ANNALES DE L'I. H. P., SECTION A

ROBERT T. JANTZEN

Conformal geometry and spatially homogeneous cosmology

Annales de l'I. H. P., section A, tome 33, nº 2 (1980), p. 121-146

<http://www.numdam.org/item?id=AIHPA_1980__33_2_121_0>

Ann. Inst. Henri Poincaré Section A :
Vol. XXXIII, n° 2, 1980, p. 121-146. *Physique théorique.*

Conformal geometry and spatially homogeneous cosmology

by

Robert T. JANTZEN
Institute of Field Physics. Department of Physics and Astronomy,
University of North Carolina, Chapel Hill, N. C. 27514

ABSTRACT. — York's splitting of the tangent spaces to the space of Riemannian metrics over a 3-manifold is examined in the context of spatially homogeneous cosmology where it is generally found to have two distinct analogues in the noncompact case. This requires a modification of the theory of minimal distortion shifts as it applies to noncompact cosmology and clarifies the roles played by the « kinematical » and « dynamical » degrees of freedom in the evolution of spatially homogeneous Cauchy data.

I. INTRODUCTION

In a previous paper [1], the « kinematical » and « dynamical » degrees of freedom of the gravitational field as described by Smarr and York [*2*] were identified for spatially homogeneous spacetimes. Taking the point of view that the spacetime is given, one may use the freedom of choice of the shift vector field compatible with the symmetry to simplify the time evolution of the conformal metric induced on the family of spatially homogeneous slices. In particular, the shift vector field can be chosen so that the only changes occurring in the conformal metric represent a change in the conformal 3-geometry. Such a shift vector field was shown in reference [*1*] to be a solution of the minimal distortion equation of York and Smarr [*2*] [*3*] but not necessarily a unique solution, even apart from the trivial nonuniqueness associated with the spatial Killing vector fields. However, if one takes the point of view that the spacetime is to be constructed using the three-plus-one evolution approach, it was essentially found in reference [1] that in those cases where the nonuniqueness is nontrivial, one cannot use the minimal distortion shift equation to eliminate *all* of the kinematical degrees of freedom. That is, one is forced to evolve some of the kinematical degrees of freedom.

This breakdown in the notion of kinematical and dynamical degrees of freedom arises because the geometry underlying the York-Smarr definitions in the compact and asymptotically flat cases does not carry over in general to the noncompact spatially homogeneous case. Instead, York's decomposition of symmetric tracefree tensors [4-6], which is fundamental to his conformal approach to the initial value problem and the theory of minimal distortion shifts, splits into two separate decompositions when the spatial homogeneity symmetry group is not semisimple. One decomposition is adapted to the initial value problem while the other is associated with the true analogues of minimal distortion shifts which are unique modulo spatial Killing vector fields. The distinction between the two decompositions has the effect of destroying the complementary relationship between kinematics and dynamics that occurs in the compact and asymptotically flat cases.

II. DECOMPOSITIONS OF SYMMETRIC TENSORS

Both the minimal distortion equation and the conformal treatment of the initial value problem depend on the well known geometry involved in the action of diffeomorphisms and conformal scalings on the space $\mathcal{M} = \mathrm{RIEM(M)}$ of Riemannian metrics on a given 3-manifold M. This has been studied for compact M [7] and for $\mathrm{M} = \mathrm{R}^3$ on the subspace of $\mathcal{M}$ corresponding to asymptotically flat metrics [8]. For comparison with the spatially homogeneous case, a brief sketch of this material will be given ignoring the appropriate functional restrictions required in each case.

The tangent space $\mathrm{T}\mathcal{M}_{\mathbf{g}}$ at a point $\mathbf{g} \in \mathcal{M}$ may be identified with the space S_2 of symmetric $\begin{pmatrix}0\\2\end{pmatrix}$ tensors on M (1). This space decomposes into a pointwise orthogonal direct sum of tracefree elements and multiples of $\mathbf{g}$:

$$(1) \qquad \mathrm{T}\mathcal{M}_{\mathbf{g}} \sim \mathrm{S}_2 = \mathrm{S}_2^{\mathrm{C}} \oplus \mathrm{S}_2^{\mathrm{TF}}$$

$$\mathrm{S}_2^{\mathrm{C}} = \left\{ h \in \mathrm{S}_2 \mid h = \frac{1}{3}\mathbf{g}\ \mathrm{Tr}\ h \right\}$$

$$\mathrm{S}_2^{\mathrm{TF}} = \{ h \in \mathrm{S}_2 \mid \mathrm{Tr}\ h = g^{ij}h_{ij} = 0 \}.$$

(1) The distinction between these spaces will usually be ignored here.

S_2^C is the subspace tangent to the orbit through $\mathbf{g}$ of the action on $\mathcal{M}$ of the group of conformal scalings while S_2^{TF} is tangent to a local slice for that action:

$$\mathbf{g} \to \phi^4 \mathbf{g} \qquad \phi > 0\,, \quad \phi \in \mathfrak{F}(M)\,. \tag{2}$$

Let the superscript « TF » stand for the projection of an element of S_2 into the tracefree subspace S_2^{TF} :

$$h^{TF} = h - \frac{1}{3}\mathbf{g}\,\mathrm{Tr}\,h \tag{3}$$

$\mathcal{M}$ has a natural metric $\mathcal{G}$ [2] :

$$\mathcal{G}(h, k) = \int_M \eta g^{ij} g^{kl} h_{ik} k_{jl} \qquad h, k \in S_2\,. \tag{4}$$

η is the volume element 3-form of the metric $\mathbf{g}$ and h and k are thought of as elements of $T\mathcal{M}_{\mathbf{g}}$. S_2^C and S_2^{TF} are trivially orthogonal with respect to $\mathcal{G}$.

Let $-\delta : S_2 \to \mathfrak{X}^*(M)$ be the 1-form valued divergence operator for S_2 for the metric $\mathbf{g}$, Kill : $\mathfrak{X}(M) \to S_2$ the Killing derivative operator on vector fields over M and L : $\mathfrak{X}(M) \to S_2^{TF}$ the conformal Killing derivative operator :

$$(\delta h)_i = - D^j h_{ji} \tag{5}$$
$$\mathrm{Kill}\,X = \pounds_X \mathbf{g} \qquad LX = (\mathrm{Kill}\,X)^{TF}\,.$$

D is the covariant derivative associated with $\mathbf{g}$. Assuming square-integrability of all the fields in the following equations, one has :

$$\mathcal{G}(h^{TF}, LX) = \langle\, \delta^{TF} h^{TF}, X \,\rangle = \int_M \eta(\delta^{TF} h^{TF})_i X^i \tag{6}$$

where the second equality defines the global scalar product of a 1-form and a vector field. δ^{TF} is the restriction of δ to S_2^{TF} and by the first equality is seen to be the covariant form of the formal L^2-adjoint of the operator L. Ran L is the tracefree projection of the tangent space to the orbit through $\mathbf{g}$ of the action on $\mathcal{M}$ of the diffeomorphism group $\mathcal{D}(M)$, while $\ker \delta^{TF}$ is the subspace of S_2^{TF} orthogonal to Ran L :

$$S_2^{TF} = \ker \delta^{TF} \oplus \mathrm{Ran}\,L\,. \tag{7}$$

Introduce the space $\mathcal{W}$ of conformal metrics on M and the space $\mathfrak{F}_d^+$ of positive scalar densities of weight 1. There is a natural diffeomorphism between $\mathcal{M}$ and $\mathfrak{F}_d^+ \times \mathcal{W}$ which may be stated in terms of the components

---

(2) More precisely, a weak Riemannian structure.

of these tensors and tensor densities in any given local frame $\{ e_i \}$ on M as follows:

$$\mathbf{g} \to (g^{1/2}, \tilde{\mathbf{g}}) \tag{8}$$
$$g = \det(g_{ij}) \qquad \tilde{g}_{ij} = g^{-1/3} g_{ij}$$

$\tilde{\mathbf{g}} \in \mathscr{W}$ is a tensor density of weight $-2/3$ which is invariant under the conformal scaling (II.2):

$$\phi^4 \mathbf{g} \to (\phi^6 g^{1/2}, \tilde{\mathbf{g}}) . \tag{9}$$

Let $\mathrm{P} : \mathscr{M} \to \mathscr{W}$ be the natural projection map defined by $\mathrm{P}(\mathbf{g}) = \tilde{\mathbf{g}}$. Its differential has $\mathrm{S}_2^{\mathrm{C}} \subset \mathrm{T}\mathscr{M}_{\mathbf{g}}$ as its kernel and acts as a diffeomorphism between $\mathrm{S}_2^{\mathrm{TF}}$ and $\mathrm{T}\mathscr{W}_{\tilde{\mathbf{g}}}$ such that the local inner product of elements of $\mathrm{S}_2^{\mathrm{TF}}$ equals the local inner product associated with the conformal metric of the image elements in $\mathrm{T}\mathscr{W}_{\tilde{\mathbf{g}}}$:

$$h_{ij} \to \tilde{h}_{ij} = g^{-1/3} h_{ij}^{\mathrm{TF}} \tag{10}$$
$$g^{ij} g^{kl} h_{ik}^{\mathrm{TF}} k_{jl}^{\mathrm{TF}} = \tilde{g}^{ij} \tilde{g}^{kl} \tilde{h}_{ik} \tilde{k}_{jl} .$$

However, the global inner product $\mathscr{G}$ does not project to a metric on $\mathscr{W}$ since one needs a volume element to integrate the local inner product against and no such object can be constructed from a conformal metric. One is therefore forced to work with the tracefree projection of $\mathrm{T}\mathscr{M}$ if one wants to have a metric available.

The action of the diffeomorphism group $\mathscr{D}(\mathrm{M})$ on $\mathscr{W}$ by dragging along is just the projection of its action on $\mathscr{M}$. A tangent $\pounds_{\mathrm{X}} \mathbf{g}$ to the orbit through $\mathbf{g}$ projects to $\pounds_{\mathrm{X}} \tilde{\mathbf{g}}$ which is tangent to the projected orbit through $\tilde{\mathbf{g}}$:

$$(\widetilde{\mathrm{LX}}) = g^{-1/3} (\pounds_{\mathrm{X}} \mathbf{g})^{\mathrm{TF}} = \pounds_{\mathrm{X}} \tilde{\mathbf{g}} . \tag{11}$$

Thus LX is the element of $\mathrm{S}_2^{\mathrm{TF}}$ which corresponds to $\pounds_{\mathrm{X}} \tilde{\mathbf{g}} \in \mathrm{T}\mathscr{W}_{\tilde{\mathbf{g}}}$. Similarly if $\mathbf{g}_t$ is a curve in $\mathscr{M}$ with tangent $h_t = d/dt\, \mathbf{g}_t = \dot{\mathbf{g}}_t$, then h_t^{TF} corresponds to the tangent to the projected curve:

$$\tilde{h}_t = g^{-1/3} h_t^{\mathrm{TF}} = (\tilde{\mathbf{g}}_t)^{\cdot} . \tag{12}$$

The orthogonal decomposition (II.7) projects to a direct sum of $\mathrm{T}\mathscr{W}_{\mathbf{g}}$ adapted to the orbits of the action of $\mathscr{D}(\mathrm{M})$ on $\mathscr{W}$ but there is no metric with respect to which it is orthogonal.

Suppose M is diffeomorphic to a spacelike Cauchy hypersurface N_0 in a spacetime $({}^4\mathrm{M}, {}^4\mathbf{g})$, with $\mathrm{N}_0 = h_0(\mathrm{M})$. By specifying the time dependent lapse function α_t and shift vector field β_t on M, one determines a foliation of spacelike hypersurfaces N_t, a diffeomorphism $h_t : \mathrm{M} \to \mathrm{N}_t$ and a curve $(\mathbf{g}_t, \mathrm{K}_t)$ in $\mathscr{M} \times \mathrm{S}_2 \sim \mathrm{T}\mathscr{M}$, where $\mathbf{g}_t$ and K_t are the pullbacks to M via h_t of the induced metric and extrinsic curvature tensor of N_t. The extrinsic curvature is related to the tangent $\dot{\mathbf{g}}_t$ to the curve $\mathbf{g}_t$ by the formula:

$$2\alpha_t \mathrm{K}_t = - \dot{\mathbf{g}}_t + \pounds_{\beta_t} \mathbf{g}_t . \tag{13}$$

Define the function $\Sigma(\mathbf{g}, K, \alpha, \beta)$ on $\mathcal{M} \times S_2 \times \mathfrak{F}(M) \times \mathfrak{X}(M)$ by:

$$\Sigma(\mathbf{g}, K, \alpha, \beta) = -2\alpha K^{TF} + L\beta \tag{14}$$
$$\Sigma_t = \Sigma(\mathbf{g}_t, K_t, \alpha_t, \beta_t) = \dot{\mathbf{g}}_t^{TF}.$$

Σ_t is just the tracefree part of the tangent $\dot{\mathbf{g}}_t$. For fixed $(\mathbf{g}, K, \alpha)$, one may pick β so that Σ is orthogonal to the projected orbits of the diffeomorphism group, i. e.:

$$\delta^{TF}\Sigma = 0. \tag{15}$$

This is the minimal distortion shift equation of York and Smarr [2] [3] [6]. Modulo conformal Killing fields of $\mathbf{g}$ (elements of ker L), the « unique » solution β is the one for which $L\beta$ cancels the component of $-2\alpha K^{TF}$ belonging to Ran L. Since varying β changes Σ along the directions tangent to the projected orbits of $\mathcal{D}(M)$, clearly those values of β which make Σ orthogonal to those directions also minimize the length of Σ:

$$S(\mathbf{g}, K, \alpha, \beta) = \mathcal{G}(\Sigma, \Sigma). \tag{16}$$

S is a minimum for a solution of the minimal distortion equation. Its variation at such a solution β_{MD} is zero:

$$S(\beta_{MD})' = 2\mathcal{G}(\Sigma(\beta_{MD}), L\beta') = 2\langle \delta^{TF}\Sigma(\beta_{MD}), \beta' \rangle = 0. \tag{17}$$

A time dependent shift vector field β_t is a minimal distortion shift if it satisfies the minimal distortion equation at each t. A minimal distortion shift vector field minimizes the global time rate of change of the conformal metric as discussed in reference [2].

The orthogonal decomposition of S_2^{TF} is also fundamental in York's conformal treatment of the initial value problem [4] [6]. The metric and the tracefree part of the extrinsic curvature K^{TF} are conformally transformed to $\bar{\mathbf{g}}$ and $\bar{K}^{TF}$ and then $\bar{K}^{TF}$ is decomposed according to the barred version of (II.7):

$$\mathbf{g} = \phi^4\bar{\mathbf{g}} \qquad K^{TF} = \phi^{-2}\bar{K}^{TF} \tag{18}$$
$$\bar{K}^{TF} = A + \bar{L}W \qquad \mathrm{Tr}\, A = \bar{\delta}A = 0.$$

The conformal transformation of K^{TF} is determined by the requirement that $\delta^{TF}K^{TF}$ also undergo a simple scaling transformation:

$$\delta^{TF}K^{TF} = \phi^{-6}\bar{\delta}^{TF}\bar{K}^{TF} \tag{19}$$

York's vector potential equation is simply:

$$\delta^{TF}K^{TF} = \phi^{-6}\bar{\delta}^{TF}\bar{L}W = \delta^{TF}(\phi^{-2}(\pounds_W(\phi^{-4}\mathbf{g}))). \tag{20}$$

When $d\phi = 0$ this becomes:

$$\delta^{TF}(-K^{TF} + L(\phi^{-6}W)) = 0. \tag{21}$$

If $d\alpha = 0$ also, then the vector potential equation reduces to the minimal distortion shift equation for $\beta = 2\alpha\phi^{-6}\mathrm{W}$:

$$0 = \delta^{\mathrm{TF}}(-2\alpha \mathrm{K}^{\mathrm{TF}} + \mathrm{L}\beta)\,. \tag{22}$$

This occurs in spatially homogeneous cosmology.

A is the freely specifiable part of K while the vector potential W is determined to within elements of ker $\bar{\mathrm{L}}$ (conformal Killing vectors of $\bar{\mathbf{g}}$) by (II.20), usually rewritten in terms of the supermomentum constraint:

$$\delta^{\mathrm{TF}}\mathrm{K}^{\mathrm{TF}} = -\frac{2}{3} d \,\mathrm{Tr}\, \mathrm{K} + j\,, \tag{23}$$

where $j = \phi^{-6}\bar{j}$ is the 1-form current of the source. Using this constraint and (II.19), the vector potential equation takes the form:

$$\bar{\delta}^{\mathrm{TF}}\bar{\mathrm{L}}\mathrm{W} = -\frac{2}{3}\phi^{6} d \,\mathrm{Tr}\, \mathrm{K} + \bar{j}\,. \tag{24}$$

The super-Hamiltonian constraint uniquely determines the conformal factor ϕ:

$$8\bar{\Delta}\phi - \bar{\mathrm{R}}\phi + \bar{\mathrm{K}}^{\mathrm{TF}\,ij}\bar{\mathrm{K}}^{\mathrm{TF}}{}_{ij}\phi^{-7} - \frac{2}{3}(\mathrm{Tr}\,\mathrm{K})^{2}\phi^{5} + 2\bar{\rho}\phi^{-3} = 0\,. \tag{25}$$

Here $\rho = \phi^{-8}\bar{\rho}$ is the energy density of the source.

The variable Tr K is not conformally transformed but is considered a kinematical variable whose choice is related to picking the initial value slice. The condition d Tr K $= 0$ simplifies the solution of (II.23)-(II.25). For example, (II.24) and (II.25) decouple, while (II.24) and (II.23) each have the form:

$$\delta^{\mathrm{TF}}\mathrm{K}^{\mathrm{TF}} = j\,. \tag{26}$$

A slicing of a spacetime by spacelike hypersurfaces on each of which the trace of the extrinsic curvature tensor is constant will be called an extrinsic time slicing [*10*].

One should not overlook the orthogonal decomposition of the full space S_2 adapted to the orbits of $\mathscr{D}(\mathrm{M})$ [*11*]:

$$\mathrm{S}_2 = \ker \delta \oplus \mathrm{Ran\ Kill}\,. \tag{27}$$

Because of the formula:

$$\delta^{\mathrm{TF}}h^{\mathrm{TF}} = \delta h + \frac{1}{3} d \,\mathrm{Tr}\, h\,, \tag{28}$$

$(\ker \delta)^{\mathrm{TF}}$ does not coincide with ker δ^{TF} except on the subspace of S_2 for which d Tr $h = 0$, so this decomposition does not project to the decomposition (II.7) [*5*]. In analogy with the minimal distortion shift vector fields, one can introduce minimal strain shift vector fields which make $\dot{\mathbf{g}}_t$ orthogonal to the orbits of $\mathscr{D}(\mathrm{M})$. These minimize the norm of $\dot{\mathbf{g}}_t$ rather than $\dot{\mathbf{g}}_t^{\mathrm{TF}}$, leading to the minimal strain shift equation [*2*]:

$$\delta(-2\alpha_t\mathrm{K}_t + \mathrm{Kill}\,\beta_t) = 0\,. \tag{29}$$

III. THE SPATIALLY HOMOGENEOUS CASE

In spatially homogeneous cosmology, M is a 3-dimensional simply connected Lie group G with Lie algebra $\mathfrak{g}$, the 3-dimensional Lie algebra of left invariant vector fields on G. Let $\tilde{\mathfrak{g}}$ be the Lie algebra of right invariant vector fields and $\sim : \mathfrak{g} \to \tilde{\mathfrak{g}}$ the natural map such that $X \in \mathfrak{g}$ and $\tilde{X} \in \tilde{\mathfrak{g}}$ have the same value at the identity of G. $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ commute as Lie subalgebras of $\mathfrak{X}(G)$ and $X \to -\tilde{X}$ is a Lie algebra isomorphism from $\mathfrak{g}$ onto $\tilde{\mathfrak{g}}$. The Lie algebra

$$\text{(1)} \qquad \mathfrak{iaut}(G) = \{ X - \tilde{X} \mid X \in \mathfrak{g} \}$$

generates the action on G of the adjoint group AD(G) = IAut(G) or group of inner automorphisms of G. This is a normal subgroup of the group Aut(G) of automorphisms of G and is canonically isomorphic to the linear adjoint group $\mathrm{Ad}(G) = \mathrm{IAut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ or group of inner automorphisms of $\mathfrak{g}$. Its Lie subalgebra

$$\text{(2)} \qquad \begin{aligned} \mathrm{ad}(\mathfrak{g}) &= \{ \mathrm{ad}(X) \mid X \in \mathfrak{g} \} \subset \mathfrak{gl}(\mathfrak{g}) \\ \mathrm{ad}(X)Y &= [X, Y] \qquad X, Y \in \mathfrak{g} \end{aligned}$$

is called the adjoint Lie algebra and consists of the inner derivations of $\mathfrak{g}$. The Lie algebra $\mathrm{der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ of derivations of $\mathfrak{g}$ contains $\mathrm{ad}(\mathfrak{g})$ as an ideal and is the Lie algebra of the group $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$. Since $\mathrm{IAut}(\mathfrak{g})$ is also a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$, both

$$\mathrm{OAut}(G) = \mathrm{Aut}(G)/\mathrm{IAut}(G) \qquad \text{and} \qquad \mathrm{OAut}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{IAut}(\mathfrak{g})$$

are also groups called the group of outer automorphisms of G and $\mathfrak{g}$ respectively. The restriction to simply connected Lie groups G is sufficient to guarantee that the natural homomorphism from OAut(G) into $\mathrm{OAut}(\mathfrak{g})$ be an isomorphism and hence $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$ [*12*]. Let $\mathfrak{aut}(G)$ be the generating Lie algebra for the action of Aut(G) on G; it contains $\mathfrak{iaut}(G)$ as an ideal. In the simply connected case considered here, the derivations of $\mathfrak{g}$ are obtained by Lie bracketing $\mathfrak{g}$ by elements of $\mathfrak{aut}(G)$. The derivation of $\mathfrak{g}$ determined by $\xi \in \mathfrak{aut}(G)$ will be denoted by $\mathrm{ad}(\xi)$. Note that $\mathrm{ad}(X - \tilde{X}) = \mathrm{ad}(X)$ for $X \in \mathfrak{g}$. Similarly $\mathfrak{aut}(G) \cong \mathrm{der}(\tilde{\mathfrak{g}})$ and one can easily show that $\mathrm{ad}_{\tilde{e}}(\xi) = \mathrm{ad}_e(\xi)$ for $\xi \in \mathfrak{aut}(G)$, where in the left member of the equality $\mathrm{ad}_{\tilde{e}}$ is the matrix representation of the map $\mathrm{ad} : \mathfrak{aut}(G) \to \mathrm{der}(\tilde{\mathfrak{g}})$ with respect to the basis $\tilde{e}$ of $\tilde{\mathfrak{g}}$.

Let $e = \{ e_a \}$ be a basis of $\mathfrak{g}$ with dual basis $\{ \omega^a \}$ of the dual space $\mathfrak{g}^*$ of left invariant 1-forms on G and with structure constant tensor components

$$\text{(3)} \qquad \begin{aligned} C^a{}_{bc} &= \omega^a([e_b, e_c]) = \varepsilon_{bcd} n^{ad} + a_f \delta^{fa}_{bc} \\ n^{ab} &= n^{(ab)} \qquad a_f n^{fa} = 0 \,. \end{aligned}$$

As in reference [1], a subscript e will indicate the matrix representation with respect to e of a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ or a Lie subgroup of $GL(\mathfrak{g})$ or of some homomorphism into these spaces, i. e. $\mathrm{ad}_e(\mathfrak{g})$, $\mathrm{Aut}_e(\mathfrak{g})$ or $\mathrm{ad}_e : \mathfrak{aut}(G) \to \mathrm{der}_e(\mathfrak{g})$. Let $\{ e^b{}_a \}$ be the natural basis of $\mathfrak{gl}(3,R)$ as described in reference [1]. Then the matrices $\{ \boldsymbol{k}_a \}$ generate the matrix adjoint Lie algebra $\mathrm{ad}_e(\mathfrak{g})$:

$$\text{(4)} \qquad \boldsymbol{k}_a = \mathrm{ad}_e(e_a) = C^b{}_{ac} e^c{}_b \qquad \mathrm{Tr}\, \boldsymbol{k}_a = 2a_a$$
$$[\boldsymbol{k}_a, \boldsymbol{k}_b] = C^c{}_{ab} \boldsymbol{k}_c .$$

When the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{ad}(\mathfrak{g})$ is an isomorphism, $\{ \boldsymbol{k}_a \}$ is a basis of $\mathrm{ad}_e(\mathfrak{g})$.

The configuration space $\mathscr{M} = \mathrm{RIEM}(G)$ is now replaced by its left invariant subspace $\mathscr{M}(\mathfrak{g})$, naturally thought of as the space of positive definite inner products on the Lie algebra $\mathfrak{g}$:

$$\text{(5)} \qquad \mathbf{g} = g_{ab} \omega^a \otimes \omega^b \in \mathscr{M}(\mathfrak{g})$$
$$\boldsymbol{g} = g_{ab} e^b{}_a \in \mathscr{M}_3 \subset GL(3,R) .$$

$\mathscr{M}_3$ is the submanifold of $GL(3,R)$ consisting of the matrices of positive definite inner products on R^3. This correspondence between $\mathscr{M}(\mathfrak{g})$ and $\mathscr{M}_3$ in a given basis e is very useful.

The Lie subgroup of $\mathscr{D}(G)$ which maps $\mathscr{M}(\mathfrak{g})$ into itself is the semi-direct product group

$$\text{(6)} \qquad \mathscr{D}(\mathfrak{g}) = L(G) \times \mathrm{Aut}(G) = R(G) \times \mathrm{Aut}(G)$$

where $L(G)$ [respectively $R(G)$] is the group of left [respectively right] translations of G into itself. The equality of these semi-direct product groups follows from the fact that $\mathrm{Aut}(G)$ contains the inner automorphisms:

$$\text{(7)} \qquad \mathrm{AD}_u = L_u \circ R_{u^{-1}} = R_{u^{-1}} \circ L_u \qquad u \in G$$
$$L_{u_1}(u_2) = u_1 u_2 = R_{u_2}(u_1) \qquad u_1, u_2 \in G$$
$$L_u \circ h = R_u \circ (\mathrm{AD}_u \circ h) \qquad h, \mathrm{AD}_u \circ h \in \mathrm{Aut}(G) .$$

For the same reason one has equality of the corresponding generating Lie algebras [3]:

$$\text{(8)} \qquad \mathfrak{X}(\mathfrak{g}) = \tilde{\mathfrak{g}} \oplus \mathfrak{aut}(G) = \mathfrak{g} \oplus \mathfrak{aut}(G) .$$

The shift vector field must be restricted to this subspace of $\mathfrak{X}(G)$ in order to restrict $\mathscr{M}$ to $\mathscr{M}(\mathfrak{g})$ and $\mathscr{D}(G)$ to $\mathscr{D}(\mathfrak{g})$.

Let $S_2(\mathfrak{g})$ be the left invariant subspace of S_2, namely symmetric $\binom{0}{2}$ tensors over $\mathfrak{g}$. The metric (II.4) is the pointwise inner product of elements of S_2 integrated against the volume element of $\mathbf{g}$. When evaluated on

[3] $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ are the generating Lie algebras for the left and right translations respectively.

elements of $S_2(\mathfrak{g})$ for $\mathbf{g} \in \mathcal{M}(\mathfrak{g})$, the pointwise inner product is constant and hence (II.4) factors into the product of that constant and the volume of the group G which will be infinite for noncompact groups. One must therefore use instead the pointwise inner product itself as the appropriate analogue of (II.4) on $\mathcal{M}(\mathfrak{g})$ ([4]):

$$(9) \qquad \mathcal{G}(h, k) = g^{ab}g^{cd}h_{ac}k_{bd} \qquad h, k \in S_2(\mathfrak{g}) .$$

This corresponds to a familiar Riemannian metric on the finite dimensional manifold $\mathcal{M}(\mathfrak{g})$ whose tangent spaces are isomorphic to $S_2(\mathfrak{g})$. Each $\mathbf{g} \in \mathcal{M}(\mathfrak{g})$ induces the orthogonal decomposition (II.7) of $S_2(\mathfrak{g})$:

$$(10) \qquad S_2(\mathfrak{g}) = S_2(\mathfrak{g})^{C} \oplus S_2(\mathfrak{g})^{TF}$$

The distinction between $T\mathcal{M}(\mathfrak{g})_{\mathbf{g}}$ and $S_2(\mathfrak{g})$ and various subspaces of these spaces will usually be overlooked in this discussion.

The metric induced on $\mathcal{M}_3$ by the correspondence (III.5) will be denoted by the same symbol $\mathcal{G}$:

$$(11) \qquad \mathcal{G} = g^{a(c}g^{d)b}dg_{ab} \otimes dg_{cd} .$$

Here $g_{ab} = g_{ba}$ are interpreted as the natural component functions on $\mathcal{M}_3$. Let $\mathcal{W}_3$ be the unimodular submanifold of $\mathcal{M}_3$ ([5]) and let $\tilde{g}_{ab} = g^{-1/3}g_{ab}$ where $g = \det \boldsymbol{g}$. The correspondence (III.5) projects to a similar correspondence between the left invariant subspace $\mathcal{W}(\mathfrak{g})$ of $\mathcal{W}$ and $\mathcal{W}_3$. Since $\mathcal{G}$ is now just the local inner product, it does project to a metric $\mathcal{G}$ on $\mathcal{W}(\mathfrak{g})$. The corresponding metric on $\mathcal{W}_3$ is just the restriction of $\mathcal{G}$ to that submanifold, again denoted by $\mathcal{G}$:

$$(12) \qquad \mathcal{G} = \tilde{g}^{a(c}\tilde{g}^{d)b}d\tilde{g}_{ab} \otimes d\tilde{g}_{cd}$$

Here $\tilde{g}_{ab} = \tilde{g}_{ba}$ are interpreted as the restrictions to $\mathcal{W}_3$ of functions on $\mathcal{M}_3$. The decomposition (III.10) now projects to an orthogonal decomposition of $T\mathcal{W}(\mathfrak{g})_{\mathbf{g}}$. However, this discussion will continue in terms of $\mathcal{M}(\mathfrak{g})$ as in the general setting.

Since L(G) acts as the identity on $\mathcal{M}(\mathfrak{g})$ under dragging along, it is sufficient to consider the action of Aut(G) on $\mathcal{M}(\mathfrak{g})$. This coincides with the natural action on $\mathcal{M}(\mathfrak{g})$ of the isomorphic group Aut($\mathfrak{g}$). The elements of S_2 corresponding to the generators of this action at $\mathbf{g} \in \mathcal{M}(\mathfrak{g})$ were shown in reference [1] to be:

$$(13) \qquad \begin{array}{ll} \text{Kill}\,\xi = -2A^{\#} & A^{\#}{}_{ab} = g_{c(a}A^{c}{}_{b)} \\ \xi \in \mathfrak{aut}(G) & A = A^{a}{}_{b}e^{b}{}_{a} = \mathrm{ad}_e(\xi) \in \mathrm{der}_e(\mathfrak{g}) . \end{array}$$

([4]) In the compact case, this differs from (II.4) only by a constant for a given $\mathbf{g} \in \mathcal{M}(\mathfrak{g})$ and $h, k \in S_2(\mathfrak{g})$.

([5]) $\mathcal{M}_3$ and $\mathcal{W}_3$ were denoted by $\mathcal{M}$ and $\overline{\mathcal{U}}$ in [1].

In other words the subspace of $S_2(\mathfrak{g}) \sim T\mathscr{M}(\mathfrak{g})_{\mathbf{g}}$ tangent to the orbit through $\mathbf{g}$ of the action of either Aut(G) or Aut(g) is the image of the map $\# : \text{der}(\mathfrak{g}) \to S_2(\mathfrak{g})$. The operator L acts as a linear transformation from $\mathfrak{X}(\mathfrak{g})$ into $S_2(\mathfrak{g})^{TF}$ with $\tilde{\mathfrak{g}} \subset \ker L$ and $L\xi = -2A^{\# TF}$ for $\xi \in \mathfrak{aut}(G)$.

Note that Kill and L are related by the formula:

$$\text{(14)} \quad \text{Kill}\,\xi = L\xi + \frac{1}{3}\mathbf{g}\,\text{Tr Kill}\,\xi = L\xi - \frac{2}{3}\mathbf{g}\,\text{Tr ad}(\xi) \qquad \xi \in \mathfrak{aut}(G)\,.$$

In the semisimple case [Bianchi types VIII and IX],

$$\text{Aut}(\mathfrak{g}) = \text{IAut}(\mathfrak{g}) = \text{SAut}(\mathfrak{g}) \qquad \text{and} \qquad \text{der}(\mathfrak{g}) = \text{ad}(\mathfrak{g}) = \mathfrak{sad}(\mathfrak{g})$$

is tracefree, so Kill and L coincide on $\mathfrak{aut}(G)$ and hence on $\mathfrak{X}(\mathfrak{g})$ since $\tilde{\mathfrak{g}}$ is in the kernel of both operators.

The kernels of Kill and L within $\mathfrak{X}(\mathfrak{g})$ for $\mathbf{g} \in \mathscr{M}(\mathfrak{g})$ always contain $\tilde{\mathfrak{g}}$, the « spatial homogeneity » Killing Lie algebra:

$$\text{(15)} \qquad \begin{aligned} \ker \text{Kill} \cap \mathfrak{X}(\mathfrak{g}) &= (\ker \text{Kill} \cap \mathfrak{aut}(G)) \oplus \tilde{\mathfrak{g}} \\ \ker L \cap \mathfrak{X}(\mathfrak{g}) &= (\ker L \cap \mathfrak{aut}(G)) \oplus \tilde{\mathfrak{g}}\,. \end{aligned}$$

Let $I_{\mathbf{g}} \subset \text{Aut}(\mathfrak{g})$ be the isotropy group at $\mathbf{g} \in \mathscr{M}(\mathfrak{g})$ of the action of Aut(g) on that space and let $\mathfrak{i}_{\mathbf{g}} \subset \text{der}(\mathfrak{g})$ be its Lie algebra. [Similarly let $I_g = (I_{\mathbf{g}})_e$ and $\mathfrak{i}_g = (\mathfrak{i}_{\mathbf{g}})_e$ be their matrix representations with respect to a basis e, namely the isotropy group and its Lie algebra at $g \in \mathscr{M}_3$ for the action of $\text{Aut}_e(\mathfrak{g})$ on $\mathscr{M}_3$.] ad is an isomorphism from $\ker \text{Kill} \cap \mathfrak{aut}(G)$ onto $\mathfrak{i}_{\mathbf{g}}$. The first space is the space of Killing vector fields of $\mathbf{g}$ contained in $\mathfrak{aut}(G)$. In reference [*1*] it was shown that the isotropy group $I_{\mathbf{g}}$ is generically nontrivial only for Bianchi types I, II and V where the generic dimension of this group is 3, 1 and 1 respectively. When $g = \mathbf{1}$ in the type I [abelian] case, $I_g = SO(3,R)$. When g is proportional to $\mathbf{1}$ in the other two cases, I_g is the subgroup of SO(3,R) corresponding to rotations about the third axis.

The elements of $\ker L \cup \mathfrak{aut}(G)$ are homothetic Killing vector fields of $\mathbf{g} \in \mathscr{M}(\mathfrak{g})$. Since $L\xi = A^{\# TF} = 0$ for $\xi \in \mathfrak{aut}(G)$ implies $A^{\#} \in S_2(\mathfrak{g})^C$, the space $\text{der}(\mathfrak{g})^{\#} \cap S_2(\mathfrak{g})^C$ must be nonempty for a nontrivial homothetic Killing vector field ([6]) to exist. In reference [*1*] this space was shown to be nonempty only in the abelian case where a single nontrivial linearly independent homothetic Killing vector field ξ exists which generates pure dilations in the cartesian coordinates associated with a basis e for which $g = \mathbf{1}$; in this basis one may pick $\text{ad}_e(\xi) = \mathbf{1}$. That only one nontrivial homothetic Killing vector field belongs to ker L in the abelian case is a reflection of the fact that a Lie algebra of homothetic Killing vector fields can contain at most one linearly independent nontrivial element [*13*]. In all other cases $\ker L \cap \mathfrak{X}(\mathfrak{g}) = \ker \text{Kill} \cap \mathfrak{X}(\mathfrak{g})$.

([6]) A homothetic Killing vector field which is not also a Killing vector field.

The operator δ acts as a linear transformation from $S_2(\mathfrak{g})$ into $\mathfrak{g}^*$. Since $d\,\mathrm{Tr}\, h = 0$ for all $h \in S_2(\mathfrak{g})$, $S_2(\mathfrak{g})^C$ is contained in $\ker \delta$. If $h \in S_2(\mathfrak{g})$, the components of its divergence with respect to a basis e of $\mathfrak{g}$ are given by the formula:

$$(16) \qquad -(\delta h)_a = D^b h_{ba} = (\delta_a)^c{}_b h^b{}_c = \mathscr{G}(\delta_a{}^\#, h)$$

$$\boldsymbol{\delta}_a = (\delta_a)^c{}_b e^b{}_c = (C^b{}_{ac} - 2a_c \delta^b{}_a) e^c{}_b \qquad \mathrm{Tr}\, \boldsymbol{\delta}_a = 0 .$$

Thus $\ker \delta^{TF}$ is the subspace of $S_2(\mathfrak{g})^{TF}$ orthogonal to span $\{ \delta_a{}^\# \}$.

Except in the semisimple case [Bianchi types VIII and IX], the analogue of the decomposition (II.7) is not valid, nor is there any analogue of (II.6) or of the second equality in (II.17) except in the compact case [Bianchi type IX]. Instead one has two separate orthogonal decompositions:

$$(17) \qquad S_2(\mathfrak{g})^{TF} = (\mathrm{Ran}\, L)^\perp \oplus \mathrm{Ran}\, L$$

$$S_2(\mathfrak{g})^{TF} = \ker \delta^{TF} \oplus (\ker \delta^{TF})^\perp$$

$$(\mathrm{Ran}\, L)^\perp \subseteq \ker \delta^{TF} \qquad (\ker \delta^{TF})^\perp \subseteq \mathrm{Ran}\, L .$$

The first might be called the minimal distortion shift (MDS) decomposition and the second the initial value problem (IVP) decomposition. The equalities in the third line hold only in the semisimple case. Similarly one has a minimal strain shift (MSS) decomposition of the whole space:

$$(18) \qquad S_2(\mathfrak{g}) = (\mathrm{Ran\, Kill})^\perp \oplus \mathrm{Ran\, Kill}$$

$$(\mathrm{Ran\, Kill})^\perp \subseteq \ker \delta .$$

The equality holds only in the semisimple case where the situation is analogous to (II.27). In this case the MDS and IVP decompositions coincide and are compatible with the MSS decomposition in the sense that the tracefree projection of the latter yields the former. In the abelian case, $\mathrm{Ran}\, L = S_2(\mathfrak{g})^{TF}$ and $\mathrm{Ran\, Kill} = S_2(\mathfrak{g})$ so $(\mathrm{Ran\, Kill})^\perp = \{0\} = (\mathrm{Ran}\, L)^\perp$.

Let $({}^4M, {}^4\mathbf{g})$ be a spatially homogeneous spacetime with G as an isometry group acting simply transitively on spacelike hypersurfaces and let $N_t = h_t(G)$ be a foliation whose elements are these hypersurfaces, each diffeomorphic to G (7). The lapse function α_t for this foliation must be a constant function G for each t and in order to restrict $(\mathbf{g}_t, K_t)$ to $\mathscr{M}(\mathfrak{g}) \times S_2(\mathfrak{g})$, the shift vector field β_t must be confined to $\mathfrak{X}(\mathfrak{g})$.

Consider $\Sigma(\mathbf{g}, K, \alpha, \beta)$ of (II.14) as a function on $\mathscr{M}(\mathfrak{g}) \times S_2(\mathfrak{g}) \times \mathfrak{F}(\mathfrak{g}) \times \mathfrak{X}(\mathfrak{g})$, where $\mathfrak{F}(\mathfrak{g})$ is the space of constant functions on G. For fixed $(\mathbf{g}, K, \alpha)$, the function

$$(19) \qquad S(\mathbf{g}, K, \alpha, \beta) = \mathscr{G}(\Sigma, \Sigma)$$

is a quadratic function on the vector space $\mathfrak{X}(\mathfrak{g})$ constant along the directions corresponding to $\ker L$, i. e. it projects to a quadratic function on the quotient space $\mathfrak{X}(\mathfrak{g})/\ker L$. It is a minimum for those β such that $\Sigma(\beta)$

(7) This is an extrinsic time slicing.

lies in $(\mathrm{Ran\ L})^{\perp}$. Although this determines β only modulo ker L, $\Sigma(\beta)$ is uniquely determined. Let $\mathrm{TMDS}(\mathbf{g}, \mathrm{K}, \alpha) \subset \mathfrak{X}(\mathfrak{g})$ be the equivalence class of such β's, projecting onto a unique point in $\mathfrak{X}(\mathfrak{g})/\ker \mathrm{L}$. Since $(\mathrm{Ran\ L})^{\perp} \subseteq \ker \delta^{\mathrm{TF}}$, $\mathrm{TMDS}(\mathbf{g}, \mathrm{K}, \alpha)$ is contained in the solution space of the minimal distortion equation (II.15). Elements of $\mathrm{TMDS}(\mathbf{g}_t, \mathrm{K}_t, \alpha_t)$ might be called true minimal distortion shift vector fields, since it is only this class of minimal distortion shift vector fields which are the analogues of such vector fields in the compact and asymptotically flat cases. Similarly true minimal strain shift vector fields are those which make $\dot{\mathbf{g}}_t$ orthogonal to Ran Kill and minimize the norm of $\dot{\mathbf{g}}_t$. These are defined modulo ker Kill.

In the semisimple case where Kill and L coincide on $\mathfrak{X}(\mathfrak{g})$, the true minimal strain and minimal distortion shift vector fields coincide and are in fact determined by the minimal distortion/minimal strain equation. In the abelian case, the true minimal distortion shifts make $\dot{\mathbf{g}}_t{}^{\mathrm{TF}} = 0$ and the true minimal strain shifts make $\dot{\mathbf{g}}_t = 0$, hence the latter class of shifts automatically belongs to the first class. This is due entirely to the existence of a nontrivial homothetic Killing vector field. The additional freedom in the minimal distortion shifts relative to the minimal strain shifts is associated with the dilation freedom accompanying the existence of this vector field. It seems reasonable to fix this freedom by choosing the minimal strain condition in this case, leaving the remaining freedom in the true minimal distortion shift entirely in ker Kill as in the other Bianchi types ([8]). This condition will be assumed in further discussion of minimal distortion shifts when referring to the abelian case. In those cases where G is neither abelian nor semisimple, the true minimal distortion shifts and true minimal strain shifts do not intersect.

In reference [*1*] a local slice for the action of Aut(g) on $\mathscr{M}(\mathfrak{g})$ was described for each Bianchi type Lie group G in terms of the action of $\mathrm{Aut}_e(\mathfrak{g})$ on $\mathscr{M}_3$ for a canonical basis e of $\mathfrak{g}$ [defined in that reference]. At a point g in the diagonal submanifold $\mathscr{M}_{\mathrm{D}}$ of $\mathscr{M}_3$, the subspace of the tangent space corresponding to off-diagonal symmetric matrices is contained in the span of the generating vector fields for the action of $\mathrm{Aut}_e(\mathfrak{g})$ on $\mathscr{M}_3$. A local slice for this action was chosen to be a submanifold of $\mathscr{M}_{\mathrm{D}}$ such that the tracefree projections of its tangent spaces were orthogonal to the diagonal part of the tracefree projection of the orbit of $\mathrm{Aut}_e(\mathfrak{g})$ through each point of that submanifold and hence to the entire tracefree projection of that orbit ([9]). In other words the tracefree projection of the

([8]) Smarr and York impose this condition in their treatment of an abelian spatially homogeneous spacetime as an example in reference [*3*].

([9]) Diagonal and off-diagonal symmetric matrices thought of as belonging to $\mathrm{T}(\mathscr{M}_3)_g$ are orthogonal for $g \in \mathscr{M}_{\mathrm{D}}$ so the tangent space to the slice is automatically orthogonal to the off-diagonal subspace of the full tangent space.

tangent space of this local slice is exactly $(\text{Ran L})^\perp$ [almost everywhere]. Using the action of $\text{Aut}_e(\mathfrak{g})$, one can drag this local slice over $\mathcal{M}_3$ and determine a local slice through every point of $\mathcal{M}_3$ which has the property that the tracefree projection of its tangent space is $(\text{Ran L})^\perp$ [almost everywhere] ([10]). This property is not destroyed under dragging along by $\text{Aut}_e(\mathfrak{g})$ since it is necessarily a subgroup of the full group GL(3,R) of isometries of $(\mathcal{M}_3, \mathcal{G})$. For given initial data $(\mathbf{g}_0, \text{K}_0)$, solving the super-Hamiltonian and supermomentum constraints, a choice of shift vector field β_t which keeps $\boldsymbol{g}_t$ in the local slice through $\boldsymbol{g}_0$ is therefore a true minimal distortion shift vector field. Of course it is sufficient to consider initial data such that $\boldsymbol{g}_0$ lies in the diagonal local slice since initial data differing by an automorphism generates an isometric spacetime. The minimal distortion shift vector field described in reference [*1*] is therefore a true minimal distortion shift vector field.

The IVP decomposition of $S_2(\mathfrak{g})^{\text{TF}}$ is also related to the orbits of the action of a group on $\mathcal{M}(\mathfrak{g})$. In the class A case where a_b vanishes, $\boldsymbol{\delta}_a = \boldsymbol{k}_a$ and so $(\ker \delta^{\text{TF}})^\perp = \text{span}\{ \delta_a{}^\# \}$ is the tangent space to the orbits of the linear adjoint group Ad(G) = IAut($\mathfrak{g}$) which is a unimodular group. In the class B case where a_b does not vanish, the tracefree matrices $\{ \boldsymbol{\delta}_a \}$ also generate a Lie algebra, say $\text{dad}_e(\mathfrak{g})$, since the following relation holds:

$$[\boldsymbol{\delta}_a, \boldsymbol{\delta}_b] = d^c{}_{ab}\boldsymbol{\delta}_c \tag{20}$$

$$d^c{}_{ab} = C^c{}_{ab} + 2a_f\delta^{fc}_{ab} = \varepsilon_{abd}n^{cd} + 3a_f\delta^{fc}_{ab}\,.$$

Thus $(\ker \delta^{\text{TF}})^\perp$ is the tangent space to the orbits of the unimodular group generated by dad($\mathfrak{g}$).

The structure constant tensor components $d^c{}_{ab}$ differ from $C^c{}_{ab}$ by the transformation $a_f \to 3a_f$ or $h \to 1/9\, h$, where h is the invariant defined in the class B case when the rank of $\boldsymbol{n} = n^{ab}\boldsymbol{e}^b{}_a$ is two by the relation:

$$a_a a_b = \frac{1}{2} h\varepsilon_{acd}\varepsilon_{bfg}n^{cf}n^{dg}\,. \tag{21}$$

The only class B type whose adjoint representation is degenerate [not an isomorphism] is type III = VI_{-1} for which span $\{ \boldsymbol{k}_a \}$ has dimension two ([11]). It therefore follows that the adjoint representation of dad($\mathfrak{g}$) has dimension two for type $\text{VI}_{-1/9}$; in fact dad($\mathfrak{g}$) is itself 2-dimensional in this case. This is easily seen by inspection of the expressions for $\{ \boldsymbol{k}_a \}$

([10]) Namely on the open submanifold of $\mathcal{M}_3$ on which $(\text{Ran L})^\perp$ assumes its maximal dimension, the remainder of $\mathcal{M}_3$ being a set of measure zero. On this open submanifold the above discussion essentially shows that $S_2(\mathfrak{g})^C \oplus (\text{Ran L})^\perp$ is a completely integrable or involutive distribution, having a family of integral submanifolds which are slices for the action of $\text{Aut}_e(\mathfrak{g})$ on that open submanifold, except in the abelian case where $\text{Aut}_e(\mathfrak{g})$ = GL(3,R) acts transitively on $\mathcal{M}_3$.

([11]) The subscript on the Roman numeral is the value of h.

and $\{ \boldsymbol{\delta}_a \}$ when the structure constant tensor components are in standard diagonal form, i. e. when $\boldsymbol{n} = \mathrm{diag}(n^{(1)}, n^{(2)}, n^{(3)})$, $a_b = a\delta^3{}_b$ and $a^2 = hn^{(1)}n^{(2)}$:

$$(22) \qquad \begin{aligned} \boldsymbol{k}_1 &= - n^{(2)}\boldsymbol{e}^3{}_2 + n^{(3)}\boldsymbol{e}^2{}_3 - a\boldsymbol{e}^3{}_1 \\ \boldsymbol{k}_2 &= - n^{(3)}\boldsymbol{e}^1{}_3 + n^{(1)}\boldsymbol{e}^3{}_1 - a\boldsymbol{e}^3{}_2 \\ \boldsymbol{k}_3 &= - n^{(1)}\boldsymbol{e}^2{}_1 + n^{(2)}\boldsymbol{e}^1{}_2 + a(\boldsymbol{e}^1{}_1 + \boldsymbol{e}^2{}_2) \\ & \quad an^{(3)} = 0 \\ \boldsymbol{\delta}_1 &= - n^{(2)}\boldsymbol{e}^3{}_2 + n^{(3)}\boldsymbol{e}^2_3 - 3a\boldsymbol{e}^3{}_1 \\ \boldsymbol{\delta}_2 &= - n^{(3)}\boldsymbol{e}^1{}_3 + n^{(1)}\boldsymbol{e}^3{}_1 - 3a\boldsymbol{e}^3{}_2 \\ \boldsymbol{\delta}_3 &= - n^{(1)}\boldsymbol{e}^2{}_1 + n^{(2)}\boldsymbol{e}^1{}_2 + a(\boldsymbol{e}^1{}_1 + \boldsymbol{e}^2{}_2 - 2\boldsymbol{e}^3{}_3) . \end{aligned}$$

Except for types III and $\mathrm{VI}_{-1/9}$, span $\{ \boldsymbol{k}_1, \boldsymbol{k}_2 \}$ = span $\{ \boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \}$. For the first type, span $\{ \boldsymbol{k}_1, \boldsymbol{k}_2 \}$ is 1-dimensional as is span $\{ \boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \}$ for the second type. In all class B cases these belong to an abelian 2-dimensional Lie subalgebra of the special automorphism matrix Lie algebra $\mathfrak{saut}_e(\mathfrak{g})$. The matrices $\boldsymbol{k}_3$ and $\boldsymbol{\delta}_3$ differ only in their diagonal components:

$$(23) \qquad \begin{aligned} \boldsymbol{k}_3 &= \boldsymbol{k}_3^0 + a\mathbf{I}^{(3)} & \mathbf{I}^{(3)} &= \mathrm{diag}(1, 1, 0) \\ \boldsymbol{\delta}_3 &= \boldsymbol{k}_3 - 2a\boldsymbol{e}^3{}_3 = \boldsymbol{k}_3^0 + a\boldsymbol{e}_+ & \boldsymbol{e}_+ &= \mathrm{diag}(1, 1, -2) \end{aligned}$$

$\boldsymbol{k}_3^0$ is the off-diagonal part of $\boldsymbol{k}_3$. $\boldsymbol{\delta}_3$ is the matrix of a « projective automorphism » of $\mathfrak{g}$, since the subgroup it generates acts on $C^a{}_{bc}$ as a uniform scaling.

Thus $(\ker \delta^{\mathrm{TF}})^\perp$ is generically 3-dimensional for all class B types except type $\mathrm{VI}_{-1/9}$ where it is 2-dimensional. The class A types with degenerate adjoint representations are types I and II. The adjoint Lie algebra has dimension zero and two respectively in these cases. Therefore $(\ker \delta^{\mathrm{TF}})^\perp$ is 3-dimensional for all Bianchi types except I, II and $\mathrm{VI}_{-1/9}$ and $\delta^{\mathrm{TF}} : (\ker \delta^{\mathrm{TF}})^\perp \rightarrow \mathfrak{g}^*$ is a vector space isomorphism which can be inverted. Since δ^{TF} is always a 1-1 linear map when restricted to $(\ker \delta^{\mathrm{TF}})^\perp$, it can also be inverted from its 2-dimensional image in $\mathfrak{g}^*$ to the space $(\ker \delta^{\mathrm{TF}})^\perp$ for the exceptional types II and $\mathrm{VI}_{-1/9}$. In the type I case, $S_2(\mathfrak{g})^{\mathrm{TF}} = \ker \delta^{\mathrm{TF}}$ and $(\ker \delta^{\mathrm{TF}})^\perp = \{ 0 \}$.

The supermomentum constraint (II.26) is said to be integrable in the vacuum case if $\ker \delta = S_2(\mathfrak{g})^{\mathrm{C}} \oplus \ker \delta^{\mathrm{TF}}$ is an involutive distribution [almost everywhere] on $\mathcal{M}(\mathfrak{g})$. Then the vacuum constraint $\delta^{\mathrm{TF}}\mathrm{K}^{\mathrm{TF}} = 0$ for zero shift vector field simply requires that each solution curve $\mathbf{g}_t$ of the dynamical Einstein equations be contained in an integral submanifold of the distribution in question. In the nonvacuum case, the motion orthogonal to these submanifolds is determined by the supermomentum constraint and is excited only by a nonzero source current. Since $(\ker \delta)^\perp \subset \mathrm{Ran\ Kill}$, one can pick a nonzero shift vector field to remove this motion orthogonal to the given family of submanifolds. Such a shift vector field might be called a dynamical shift vector field and is entirely determined by the supermomentum constraint as discussed in a following paragraph. Using the local coordinates of reference [1], one can easily see that ker δ is an involutive distribution only in the class A case. For all class A Bianchi types except I and II, $\mathscr{M}_{\mathrm{D}}$ is an integral submanifold of the corresponding distribution on $\mathscr{M}_3$ [assuming the correspondence between $\mathscr{M}(\mathfrak{g})$ and $\mathscr{M}_3$ is determined by a canonical basis e of $\mathfrak{g}$] and the others may be obtained from it under the action of the 3-dimensional special automorphism group. The dynamical shifts are those which diagonalize $\boldsymbol{g}_t$ modulo constant automorphisms. In the semisimple case, all of the special shifts considered in this paper coincide. For Bianchi type II, $\mathscr{M}_{\mathrm{S}(3)}$ is such an integral submanifold ([12]) and the others may be obtained from it under the action of the 2-dimensional group of inner automorphisms. In the type I case, ker δ is the whole tangent space.

The conformal approach to the initial value problem described in section II is in fact unnecessary in the spatially homogeneous case. For constant conformal factors ϕ, the super-Hamiltonian constraint (II.25) reduces to a polynomial equation in ϕ which is equivalent to a cubic equation in the variable ϕ^4 that has received much attention in the study of the solutions of that constraint [14]. However, in the spatially homogeneous case this constraint can be trivially solved for the variable Tr K without introducing a conformal transformation, namely (II.25) with $\phi = 1$. This does not mean that the conformal approach is not dynamically important. Often the Einstein equations are rewritten in terms of the variables $(g_t, \tilde{\mathbf{g}}_t)$ which is equivalent to setting $\phi_t{}^{12} = g_t$ and $\bar{\mathbf{g}}_t = \tilde{\mathbf{g}}_t$ in a given frame on G. This approach is usually accompanied by a choice of intrinsic time which the super-Hamiltonian constraint makes preferable to the extrinsic time variable $\tau = \frac{4}{3}\mathrm{Tr}\,\mathrm{K}$ [10]. For example, Misner's choice [15] $\Omega = -\frac{1}{6}\ln g_t = t$ determines the lapse function in terms of the divergence of the shift vector field [often zero] and Tr K_t, the latter of which is itself determined by the super-Hamiltonian constraint:

$$1 = \dot{\Omega} = -\frac{1}{6}(\ln g_t)^{\cdot} = \frac{1}{3}(\alpha_t\,\mathrm{Tr}\,\mathrm{K}_t + \delta\beta_t)\,. \tag{24}$$

This effectively removes the conformal scaling degree of freedom associated with g_t and Tr K_t from the dynamics by incorporating it into the choice of time. Of course the equations of motion then become explicitly time dependent.

Similarly the conformal transformation is superfluous in the solution of the supermomentum constraint (II.26) ([13]) which amounts to inverting

([12]) See reference [1].

([13]) However, the choice of variables $(g_t, \tilde{\mathbf{g}}_t)$ reintroduces the conformal transformation into this constraint.

the 1-1 map δ^{TF} from $(\ker \delta^{TF})^{\perp}$ onto its image in $\mathfrak{g}^*$ to which the source current must be confined. The freely specifiable part of K^{TF} is its component in $\ker \delta^{TF}$:

$$(25) \qquad K^{TF} = A + B \qquad A \in \ker \delta^{TF} \qquad B \in (\ker \delta^{TF})^{\perp}$$
$$\delta^{TF}K^{TF} = \delta^{TF}B = j \in \mathrm{Ran}\, \delta^{TF}.$$

The decomposition of K^{TF} into a divergence free part and the conformal Killing derivative of a vector potential W is no longer particularly relevant in the nonsemisimple case since this does not correspond to the IVP decomposition. Since $(\ker \delta^{TF})^{\perp} \subseteq \mathrm{Ran}\, L$, one can find a subspace of $\mathfrak{X}(\mathfrak{g})$ of solutions of the vector potential equation whose image by the map L is $(\ker \delta^{TF})^{\perp}$ but this step is unnecessary. As with the nearly identical minimal distortion shift equation, the solution space is simply too large in general. However, as pointed out in the previous section for the class A case, the component B of K^{TF} can be taken as the image by L of a vector potential $W \in \mathfrak{iaut}(G)$ which is related to a dynamical shift vector $\beta = 2\alpha W$.

For all class A Bianchi types except I and II, the special and inner automorphism groups coincide and $(\ker \delta^{TF})^{\perp}$ is tangent to the orbits of $\mathrm{SAut}(\mathfrak{g}) = \mathrm{IAut}(\mathfrak{g})$. In the semisimple case these groups coincide with $\mathrm{Aut}(\mathfrak{g})$. Let L_s be the restriction of L to $\tilde{\mathfrak{g}} \oplus \mathfrak{saut}(G) = \tilde{\mathfrak{g}} \oplus \mathfrak{g}$. For these Bianchi types one has the orthogonal decomposition:

$$(26) \qquad S_2(\mathfrak{g})^{TF} = (\ker \delta^{TF}) \oplus \mathrm{Ran}\, L_s .$$

These two subspaces are almost everywhere tangent to submanifolds of $\mathcal{M}(\mathfrak{g})$ ([14]), except where the dimension of the isotropy group of the action of $\mathrm{SAut}(\mathfrak{g})$ on $\mathcal{M}(\mathfrak{g})$ does not assume its minimum value. Using a canonical basis e of $\mathfrak{g}$, this discussion can be projected onto $\mathcal{M}_3$. In reference [*1*], an obvious parametrization of $\mathcal{M}_3$ adapted to these two complementary families of submanifolds was introduced and the supermomentum constraint was explicitly solved as indicated above. In fact, the variables associated with the orbits of $\mathrm{SAut}_e(\mathfrak{g})$ were completely eliminated from the zero shift dynamics, corresponding to the choice of a [not necessarily true] minimal distortion shift vector field β_t which kept g_t diagonal, i. e. a dynamical shift vector field. The nonuniqueness of β_t can even be eliminated by considering only left invariant shift vector fields in this case. For the semisimple Bianchi types the situation is analogous to that described in section II and all of the automorphism degrees of freedom can be eliminated from the zero shift dynamics, corresponding to the choice of the unique left invariant true minimal distortion/strain dynamical shift vector field. For Bianchi types VI_0 and VII_0, one automorphism degree of freedom

---

([14]) This is clearly related to the integrability of the vacuum supermomentum constraint in the class A case.

associated with the 1-dimensional subspace Ran L $\cap$ ker δ^{TF} remains in the dynamics as an effective dynamical degree of freedom. This occurs since the supermomentum constraint fails to determine the true minimal distortion shift vector fields for these two types.

For all the class B types except type V, a similar parametrization of $\mathcal{M}_3$ was introduced involving the 3-dimensional special automorphism group $SAut_e(\mathfrak{g})$. The associated complementary families of submanifolds are almost everywhere adapted to the orthogonal direct sum ([15]):

$$S_2(\mathfrak{g})^{TF} = (\text{Ran } L_s)^{\perp} \oplus \text{Ran } L_s \,, \tag{27}$$

where L_s is the restriction of L to $\tilde{\mathfrak{g}} \oplus \mathfrak{saut}(G)$ as before. Although Ran L_s is not orthogonal to ker δ^{TF} here, $S_2(\mathfrak{g})^{TF}$ can be decomposed into a direct sum of these two subspaces and therefore δ^{TF} : Ran $L_s \to \mathfrak{g}^*$ is invertible on its image. Thus if $K^{TF} = A + B$ with $A \in (\text{Ran } L_s)^{\perp}$ and $B \in \text{Ran } L_s$, one can determine B in terms of the source current and $\delta^{TF}A$ by inverting the equation:

$$\delta^{TF}B = j - \delta^{TF}A \in \text{Ran } \delta^{TF} \,. \tag{28}$$

This enables one to eliminate the special automorphism degrees of freedom from the zero shift dynamics using the supermomentum constraint, corresponding to the choice of a shift vector field β_t such that $B_t = 2\alpha_t L\beta_t$ which keeps g_t diagonal. This is not quite a minimal distortion shift vector field. Again one must solve evolution type equations for the remaining automorphism degree of freedom in order to determine the true minimal distortion shift vector fields, i. e. the supermomentum constraint fails to determine them completely. Bianchi type $VI_{-1/9}$ is an exception to this picture in that Ran $\delta^{TF} \cap \mathfrak{g}^*$ is only 2-dimensional and only two automorphism degrees of freedom can be eliminated.

For the remaining Bianchi types I, II and V, the automorphism group is too large and there is no unique choice of two complementary families of submanifolds adapted to a useful decomposition of $S_2(\mathfrak{g})^{TF}$. This is discussed in reference [*1*].

It is helpful to list the generic dimensions of the various spaces which have been discussed for each Bianchi type ([16]). This is done in table 1. In this table L and δ^{TF} are understood to be restricted to $\mathfrak{X}(\mathfrak{g})$ and $S_2(\mathfrak{g})^{TF}$ respectively.

The dimensions in the first column were referred to in reference [*1*] as the number of dynamical degrees of freedom for each Bianchi type. The

([15]) The vacuum supermomentum constraint is not integrable in the class B case, explaining why the two complementary families are adapted to this decomposition rather than one involving ker δ^{TF}.

([16]) These dimensions change on a set of measure zero in $\mathcal{M}(\mathfrak{g})$ on which dim (ker $L \cap \mathfrak{X}(\mathfrak{g})$) does not assume its minimum value.

TABLE 1.

	$(\text{Ran L})^{\perp}$	Ran L	ker δ^{TF}	$(\ker \delta^{TF})^{\perp}$	$(\ker \text{L})/\tilde{\mathfrak{g}}$	Ran δ^{TF}	Ran L$\cap$ker δ^{TF}
IX, VIII	2	3	2	3	0	3	0
VII_0, VI_0 $VII_{h\neq 0}$, $VI_{h\neq -1/9,0}$ IV	1	4	2	3	0	3	1
$VI_{-1/9}$	1	4	3	2	0	2	2
V	0	5	2	3	1	3	2
II	0	5	3	2	1	2	3
I	0	3	5	0	4	0	5

sum of the dimension of Ran L and (ker L)/$\tilde{\mathfrak{g}}$ equals the dimension of the automorphism group, while the sum of the dimensions of (ker L)/$\tilde{\mathfrak{g}}$ and Ran L $\cap$ ker δ^{TF} equals the dimension of the subspace of $\mathfrak{aut}$(G) consisting of solutions of the minimal distortion equation. The dimensions in the fourth column are the number of degrees of freedom which can be eliminated from the dynamics using the supermomentum constraint. The number of dynamical degrees of freedom which remain after the super-Hamiltonian constraint is then used to eliminate the conformal scaling degree of freedom is given in the third column.

York and Smarr define dynamical degrees of freedom to be present in a spacetime if the conformal 3-geometry changes along an extrinsic time slicing of the spacetime [*2*]. It was this definition, motivated more by kinematical considerations, that was used in reference [*1*] to determine the number of dynamical degrees of freedom generically present in spatially homogeneous spacetimes. However, section 7 of reference [*6*] contains a nice discussion by York of dynamical degrees of freedom in terms of the number of linearly independent velocity variables compatible with the constraints, whether or not they are integrable. When the MDS and IVP decompositions do not coincide, these two definitions no longer agree and kinematics and dynamics are no longer complementary as in section II. If the second definition were adopted, then the numbers in the third column of table 1 would be the number of gravitational dynamical degrees of freedom generically present in spatially homogeneous spacetimes.

This seems reasonable for all but the degenerate Bianchi types I, II and V. For these types, when the source energy-momentum satisfies certain conditions, fewer dynamical degrees of freedom are effectively present. For example, in the vacuum case it was shown in reference [*1*] that any solutions ($\mathbf{g}$, K) of the supermomentum constraint could be reduced to diagonal initial data ([17]) using the automorphism group. Furthermore, with zero shift vector field, this initial data remains diagonal when evolved by the dynamical Einstein equations. For types I and II the supermomentum constraint is identically satisfied by diagonal initial data and hence only the super-Hamiltonian constraint remains, leaving two effective dynamical degrees of freedom. For type V, the diagonal initial data is still subject to one (integrable) condition by the supermomentum constraint so only one effective dynamical degree of freedom is present. The time evolution of nondiagonal initial data with zero shift vector field is related to that of diagonal initial data by a time independent automorphism and so represents a spacetime isometric to one having diagonal initial data with respect to a fixed canonical basis. However, nondiagonal degrees of freedom can be excited by certain sources leading to the presence of the full number of dynamical degrees of freedom listed in the third column of table 1 for these three types. A perfect fluid can excite these modes in Bianchi types II and V, while an electromagnetic field will accomplish this in the type I case, provided that the electric and magnetic fields are not both eigenvectors of the extrinsic curvature tensor.

These three Bianchi types are distinguished from the rest by the fact that the isotropy groups of the action of Aut($\mathfrak{g}$) on $\mathcal{M}(\mathfrak{g})$ are generically nontrivial, with dimensions 2, 1 and 1 respectively for types I, II and V. Given ($\mathbf{g}$, K) which satisfy the vacuum constraints, one can transform this initial data by elements of the isotropy group $I_{\mathbf{g}}$ while leaving $\mathbf{g}$ fixed and generate initial data for an isometric spacetime. The degrees of freedom associated with the orbits of $\{ \mathbf{g} \} \times S_2(\mathfrak{g})$ under the action of $I_{\mathbf{g}}$ are therefore ignorable in the vacuum case. When a source is present, these isotropy group degrees of freedom may be excited but under conditions which do not permit a brief description.

These three Bianchi types are also an exception kinematically again due to the generically nontrivial nature of the isotropy groups $I_{\mathbf{g}}$. The fact that the true minimal distortion shift vector field [minimal strain in the abelian case] for a solution curve ($\mathbf{g}_t$, K_t) is only defined modulo ker Kill with ker Kill $\cap\ \mathfrak{aut}(G) \cong \mathfrak{i}_{\mathbf{g}_t}$ corresponds to the fact that we can act on this solution by a time dependent automorphism $A_t \in I_{\mathbf{g}_t}$ without changing the curve $\mathbf{g}_t$. Of course the curve K_t will generally change.

To discuss this further one must introduce the operator J on vector fields by defining:

$$\text{JX} = \pounds_X K \,. \tag{29}$$

([17]) That is, the matrices of components of $\mathbf{g}$ and K in a given canonical basis e of $\mathfrak{g}$ are diagonal.

If $X \in \ker$ Kill, index raising commutes with the action of Lie derivation and when $d \operatorname{Tr} K = 0$ also holds, JX is tracefree. The operator J appears in the dynamical Einstein equations when written in the form:

$$\dot{K}_t = Q(\mathbf{g}_t, K_t, \alpha_t, \rho_t, S_t) + J\beta_t . \tag{30}$$

where S is the spatial stress tensor of the source [6]. In the spatially homogeneous case, with $(\mathbf{g}, K) \in \mathcal{M}(\mathfrak{g}) \times S_2(\mathfrak{g})$, J acts as a linear transformation from $\mathfrak{X}(\mathfrak{g})$ into $S_2(\mathfrak{g})$ with $\tilde{\mathfrak{g}} \subset \ker J$. Its value on $\mathfrak{aut}(G)$ is given by the following expression in terms of a basis e of $\mathfrak{g}$:

$$(J\xi)_{ab} = -\,2K_{c(a}A^c{}_{b)} \qquad A = \mathrm{ad}_e(\xi) \qquad \xi \in \mathfrak{aut}(G) . \tag{31}$$

The true minimal distortion shifts [minimal strain in the abelian case] are defined only modulo ker Kill, but varying these shift vector fields along ker Kill will change $\dot{K}_t$ unless $\ker J \subseteq \ker$ Kill. It seems reasonable, therefore, to impose a condition to narrow this freedom to the subspace ker Kill $\cap$ ker J in a way analogous to the minimal distortion shift choice, leading to a unique curve $(\mathbf{g}_t, K_t)$ for given initial data ([18]). Suppose the true minimal distortion vector field is chosen to minimize the norm of $\dot{K}$, i. e. $\mathcal{G}(\dot{K}, \dot{K})$. Varying this function on the space of true minimal distortion shifts shows that it is minimized when $\dot{K}$ is orthogonal to J (ker Kill). One would therefore pick β_t so that $J\beta_t$ cancels the component of Q_t along J (ker Kill).

Suppose β_t is a true minimal distortion shift such that g_t belongs to the diagonal slice for the action of the canonical automorphism group $\mathrm{Aut}_e(\mathfrak{g})$ on $\mathcal{M}_3$ as described in reference [1]. In the type I case where $g_t = \mathbf{1}$ and $I_1 = SO(3,R)$, one can always use I_1 to diagonalize $\mathbf{K}_t$. The additional condition simply requires that the freedom remaining in the true minimal strain shift be used to see that $\mathbf{K}_t$ remains diagonal if it is initially so ([19]), since J (ker Kill) in this case is generically equal to the subspace of $S_2(\mathfrak{g})$ corresponding to off-diagonal matrices. This prevents one from adding an arbitrary time dependent rotation generator to the true minimal strain shift β_t, once it is chosen to satisfy the additional condition. In the type II and type V cases, g_t is proportional to $\mathbf{1}$ and $I_{g_t} = I_1$ is the subgroup of SO(3,R) corresponding to rotations about the third axis of R^3. One may always use I_1 to make $(K_{12})_t = 0$. The additional condition requires that

([18]) Since raising and lowering indices does not commute with the time derivative, there are many candidates for such a condition, depending on which valence form of the extrinsic curvature or even of the canonical momentum π whose time derivative is to be minimized with respect to the natural norm. Tracefree projections are no longer relevant at the second derivative level so there is no motivation to use tracefree norms as in the original true minimal distortion condition.

([19]) Or that it be diagonal modulo a time independent rotation.

this be true at least modulo a time independent element of I_1 and once satisfied, prevents the addition to the minimal distortion shift vector field of an arbitrary time dependent generator of rotations about the third canonical coordinate of a canonical coordinate system associated with the canonical basis of this discussion ([20]).

In the generic case for all Bianchi types, the true minimal distortion shift vector fields have been determined modulo the space

$$F = \ker \mathrm{Kill} \cap \ker J \cap \mathfrak{X}(\mathfrak{g})$$

and have a unique projection on the quotient space $\mathfrak{X}(\mathfrak{g})/F$. The space $\ker \mathrm{Kill} \cap \ker J$ is well known to be isomorphic to the Lie algebra of spacetime Killing vector fields which are tangent to the initial value slice in a spacetime generated by initial data $(\mathbf{g}, K)$ [*16*]. In the spatially homogeneous case, these spacetime Killing vector fields remain tangent to the spatially homogeneous slicing, although for some Bianchi types and very special values of $(\mathbf{g}, K)$, $\ker \mathrm{Kill} \cap \ker J$ is not entirely contained in $\mathfrak{X}(\mathfrak{g})$. Let ${}^4Z$ be a spacetime Killing vector field tangent to the spatially homogeneous foliation $N_t = h_t(G)$ with $\vec{n}$ denoting the field of unit normals to this foliation and let $Z_t \in F_t$ be the time dependent vector field induced on G by ${}^4Z$ via the diffeomorphism h_t. The condition $\pounds_{\vec{n}}\, {}^4Z = 0$ must hold for all spacetime Killing vector fields. When rewritten in three-plus-one language, this condition determines the time dependence of Z_t:

$$\dot{Z}_t = \pounds_{\beta_t} Z_t = [\beta_t, Z_t]\,. \tag{32}$$

F_t always contains $\tilde{\mathfrak{g}}$ which is isomorphic to the Lie algebra of generators of the spatial homogeneity isometry group of the spacetime. In the generic case, $F_t = \tilde{\mathfrak{g}}$. Each $Z_t \in \tilde{\mathfrak{g}}$ will be time independent only if $\beta_t \in \mathfrak{g}$. As long as β_t has a nonzero projection ([21]) into the quotient Lie algebra

$$\mathfrak{aut}(G)/\mathfrak{iaut}(G) \cong \mathfrak{oaut}(G)\,,$$

each such Z_t must be time dependent no matter how we use the remaining freedom in the choice of β_t. However, the time dependence of Z_t arising from inner automorphisms generated by the part of β_t « along » $\tilde{\mathfrak{g}}$ can in a sense be minimized by fixing this remaining freedom. This is accomplished by finding a new semi-direct sum decomposition

$$\mathfrak{X}(\mathfrak{g}) = \tilde{\mathfrak{g}} \oplus \mathfrak{X}(\mathfrak{g})^{\mathrm{RED}} \tag{33}$$

([20]) In each of these cases the additional condition turns out to agree with any of those discussed in a previous footnote. Let M be any valence form of K or π whose time derivative is to be minimized and let $\mathscr{J}X = \pounds_X M$; then $\dot{M}$ must be orthogonal to $\mathscr{J}(\ker \mathrm{Kill})$. But the covariant tensor image of $\mathscr{J}(\ker \mathrm{Kill})$ for all such M coincides with $J(\ker \mathrm{Kill})$ and the covariant tensor images of the various $\dot{M}$'s differ only on the subspace orthogonal to $J(\ker \mathrm{Kill})$ so the various conditions yield the same results.

([21]) First project β_t into $\mathfrak{aut}(G)$ using the vector space direct sum $\mathfrak{X}(\mathfrak{g}) = \tilde{\mathfrak{g}} \oplus \mathfrak{aut}(G)$.

where $\mathfrak{X}(\mathfrak{g})^{\text{RED}}$ is a Lie subalgebra of $\mathfrak{X}(\mathfrak{g})$ such that if $\beta \in \mathfrak{iaut}(G)$, its projection into $\mathfrak{X}(\mathfrak{g})^{\text{RED}}$ commutes with $\tilde{\mathfrak{g}}$. Thus any true minimal distortion shift β_t whose projection ([22]) into $\mathfrak{aut}(G)$ lies entirely in $\mathfrak{iaut}(G)$ will have a representative in $\mathfrak{X}(\mathfrak{g})^{\text{RED}}$ which will leave each Z_t time independent.

Let $\mathfrak{c} \subset \mathfrak{g}$ be the center of $\mathfrak{g}$; then $\mathfrak{c} = \tilde{\mathfrak{c}} = \mathfrak{g} \cap \tilde{\mathfrak{g}}$ is also the center of $\tilde{\mathfrak{g}}$ since the center is invariant under the action of the linear adjoint group which maps $\mathfrak{g}$ onto $\tilde{\mathfrak{g}}$ anti-isomorphically ([23]). Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{h}$ and $\tilde{\mathfrak{g}} = \mathfrak{c} \oplus \tilde{\mathfrak{h}}$; then $\text{ad} : \mathfrak{h} \to \text{ad}(\mathfrak{g})$ is an isomorphism and $\mathfrak{iaut}(G) = \{ X - \tilde{X} \mid X \in \mathfrak{h} \}$ so that $\mathfrak{h} \oplus \mathfrak{iaut}(\mathfrak{g}) = \mathfrak{h} \oplus \tilde{\mathfrak{h}}$. Finally let $\mathfrak{out}(G) \subset \mathfrak{aut}(G)$ be a Lie subalgebra such that

$$\mathfrak{aut}(G) = \mathfrak{iaut}(G) \oplus \mathfrak{out}(G) .$$

Together these direct sums imply the following result:

$$\mathfrak{X}(\mathfrak{g}) = (\mathfrak{c} \oplus \tilde{\mathfrak{h}}) \oplus (\mathfrak{h} \oplus \mathfrak{out}(G)) = \tilde{\mathfrak{g}} \oplus \mathfrak{X}(\mathfrak{g})^{\text{RED}} \tag{34}$$

$\mathfrak{X}(\mathfrak{g})^{\text{RED}} = \mathfrak{h} \oplus \mathfrak{out}(G)$ is itself a Lie algebra since $\mathfrak{out}(G)$ maps $\mathfrak{h}$ into itself under bracketing; it is in fact isomorphic to $\mathfrak{aut}(G)$ which in turn is isomorphic to the quotient Lie algebra $\mathfrak{X}(\mathfrak{g})/\tilde{\mathfrak{g}}$. $\mathfrak{X}(\mathfrak{g})^{\text{RED}}$ is the generating Lie algebra for a Lie subgroup $\mathscr{D}(\mathfrak{g})^{\text{RED}}$ of $\mathscr{D}(\mathfrak{g})$ which is isomorphic to Aut(G) and which has the same action on left invariant tensor fields as Aut(G). Moreover if $X \in \mathfrak{h}$, then $X - \tilde{X} \in \mathfrak{iaut}(G)$ has the projection X into $\mathfrak{X}(\mathfrak{g})^{\text{RED}}$ and X commutes with $\tilde{\mathfrak{g}}$.

In the semisimple case, $\mathfrak{g} = \mathfrak{h}$, $\mathfrak{c} = \{ 0 \} = \mathfrak{out}(G)$, $\mathfrak{iaut}(G) = \mathfrak{aut}(G)$ and $\mathfrak{X}(\mathfrak{g})^{\text{RED}} = \mathfrak{g}$. Thus every true minimal distortion shift vector field here has a unique left invariant representative for which all the spacetime Killing field projections $Z_t \in \tilde{\mathfrak{g}}$ are time independent. In the abelian case, $\mathfrak{g} = \mathfrak{c}$, $\mathfrak{h} = \{ 0 \} = \mathfrak{iaut}(G)$ and $\mathfrak{X}(\mathfrak{g})^{\text{RED}} = \mathfrak{out}(G) = \mathfrak{aut}(G)$. Here the inner automorphisms are trivial and every extended true minimal strain shift vector field might as well be chosen to lie in $\mathfrak{aut}(G)$. The only other cases where $\mathfrak{c}$ is nontrivial are Bianchi types III $=$ VI_{-1} and II. In the first case, assuming a canonical basis e, $\mathfrak{c} = \text{span}\{ e_1 + e_2 \}$ and one may take $\mathfrak{h} = \text{span}\{ e_1 - e_2, e_3 \}$. In the second case, $\mathfrak{c} = \text{span}\{ e_3 \}$ but $\mathfrak{h} = \text{span}\{ e_1, e_2 \}$ is not a Lie subalgebra nor can one find any Lie subalgebra $\mathfrak{h}$ to complete $\mathfrak{c}$ to a direct sum of $\mathfrak{g}$ since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{c}$.

In the type II case it was shown in [*1*] that

$$\text{der}_e(\mathfrak{g}) = \text{ad}_e(\mathfrak{g}) \oplus \text{span}\{ \mathbf{1} + \boldsymbol{e}^3{}_3 \} \oplus \mathfrak{s}_e ,$$

([22]) Use the projection of the previous footnote.

([23]) A Lie algebra anti-isomorphism $\alpha : \mathfrak{g} \to \mathfrak{h}$ is a vector space isomorphism whose composition with the vector space inversion $X \in \mathfrak{g} \to -X$ is a Lie algebra isomorphism. The matrix of the anti-isomorphism $\sim : \mathfrak{g} \to \tilde{\mathfrak{g}}$ with respect to the bases e and $\tilde{e}$ is just the inverse matrix of the adjoint representation of G [*1*].

where $\mathfrak{s}_e$ is the Lie algebra of the subgroup of SL(3,R) isomorphic to SL(2,R) which leaves the third axis of $\mathbf{R}^3$ fixed when acting on that space in the natural way. Let $\mathfrak{s} \subset \mathfrak{aut}(G)$ be the corresponding Lie subalgebra of $\mathfrak{aut}(G)$ and let $\xi \in \mathfrak{aut}(G)$ be the element such that $\mathrm{ad}_e(\xi) = 1 + e^3{}_3$. Then one may take $\mathfrak{out}(G) = \mathrm{span}\{\xi\} \oplus \mathfrak{s}$. The type V case is similar with $\mathrm{der}_e(\mathfrak{g}) = \mathrm{ad}_e(\mathfrak{g}) \oplus \mathfrak{s}_e$, where $\mathfrak{s}_e$ and $\mathfrak{s}$ have the same significance as before and one may take $\mathfrak{out}(G) = \mathfrak{s}$. In the type III case,

$$\mathrm{der}_e(\mathfrak{g}) = \mathrm{ad}_e(\mathfrak{g}) \oplus \mathrm{span}\{ e^3{}_1 + e^3{}_2, \mathbf{I}^{(3)} \} = \mathrm{ad}_e(\mathfrak{g}) \oplus \mathfrak{out}_e(\mathfrak{g});$$

let $\mathfrak{out}(G)$ be the Lie subalgebra of $\mathfrak{aut}(G)$ which maps onto $\mathfrak{out}_e(\mathfrak{g})$.

All that remains is to choose $\mathfrak{out}(G)$ for all those Bianchi types for which $\mathfrak{c}$ is trivial and the automorphism group is 4-dimensional. Here the group of outer automorphism is 1-dimensional and one may take any $\xi \in \mathfrak{aut}(G)$ not belonging to $\mathfrak{iaut}(G)$ and set $\mathfrak{out}(G) = \mathrm{span}\{\xi\}$. For example, if e is a canonical basis of $\mathfrak{g}$, one may choose $\xi \in \mathfrak{aut}(G)$ such that $\mathrm{ad}_e(\xi) = \mathbf{I}^{(3)}$.

The true minimal distortion shifts have a unique representative in the subspace $\mathfrak{X}(\mathfrak{g})^{\mathrm{RED}}$ and this representative induces as little time dependence in the spacetime Killing vector field projections $Z_t \in \tilde{\mathfrak{g}}$ due to the inner automorphisms as qualitatively possible. The Lie subalgebra $\mathfrak{h}$ is defined only modulo $\mathfrak{c}$ but since $\mathfrak{c}$ commutes with $\tilde{\mathfrak{g}}$, the nonuniqueness in $\mathfrak{h}$ has no effect on the time dependence of each $Z_t \in \tilde{\mathfrak{g}}$. However, since $\mathfrak{out}(G)$ is defined only modulo $\mathfrak{iaut}(G)$, different choices lead each Z_t to differ by a time dependent inner automorphism unless β_t projects to the zero element of $\mathfrak{oaut}(G) = \mathfrak{aut}(G)/\mathfrak{iaut}(G)$, in which case β_t has a representative in the subspace of $\mathfrak{X}(\mathfrak{g})^{\mathrm{RED}}$ which commutes with $\tilde{\mathfrak{g}}$ and for this representative each $Z_t \in \tilde{\mathfrak{g}}$ is time independent. This seems to be the best one can do. In any case uniqueness has been restored to the minimal distortion shift vector fields in spatially homogeneous cosmology, although in the non-abelian nonsemisimple case, the choice that accomplishes this is somewhat arbitrary ([24]).

IV. ADDITIONAL SYMMETRY : THE NONGENERIC CASE

The complete group $G^C \subset \mathscr{D}(G)$ of isometries of a left invariant Riemannian manifold $(G, \mathbf{g})$ may be larger than L(G). Since left translation is a simply transitive action of G on itself, the quotient group $G^C/L(G)$ is isomorphic to the isotropy group at any point of G of the action of G^C on G. This in turn is isomorphic to the linear isotropy group at any point of G. Since the matrix representation of the latter group with respect to an orthonormal basis is a subgroup of SO(3,R), the dimension of $G^C/L(G)$ must equal the dimension of one of these subgroups, namely 0, 1 or 3. The

([24]) Bianchi type II is the sole exception to these statements.

same dimensional restrictions hold for any subgroup of G^C containing L(G).

The semi-direct product group $L(G) \times I_{\mathbf{g}} = G^C \cap \mathscr{D}(\mathfrak{g})$ is the isometry subgroup of (G, **g**) within $\mathscr{D}(\mathfrak{g})$. Since $\ker \text{Kill} \cap \mathfrak{aut}(G) \cong \mathfrak{i}_{\mathbf{g}}$, the function $r = \dim (\ker \text{Kill} \cap \mathfrak{aut}(G))$ on $\mathscr{M}(\mathfrak{g})$ can at most assume the values 0, 1 or 3; its value at **g** is just the dimension of the isotropy group $I_{\mathbf{g}}$ of the action of Aut(g) on $\mathscr{M}(\mathfrak{g})$. Let $r^{-1}(r_0) \subset \mathscr{M}(\mathfrak{g})$ be the submanifold on which r assumes the value r_0. When r assumes more than one value on $\mathscr{M}(\mathfrak{g})$, it stratifies the space [*17*]. Each stratum $r^{-1}(r_0)$ consists of all left invariant metrics which have an r_0-dimensional isometry subgroup of Aut(G); the isotropy groups at all points of $r^{-1}(r_0)$ are conjugate subgroups of Aut(g). Table 2 lists the values which r actually assumes for each Bianchi type [*1*]:

TABLE 2.

	0	1	3
IX	X	X	X
VIII	X	X	
VII_0, $VII_{h \neq 0}$	X	X	
VI_0, $VI_{h \neq 0}$, IV	X		
V		X	
II		X	
I			X

r assumes more than one value only for Bianchi types VII_0, $VII_{h \neq 0}$, VIII and IX. Let r_{max} and r_{min} be the maximum and minimum values of r for each Bianchi type.

In the type V case there are always two additional linearly independent Killing vector fields not belonging to $\mathfrak{X}(\mathfrak{g})$ since (G, **g**) is a Riemannian manifold of constant negative curvature for every $\mathbf{g} \in \mathscr{M}(\mathfrak{g})$ and G^C is always 6-dimensional. The same is true for Bianchi types $VII_{h \neq 0}$ and VII_0 for $\mathbf{g} \in r^{-1}(1)$ except that (G, **g**) is flat in the latter case rather than having constant negative curvature as in the former case. In the type $VI_{-1} = III$ case, G^C is always 4-dimensional so there is always a single linearly independent Killing vector field not belonging to $\mathfrak{X}(\mathfrak{g})$. For the remaining types G^C is always contained in $\mathscr{D}(\mathfrak{g})$.

The additional spatial symmetries which are also spacetime symmetries are associated with the function $s = \dim (\ker \text{Kill} \cap \ker J \cap \mathfrak{aut}(G))$ on $\mathscr{M}(\mathfrak{g}) \times S_2(\mathfrak{g})$. Let $s^{-1}(s_0) \subset \mathscr{M}(\mathfrak{g}) \times S_2(\mathfrak{g})$ contain those (**g**, K) for which s assumes the value s_0. For a given Bianchi type, s assumes all the values 0, 1, 3 such that $s \leq r_{max}$. Any solutions of the super-Hamiltonian and supermomentum constraints in $s^{-1}(s_0)$ will generate a spacetime having an s_0-dimensional subgroup of Aut(G) as an isometry subgroup; these are all conjugate subgroups of Aut(G). For Bianchi types V, $\mathrm{VII}_{h\neq 0}$ and VII_0, the additional Killing fields not in $\mathfrak{X}(\mathfrak{g})$ are spacetime Killing vector fields only for Friedmann initial data. The single such additional Killing vector field in the type III case is a spacetime Killing vector field only for initial data corresponding to the negative curvature analogues of the Kantowski-Sachs spacetimes [*1*].

The term « generic » has been used to refer to $r^{-1}(r_{\min}) \subset \mathcal{M}(\mathfrak{g})$ and $s^{-1}(0) \subset \mathcal{M}(\mathfrak{g}) \times S_2(\mathfrak{g})$ depending on the context. The first is relevant to purely spatial matters as in table 1, while the second is relevant to spacetime matters as in shift vector field considerations. Except for Bianchi types $\mathrm{VII}_{h\neq 0}$, VII_0, VIII and IX, the values given in table 1 always hold. For type $\mathrm{VII}_{h\neq 0}$, only the kinematical values change, namely those values involving L, and the change in value to within an obvious sign is $r - r_{\min}$. For types VII_0, VIII and IX, all of the values in table 1 change by this amount.

For these three Bianchi types, the dimension of the distribution ker δ increases by the amount $r_0 - r_{\min}$ on the submanifold $r^{-1}(r_0)$ with $r_0 > r_{\min}$, i. e. it has singularities on the nongeneric subspace of $\mathcal{M}(\mathfrak{g})$. Viewed as a subspace of $\mathcal{M}(\mathfrak{g}) \times S_2(\mathfrak{g})$, ker δ is the solution space of the vacuum supermomentum constraint. The singularities in the distribution ker δ correspond exactly to the bifurcation of this constraint space which occurs each time the function s increases in value on ker δ. This bifurcation is described for the general spatially compact case by Fischer and Marsden [*18*]. Recall that $(\ker \delta)^{\perp}$ is tangent to the orbits of the group $\mathrm{Dad}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ generated by the Lie algebra $\mathrm{dad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. The dimension of ker δ will therefore increase when these orbits decrease in dimension. This only occurs for the three Bianchi types VII_0, VIII and IX where $\mathrm{Dad}(\mathfrak{g}) = \mathrm{IAut}(\mathfrak{g})$. These are the only types for which $\mathrm{Dad}(\mathfrak{g})$ is compact or has a compact subgroup and the bifurcations occur when the orbits of the compact subgroups decrease in dimension.

True minimal distortion shift vector fields were discussed in section III for the generic case where s assumes the value 0 on a solution curve $(\mathbf{g}_t, \mathrm{K}_t)$ of the Einstein equations. These vector fields were fixed modulo the space $\ker \mathrm{Kill} \cap \ker \mathrm{J} \cap \mathfrak{X}(\mathfrak{g})$ which generically equals $\tilde{\mathfrak{g}}$ and then intersected with a Lie subalgebra $\mathfrak{X}(\mathfrak{g})^{\mathrm{RED}} \cong \mathfrak{aut}(\mathrm{G})$ to provide a unique representative (25). When s has a value $s_0 > 0$, the degeneracy space

$$\ker \mathrm{Kill} \cap \ker \mathrm{J} \cap \mathfrak{aut}(\mathrm{G}) \cong \ker \mathrm{Kill} \cap \ker \mathrm{J} \cap \mathfrak{X}(\mathfrak{g})^{\mathrm{RED}}$$

becomes nontrivial and the true minimal distortion shift representative in $\mathfrak{X}(\mathfrak{g})^{\mathrm{RED}}$ loses its uniqueness.

(25) This scheme failed for Bianchi type II.

ACKNOWLEDGMENT

The author wishes to thank C. W. Misner for an instructive comment which led to the present investigation and J. W. York, Jr. for providing the opportunity to pursue it.

REFERENCES

[1] R. T. JANTZEN, *Commun. math. Phys.*, t. **64**, 1979, p. 211-232.
[2] L. SMARR, J. W. YORK Jr., *Phys. Rev.*, t. **D 17**, 1978, p. 2529-2549 ; *Phys. Rev.*, t. **D 17**, 1978, p. 1945-1956.
[3] J. W. YORK, Jr., N. O'MURCHADHA, *Bull. Amer. Phys. Soc.*, t. **19**, 1974, p. 509 ; J. W. YORK Jr., L. SMARR, *Bull. Amer. Phys. Soc.*, t. **20**, 1975, p. 544.
[4] J. W. YORK Jr., *J. Math. Phys.* t. **14**, 1973, p. 456-464.
[5] J. W. YORK Jr., *Ann. Inst. Henri Poincaré*, t. **21**, 1974, p. 319-332.
[6] J. W. YORK Jr., *Kinematics and dynamics of general relativity*. In: *Sources of gravitational radiation* (ed. L. Smarr). Cambridge University Press, 1979.
[7] A. E. FISCHER, J. E. MARSDEN, *Can. J. Math.*, t. **29**, 1977, p. 193-209.
[8] M. CANTOR, *Comp. Math.*, t. **38**, 1979, p. 3-35.
[9] A. FISCHER, J. E. MARSDEN, *Topics in the dynamics of general relativity*. In: *Proceedings of the Enrico Fermi Summer School of the Italian Physical Society* (ed. J. Ehlers), Varenna *(to appear)*.
[10] J. W. YORK Jr., *Phys. Rev. Lett.*, t. **28**, 1972, p. 1082-1085.
[11] M. BERGER, D. EBIN, *J. Diff. Geom.*, t. **3**, 1969, p. 379-392.
[12] F. W. WARNER, *Foundations of differentiable manifolds and Lie groups*. Glenview, Ill., Scott, Foresman and Co., 1971.
[13] D. M. EARDLEY, *Commun. math. Phys.*, t. **37**, 1974, p. 287-309.
[14] N. O'MURCHADHA, J. W. YORK Jr., *J. Math. Phys.*, t. **14**, 1973, p. 1551-1557.
Y. CHOQUET-BRUHAT, J. W. YORK Jr., *The Cauchy problem*. In: *Einstein Centenary Volume* (eds. P. Bergman, J. Goldberg, A. Held). Plenum Press, 1979.
A. LICHNEROWICZ, *J. Maths pures et appliquées*, t. **23**, 1944, p. 37-63.
Y. CHOQUET-BRUHAT, *C. R. Acad. Sci.*, t. **274**, 1972, p. 682-684 and *Acta di Matematica*, t. **XII**, 1973, p. 317-325.
A. CHALJUB-SIMON, Y. CHOQUET-BRUHAT, Solution of the Lichnerowicz equation on an asymptotically euclidean manifold. *G. R. G. Journal*, to appear.
[15] C. W. MISNER, *Phys. Rev. Lett.*, t. **22**, 1969, p. 1071-1074.
M. P. RYAN Jr., L. C. SHEPLEY, *Homogeneous Relativistic Cosmologies*. Princeton University Press, 1975.
[16] V. MONCRIEF, *J. Math. Phys.*, t. **16**, 1975, p. 493-498.
[17] A. E. FISCHER, *The Theory of Superspace*. In: *Relativity* (eds. M. Carmeli, S. I. Fickler, L. Witten). New York, Plenum Press, 1970.
[18] A. E. FISCHER, J. E. MARSDEN, *Symmetry breaking in general relativity* (to appear).

(Manuscrit reçu le 6 août 1979)