat{r}_{aa'}^{-3}\widehat{\omega}^\sigma_{aa'}\widehat{\omega}^\beta_{aa'}\widehat{\pi}^\alpha_{aa'} + e_{a'}\widehat{\pi}_{aa'}^{-2}\widehat{r}_{aa'}^{-3}(\widehat{r}_{aa'}^{-1}\widehat{\omega}^\sigma_{aa'}\widehat{l}^\beta_{aa'} - \widehat{\pi}_{aa'}\widehat{\kappa}^{\beta\sigma}_{aa'})\widehat{\pi}^\alpha_{aa'} \\ &+ \frac{3}{2}\lambda_{a'}e_{a'}s_{a'}\widehat{\pi}_{aa'}^{-3}\widehat{r}_{aa'}^{-4}(\widehat{\omega}_{aa'}\eta^{\alpha\beta\mu\nu} + \widehat{\omega}^\beta_{aa'}\eta^{\alpha\sigma\mu\nu})\widehat{\pi}_{aa'\mu}\widehat{\gamma}_{aa'\nu} \\ &- 6\lambda_{a'}e_{a'}s_{a'}\widehat{\pi}_{aa'}^{-4}\widehat{r}_{aa'}^{-5}\widehat{\omega}^\sigma_{aa'}\widehat{\omega}^\beta_{aa'}\eta^{\alpha\mu\nu\rho}\widehat{l}_{aa'\mu}\widehat{\pi}_{aa'\nu}\widehat{\gamma}_{aa'\rho} \\ &- \frac{3}{2}\lambda_{a'}e_{a'}s_{a'}\widehat{\pi}_{aa'}^{-3}\widehat{r}_{aa'}^{-5}(\widehat{r}_{aa'}^{-1}\widehat{\omega}^\sigma_{aa'}\widehat{l}^\beta_{aa'} - \widehat{\pi}_{aa'}\widehat{\kappa}^{\beta\sigma}_{aa'})\eta^{\alpha\mu\nu\rho}\widehat{l}_{aa'\mu}\widehat{\pi}_{aa'\nu}\widehat{\gamma}_{aa'\rho},\end{aligned} \tag{3.9}$$

where, to simplify writing, these notations have been used:

$$\widehat{\omega}^{\sigma}_{aa'} \equiv \varepsilon\widehat{\pi}^{\sigma}_{aa'} + \widehat{\pi}_{aa'}\widehat{r}^{-1}_{aa'}\widehat{l}^{\sigma}_{aa'} \tag{3.10 a}$$

$$\widehat{\kappa}^{\beta\sigma}_{aa'} \equiv \eta^{\beta\sigma} - \varepsilon\widehat{\pi}^{-1}_{aa'}\widehat{r}^{-1}_{aa'}\widehat{\pi}^{\beta}_{aa'}\widehat{l}^{\sigma}_{aa'}. \tag{3.10 b}$$

We could now simply substitute (3.8) and (3.9) in the second members of (2.21 *b*) and (2.26) to obtain the desired functions $W^{\alpha(1)}_a$ and $Z^{\alpha(1)}_a$. Note that the presence of the tensor $\eta_{\rho\sigma\mu\nu}$ in the second term of the right hand side of (2.21 *b*) makes the antisymmetrization of (3.9) unnecessary.

Now let us consider equations (3.3) to the first order in g. In view of (3.6) we have the trivial result:

$$\pi^{\rho}_{a'}\frac{\partial\theta^{\alpha(1)}_a}{\partial x^{a'\rho}} = 0 \quad , \quad \pi^{\rho}_{a'}\frac{\partial\Delta^{\alpha(1)}_a}{\partial x^{a'\rho}} = 0. \tag{3.11}$$

Our object, then, is to solve these equations (3.11) with the « boundary conditions » imposed by functions $W^{\alpha(1)}_a$ and $Z^{\alpha(1)}_a$. Now, as is already known [*12*], there exists a unique solution of (3.11) satisfying these « boundary conditions », which is given by:

$$\theta^{\alpha(1)}_a(x^{\beta}_b, \pi^{\mu}_c, \gamma^{\nu}_d) = \overset{*}{W}{}^{\alpha(1)}_a \quad , \quad \Delta^{\alpha(1)}_a(x^{\beta}_b, \pi^{\mu}_c, \gamma^{\nu}_d) = \overset{*}{Z}{}^{\alpha(1)}_a, \tag{3.12}$$

where $\overset{*}{W}{}^{\alpha(1)}_a$ and $\overset{*}{Z}{}^{\alpha(1)}_a$ are obtained from $W^{\alpha(1)}_a$ and $Z^{\alpha(1)}_a$ respectively by making the following substitutions:

$$\left\{\begin{array}{ll} \widehat{l}^{\alpha}_{aa'} \to x^{\alpha}_{aa'} - \pi^{-2}_{a'}\{-(x_{aa'}\pi_{a'}) + \varepsilon\pi_{a'}r_{aa'}\}\pi^{\alpha}_{a'} & (3.13\,a) \\ \widehat{\pi}^{\alpha}_{aa'} \to \pi^{\alpha}_{a'} & (3.13\,b) \\ \widehat{\gamma}^{\alpha}_{aa'} \to \gamma^{\alpha}_{a'} & (3.13\,c) \end{array}\right.$$

where we have made:

$$x^{\alpha}_{aa'} \equiv x^{\alpha}_a - x^{\alpha}_{a'} \tag{3.14 a}$$

$$r_{aa'} \equiv +\{x^2_{aa'} + \pi^{-2}_{a'}(x_{aa'}\pi_{a'})^2\}^{1/2}. \tag{3.14 b}$$

Note that, in particular, we have:

$$\widehat{r}_{aa'} \to r_{aa'} \tag{3.15 a}$$

$$\widehat{\omega}^{\alpha}_{aa'} \to \pi_{a'}r^{-1}_{aa'}[x^{\alpha}_{aa'} + \pi^{-2}_{a'}(x_{aa'}\pi_{a'})\pi^{\alpha}_{a'}] \tag{3.15 b}$$

$$\begin{aligned} \widehat{r}^{-1}_{aa'}\widehat{\omega}^{\sigma}_{aa'}l^{\beta}_{aa'} - \widehat{\pi}_{aa'}\widehat{\kappa}^{\beta\sigma}_{aa'} \to & - \pi_{a'}(\eta^{\beta\sigma} + \pi^{-2}_{a'}\pi^{\beta}_{a'}\pi^{\sigma}_{a'}) \\ & + \pi_{a'}r^{-2}_{aa'}[x^{\sigma}_{aa'} + \pi^{-2}_{a'}(x_{aa'}\pi_{a'})\pi^{\sigma}_{a'}][x^{\beta}_{aa'} + \pi^{-2}_{a'}(x_{aa'}\pi_{a'})\pi^{\beta}_{a'}] \end{aligned} \tag{3.15 c}$$

whereupon, looking at (3.8) and (3.9), we may assert that functions (3.12) do not depend on ε, i. e. in the order considered the result is independent of whether we have taken retarded or advanced potentials.

Carrying out the substitutions (3.13) and (3.15) in (3.8) and (3.9), and bringing this result to (2.21 b) and (2.26), we finally obtain, after a tedious calculation, the following expressions for $\theta_a^{\alpha(1)}$ and $\Delta_a^{\alpha(1)}$:

$$
\begin{aligned}
\theta_a^{\alpha(1)} = & \frac{1}{c}\pi_{a'}^{-1}r_{aa'}^{-3}(kh_{aa'}^{\alpha} - z_a t_{a'}^{\alpha}) + \frac{1}{c}\lambda_{a'}s_{a'}\pi_{a'}^{-2}r_{aa'}^{-3}\eta^{\alpha\rho\mu\nu}\pi_{a\rho}\pi_{a'\mu}\gamma_{a'\nu} \\
& - \frac{3}{2c}\lambda_{a'}s_{a'}\pi_{a'}^{-4}\Lambda^2 z_a r_{aa'}^{-5}\eta^{\alpha\rho\mu\nu}(h_{aa'\rho} + z_a\pi_{a\rho})\pi_{a'\mu}\gamma_{a'\nu} \\
& - \frac{3}{2c}\lambda_{a'}s_{a'}\pi_{a'}^{-2}r_{aa'}^{-5}\eta_{\sigma\rho\mu\nu}h_{aa'}^{\sigma}\pi_a^{\rho}\pi_{a'}^{\mu}\gamma_{a'}^{\nu}(h_{aa'}^{\alpha} + \pi_{a'}^{-2}z_a t_a^{\alpha}) \\
& + \frac{1}{2c}\lambda_a s_a \pi_a^{-1}\pi_{a'}^{-1}r_{aa'}^{-3}\eta^{\alpha\rho\mu\nu}\pi_{a\rho}\pi_{a'\mu}\gamma_{a\nu} \\
& - \frac{3}{2c}\lambda_a s_a \pi_a^{-1}\pi_{a'}^{-1}r_{aa'}^{-5}\eta_{\sigma\rho\mu\nu}h_{aa'}^{\sigma}\pi_a^{\rho}\pi_{a'}^{\mu}\gamma_a^{\nu}(h_{aa'}^{\alpha} - \pi_a^{-2}\pi_{a'}^{-2}kz_a t_{a'}^{\alpha}) \\
& + \frac{3}{4c}\lambda_a\lambda_{a'}s_a s_{a'}\pi_a^{-1}\pi_{a'}^{-2}kr_{aa'}^{-5}[(h_{aa'}\gamma_{a'}) + z_a(\pi_a\gamma_{a'})]\gamma_a^{\alpha} \\
& + \frac{3}{4c}\lambda_a\lambda_{a'}s_a s_{a'}\pi_a^{-1}\pi_{a'}^{-2}r_{aa'}^{-5}[k(h_{aa'}\gamma_a) - \pi_a^2 z_a(\pi_{a'}\gamma_a)] \,.\, [\gamma_{a'}^{\alpha} + \pi_a^{-2}(\pi_a\gamma_{a'})\pi_a^{\alpha}] \\
& + \frac{3}{4c}\lambda_a\lambda_{a'}s_a s_{a'}\pi_a^{-1}\pi_{a'}^{-2}r_{aa'}^{-5}\,\{\, (\pi_a\gamma_{a'})(\pi_{a'}\gamma_a) + k(\gamma_a\gamma_{a'}) - 5r_{aa'}^{-2}[(h_{aa'}\gamma_{a'}) + z_a(\pi_a\gamma_{a'})] \\
& \qquad .\,[k(h_{aa'}\gamma_a) - \pi_a^2 z_a(\pi_{a'}\gamma_a)]\,\}\,(h_{aa'}^{\alpha} - \pi_a^{-2}\pi_{a'}^{-2}kz_a t_{a'}^{\alpha}) \qquad (3.16\,a)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_a^{\alpha(1)} = & \pi_a^{-2}\gamma_{a\rho}\theta_a^{\rho(1)}\pi_a^{\alpha} - \frac{1}{2c}\lambda_a\pi_{a'}^{-1}r_{aa'}^{-3}[(\pi_{a'}\gamma_a)h_{aa'}^{\alpha} - \pi_a^{-2}(h_{aa'}\gamma_a)t_{a'}^{\alpha}] \\
& - \frac{1}{2c}\lambda_a\lambda_{a'}s_{a'}\pi_{a'}^{-2}r_{aa'}^{-3}\eta^{\alpha\rho\mu\nu}\pi_{a'\rho}\gamma_{a\mu}\gamma_{a'\nu} \\
& - \frac{3}{4c}\lambda_a\lambda_{a'}s_{a'}\pi_{a'}^{-2}r_{aa'}^{-5}[(h_{aa'}\gamma_a) - \pi_{a'}^{-2}kz_a(\pi_{a'}\gamma_a)]\eta^{\alpha\rho\mu\nu}(h_{aa'\rho} + z_a\pi_{a\rho})\pi_{a'\mu}\gamma_{a'\nu} \\
& - \frac{1}{2c}\lambda_a\lambda_{a'}s_{a'}\pi_a^{-2}\pi_{a'}^{-2}r_{aa'}^{-3}\eta_{\sigma\rho\mu\nu}\pi_a^{\sigma}\pi_{a'}^{\rho}\gamma_a^{\mu}\gamma_{a'}^{\nu}\,.\,\pi_a^{\alpha} \\
& + \frac{3}{4c}\lambda_a\lambda_{a'}s_{a'}\pi_a^{-2}\pi_{a'}^{-2}r_{aa'}^{-5}[(h_{aa'}\gamma_a) - \pi_{a'}^{-2}kz_a(\pi_{a'}\gamma_a)]\eta_{\sigma\rho\mu\nu}h_{aa'}^{\sigma}\pi_{a'}^{\mu}\pi_a^{\rho}\gamma_{a'}^{\nu}\,.\,\pi_a^{\alpha} \\
& + \frac{3}{4c}\lambda_a\lambda_{a'}s_{a'}\pi_{a'}^{-2}r_{aa'}^{-5}\eta_{\sigma\rho\mu\nu}(h_{aa'}^{\sigma} + z_a\pi_a^{\sigma})\pi_{a'}^{\rho}\gamma_a^{\mu}\gamma_{a'}^{\nu}\,.\,(h_{aa'}^{\alpha} - \pi_a^{-2}\pi_{a'}^{-2}kz_a t_{a'}^{\alpha}), \qquad (3.16\,b)
\end{aligned}
$$

where the following definitions have been used:

$$k \equiv -(\pi_a\pi_{a'}) \quad , \quad \Lambda^2 \equiv k^2 - \pi_a^2\pi_{a'}^2 \qquad (3.17\,a)$$

$$z_a \equiv \Lambda^{-2}[\pi_{a'}^2(x_{aa'}\pi_a) - k(x_{aa'}\pi_{a'})] \qquad (3.17\,b)$$

$$h_{aa'}^{\alpha} \equiv x_{aa'}^{\alpha} - z_a\pi_a^{\alpha} + z_{a'}\pi_{a'}^{\alpha} \qquad (3.17\,c)$$

$$t_a^{\alpha} \equiv \pi_{a'}^2\pi_a^{\alpha} - k\pi_{a'}^{\alpha} \qquad (3.17\,d)$$

$$(\mathrm{AB}) \equiv \mathrm{A}^{\rho}\mathrm{B}_{\rho}, \quad \text{for every pair of 4-vectors.} \qquad (3.17\,e)$$

Note that with these new variables quantity (3.14 *b*) is written as follows:

$$r_{aa'} = + \{ h_{aa'}^2 + \Lambda^2 \pi_{a'}^{-2} z_a \}^{1/2}. \tag{3.18}$$

We should also point out that in the construction of $\theta_a^{\alpha(1)}$ and $\Delta_a^{\alpha(1)}$ we have already taken into account relations (3.4) in order to simplify writing.

4. CONSERVED MOMENTA AND FIRST ORDER SIMPLECTIC FORM

As we stated in the introduction, the purpose of this Section is to construct the Linear Momentum, the Angular Momentum and the Simplectic Form corresponding to the dynamical system defined by expressions (3.16). As this involves a direct application of the results and methods put forward in the last Section of BM, we refer the reader to the said article for any matters of theory or detail which may remain unclear in the present work. We would also point out that the final explicit results are excessively long, for which reason we shall limit ourselves to writing them out in the most compact possible form. To be precise, we shall write the final expressions in a simple integral form, a table of the integrals involved in the expressions being given in Appendix 1. Thus the reader will be able to analyze the behaviour of the required quantities without great difficulty.

A. Linear momentum.

Let us suppose that the total Linear Momentum or Energy Momentum vector P^μ of the system being considered admits of a development analogous to (3.6), that is:

$$P^\mu = \sum_{n=0}^{\infty} g^n P^{\mu(n)}, \tag{4.1}$$

where the zero order corresponds to the Linear Momentum of free particles:

$$P^{\mu(0)} \equiv \pi_a^\mu + \pi_{a'}^\mu. \tag{4.2}$$

Then, using (4.2) and formulae (4.45) and (4.46 *b*) of BM, order one of series (4.1) is written as follows:

$$P^{\mu(1)} = \int_0^{-\infty} d\tau . \varphi_\tau^* \theta_a^{\mu(1)} + \int_0^{-\infty} d\tau . \varphi_\tau^* \theta_{a'}^{\mu(1)}, \tag{4.3}$$

where φ_τ^* represents the « reciprocal image » transformation [*13*] of transformation φ_τ defined by:

$$\varphi_\tau : \begin{cases} x_a^\alpha \rightarrow x_a^\alpha + \pi_a^\alpha \tau \\ \pi_a^\alpha \rightarrow \pi_a^\alpha \\ \gamma_a^\alpha \rightarrow \gamma_a^\alpha . \end{cases} \tag{4.4}$$

To carry out the integrations in (4.3) it should be remembered that, in general, if φ is a transformation of a manifold V onto itself and $f(y)$ represents a function on V, the function φ^*f is defined as follows:

$$(\varphi^*f)(y) \equiv f[\varphi(y)] \quad , \quad \forall y \in \mathrm{V} \tag{4.5}$$

Consequently, the integrands of (4.3) will be obtained from (3.16 *a*) by carrying out the substitutions indicated by (4.4). Now, since the variables $(\pi^\alpha_a, \gamma^\beta_b)$ remain unchanged, only the quantities in (3.16 *a*) containing the variables x^α_a will change, that is $(z_a, h^\alpha_{aa'}, r_{aa'})$. But according to (3.17 *b*, *c*) and (4.4), the following is trivially obtained:

$$\varphi^*_\tau z_a \;= z_a + \tau \tag{4.6 a}$$

$$\varphi^*_\tau h^\alpha_{aa'} = h^\alpha_{aa'} \tag{4.6 b}$$

and consequently, following (3.18),

$$\varphi^*_\tau r_{aa'} = + \left\{ h^2_{aa'} + \Lambda^2 \pi^{-2}_{a'}(z_a + \tau)^2 \right\}^{1/2} \equiv r_{aa'}(\tau). \tag{4.6 c}$$

It turns out, then, that if we designate symbolically the second member of (3.16 *a*) by $\theta^{\alpha(1)}_a(z_a, r_{aa'})$, expression (4.3) of total Linear Momentum to the first order is rewritten as follows:

$$\mathrm{P}^{\mu(1)} = \int_0^{-\infty} \theta^{\mu(1)}_a[z_a + \tau, r_{aa'}(\tau)].d\tau + \int_0^{-\infty} \theta^{\mu(1)}_{a'}[z_{a'} + \tau, r_{a'a}(\tau)].d\tau. \tag{4.7}$$

It would now be sufficient to carry out the corresponding integrations in order to obtain explicitly the required result. Observing (3.16 *a*) it can be seen that these integrals are all immediate and converging, yet we shall write them out in Appendix 1 for a rapid visualization of the behaviour of $\mathrm{P}^{\mu(1)}$.

Note, finally, that at the order under consideration the calculation of Linear Momentum does not require the knowledge of the « precessions » (3.16 *b*).

B. **Angular momentum.**

As with Linear Momentum, let us assume that the total Angular Momentum $\mathrm{J}^{\lambda\mu} = - \mathrm{J}^{\mu\lambda}$ of the system of two EP-MD's admits of a development in series of the following type:

$$\mathrm{J}^{\lambda\mu} = \sum_{n=0}^{\infty} g^n \mathrm{J}^{\lambda\mu(n)}, \tag{4.8}$$

where $\mathrm{J}^{\lambda\mu(0)}$ represents the Angular Momentum relative to a system of free particles, which, according to formula (4.20 *c*) of BM, is written as follows:

$$\mathrm{J}^{\lambda\mu(0)} = x^\lambda_a \pi^\mu_a - x^\mu_a \pi^\lambda_a + s_a \pi^{-1}_a \Gamma^{\lambda\mu}_a + x^\lambda_{a'} \pi^\mu_{a'} - x^\mu_{a'} \pi^\lambda_{a'} + s_{a'} \pi^{-1}_{a'} \Gamma^{\lambda\mu}_{a'}, \tag{4.9}$$

with: $$\Gamma_a^{\lambda\mu} \equiv \eta^{\lambda\mu\nu\sigma}\pi_{a\nu}\gamma_{a\sigma}. \tag{4.10}$$

In view of this, and of the techniques developed in the last part of BM [14], we conclude that the formula analogous to (4.3) for Angular Momentum is the following [15]:

$$J^{\lambda\mu(1)} = \sum_{a=1}^{2} \int_0^{-\infty} d\tau . \varphi_\tau^* \{ x_a^\lambda \theta_a^{\mu(1)} - x_a^\mu \theta_a^{\lambda(1)} + s_a \pi_a^{-1} \eta^{\lambda\mu\nu\sigma}(\theta_{a\nu}^{(1)}\gamma_{a\sigma} + \pi_{a\nu}\Delta_{a\sigma}^{(1)}) \}. \tag{4.11}$$

If we now utilize (4.4) and the results in (4.6) as well as the considerations previously made for Linear Momentum, we find the following expression for first order Angular Momentum:

$$\begin{aligned} J^{\lambda\mu(1)} = & \sum_{a=1}^{2} \int_0^{-\infty} \{ x_a^\lambda . \theta_a^{\mu(1)}[z_a+\tau, r_{aa'}(\tau)] - x_a^\mu . \theta_a^{\lambda(1)}[z_a+\tau, r_{aa'}(\tau)] \} d\tau \\ & + \sum_{a=1}^{2} \int_0^{-\infty} \{ \pi_a^\lambda . \theta_a^{\mu(1)}[z_a+\tau, r_{aa'}(\tau)] - \pi_a^\mu . \theta_a^{\lambda(1)}[z_a+\tau, r_{aa'}(\tau)] \} \tau d\tau \\ & + \sum_{a=1}^{2} s_a \pi_a^{-1} \eta^{\lambda\mu\nu\sigma} \int_0^{-\infty} \{ \gamma_{a\sigma} . \theta_{a\nu}^{(1)}[z_a+\tau, r_{aa'}(\tau)] + \pi_{a\nu} . \Delta_{a\nu}^{(1)}[z_a+\tau, r_{aa'}(\tau)] \} . d\tau \end{aligned} \tag{4.12}$$

Again we only need to carry out all the integrations in order to obtain explicitly the required result. Here too, observing (3.16), it is seen that all the integrals are immediate. However, there appears a small difficulty respecting the convergence of the second term of (4.12). In fact, some of the integrals in this term are divergent if considered one by one, but the sum over the two particles leads to a finite result. To illustrate this phenomenon let us consider the following expression extracted from the second term of (4.12):

$$\begin{aligned} I^{\lambda\mu} \equiv & \pi_{a'}^{-1}(\pi_a^\lambda t_{a'}^\mu - \pi_a^\mu t_{a'}^\lambda) \int_0^{-\infty} \tau (z_a + \tau)[h_{aa'}^2 + \Lambda^2 \pi_{a'}^{-2}(z_a + \tau)^2]^{-3/2} d\tau \\ & + \pi_{a'}^{-1}(\pi_{a'}^\lambda t_a^\mu - \pi_{a'}^\mu t_a^\lambda) \int_0^{-\infty} \tau (z_{a'} + \tau)[h_{aa'}^2 + \Lambda^2 \pi_a^{-2}(z_{a'} + \tau)^2]^{-3/2} d\tau. \end{aligned} \tag{4.13}$$

It is obvious that each of the integrals by itself is divergent, yet a simple calculation shows that the whole expression takes the value:

$$I^{\lambda\mu} = - \pi_a^2 \pi_{a'}^2 \Lambda^{-3} (\pi_a^\lambda \pi_{a'}^\mu - \pi_a^\mu \pi_{a'}^\lambda) . \ln \frac{\pi_a^2 \Lambda z_a + \pi_a \pi_{a'} r_{aa'}}{\pi_{a'}^2 \Lambda z_{a'} + \pi_a \pi_{a'} r_{a'a}} . \tag{4.14}$$

C. Simplectic form.

This time we start out from hypothesis (4.41) of BM adapted to the present situation, namely, the assumption that the simplectic form Ω associated with our dynamic system admits of a development in series of powers as follows:

$$\Omega = \sum_{n=0}^{\infty} g^n \Omega^{(n)}, \tag{4.15}$$

where $\Omega^{(0)}$ is the simplectic form associated with a system of free particles, given by expression (4.16) of BM, i. e.:

$$\Omega^{(0)} \equiv \sum_{a=1}^{2} dx_a^\rho \wedge d\pi_{a\rho} + \frac{1}{2}\sum_{a=1}^{2} s_a \pi_a^{-3} \Gamma_{a\lambda\mu} d\Gamma_a^{\lambda\rho} \wedge d\Gamma_a{}^\mu{}_\rho. \tag{4.16}$$

Now using trivially equation (4.44) of BM, we get the following expression for the simplectic form to the first order of approximation:

$$\Omega^{(1)} = \sum_{a=1}^{2} \int_0^{-\infty} d\tau \,.\, \varphi_\tau^* \mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{I}(1)})\Omega^{(0)}, \tag{4.17}$$

where $\mathscr{L}(\ )$ represents the Lie derivative operator and $\vec{\mathrm{H}}_a^{\mathrm{I}(1)}$ the following vector field:

$$\vec{\mathrm{H}}_a^{\mathrm{I}(1)} \equiv \theta_a^{\rho(1)} \frac{\partial}{\partial \pi^{a\rho}} + \Delta_a^{\rho(1)} \frac{\partial}{\partial \gamma^{a\rho}}. \tag{4.18}$$

Let us first calculate the Lie derivative appearing in the integrand of (4.17). Taking into consideration (4.18), (4.16) and (4.10), as well as the fact that the Lie derivative commutes with the external differential, we have the following result [16]:

$$\mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{I}(1)})\Omega^{(0)} = dx_a^\rho \wedge d\theta_{a\rho}^{(1)} + \frac{1}{2} s_a \pi_a^{-3} \mathrm{B}_{a\lambda\mu} d\Gamma_a^{\lambda\rho} \wedge d\Gamma_a{}^\mu{}_\rho$$
$$+ s_a \pi_a^{-3} \Gamma_{a\lambda\mu} d\mathrm{B}_a^{\lambda\rho} \wedge d\Gamma_a{}^\mu{}_\rho, \tag{4.19}$$

where the following notation has been used:

$$\mathrm{B}_a^{\lambda\rho} \equiv \mathscr{L}(\vec{\mathrm{H}}_a^{\mathrm{I}(1)})\Gamma_a^{\lambda\rho} = \eta^{\lambda\rho\nu\sigma}(\theta_{a\nu}^{(1)}\gamma_{a\sigma} + \pi_{a\nu}\Delta_{a\sigma}^{(1)}). \tag{4.20}$$

In accordance with this, and considering that the transformation φ_τ^* also commutes with the exterior differential [16], we find that the expression (4.17) of $\Omega^{(1)}$ can be written as follows:

$$\Omega^{(1)} = \sum_{a=1}^{2} dx_a^\rho \wedge d\int_0^{-\infty} \theta_{a\rho}^{(1)}[z_a + \tau, r_{aa'}(\tau)] \,.\, d\tau$$
$$+ \sum_{a=1}^{2} d\pi_a^\rho \wedge d\int_0^{-\infty} \tau \,.\, \theta_{a\rho}^{(1)}[z_a + \tau, r_{aa'}(\tau)] \,.\, d\tau$$

$$+ \frac{1}{2} \sum_{a=1}^{2} s_a \pi_a^{-3} \left\{ \int_0^{-\infty} \mathrm{B}_{a\lambda\mu}[z_a + \tau, r_{aa'}(\tau)] . d\tau \right\} d\Gamma_a^{\lambda\rho} \wedge d\Gamma_a{}^{\mu}{}_{\rho}$$

$$+ \sum_{a=1}^{2} s_a \pi_a^{-3} \Gamma_{a\lambda\mu} d \left\{ \int_0^{-\infty} \mathrm{B}_{a\lambda\mu}[z_a + \tau, r_{aa'}(\tau)] . d\tau \right\} \wedge d\Gamma_a{}^{\mu}{}_{\rho}, \quad (4.21)$$

where, in according to (4.20), the following obvious notation has been used:

$$\mathrm{B}_a^{\lambda\mu}[z_a + \tau, r_{aa'}(\tau)] \equiv \eta^{\lambda\mu\nu\sigma} \{ \theta_{a\nu}^{(1)}[z_a + \tau, r_{aa'}(\tau)] . \gamma_{a\sigma} + \pi_{a\nu} . \Delta_{a\sigma}^{(1)}[z_a + \tau, r_{aa'}(\tau)] \}. \quad (4.22)$$

As previously, we need only to carry out all the integrations appearing in (4.21) to obtain explicitly the required result. Again, as in the previous Subsection, the convergence of the second term of (4.21) is assured if taken as a whole, that is, the sum over the two particles.

5. DEVELOPMENTS IN 1/c. BREIT'S CLASSICAL HAMILTONIAN

Let us use F to designate an arbitrary function of the arguments $(x_a^\lambda, \pi_b^\mu, \gamma_c^\nu)$ relative to the two particles of the dynamic system being considered. Following the terminology and interpretations of Section 3A of BM, we shall call *projection* of F, designated $\overline{\mathrm{F}}$, the restriction of this function to the following values of its arguments:

$$x_1^0 = x_2^0 = \mathrm{cte} \quad (5.1\,a)$$

$$\pi_a^0 = \overline{\pi}_a^0 \equiv m_a c (1 - v_a^2/c^2)^{-1/2} \quad , \quad \pi_a^i = \overline{\pi}_a^i \equiv m_a v_a^i (1 - v_a^2/c^2)^{-1/2} \quad (5.1\,b)$$

$$\gamma_a^0 = \overline{\gamma}_a^0 \equiv m_a^{-1} c^{-1} \overline{\pi}_{ai} \alpha_a^i \quad , \quad \gamma_a^i = \overline{\gamma}_a^i \equiv \left\{ \delta_j^i + \frac{\overline{\pi}_a^i \overline{\pi}_{aj}}{m_a c (m_a c + \overline{\pi}_a^0)} \right\} \alpha_a^j \quad (5.1\,c)$$

where m_a represents the masses of the particles considered, and where (v_a^i, α_b^j) represent the velocities and instantaneous spins respectively of these particles (see Section 2 of BM). So we shall have by definition:

$$\overline{\mathrm{F}}(x_a^i, v_b^j, \alpha_c^l ; m_d) \equiv \mathrm{F}(x_1^0 = x_2^0 = \mathrm{cte}, x_a^i, \pi_b^\mu = \overline{\pi}_b^\mu, \gamma_c^\nu = \overline{\gamma}_c^\nu). \quad (5.2)$$

The basic purpose of this Section is to show that the results of the previous Section lead, in particular, to obtaining the non-quantic version of Breit's well-known Hamiltonian [4]. To this end we shall consider first the projection $\overline{\mathrm{P}}^{\mu[1]} \equiv \overline{\mathrm{P}}^{\mu(0)} + g\overline{\mathrm{P}}^{\mu(1)}$, according to definition (5.2), of the Energy Momentum vector at the first order, given by formulae (4.2) and (4.7). A somewhat laborious but direct calculation shows that if we carry out a development in power series of $1/c$ of this projection $\overline{P}^{\mu[1]}$, the following results are obtained for the time component and the space components respectively (see Appendix 2):

$$
\begin{aligned}
c.\overline{P}^{0[1]}-(m_a+m_{a'})c^2 \approx\ & \frac{1}{2}m_a v_a^2+\frac{1}{2}m_{a'}v_{a'}^2+\frac{3}{8c^2}m_a v_a^4+\frac{3}{8c^2}m_{a'}v_{a'}^4 \\
& +gr^{-1}+\frac{1}{2c^2}gr^{-1}\{(\vec{v}_a\vec{v}_{a'})+r^{-2}(\vec{x}_{aa'}\vec{v}_a)(\vec{x}_{aa'}\vec{v}_{a'})\} \\
& -\frac{1}{2c^2}g\lambda_a s_a m_a^{-1}r^{-3}(\vec{x}_{aa'}\wedge\vec{v}_a).\vec{\alpha}_a+\frac{1}{2c^2}g\lambda_{a'}s_{a'}m_{a'}^{-1}r^{-3}(\vec{x}_{aa'}\wedge\vec{v}_{a'}).\vec{\alpha}_{a'} \\
& +\frac{1}{4c^2}g\lambda_a\lambda_{a'}s_a s_{a'}m_a^{-1}m_{a'}^{-1}r^{-3}\{(\vec{\alpha}_a\vec{\alpha}_{a'})-3r^{-2}(\vec{x}_{aa'}\vec{\alpha}_a)(\vec{x}_{aa'}\vec{\alpha}_{a'})\} \qquad (5.3\,a)
\end{aligned}
$$

$$
\begin{aligned}
\overline{P}^{i[1]} \approx\ & m_a v_a^i+m_{a'}v_{a'}^i+\frac{1}{2c^2}m_a v_a^2 v_a^i+\frac{1}{2c^2}m_{a'}v_{a'}^2 v_{a'}^i \\
& +\frac{1}{2c^2}gr^{-1}\{v_a^i+v_{a'}^i+r^{-2}[(\vec{x}_{aa'}\vec{v}_a)+(\vec{x}_{aa'}\vec{v}_{a'})]x_{aa'}^i\} \\
& +\frac{1}{2c^2}g\lambda_a s_a m_a^{-1}r^{-3}(\vec{x}_{aa'}\wedge\vec{\alpha}_a)^i-\frac{1}{2c^2}g\lambda_{a'}s_{a'}m_{a'}^{-1}r^{-3}(\vec{x}_{aa'}\wedge\vec{\alpha}_{a'})^i, \qquad (5.3\,b)
\end{aligned}
$$

where the symbol $\approx$ represents an equality up to higher order terms in $1/c$ and where the standard three-dimensional vectorial notation has been used. We have also taken:

$$x_{aa'}^i \equiv x_a^i - x_{a'}^i \quad , \quad r \equiv +\sqrt{x_{aa'}^i x_{aa'i}} \qquad (5.4)$$

The expression (5.3 a) represents the Energy of the system to the order being considered, and in it may be observed the kinematic contribution and the interaction contribution. The latter is composed of an independent spin part [*17*] and another part due to the presence of the spins, in its turn composed of a spin-orbit contribution and a spin-spin contribution. Analogous considerations can be made in relation to the expression (5.3 b), which represents the Momentum of the system to the order considered. Note, however, that in this case there is no spin-spin contribution.

Let us consider the restriction, in line with (5.1), of the simplectic form $\Omega^{[1]} = \Omega^{(0)} + g\Omega^{(1)}$, given by formulae (4.16) and (4.21). A calculation similar to that which has allowed us to obtain (5.3) shows that up to the order $1/c^2$ a possible subset of canonical coordinates (see Section 4A of BM) of the said restriction $\overline{\Omega}^{[1]}$ is given by:

$$\vec{q}_a \approx \vec{x}_a + \frac{1}{2c^2}s_a m_a^{-1}\vec{v}_a\wedge\vec{\alpha}_a \qquad (5.5\,a)$$

$$
\begin{aligned}
\vec{p}_a \approx\ & m_a\vec{v}_a+\frac{1}{2c^2}m_a v_a^2\vec{v}_a+\frac{1}{2c^2}gr^{-1}\{\vec{v}_{a'}+r^{-2}(\vec{x}_{aa'}\vec{v}_{a'})\vec{x}_{aa'}\} \\
& -\frac{1}{2c^2}g\lambda_{a'}s_{a'}m_{a'}^{-1}r^{-3}(\vec{x}_{aa'}\wedge\vec{\alpha}_{a'}) \qquad (5.5\,b)
\end{aligned}
$$

where once again the standard three-dimensional vectorial notation has been used. Note first that (5.5 *a*) does not contain the coupling constant g, that is, it is an expression independent of the interaction. It is in fact the development of the corresponding expression for a system of free particles [see formula (4.24 *b*) of BM]. Note also that if we add (5.5 *b*) for the two particles of the system we obtain the Momentum (5.3 *b*).

Let us now write out Energy (5.3 *a*) in terms of variables $(\vec{q}_a, \vec{p}_b)$. Up to the order considered, we get from (5.5):

$$\vec{x}_a \approx \vec{q}_a - \frac{1}{2c^2} s_a m_a^{-2} \vec{p}_a \wedge \vec{\alpha}_a \tag{5.6 a}$$

$$\vec{v}_a \approx \frac{\vec{p}_a}{m_a} - \frac{1}{2m_a^3c^2} p_a^2 \vec{p}_a - \frac{1}{2m_a m_{a'} c^2} g q^{-1} \{ \vec{p}_{a'} + q^{-2}(\vec{q}_{aa'} \cdot \vec{p}_{a'})\vec{q}_{aa'} \} + \frac{1}{2m_a m_{a'} c^2} g \lambda_{a'} s_{a'} q^{-3} (\vec{q}_{aa'} \wedge \vec{\alpha}_{a'}) \tag{5.6 b}$$

where we have taken

$$\vec{q}_{aa'} \equiv \vec{q}_a - \vec{q}_{a'} \quad , \quad q \equiv \sqrt{\vec{q}^{\,2}_{aa'}}. \tag{5.7}$$

Now substituting (5.6) in (5.3 *a*), and designating this expression H, we finally obtain:

$$\begin{aligned} \mathrm{H} = {} & \frac{1}{2m_a} p_a^2 + \frac{1}{2m_{a'}} p_{a'}^2 - \frac{1}{8m_a^3c^2} p_a^4 - \frac{1}{8m_{a'}^3c^2} p_{a'}^4 \\ & + g q^{-1} - \frac{1}{2m_a m_{a'} c^2} g q^{-1} \{ (\vec{p}_a \vec{p}_{a'}) + q^{-2} (\vec{q}_{aa'} \vec{p}_a)(\vec{q}_{aa'} \vec{p}_{a'}) \} \\ & - \frac{1}{2m_a^2c^2} (\lambda_a - 1) g s_a q^{-3} (\vec{q}_{aa'} \wedge \vec{p}_a) . \vec{\alpha}_a + \frac{1}{2m_{a'}^2c^2} (\lambda_{a'} - 1) g s_{a'} q^{-3} (\vec{q}_{aa'} \wedge \vec{p}_{a'}) \\ & - \frac{1}{2m_a m_{a'} c^2} g q^{-3} \{ \lambda_{a'} s_{a'} (\vec{q}_{aa'} \wedge \vec{p}_a) . \vec{\alpha}_{a'} - \lambda_a s_a (\vec{q}_{aa'} \wedge \vec{p}_{a'}) . \vec{\alpha}_a \} \\ & + \frac{1}{4m_a m_{a'} c^2} \lambda_a \lambda_{a'} s_a s_{a'} g q^{-3} \{ (\vec{\alpha}_a \vec{\alpha}_{a'}) - 3q^{-2} (\vec{x}_{aa'} \vec{\alpha}_a)(\vec{x}_{aa'} \vec{\alpha}_{a'}) \}. \end{aligned} \tag{5.8}$$

As can be seen, if we here make $\lambda_b = 2$ and $s_b = \frac{1}{2}\hbar (h \equiv 2\pi\hbar$ is Planck's constant) we formally obtain Breit's Hamiltonian [*4*] except for Darwin's contact term [*18*]. Note that in this context the classical equivalent of the quantum mechanical position operator is not the classical position $\vec{x}_a$ but the canonical coordinate $\vec{q}_a$ given by (5.5 *a*). This fact fits in with the quantification program proposed by one of us [*19*] for the case of spin zero particles in interaction, where the classical positions are not canonical coordinates either.

An extremely interesting aspect of hamiltonian (5.8) is that in principle it is valid for particles with arbitrary giromagnetic coefficient.

APPENDIX 1

Below we give a table of the fundamental integrals involved in (4.7), (4.12) and (4.21):

$$\int_0^{-\infty} r_{aa'}^{-3}(\tau).d\tau = -h_{aa'}^{-2}(z_a r_{aa'}^{-1} + \Lambda^{-1}\pi_{a'}) \tag{A1.1}$$

$$\int_0^{-\infty} (z_a + \tau).r_{aa'}^{-3}(\tau).d\tau = \Lambda^{-2}\pi_{a'}^2 r_{aa'}^{-1} \tag{A1.2}$$

$$\int_0^{-\infty} r_{aa'}^{-5}(\tau).d\tau = -h_{aa'}^{-4}\left(z_a r_{aa'}^{-1} - \frac{1}{3}\Lambda^2\pi_{a'}^{-2}z_a^3 r_{aa'}^{-3} + \frac{2}{3}\Lambda^{-1}\pi_{a'}\right) \tag{A1.3}$$

$$\int_0^{-\infty} (z_a + \tau).r_{aa'}^{-5}(\tau).d\tau = \frac{1}{3}\Lambda^{-2}\pi_{a'}^2 r_{aa'}^{-3} \tag{A1.4}$$

$$\int_0^{-\infty} (z_a + \tau)^2.r_{aa'}^{-5}(\tau).d\tau = -\frac{1}{3}h_{aa'}^{-2}(z_a^3 r_{aa'}^{-3} + \Lambda^{-3}\pi_{a'}^3) \tag{A1.5}$$

$$\int_0^{-\infty} r_{aa'}^{-7}(\tau).d\tau = -h_{aa'}^{-6}\left(z_a r_{aa'}^{-1} - \frac{2}{3}\Lambda^2\pi_{a'}^{-2}z_a^3 r_{aa'}^{-3} + \frac{1}{5}\Lambda^4\pi_{a'}^{-4}z_a^5 r_{aa'}^{-5} + \frac{8}{15}\Lambda^{-1}\pi_{a'}\right) \tag{A1.6}$$

$$\int_0^{-\infty} (z_a + \tau).r_{aa'}^{-7}(\tau).d\tau = \frac{1}{5}\Lambda^{-2}\pi_{a'}^2 r_{aa'}^{-5} \tag{A1.7}$$

$$\int_0^{-\infty} (z_a + \tau)^2.r_{aa'}^{-7}(\tau).d\tau = -h_{aa'}^{-4}\left(\frac{1}{3}z_a^3 r_{aa'}^{-3} - \frac{1}{5}\Lambda^2\pi_{a'}^{-2}z_a^5 r_{aa'}^{-5} + \frac{2}{15}\Lambda^{-3}\pi_{a'}^3\right) \tag{A1.8}$$

$$\int_0^{-\infty} (z_a + \tau).r_{aa'}^{-7}(\tau).d\tau = \Lambda^{-4}\pi_{a'}^4\left(\frac{1}{3}r_{aa'}^{-3} - \frac{1}{5}h_{aa'}^2 r_{aa'}^{-5}\right) \tag{A1.9}$$

APPENDIX 2

Here we write the developments in series of powers of $1/c$ of the fundamental quantities involved in the calculation of (5.3) and (5.5):

$$\bar{\pi}_a^0 \approx m_a c\left(1 + \frac{1}{2c^2} v_a^2 + \frac{3}{8c^4} v_a^4\right) \tag{A2.1}$$

$$\bar{\pi}_a^i \approx m_a v_a^i\left(1 + \frac{1}{2c^2} v_a^2\right) \tag{A2.2}$$

$$\bar{k} \approx m_a m_{a'} c^2\left(1 + \frac{1}{2c^2}\beta + \frac{1}{8c^4}\delta\right) \tag{A2.3}$$

$$\bar{t}_a^0 \approx \frac{1}{2} m_a m_{a'}^2 c \left\{ v_a^2 - v_{a'}^2 - \beta + \frac{1}{4c^2}(3v_a^4 - 3v_{a'}^4 - 2\beta v_{a'}^2 - \delta) \right\} \tag{A2.4}$$

$$\bar{t}_a^i \approx m_a m_{a'}^2 c^2 \left\{ v_a^i - v_{a'}^i + \frac{1}{2c^2}(v_a^2 v_a^i - v_{a'}^2 v_{a'}^i - \beta v_{a'}^i) \right\} \tag{A2.5}$$

$$\bar{\Lambda}^2 \approx m_a^2 m_{a'}^2 c^2 \beta \left\{ 1 + \frac{1}{4c^2}\beta^{-1}(\beta^2 + \delta) \right\} \tag{A2.6}$$

$$\bar{r}_{aa'} \approx r\left\{ 1 + \frac{1}{2c^2} r^{-2}(\vec{x}_{aa'}\vec{v}_{a'})^2 \right\} \tag{A2.7}$$

$$\bar{z}_a \approx m_a^{-1}\beta^{-1}\mu \left\{ 1 + \frac{1}{4c^2}\beta^{-1}\mu^{-1}[2\beta\omega_a - \mu(\beta^2 + \delta)] \right\} \tag{A2.8}$$

$$\bar{h}_{aa'}^0 \approx \frac{1}{2c}\beta^{-1}\mu\{ v_{a'}^2 - v_a^2 + \mu^{-1}(\omega_{a'} - \omega_a)\} + 0\left(\frac{1}{c^3}\right) \tag{A2.9}$$

$$\begin{aligned}\bar{h}_{aa'}^i \approx x_{aa'}^i &- \beta^{-1}\mu \left\{ 1 + \frac{1}{4c^2}\langle 2v_a^2 + \beta^{-1}\mu^{-1}[2\beta\omega_a - \mu(\beta^2 + \delta)]\rangle \right\} v_a^i \\ &+ \beta^{-1}\mu \left\{ 1 + \frac{1}{4c^2}\langle 2v_{a'}^2 + \beta^{-1}\mu^{-1}[2\beta\omega_{a'} - \mu(\beta^2 + \delta)]\rangle \right\} v_{a'}^i\end{aligned} \tag{A2.10}$$

$$\bar{h}_{aa'}^2 \approx r^2 - \beta^{-1}\mu^2 + 0\left(\frac{1}{c^2}\right) \tag{A2.11}$$

$$\bar{y}_a^0 \approx \frac{1}{c}(\vec{v}_a\vec{\alpha}_a) + 0\left(\frac{1}{c^3}\right) \tag{A2.12}$$

$$\bar{y}_a^i \approx \alpha_a^i + 0\left(\frac{1}{c^2}\right) \tag{A2.13}$$

where the following notations have been used:

$$\beta \equiv v_a^2 + v_{a'}^2 - 2(\vec{v}_a\vec{v}_{a'}) \tag{A2.14 a}$$

$$\delta \equiv 3(v_a^4 + v_{a'}^4) + 2v_a^2 v_{a'}^2 - 4(\vec{v}_a\vec{v}_{a'})(v_a^2 + v_{a'}^2) \tag{A2.14 b}$$

$$\mu \equiv (\vec{x}_{aa'}\vec{v}_a) - (\vec{x}_{aa'}\vec{v}_{a'}) \tag{A2.14 c}$$

$$\omega_a \equiv (\vec{x}_{aa'}\vec{v}_a)v_a^2 - (\vec{x}_{aa'}\vec{v}_{a'})(\beta + v_{a'}^2) \tag{A2.14 d}$$

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[2] V. Bargmann, L. Michel and V. L. Telegdi, *Phys. Rev. Lett.*, t. **2**, 1959, p. 435.

[3] See for example: L. Bel, A. Salas and J. M. Sanchez-Ron, *Phys. Rev.*, t. **D 7**, 1973, p. 1099; L. Bel and J. Martin, *Phys. Rev.*, t. **D 8**, 1973, p. 4347; L. Bel, in *Journées Relativistes de Toulouse*, Université de Toulouse, Département de Mathématiques, 1974 ; L. Bel et X. Fustero, *Ann. Inst. H. Poincaré*, t. **25**, 1976, p. 411.

[4] G. Breit, *Phys. Rev.*, t. **34**, 1929, p. 553; *Phys. Rev.*, t. **36**, 1930, p. 383. A deduction of this Hamiltonian by Quantum Electrodynamic procedures may be found, *in* : Landau et Lifchitz, *Théorie Quantique Relativiste* (première partie), éditions Mir, Moscou, 1972.

[5] A different classical derivation of the equations of motion and lagrangians relative to this Hamiltonian may be found, *in* : J. Llosa, Tesis doctoral, Universitat de Barcelona, 1978; X. Fustero and E. Verdaguer, *preprint*, Universitat Autonoma de Barcelona, Spain, 1979.

[6] It is assumed that the curve is time-like and future oriented. We take signature + 2 for $\mathscr{M}_4$. Einstein's summation convention will be utilized for all kinds of indices; these will always be placed in the appropriate position (« covariant » or « contravariant ») to respect the said convention.

[7] This condition eliminates the presence of an electric dipolar moment.

[8] We use the convention $\eta_{0123} = +1$, and consequently $\eta^{0123} = -1$.

[9] We here consider the advanced propagator as an open possibility.

[10] See for example: J. Cohn and H. Wiebe, *J. Math. Phys.*, t. **17**, 1976, p. 1496.

[11] Actually Bargmann, Michel and Telegdi only use these equations for the case of a homogeneous electromagnetic field. It should also be pointed out that similar Eqs. appear in the following: L. H. Thomas, *Phil. Mag.*, t. **3**, 1927, p. 1; J. Frenkel, *Z. Physik*, t. **37**, 1926, p. 243; H. A. Kramers, *Quantum Mechanics*, North Holland Publishing Co., Amsterdam, 1957. In Kramer's equations there appear, nevertheless, certain inconsistencies, as is pointed out in Bargmann, Michel and Teledgi.

[12] See for example: L. Bel and X. Fustero (ref. 3).

[13] See for example: Y. Choquet-Bruhat, *Géométrie Différentielle et Systèmes Extérieurs*, éd. Dunod, Paris, 1968.

[14] In particular formulae (4.14 *b*) and (4.34) of BM.

[15] For the case of spinless particles consult ref. 12.

[16] For more details on Exterior Calculus techniques consult ref. 13.

[17] This part is already obtained from Darwin's well-known Lagrangian: C. G. Darwin, *Phil. Mag.*, t. **39**, 1920, p. 537.

[18] This term, which contains a Dirac « delta », is purely quantum mechanical. Consult the refs. at 4.

[19] L. Bel, *Contribution to Differential Geometry and Relativity*, Cahen and Flato (eds.), D. Reidel Publishing Co., Dordrecht, Holland, 1976.

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