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Analytic scattering theory of quantum mechanical three-body systems

by

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ABSTRACT. — We consider a three-body Schrödinger operator $H = H_0 + V$ in $L^2(R^{2n})$, where $V = \sum_\alpha V_\alpha$, and each V_α is a dilation-analytic two-body interaction decreasing faster than $r^{-\beta}$, where $\beta > 1$ for negative energies and $\beta > 2$ for positive energies. Together with H we consider the associated self-adjoint analytic family of operator given in momentum space by $H(z) = z^2 H_0 + V(z)$, $|\operatorname{Arg} z| < a$.

We develop the stationary scattering theory for the pair $(H_0(e^{i\varphi}), H(e^{i\varphi}))$ for each $\varphi \in (-a, a)$ and each threshold λ of the System. The local inverse wave operators are constructed and asymptotic completeness proved. The full S-matrix $\mathcal{S}(\mu)$ and for $\varphi \neq 0$ the channel S-matrices

$$\mathcal{S}_\lambda(\varphi, \rho^2/2n_\alpha) = \mathcal{S}_\lambda(\rho e^{i\varphi})$$

are expressed in terms of boundary values of the resolvent. It is proved that for each λ the function $\mathcal{S}_\lambda(\rho e^{i\varphi})$ is an analytic continuation into the lower half-plane of the diagonal element $[\mathcal{S}_{\lambda,\lambda}^{-1}(\lambda + \rho^2/2n_\alpha)]^*$ with poles at most at resolvent resonances and, under some reasonable assumptions, precisely at these resonances.

INTRODUCTION

During recent years there has been a significant development in the mathematical theory of the quantum-mechanical three-body problem.

Based on symmetrized versions of Faddeev's equations, Ginibre and Moulin [*14*], Thomas [*34*], Howland [*18*], Kato [*24*] and Yajima [*38*] through various approaches using Hilbert space techniques greatly simplified the original work of Faddeev [*13*]. The assumptions on the potentials were more explicit and somewhat weaker, but essentially with the same $r^{-2-\varepsilon}$ decay. Mourre [*29*] obtained certain generalizations to potentials decaying as $r^{-1-\varepsilon}$. Hagedorn [*15*] has treated the four-body problem, using a modified version of Yakubovski's equations. The general n-body problem has been discussed first by Hepp [*17*], and recently Sigal [*31*] has treated this problem, using Berezin's equations.

On the other hand, the dilation-analytic theory of many-body Schrödinger operators ([*3*], [*5*], [*30*], [*35*]) suggests the possibility of getting a deeper insight into the analytic structure of the wave- and scattering operators. Works of Hagedorn [*16*], Sigal [*32*] and van Winter [*37*] are contributions in this direction. A dilation-analytic scattering theory for the two-body problem was developed in [*8*], where an analytic continuation of the S-matrix with poles at resonances was obtained. A more detailed exposition of the results on the three-body problem contained in this paper has been given in [*10*]. The results have been extended to many-body systems below the smallest three-body threshold, using the Weinberg-van Winter equation [*11*].

The present work is aimed at understanding the analytic structure of the three-body problem, which gets its clearest expression in the analytic continuation properties of certain diagonal elements of the S-matrix. This is achieved through a combination of the abstract stationary theory developed by Howland, Kato and Yajima with the dilation-analytic theory. Yajima's approach, utilizing Kuroda's spectral trace formalism, is particularly suited for this purpose, since there is a simple connection between the trace and dilation operators. This leads to an explicit expression for the analytically continued diagonal elements of the S-matrix (Theorem 7.4).

We work with two classes of dilation-analytic potentials. For scattering at negative energies we allow not necessarily local potentials decaying faster than $r^{-1-\varepsilon}$ and for scattering at positive energies local potentials decaying faster than $r^{-2-\varepsilon}$.

It is convenient in the present context to work in momentum representation. We consider the analytic family of operators H(z) introduced in [*5*], where $z = \rho e^{i\varphi}$ is a complex dilation parameter varying in an angle $\mathcal{O} = \{ z = \rho e^{i\varphi} \mid \rho > 0, -a < \varphi < a \}$. As shown in [*5*], the essential spectrum of H(z) consists of a set of « cuts » $\{ \lambda_\alpha + e^{2i\varphi}\mathrm{R}^+ \}$ starting at thresholds λ_α, and the discrete spectrum consists of eigenvalues of H and non-real eigenvalues, called resonances.

The basic resolvent equations described in section 2 are extensions of the equations of [*38*] to the dilated operators, allowing for the possibility of an infinite number of thresholds. A central feature is the construction of an analytic, operator-valued function $A(z, \zeta)$ with compact square and singular points at discrete eigenvalues of $H(z)$.

We proceed to establish a limiting absorption principle for the operator $H(z)$ for each fixed z in $\mathcal{O}$. For this purpose we introduce different topologies for negative and positive energy and treat separately these two cases in sections 3 and 4. In this connection we identify the singular points of the boundary values $A_+(z, \lambda)$ for $\varphi < 0$ with resonances and for $\varphi = 0$ with eigenvalues of H. The basic analyticity properties of the boundary values of the resolvent are derived as well as the connection between these operators for $\varphi \neq 0$ and $\varphi = 0$.

After giving some facts on trace operators in section 5, we develop in section 6 the scattering theory of the dilated operators $H(e^{i\varphi})$ as well as that of H itself. For $\varphi \neq 0$ this involves the construction of a local spectral measure for $H(e^{i\varphi})$. The local inverse wave operators $F_{\lambda_\alpha \pm}(\varphi, \Delta)$ are then defined in terms of the boundary values of the resolvent for bounded Borel subsets of the cut $\{ \lambda_\alpha + e^{2i\varphi}\mathbf{R}^+ \}$ associated with each threshold λ_α, and their basic properties are established. For $\varphi = 0$ we construct the inverse wave operators and prove asymptotic completeness (Theorem 6.9). Without the analyticity assumptions this yields a generalisation of results of [*38*], however in this generality the singular points are only known to lie in a closed set of measure 0.

In section 7 we discuss the scattering matrix and its analyticity properties. For $\varphi = 0$ the S-matrix is a unitary matrix of operators $\mathcal{S}_{\lambda_\beta,\lambda_\gamma}(E)$, where λ_β, λ_γ vary over all thresholds below E. For $\varphi \neq 0$ we define for each λ_α the S-matrix $\mathcal{S}_{\lambda_\alpha}(\varphi, \mu)$ associated with the scattering operators $F_{\lambda_\alpha +}(\varphi, \Delta)F_{\lambda_\alpha -}^{-1}(\varphi, \Delta)$. It is proved in Theorem 7.4, that the function $\mathcal{S}_{\lambda_\alpha}(\rho e^{i\varphi}) = \mathcal{S}_{\lambda_\alpha}(\varphi, \rho^2/2n_\alpha)$ is a meromorphic function of $z = \rho e^{i\varphi}$ in $\mathcal{O}^-$ with poles at most at resonances, and that $\mathcal{S}_{\lambda_\alpha}^{-1*}(\rho e^{i\varphi})$ has as its boundary value for $\varphi \to 0_+$ the diagonal element $[\mathcal{S}(\lambda_\alpha + \rho^2/2n_\alpha)]_{(\lambda_\alpha,\lambda_\alpha)}$ of the S-matrix, provided $\lambda_\alpha + \rho^2/2n_\alpha$ is smaller than the next threshold. A similar result holds for the S-matrix $\mathcal{S}_0(\rho e^{i\varphi})$ associated with the 0-channel.

We finally investigate the question, whether every resolvent resonance is a pole of the analytically continued diagonal elements of the S-matrix. For the lowest threshold the answer is positive, as for the two-body problem, and there are no embedded eigenvalues on the corresponding cut. For the higher thresholds the problem is complicated by the possibility of embedded eigenvalues of the dilated operators, and the possibility that the S-matrix is regular at a resolvent resonance cannot be ruled out. However, if a resonance appears only on one side of the cut, in which case it does not turn into an embedded eigenvalue, then it is a pole of the S-matrix (Theorems 7.7, 7.11). There is also the possibility of a degenerate resolvent resonance with part of the null space corresponding to an embedded eigenvalue (and the resonance appearing on the other side of the cut) and part of the null space giving rise to a pole of the S-matrix. This situation is dealt with in Theorems 7.9 and 7.11.

Throughout this work we have assumed that non-zero thresholds are simple eigenvalues of two-body subsystems. In [*10*] we have indicated the extension to the important case when they are degenerate, as it occurs when a potential is rotation-symmetric or when two particles are identical (cf. [*6*]).

1. DEFINITIONS, ASSUMPTIONS AND BASIC RESULTS

We consider a system of three particles, denoted by 1, 2, 3, in n-dimensional space R^n. The mass, position and momentum of particle i are denoted by m_i, x_i and k_i. The pairs (ij) are denoted by α, β, etc. We use the notation $R^+ = (0, \infty)$, $\bar{R}^+ = [0, \infty)$.

For any permutation (ijk) of $(1\ 2\ 3)$ with $\alpha = (ij)$ we set

$$
\begin{aligned}
m_\alpha^{-1} &= m_i^{-1} + m_j^{-1}, \qquad n_\alpha^{-1} = (m_i + m_j)^{-1} + m_k^{-1} \\
x_\alpha &= x_j - x_i, \qquad y_\alpha = x_k - (m_i + m_j)^{-1}(m_i x_i + m_j x_j) \\
k_\alpha &= (m_i + m_j)^{-1}(m_j k_i - m_i k_j) \\
p_\alpha &= (m_i + m_j + m_k)^{-1}((m_i + m_j)k_k - m_k(k_i + k_j)) .
\end{aligned}
$$

The free Hamiltonian in the center-of-mass frame is given for any α in momentum representation by

$$H_0 = (2m_\alpha)^{-1}k_\alpha^2 + (2n_\alpha)^{-1}p_\alpha^2$$

and in position representation by the substitution $k_\alpha \to -i\hbar\nabla_{x_\alpha}$, $p_\alpha \to -i\hbar\nabla_{y_\alpha}$.

We set

$$R_0(z) = (H_0 - z)^{-1} \qquad \text{for} \qquad z \notin \bar{R}^+ .$$

We shall work mainly in momentum representation and only occasionally in position representation. Any definition or statement containing the index α is meant to hold for $\alpha = (12), (23), (31)$. We distinguish the spaces of vectors x_α, y_α, etc. by the notation $R^n_{x_\alpha}$, $R^n_{y_\alpha}$, etc. The basic Hilbert space $L^2(R^{2n})$ is denoted by $\mathscr{H}$. The weighted Sobolev spaces with weight δ and differentiability parameter s is denoted by $H^{s,\delta}(R^{2n})$, etc., and $H^s = H^{s,0}$.

For a discussion of the basic properties of Sobolev spaces we refer to [*26*].

For any pair of Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ we denote by $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ the space of all bounded linear operators from $\mathscr{H}_1$ into $\mathscr{H}_2$ and by $\mathscr{C}(\mathscr{H}_1, \mathscr{H}_2)$ the subspace of all compact operators.

We denote by S^{m-1} the unit sphere in R^m and identify $L^2(R^m)$ with $L^2(R^+, L^2(S^{m-1}); \rho^{m-1})$, writing $f(\rho, \cdot)$ for $f \in L^2(R^m)$.

For $f \in C_0^\infty(R^m)$ we set $\gamma(\rho)f = f(\rho, \cdot)$.

The dilation operator $U(\rho)$ is defined for $\rho \in R^+$, $s, \delta \in R$, on $H^{s,\delta}(R^{2n})$ by

$$(U(\rho)f)(p) = \rho^n f(\rho p).$$

Considered as operators on $\mathscr{H}$ the $U(\rho)$ form a unitary representation of $(R^+, \cdot)$ on $\mathscr{H}$. Similarly we define the operators $U_\alpha(\rho)$ on $H^{s,\delta}(R^n_{p_\alpha})$ by

$$(U_\alpha(\rho)f)(p_\alpha) = \rho^{n/2} f(\rho p_\alpha).$$

For any operator A in $\mathscr{H}$ we set

$$A(\rho) = U(\rho)AU^{-1}(\rho), \qquad \rho \in R^+.$$

The operators $U(\rho)$ and $\gamma(\rho)$ are connected by

$$\gamma(\rho) = \rho^{-n}\gamma(1)U(\rho). \tag{1.1}$$

For $0 < a < \dfrac{\pi}{2}$, let $\mathcal{O} = \mathcal{O}_a$ denote the angular region

$$\mathcal{O}_a = \{ z = \rho e^{i\varphi} \mid \rho \in R^+, \ -a < \varphi < a \}.$$

We use the notation

$$(\langle \phi \mid f)(y_\alpha) = \int_{R^n_{x_\alpha}} f(x_\alpha, y_\alpha)\phi(x_\alpha)dx_\alpha,$$

whenever the right hand side is defined, and

$$(\mid \phi \rangle g)(x_\alpha, y_\alpha) = \phi(x_\alpha)g(y_\alpha).$$

Assumptions on the interactions

We shall consider the following two sets of conditions on the interactions V_α.

A I *i*). There exists $s > \frac{1}{2}$ such that

$$V_\alpha = (1 - \Delta_{k_\alpha})^{-s/2} W_\alpha (1 - \Delta_{k_\alpha})^{-s/2},$$

where W_α is a symmetric operator in $L^2(R^n_{k_\alpha})$ with the property

$$W_\alpha \in \mathscr{C}(H^{0,2}(R^n_{k_\alpha}), L^2(R^n_{k_\alpha})) \cap \mathscr{C}(H^{s,2}(R^n_{k_\alpha}), H^s(R^n_{k_\alpha})).$$

A I *ii*). The function $W_\alpha(\rho)$ has an analytic extension to $\mathcal{O}$, considered as a function with values in $\mathscr{C}(H^{0,2}(R^n_{k_\alpha}), L^2(R^n_{k_\alpha}))$ as well as in $\mathscr{C}(H^{s,2}(R^n_{k_\alpha}), H^s(R^n_{k_\alpha}))$.

A II *i*). The interaction V_α is convolution with a function $v_\alpha(k_\alpha)$, whose inverse Fourier transform $\mathscr{F}^{-1}v_\alpha$ is a real valued function on $R^n_{x_\alpha}$ such that for some $s > 1$

$$\mathscr{F}^{-1}v_\alpha = (1 + x_\alpha^2)^{-s}(u_{\alpha_1} + w_{\alpha_1})(u_{\alpha_2} + w_{\alpha_2}),$$

where

$$u_{\alpha i} \in L^\infty(R^n{}_{x_\alpha}), \qquad w_{\alpha i} \in L^p(R^n_{x_\alpha}), \qquad i = 1, 2,$$

for some $p > n$ if $n \geq 4$ and for $p = 4$ if $n = 3$.

A II *ii*). The $L^\infty(R^n_{x_\alpha})$—valued and $L^p(R^n_{x_\alpha})$— valued functions $u_{\alpha i}(\rho)$ and $w_{\alpha i}(\rho)$ have analytic extensions $u_{\alpha i}(z)$ and $w_{\alpha i}(z)$ to $\mathcal{O}$ for $i = 1, 2$.

REMARK. — It is well known that for $\varepsilon > 0$ the function

$$(1 + x_\alpha^2)^{-\varepsilon}(u_{\alpha 1} + w_{\alpha 1})(u_{\alpha 2} + w_{\alpha 2}),$$

where $u_{\alpha i}$, $w_{\alpha i}$ satisfy the conditions of A II *i*), defines a multiplication operator in $\mathscr{C}(H^2(R^n_{x_\alpha}), L^2(R^n_{x_\alpha}))$ and hence in $\mathscr{C}(H^{2,s}(R^n_{x_\alpha}), H^{0,s}(R^n_{x_\alpha}))$, so A II *i*) implies A I *i*) and A II *ii*) implies A I *ii*).

We shall make use of the following factorizations of $V_\alpha(z)$:

$$V_\alpha(z) = A_\alpha(z)B_\alpha(z),$$

where in case I

$$A_\alpha(z) = (1 - z^2\Delta_{k_\alpha})^{-s/2}, \qquad B_\alpha(z) = W_\alpha(z)(1 - z^2\Delta_{k_\alpha})^{-s/2}$$

and in case II

$$A_\alpha(z) = (1 - z^2\Delta_{k_\alpha})^{-s/2}(\hat{U}_{\alpha 1}(z) + \hat{W}_{\alpha 1}(z)),$$
$$B_\alpha(z) = (1 - z^2\Delta_{k_\alpha})^{-s/2}(\hat{U}_{\alpha 2}(z) + \hat{W}_{\alpha 2}(z)),$$

where $\hat{U}_{\alpha i}(z)$ and $\hat{W}_{\alpha i}(z)$ are the operators of convolution by $\mathscr{F}u_{\alpha i}(z)$ and $\mathscr{F}w_{\alpha i}(z)$, $i = 1, 2$.

The two-body case

The free Hamiltonian $h_{0\alpha}$ is the self-adjoint operator in $L^2(R^n_{k_\alpha})$ defined by

$$h_{0\alpha} = k_\alpha^2/2m_\alpha, \qquad \mathscr{D}(h_{0\alpha}) = H^{0,2}(R^n_{x_\alpha}).$$

The assumption A I *i*) implies that V_α is $h_{0\alpha}$-ε-bounded, and hence the total Hamiltonian h_α defined by

$$h_\alpha = h_{0\alpha} + V_\alpha, \qquad \mathscr{D}(h_\alpha) = H^{0,2}(R^n_{x_\alpha})$$

is self-adjoint by the Rellich-Kato criterion, see [*23*].

Moreover, by A I *ii*) a self-adjoint analytic family $h_\alpha(z)$ is defined for $z \in \mathcal{O}$ by

$$h_\alpha(z) = z^2 h_{0\alpha} + V_\alpha(z).$$

For the basic results on the operators $h_\alpha(z)$ we refer to [*3*], [*5*], [*8*] and [*12*].

We shall make the following simplifying assumption in both case I and II:

A I *iii*) and A II *iii*). For $z \in \mathcal{O}$ all discrete eigenvalues of $h_\alpha(z)$ are simple, and $h_\alpha(z)$ and $h_\beta(z)$ have no common eigenvalue for $\alpha \neq \beta$.

In case II we shall need some further assumptions. Let

$$q_\alpha(\lambda) = B_\alpha(h_{\alpha 0} - \lambda)^{-1}A_\alpha, \qquad \lambda \notin \overline{R^+}$$

By a result of Kato [*22*], under assumption II *i*) the $\mathscr{C}(L^2(R^n_{k_\alpha}))$-valued function $q_\alpha(\lambda)$ has a limit $q_\alpha(0)$ for $\lambda \to 0$.

We now make the assumptions

A II *iv*) $\qquad \mathscr{N}(q_\alpha(0)) = 0$

A II v) $\qquad h_\alpha$ has no positive eigenvalues.

Condition A II iv) is satisfied « generically », and Condition A II v) is satisfied under weak regularity conditions on the potential, see [*1*].

2. BASIC EQUATIONS FOR THE THREE-BODY PROBLEM

It follows from the assumption A I i), that $V = \sum_\alpha V_\alpha$ is H_0-ε-bounded, and by the Rellich-Kato criterion the total Hamiltonian

$$H = H_0 + V$$

is self-adjoint on $\mathscr{D}(H) = \mathscr{D}(H_0) = H^{0,2}(R^{2n})$.

Moreover, by assumption A I ii) a self-adjoint analytic family $H(z)$ is defined for $z \in \mathcal{O}$ by

$$H(z) = z^2 H_0 + \sum_\alpha V_\alpha(z).$$

For the spectral properties of $H(z)$ we refer to [*5*]. We only recall here that the non-real discrete spectrum of $H(z)$, denoted by $\sigma_d^c(\varphi)$, is z-independent unless « absorbed » by the essential spectrum $\sigma_e(\varphi)$ which consists of a set of parallel half-lines with directions $e^{2i\varphi}$, starting from thresholds, i. e. points of $\bigcup_\alpha \sigma_d(h_\alpha(z)) \cup \{ 0 \}$.

DÉFINITION 2.1. — For $\lambda_\alpha^i \in \sigma_d(h_\alpha)$ we define the set $\mathscr{R}_\alpha^i$ of three-body resonances $\mathscr{R}_\alpha^i$ related to the threshold λ_α^i by

$$\mathscr{R}_\alpha^i = \bigcup_{-a<\varphi<0} \{ \lambda \in \sigma_d^c(\varphi) \mid \lambda \text{ is between } \lambda_\alpha^i + e^{2i\varphi}R^+ \text{ and } \lambda_\alpha^{i'} + e^{2i\varphi}R^+ \}$$

where $\lambda_\alpha^{i'} = \min_\delta \min_j \{ \lambda_\delta^j > \lambda_\alpha^i \}$.

The conjugate set $\overline{\mathscr{R}_\alpha^i}$ is given by the same expression with $-a < \varphi < 0$ replaced by $0 < \varphi < a$.

The set $\mathscr{R}_{\lambda_\alpha}$ of three-body resonances related to the « resonance threshold » λ_α is defined in a similar way.

Similarly we define $\mathscr{R}_\alpha^{i'}$ and $\mathscr{R}'_{\lambda_\alpha}$ as the union over φ of the set of resonances lying on the negative side of $\lambda_\alpha^i + e^{2i\varphi}R^+$ and $\lambda_\alpha + e^{2i\varphi}R^+$ respectively.

The set $\mathscr{R}_0$ of resonances related to the free channel is defined by

$$\mathscr{R}_0 = \bigcup_{-a<\varphi<0} \{ \lambda \in \sigma_d^c(\varphi) \mid \lambda \text{ is between } R^+ \text{ and } e^{2i\varphi}R^+ \}$$

We shall now describe a decomposition formula for the resolvent

$R(z, \zeta) = (H(z) - \zeta)^{-1}$ extending the procedure of Yajima [38] to the present situation. We set

$$R_\alpha(z, \zeta) = (H_\alpha(z) - \zeta)^{-1}, \qquad \zeta \in \rho(H_\alpha(z)),$$

and decompose $R_\alpha(z, \zeta)$ as follows.

Let $E \in \bigcup_\alpha \sigma_d(h_\alpha)$ be fixed, and let

$$\lambda_\alpha^1 < \lambda_\alpha^2 < \ldots < \lambda_\alpha^{n_\alpha} < E$$

be the eigenvalues of h_α below E with the corresponding eigenfunctions of $h_\alpha(z)$

$$\phi_\alpha^1(z), \phi_\alpha^2(z), \ldots, \phi_\alpha^{n_\alpha}(z)$$

chosen in accordance with a result of [3] such that $\phi_\alpha^i(z)$ is analytic in $\mathcal{C}$. and such that

$$\| \phi_\alpha^i(1) \| = 1, \qquad i = 1, \ldots, n_\alpha.$$

We use the shorthand notation

$$\sum_{\alpha,i} = \sum_\alpha \sum_{i=1}^{n_\alpha}$$

and for a quantity X depending on $\lambda_\alpha = \lambda_\alpha^i$ we write $X_\alpha^i = X_{\lambda_\alpha}$. If $\sigma_d(h_\alpha(z))$ is finite for all α, in particular in case II also $E = 0$ is allowed, in which case we let $\{ \lambda_\alpha^i \}_{i=1}^{n_\alpha} = \sigma_d(h_\alpha(z))$. We choose moreover $\phi_\alpha^i(z)$ as above for $\lambda_\alpha^i \in \sigma_d^r(\varphi)$, $\phi_\alpha^i(z)$ analytic in $\{ z \in \mathcal{O} \mid 2\varphi \lessgtr \operatorname{Arg} \lambda_\alpha^i \}$ for $\lambda_\alpha^i \in \sigma_d^c(\varphi)$ and $\operatorname{Arg} \lambda_\alpha^i \lessgtr 0$. Let

$$P_\alpha^i(z) = | \phi_\alpha^i(z) \rangle \langle \phi_\alpha^i(\bar{z}) |, \qquad i = 1, \ldots, n_\alpha,$$

$$P_\alpha^E(z) = \sum_{i=1}^{n_\alpha} P_\alpha^i(z)$$

$$R_\alpha^E(z, \zeta) = R_\alpha(z, \zeta)(1 - P_\alpha^E(z))$$

$$h_{\alpha 0}^i(z) = \lambda_\alpha^i + \frac{z^2 p_\alpha^2}{2n_\alpha}$$

$$r_\alpha^i(z, \zeta) = (h_{\alpha 0}^i(z) - \zeta)^{-1}, \qquad \zeta \in C \setminus \{ \lambda_\alpha^i + e^{2i\varphi} R^+ \}.$$

Then

$$R_\alpha(z, \zeta) = R_\alpha^E(z, \zeta) + \sum_{\alpha,i} | \phi_\alpha^i(z) \rangle r_\alpha^i(z, \zeta) \langle \phi_\alpha^i(\bar{z}) |.$$

Let $Q_{\alpha\beta}(z, \zeta)$ be the $(n_\alpha + 1) \times (n_\beta + 1)$ matrix of operators defined by

$$Q_{\alpha\beta}(z, \zeta) = \begin{Bmatrix} B_\alpha(z) R_\beta^E(z, \zeta) A_\beta(z) & B_\alpha(z) | \phi_\beta^1(z) \rangle & \ldots B_\alpha(z) | \phi_\beta^{n_\beta}(z) \rangle \\ r_\alpha^1(z, \zeta) \langle \phi_\alpha^1(\bar{z}) | V_\alpha(z) R_\beta^E(z, \zeta) A_\beta(z) & r_\alpha^1(z, \zeta) \langle \phi_\alpha^1(\bar{z}) | V_\alpha(z) | \phi_\beta^1(z) \rangle & \ldots r_\alpha^1(z, \zeta) | \phi_\alpha^1(\bar{z}) | V_\alpha(z) | \phi_\beta^{n_\beta}(z) \rangle \\ \vdots & \vdots & \vdots \\ r_\alpha^{n_\alpha}(z, \zeta) \langle \phi^{n_\alpha}(\bar{z}) | V_\alpha(z) R_\beta^E(z, \zeta) A_\beta(z) & r_\alpha^{n_\alpha}(z, \zeta) \langle \phi_\alpha^{n_\alpha}(\bar{z}) | V_\alpha(z) | \phi_\beta^1(z) \rangle & \ldots r_\alpha^{n_\alpha}(z, \zeta) \langle \phi_\alpha^{n_\alpha}(\bar{z}) | V_\alpha(z) | \phi_\beta^{n_\beta}(z) \rangle \end{Bmatrix} \tag{2.1}$$

The matrix of operators $A(z, \zeta)$ is defined for $\zeta \in C \setminus \sigma_e(\varphi)$ by

$$A(z,\zeta) = \begin{Bmatrix} 0 & Q_{\alpha\beta}(z,\zeta) & Q_{\alpha\gamma}(z,\zeta) \\ Q_{\beta\alpha}(z,\zeta) & 0 & Q_{\beta\gamma}(z,\zeta) \\ Q_{\gamma\alpha}(z,\zeta) & Q_{\gamma\beta}(z,\zeta) & 0 \end{Bmatrix} \tag{2.2}$$

Let

$$\tilde{\mathscr{H}}^{s,\delta} = \sum_{\alpha} \oplus \left\{ \mathscr{H} \oplus \sum_{i=1}^{n_\alpha} H^{s,\delta}(R^n_{p_\alpha}) \right\}, \qquad \tilde{\mathscr{H}} = \tilde{\mathscr{H}}^{0,0}$$

The elements of $\tilde{\mathscr{H}}^{s,\delta}$ are denoted by

$$\tilde{u} = \sum_{\alpha} \oplus \left(u_\alpha \oplus \sum_{i=1}^{n_\alpha} \oplus \sigma^i_\alpha \right),$$

and we set

$$u_\alpha = \Theta_{\alpha,0}\tilde{u}, \qquad \sigma^i_\alpha = \Theta_{\alpha,i}\tilde{u}.$$

It follows from Lemma 3.2 below, that

$$A(z,\zeta) \in \mathscr{B}(\tilde{\mathscr{H}}), \qquad A^2(z,\zeta) \in \mathscr{C}(\tilde{\mathscr{H}}). \tag{2.3}$$

Moreover it is easy to see that

$$\| A^2(z,\zeta) \| \to 0, \qquad \text{for} \qquad \operatorname{Re} e^{-2i\varphi}\zeta \to -\infty.$$

Since $A(z, \zeta)$ is analytic for $\zeta \in \rho_e(\varphi)$, it follows that $(1 + A(z, \zeta))^{-1}$ exists for $\zeta \in \rho_e(\varphi)$ except for a discrete set s. The following result shows that $s = \sigma_d(\varphi)$.

LEMMA 2.2. — A point $\lambda \in \rho_e(\varphi)$ is an eigenvalue of $H(z)$ if and only if -1 is an eigenvalue of $A(z, \lambda)$, and the null spaces $\mathscr{N}(H(z) - \lambda)$ and $\mathscr{N}(1 + A(z, \lambda))$ are isomorphic via the maps $K(z, \lambda)$ and its inverse $L(z, \lambda)$ defined by

$$\Psi = K(z,\lambda)\Phi = \sum_{\alpha} - R_\alpha(z,\lambda)A_\alpha(z)u_\alpha \tag{2.4}$$

for

$$\Phi = \sum_{\alpha} \oplus \left\{ u_\alpha \oplus \sum_{i=1}^{n_\alpha} \oplus \sigma^i_\alpha \right\} \in \mathscr{N}(1 + A(z,\lambda));$$

$$\Phi = L(z,\lambda)\Psi = \sum \oplus \{ B_\alpha(z)(1 + R_0(z,\lambda)V_\alpha(z)) \} \Psi$$

$$\oplus \sum_{i=1}^{n_\alpha} \oplus \{ r^i_\alpha(z,\lambda) \langle \phi^i_\alpha(\bar{z}) \mid V_\alpha(z)(1 + R_0(z,\lambda)V_\alpha(z))\Psi \} \tag{2.5}$$

for $\Psi \in \mathscr{N}(H(z) - \lambda)$.

Proof. — This follows by a straightforward algebraic verification, using the *2nd* resolvent equations.

We define $H(z, \zeta) \in \mathscr{B}(\mathscr{H}, \tilde{\mathscr{H}})$ for $\zeta \in \rho_e(\varphi)$ and $K(z, \zeta) \in \mathscr{B}(\mathscr{H}, \tilde{\mathscr{H}})$ for $\zeta \in \rho(\varphi)$ by

$$\Theta_{\alpha,0}H(z, \zeta) = \sum_{\beta \neq \alpha} B_\alpha(z)R_0(z, \zeta)V_\beta(z)R_\beta(z, \zeta) \tag{2.6}$$

$$\Theta_{\alpha,i}H(z, \zeta) = r_\alpha^i(z, \zeta) \langle \phi_\alpha^i(\bar{z}) \,|\, A_\alpha(z)\Theta_{\alpha,0}H(z, \zeta). \tag{2.7}$$

$$K(z, \zeta) = (1 + A(z, \zeta))^{-1}H(z, \zeta). \tag{2.8}$$

Then it is clear from (2.1), that

$$\Theta_{\alpha,i}K(z, \zeta) = r_\alpha^i(z, \zeta) \langle \phi_\alpha^i(\bar{z}) \,|\, A_\alpha(z)\Theta_{\alpha,0}K(z, \zeta) \tag{2.9}$$

The following basic formula is derived in the same way as Theorem 3.2 of Yajima [*38*], see also [*10*] for details.

LEMMA 2.3. — Under the assumptions A I *i)-iii)* we have for $z \in \mathscr{C}$, $\zeta \in \rho(H(z))$, $E \in \bigcup_\alpha \sigma_d(h_\alpha)$,

$$R(z, \zeta) = R_0(z, \zeta)Y_E(z, \zeta) + \sum_{\alpha,i} \{ \,|\, \phi_\alpha^i(z) \rangle r_\alpha^i(z, \zeta)Y_\alpha^i(z, \zeta) \} \tag{2.10}$$

where

$$Y_E(z, \zeta) = -2I + \sum_\alpha \left(1 - \sum_{i=1}^{n_\alpha} P_\alpha^i(z) - V_\alpha(z)R_\alpha^E(z, \zeta)\right)(1 + A_\alpha(z)\Theta_{\alpha,0}K(z, \zeta)) \tag{2.11}$$

$$Y_\alpha^i(z, \zeta) = \langle \phi_\alpha^i(\bar{z}) \,| + \langle \phi_\alpha^i(\bar{z}) \,|\, A_\alpha(z)\Theta_{\alpha,0}K(z, \zeta). \tag{2.12}$$

Under the assumptions A II *i)-iv)* the above decomposition is also valid for $E = 0$, in which case we let

$$\{ \lambda_\alpha^i \}_{i=1}^{n_\alpha} = \sigma_d(h_\alpha(z)).$$

DEFINITION 2.4. — $\mathscr{H}_1^s = H^s(R^{2n}) \oplus \sum_{\alpha,i} \oplus H^s(R_{p_\alpha}^n), \mathscr{H}_1 = \mathscr{H}_1^0$ with the elements

$$\bar{u} = (u_0, \tau_\alpha^1, \ldots, \tau_\alpha^{n_\alpha}, \tau_\beta^1, \ldots, \tau_\beta^{n_\beta}, \tau_\gamma^1, \ldots, \tau_\gamma^{n_\gamma}).$$

$$R_1(z, \zeta) = R_0(z, \zeta) \oplus \sum_{\alpha,i} \oplus r_\alpha^i(z, \zeta) \in \mathscr{B}(\mathscr{H}_1)$$

for

$$\zeta \in \mathrm{C} \setminus \left(\{ e^{2i\varphi}\overline{\mathrm{R}^+} \} \cup \left\{ \bigcup_{\alpha,i} (\lambda_\alpha^i + e^{2i\varphi}\mathrm{R}^+) \right\} \right)$$

$$\mathrm{J}(z)\bar{u} = u_0 + \sum_{\alpha,i} \{ \,|\, \phi_\alpha^i(z) \rangle\, \tau_\alpha^i \} \in \mathscr{B}(\mathscr{H}_1, \mathscr{H})$$

$$\mathrm{W}_\alpha^i(z) = \sum_{\beta \neq \alpha} \mathrm{V}_\beta(z) \,|\, \phi_\alpha^i(z) \rangle \in \mathscr{B}(\mathrm{H}^{0,2}(\mathrm{R}^n_{p_\alpha}), \mathscr{H})$$

$$\mathrm{W}(z)\bar{u} = \mathrm{V}(z)u_0 + \sum_{\alpha,i} \mathrm{W}_\alpha^i(z)\tau_\alpha^i$$

$$\mathrm{G}_\alpha^i(z, \zeta) = |\, \phi_\alpha^i(z) \rangle + \mathrm{W}_\alpha^i(z) r_\alpha^i(z, \zeta) \in \mathscr{B}(\mathrm{L}^2(\mathrm{R}^n_{p_\alpha}), \mathscr{H})$$

for

$$\zeta \in \mathrm{C} \setminus \{ \lambda_\alpha^i + e^{2i\varphi}\overline{\mathrm{R}^+} \}$$

$$\mathrm{G}_0(z, \zeta) = \mathrm{I} + \mathrm{V}(z)\mathrm{R}_0(z, \zeta) \in \mathscr{B}(\mathscr{H}) \quad \text{for} \quad \zeta \in \mathrm{C} \setminus e^{2i\varphi}\overline{\mathrm{R}^+}$$

$$\mathrm{G}(z, \zeta) = \mathrm{G}_0(z, \zeta) + \sum_{\alpha,i} \mathrm{G}_\alpha^i(z, \zeta) \in \mathscr{B}(\mathscr{H}_1, \mathscr{H})$$

for

$$\zeta \in \mathrm{C} \setminus \left(\{ e^{2i\varphi}\overline{\mathrm{R}^+} \} \cup \left\{ \bigcup_{\alpha,i} (\lambda_\alpha^i + e^{2i\varphi}\overline{\mathrm{R}^+}) \right\} \right)$$

$$\mathrm{Y}(z, \zeta)u = \{ \mathrm{Y}_\mathrm{E}(z, \zeta)u, \mathrm{Y}_\alpha^1(z, \zeta)u, \ldots, \mathrm{Y}_\alpha^{n_\alpha}(z, \zeta)u, \mathrm{Y}_\beta^1(z, \zeta)u, \ldots, \mathrm{Y}_\beta^{n_\beta}(z, \zeta)u, \mathrm{Y}_\gamma^1(z, \zeta)u, \ldots, \mathrm{Y}_\gamma^{n_\gamma}(z, \zeta)u \}$$

Thus

$$\mathrm{Y}(z, \zeta) \in \mathscr{B}(\mathscr{H}, \mathscr{H}_1) \quad \text{for} \quad \zeta \in \rho(\mathrm{H}(z)).$$

We have the following identities.

LEMMA 2.5.

$$\mathrm{G}(z, \zeta) = \mathrm{J}(z) + \mathrm{W}(z)\mathrm{R}_1(z, \zeta),$$

$$\zeta \in \mathrm{C} \setminus \left(e^{2i\varphi}\overline{\mathrm{R}^+} \cup \left\{ \bigcup_\alpha \bigcup_{i=1}^{n_\alpha} (\lambda_\alpha^i + e^{2i\varphi}\overline{\mathrm{R}^+}) \right\} \right) \tag{2.13}$$

$$\mathrm{R}(z, \zeta)\mathrm{G}(z, \zeta) = \mathrm{J}(z)\mathrm{R}_1(z, \zeta), \qquad \zeta \in \rho(\mathrm{H}(z)) \tag{2.14}$$

$$\mathrm{R}(z, \zeta) = \mathrm{J}(z)\mathrm{R}_1(z, \zeta)\mathrm{Y}(z, \zeta), \qquad \zeta \in \rho(\mathrm{H}(z)) \tag{2.15}$$

$$\mathrm{G}(z, \zeta)\mathrm{Y}(z, \zeta) = \mathrm{I}, \qquad \zeta \in \rho(\mathrm{H}(z)) \tag{2.16}$$

Proof. — (2.13) holds by definition, and (2.15) is a restatement of Lemma 2.3. A simple verification using the *2'nd* resolvent equation shows (2.14), and (2.16) follows from (2.14) and (2.15).

3. LIMITS ON THE CONTINUOUS SPECTRUM FOR NEGATIVE ENERGIES

In this section we make the assumptions A I *i)-iii)* and investigate in suitable topologies the limits of the operators H(z, ζ), A(z, ζ) and Y(z, ζ) on the part of the continuous spectrum given by

$$\sigma_e(\varphi) \setminus (\{ e^{2i\varphi}\mathrm{R}^+ \} \cup \Sigma(\varphi)) = \bigcup_{\lambda \in \Sigma(\varphi) \setminus \{0\}} \{ \lambda + e^{2i\varphi}\mathrm{R}^+ \} \qquad \text{for} \qquad \varphi \neq 0$$

and

$$(\lambda_e, 0) \setminus \Sigma(0) \qquad \text{for} \qquad \varphi = 0 .$$

The variable $z \in \mathcal{O}$ is written as $z = \rho e^{i\varphi}$. For $\lambda_\alpha \in \sigma_d(h_\alpha(z))$ we set $\lambda = \lambda_\alpha + \mu e^{2i\varphi}$.

The statement

$$\mathrm{X}_\pm(z, \lambda_\alpha + \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{X}(z, \lambda_\alpha + \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{K}_1, \mathscr{K}_2)$$

means that the operator-valued functions X$(z, \lambda_\alpha + \mu e^{2i(\varphi \pm \varepsilon)})$ with values in $\mathscr{B}(\mathscr{K}_1, \mathscr{K}_2)$, where $\mathscr{K}_1$ and $\mathscr{K}_2$ are Hilbert spaces, converges as $\varepsilon \downarrow 0$ in the uniform operator topology of $\mathscr{B}(\mathscr{K}_1, \mathscr{K}_2)$ to limits, which are denoted by $\mathrm{X}_\pm(z, \lambda_\alpha + \mu e^{2i\varphi})$. The convergence is always uniform on compact subsets of the set of real numbers μ for which the limits exist.

The following results are proved in [*10*].

LEMMA 3.1. — For $\frac{1}{2} < s' \leqq s$, all α and $\lambda_\alpha \in \sigma_{d\alpha}(\varphi)$, $\rho, \mu > 0$ for $\varphi \neq 0$ and $\mu \in (0, \lambda^i_{\alpha'} - \lambda^i_\alpha)$ for $\varphi = 0$,

$$\mathrm{H}_\pm(z, \lambda_\alpha + \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{H}(z, \lambda_\alpha + \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathrm{H}^s(\mathrm{R}^{2n}), \tilde{\mathscr{H}}^{-s',2}) \quad (3.1)$$

LEMMA 3.2. — For $\frac{1}{2} < s' \leqq s$, all α and $\lambda_\alpha \in \sigma_{d\alpha}(\varphi)$, $\rho, \mu > 0$ for $\varphi \neq 0$, $\mu \in (\lambda^i_\alpha, \lambda^{i'}_\alpha)$ for $\varphi = 0$,

$$\mathrm{A}_\pm(z, \lambda) = \lim_{\varepsilon \downarrow 0} \mathrm{A}(z, \lambda_\alpha + \mu e^{2i(\varphi \pm \varepsilon)}) = \lim_{\varepsilon \downarrow 0} \mathrm{A}(\rho e^{i(\varphi \mp \varepsilon)}, \lambda) \in \mathscr{B}(\tilde{\mathscr{H}}^{-s,2})$$

Moreover, for $\frac{1}{2} < s' < s$, $\mathrm{A}^2_\pm(z, \lambda) \in \mathscr{C}(\tilde{\mathscr{H}}^{-s',2})$.

The following result is established in [*10*], using the idea of the proof of [*8*], Lemma 2.4.

LEMMA 3.3. — For $\lambda^i_\alpha \in \sigma_d(h_\alpha)$ and $\mu \in \mathrm{R}^+$ for $\varphi \neq 0$, $\mu \in (\lambda^i_\alpha, \lambda^{i'}_\alpha)$ for $\varphi = 0$

$$\mathscr{N}(1 + \mathrm{A}_{\underset{(-)}{+}}(z, \lambda^i_\alpha + \mu e^{2i\varphi})) \neq \{0\}$$

if and only if $\lambda^i_\alpha + \mu e^{2i\varphi} \in \mathscr{R}^i_\alpha \cup \sigma_p(\mathrm{H}) \cup \bar{\mathscr{R}}^{i'}_\alpha(\mathscr{R}^{i'}_\alpha \cup \sigma_p(\mathrm{H}) \cup \bar{\mathscr{R}}^i_\alpha)$. Moreover, for $0 < \varepsilon \leqq \varepsilon_0$

$$\dim \mathscr{N}(1 + \mathrm{A}_{\underset{(-)}{+}}(z, \lambda^i_\alpha + \mu e^{2i\varphi})) = \dim \mathscr{N}(1 + \mathrm{A}(\rho e^{i(\varphi \mp \varepsilon)}, \lambda^i_\alpha + \mu e^{2i\varphi})) .$$

(For $\lambda^i_\alpha = \lambda_e$, $\bar{\mathscr{R}}^{i'}_\alpha = \mathscr{R}^{i'}_\alpha = \emptyset$).

The analogous results hold for $\lambda_\alpha \in \mathscr{R}_\alpha$ with $\lambda^i_\alpha, \mathscr{R}^i_\alpha, \mathscr{R}^{i'}_\alpha$ replaced by $\lambda_\alpha, \mathscr{R}_{\lambda_\alpha}, \mathscr{R}'_{\lambda_\alpha}$.

From Lemmas 3.1-3.3 we obtain

LEMMA 3.4. — Let $\lambda^i_\alpha \in \sigma_d(h_\alpha)$, $\mu \in \mathrm{R}^+$, and assume that $\lambda = \lambda^i_\alpha + \mu e^{2i\varphi}$ satisfies

$$\begin{array}{lll} \lambda \notin \mathscr{R}^i_\alpha(\overline{\mathscr{R}}^i_\alpha) & \text{for} & \varphi \underset{(>)}{<} 0 \\ \lambda \in (\lambda^i_\alpha, \lambda^{i'}_\alpha) \setminus \sigma_p(\mathrm{H}) & \text{for} & \varphi = 0 \\ \lambda \notin \overline{\mathscr{R}}^{i'}_\alpha(\mathscr{R}^{i'}_\alpha) & \text{for} & \varphi \underset{(<)}{>} 0, \quad \lambda^i_\alpha > \lambda_e . \end{array}$$

Then for all $\beta, j = 1, \ldots, n_\beta$,

$$\mathrm{Y}^j_{\beta\underset{(-)}{+}}(z, \lambda) = \lim_{\varepsilon \downarrow 0} \mathrm{Y}^j_\beta(z, \lambda^i_\alpha + \mu e^{2i(\varphi(\pm)\varepsilon)}) \in \mathscr{B}(\mathrm{H}^s(\mathrm{R}^{2n}), \mathrm{H}^s(\mathrm{R}^n_{p_\beta})) \quad (3.2)$$

$$\mathrm{Y}_{\mathrm{E}\underset{(-)}{+}}(z, \lambda) = \lim_{\varepsilon \downarrow 0} \mathrm{Y}_\mathrm{E}(z, \lambda^i_\alpha + \mu e^{2i(\varphi(\pm)\varepsilon)}) \in \mathscr{B}(\mathrm{H}^s(\mathrm{R}^{2n})) . \quad (3.3)$$

We shall now establish the existence of limits on the continuous spectrum $\sigma_e(\varphi)$ for $\varphi \neq 0$ of the various operators discussed above in a different topology, based on the concept of smoothness. These limits will be utilized for the construction of a spectral measure of H(z) for $\varphi \neq 0$. We denote by Δ a Borel set, such that $\bar{\Delta}$ is a compact subset of R^+. We refer to [*10*] for the proof of the following Lemma, which utilizes results of [*22*], [*27*].

LEMMA 3.5. — For $\lambda^i_\alpha \in \sigma_d(h_\alpha)$, $\frac{1}{2} < s' \leqq s$, the following limits exists in $\mathscr{B}(\mathscr{H}, \mathrm{L}^2(\Delta, \tilde{\mathscr{H}}^{-s',2})$,

$$\mathrm{H}_\pm(z, \lambda^i_\alpha + \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{H}(z, \lambda^i_\alpha + \mu e^{2i(\varphi \pm \varepsilon)}) .$$

The same holds with λ^i_α replaced by $\lambda_\alpha \in \sigma_{d\alpha}(\varphi)$.

From Lemmas 3.2, 3.3 and 3.5 we obtain

LEMMA 3.6. — Let $\lambda^i_\alpha \in \sigma_d(h_\alpha)$, and assume that

$$\begin{array}{lll} (\lambda^i_\alpha + e^{2i\varphi}\bar{\Delta}) \cup \mathscr{R}^i_\alpha(\overline{\mathscr{R}}^i_\alpha) = \emptyset & \text{for} & \varphi \underset{(>)}{<} 0 \\ \bar{\Delta} \subset (\lambda^i_\alpha, \lambda^{i'}_\alpha) \setminus \sigma_p(\mathrm{H}) & \text{for} & \varphi = 0 \\ (\lambda^i_\alpha + e^{2i\varphi}\bar{\Delta}) \cup \overline{\mathscr{R}}^{i'}_\alpha(\mathscr{R}^{i'}_\alpha) = \emptyset & \text{for} & \varphi \underset{(<)}{>} 0, \quad \lambda^i_\alpha > \lambda_e . \end{array}$$

Then the following limits exist for all $\beta, j = 1, \ldots, n_\beta$,

$$\mathrm{Y}^j_{\beta\underset{(-)}{+}}(z, \lambda^i_\alpha + \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{Y}^j_\beta(z, \lambda^i_\alpha + \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{H}, \mathrm{L}^2(\Delta, \mathrm{H}^s(\mathrm{R}^n_{p_\beta}))$$

$$\mathrm{Y}_{\mathrm{E}\underset{(-)}{+}}(z, \lambda^i_\alpha + \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{Y}_\mathrm{E}(z, \lambda^i_\alpha + \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{H}, \mathrm{L}^2(\Delta, \mathscr{H})) .$$

We shall now derive the limits in suitable topologies of the operators G(z, ζ) and extend the basic equations given by Lemma 2.5 to the boundary values of the operators G(r, ζ), Y(z, ζ) and $\mathrm{R}_1(z, \zeta)$ on the continuous spectrum.

LEMMA 3.7. — For $\lambda_\alpha^i \in \sigma_d(h_\alpha)$, $\lambda_\alpha \in \sigma_{d\alpha}(\varphi)$ and all β, $j = 1, \ldots, n_\beta$,

$$G_{\beta\pm}^j(z, \lambda) = \lim_{\varepsilon \downarrow 0} G_\beta^j(z, \lambda_\alpha^i + \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(H^s(R_{p\beta}^n), H^s(R^{2n})) \quad (3.4)$$

$$G_{\lambda_\alpha \pm}(z, \lambda) = \lim_{\varepsilon \downarrow 0} G_{\lambda_\alpha}(z, \lambda_\alpha + e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(H^s(R_{p\alpha}^n), H^s(R^{2n})) \quad (3.5)$$

$$G_{0\pm}(z, \lambda) = \lim_{\varepsilon \downarrow 0} G_0(z, \lambda_\alpha + \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(H^s(R^{2n})) \quad (3.6)$$

Proof. — This follows immediately from [*2*] Theorem 4, [*12*] Theorem 1 and the fact that $G_0(z, \zeta)$ is regular for $\zeta = \lambda$.

On the basis of definition 2.4, [*2*] Theorem 4.1 and Lemmas 3.4 and 3.7 we have the following result

LEMMA 3.8. — Let $\lambda = \lambda_\alpha^i + \mu e^{2i\varphi}$, where $\mu \in R^+$ for $\varphi \neq 0$, $\mu \in (0, \lambda_\alpha^{i'} - \lambda_\alpha^i)$ for $\varphi = 0$. Then

$$G_\pm(z, \lambda) = \lim_{\varepsilon \downarrow 0} G(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{H}_1^s, H^s(R^{2n})) \quad (3.7)$$

Assuming furthermore that $\lambda \notin \mathscr{R}_\alpha^i \cup \sigma_p(H)\,(\overline{\mathscr{R}}_\alpha^i \cup \sigma_p(H))$ and for $\lambda_\alpha^i > \lambda_e$ also $\lambda \notin \overline{\mathscr{R}}_\alpha^{i'}(\mathscr{R}_\alpha^{i'})$, we have

$$Y_{\underset{(-)}{+}}(z, \lambda) = \lim_{\varepsilon \downarrow 0} Y(z, \lambda_\alpha^i + \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(H^s(R^{2n}), \mathscr{H}_1^s) \quad (3.8)$$

In both cases we obtain the following basic identities from Lemmas 2.5 and 3.8.

LEMMA 3.9. — Let $\lambda = \lambda_\alpha^i + \mu e^{2i\varphi}$, where $\mu \in R^+$ for $\varphi \neq 0$, $\mu \in (0, \lambda_\alpha^{i'} - \lambda_\alpha^i)$ for $\varphi = 0$. Then

$$G_\pm(z, \lambda) = J(z) + W(z)R_{1\pm}(z, \lambda) \qquad \text{in } \mathscr{B}(\mathscr{H}_1^s, H^s(R^{2n})) \quad (3.9)$$

Assuming furthermore that $\lambda \notin \mathscr{R}_\alpha^i \cup \sigma_p(H)\,(\overline{\mathscr{R}}_\alpha^i \cup \sigma_p(H))$ and for $\lambda_\alpha^i > \lambda_e$ also $\lambda \notin \overline{\mathscr{R}}_\alpha^{i'}(\mathscr{R}_\alpha^{i'})$, we have

$$G_{\underset{(-)}{+}}(z, \lambda)Y_{\underset{(-)}{+}}(z, \lambda) = I \qquad \text{in } \mathscr{B}(H^s(R^{2n})) \quad (3.10)$$

We shall now investigate the basic analyticity properties of the operators $A_\pm(z, \lambda)$ and $Y_\pm(z, \lambda)$ and their limits for $\varphi \to 0$. Utilizing [*2*] Theorem 4.1, it is straightforward to derive the basic analyticity and limiting properties of the operators $r_{\beta\pm}^j\left(z, \lambda_\alpha^i + \dfrac{z^2}{2n_\alpha}\right)$. Moreover, from [*8*] Lemma 2.8 and a convolution representation it is easy to show (cf. [*10*]) that $B_\alpha(z)R_\beta^E\left(z, \lambda_\alpha^i + \dfrac{z^2}{2n}\right)A_\beta(z)$ is an analytic $\mathscr{B}(\mathscr{H}, H^s(R^{2n}))$-valued function of $z \in \mathscr{O} \setminus [(2n_\alpha(E - \lambda_\alpha^i))^{\frac{1}{2}}, \infty)$. This leads to

LEMMA 3.10. — For $\frac{1}{2} < s' \leq s$ the $\mathscr{B}(\tilde{\mathscr{H}}^{-s'})$-valued functions

$A_\pm\left(z, \lambda_\alpha^i + \dfrac{z^2}{2n_\alpha}\right)$ are analytic for $z \in \mathcal{O} \setminus \mathrm{R}^+$, and for $\lambda_\alpha^i = \lambda_e$ analytic for $z \in \mathcal{O} \setminus [(2n_\alpha(\lambda_e' - \lambda_e))^{\frac{1}{2}}, \infty)$. Moreover, for $\lambda_\alpha^i > \lambda_e$

$$A_\pm(\rho, \lambda_\alpha^i + \rho^2/2n_\alpha) = \lim_{\varphi \to 0\mp} A_\pm\left(z, \lambda_\alpha^i + \frac{z^2}{2n_\alpha}\right) \quad \text{in } \mathscr{B}(\tilde{\mathscr{H}}^{-s}) \tag{3.11}$$

The analogous analyticity result holds for $A_\pm\left(z, \lambda_\alpha + \dfrac{z^2}{2n_\alpha}\right)$, $\lambda_\alpha \in \sigma_{d\alpha}^c(\varphi)$.

LEMMA 3.11. — For all β and $j = 1, \ldots, n_\beta$ the $\mathscr{B}(\mathrm{H}^s(\mathrm{R}^{2n}), \mathrm{H}^s(\mathrm{R}_\beta^n))$-valued functions $Y_{\beta\,\overset{+}{(-)}}^j\left(z, \lambda_\alpha^i + \dfrac{z^2}{2n_\alpha}\right)$ and the $\mathscr{B}(\mathrm{H}^s(\mathrm{R}^{2n}))$-valued functions $Y_{E\,\overset{+}{(-)}}\left(z, \lambda_\alpha^i + \dfrac{z^2}{2n_\alpha}\right)$ are meromorphic for $z \in \mathcal{O} \setminus \mathrm{R}^+$ with poles at most at points of

$$\left\{ z \mid \lambda_\alpha^i + \frac{z^2}{2n_\alpha} \in \mathscr{R}_\alpha^i \cup \overline{\mathscr{R}}_\alpha^{i'}(\mathscr{R}_\alpha^{i'} \cup \overline{\mathscr{R}}_\alpha^i) \right\}.$$

For $\lambda_\alpha^i = \lambda_e$ these functions are meromorphic for

$$z \in \mathcal{O} \setminus [(2n_\alpha(\lambda_e' - \lambda_e))^{\frac{1}{2}}, \infty)$$

with poles at most at points of

$$\left\{ z \mid \lambda_e + \frac{z^2}{2n_\alpha} \in \mathscr{R}_{\lambda_e}(\overline{\mathscr{R}}_{\lambda_e}) \right\}.$$

Moreover, for $\lambda_\alpha^i > \lambda_e$ and $\lambda_\alpha^i + \dfrac{\rho^2}{2n_\alpha} \in (\lambda_\alpha^i, \lambda_{\alpha'}^i) \setminus \sigma_p(\mathrm{H})$,

$$Y_{\beta(\pm)}^j\left(\rho, \lambda_\alpha^i + \frac{\rho^2}{2n_\alpha}\right) = \lim_{\varphi \to 0\overset{-}{(+)}} Y_{\beta\,\overset{+}{(-)}}^j\left(z, \lambda_\alpha^i + \frac{z^2}{2n_\alpha}\right) \quad \text{in } \mathscr{B}(\mathrm{H}^s(\mathrm{R}^{2n}), \mathrm{H}^s(\mathrm{R}_{p\beta}^n))$$

$$Y_{E(\pm)}\left(\rho, \lambda_\alpha^i + \frac{\rho^2}{2n_\alpha}\right) = \lim_{\varphi \to 0\overset{-}{(+)}} Y_{E\,\overset{+}{(-)}}\left(z, \lambda_\alpha^i + \frac{z^2}{2n_\alpha}\right) \quad \text{in } \mathscr{B}(\mathrm{H}^s(\mathrm{R}^{2n}))$$

The analogous analyticity result holds for $\lambda_\alpha \in \sigma_{d\alpha}^c(\varphi)$.

Proof. — From the analyticity and limiting properties of $r_{\beta\pm}^j\left(z, \lambda_\alpha^i + \dfrac{z^2}{2n_\alpha}\right)$ it follows that the operators $H_\pm\left(z, \lambda_\alpha^i + \dfrac{z^2}{2n_\alpha}\right)$ are analytic for $z \in \mathcal{O} \setminus \mathrm{R}^+$, and for $\frac{1}{2} < s' \leqq s$, $\lambda_\alpha^i + \dfrac{\rho^2}{2n_\alpha} \in (\lambda_\alpha^i, \lambda_\alpha^{i'})$

$$H_\pm(\rho, \lambda_\alpha^i + \rho^2/2n_\alpha) = \lim_{\varphi \to 0\mp} H_\pm\left(z, \lambda_\alpha^i + \frac{z^2}{2n_\alpha}\right) \quad \text{in } \mathscr{B}(\mathrm{H}^s(\mathrm{R}^{2n}), \tilde{\mathscr{H}}^{-s',2}) \tag{3.12}$$

Then the result follows from Lemmas 3.2, 3.3, 3.10 and (3.12), using analytic Fredholm theory (cf. [33]).

4. LIMITS ON THE CONTINUOUS SPECTRUM FOR POSITIVE ENERGIES

In this section we make the assumptions A II *i*)-*v*) and investigate in certain topologies introduced by Yajima [38] the limits of the operators $H(z, \zeta)$, $A(z, \zeta)$ and $Y(z, \zeta)$ on the part of the continuous spectrum given by $\{ e^{2i\varphi} R^+ \}$.

DEFINITION 4.1. — Let $\mu_0 = \min_\alpha | \lambda_\alpha^n |$, $\lambda_0 \in R^+$

$$I_{\lambda_0} = \left(\max \left\{ 0, \lambda_0 - \frac{\mu_0}{8} \right\}, \lambda_0 + \frac{\mu_0}{8} \right),$$

$$\mu' = \min \left\{ \mu \mid \mu e^{2i\varphi} \in \bigcup_\alpha \mathcal{R}_\alpha \right\} \quad \text{for } \varphi \neq 0,$$

$$I_\varphi = I_{\lambda_0} \quad \text{for } \varphi = 0$$

$$I_\varphi = R^+ \quad \text{for } \varphi < 0 \quad \text{if } e^{2i\varphi} R^+ \cap \left(\bigcap_\alpha \mathcal{R}_\alpha \right) = \varnothing$$

$$I_\varphi = (0, \mu') \quad \text{for } \varphi < 0 \quad \text{if } e^{2i\varphi} R^+ \cap \left(\bigcap_\alpha \mathcal{R}_\alpha \right) \neq \varnothing$$

$$I_\varphi = I_{-\varphi} \quad \text{for } \varphi > 0$$

$$L^2_{\rho,\lambda_0}(R^n_{p_\alpha}) = \left\{ f \in L^2(R^n_{p_\alpha}) \mid f(p_\alpha) = 0 \quad \text{for} \quad \frac{p_\alpha^2}{2n_\alpha} \geqq \left(\lambda_0 + \frac{\mu_0}{2} \right) \rho^{-2} \right\},$$

$$\mathcal{H}_{\rho,\lambda_0} = \left\{ f \in \mathcal{H} \mid f(k_\alpha, p_\alpha) = 0 \quad \text{for } \frac{k_\alpha^2}{2m_\alpha} + \frac{p_\alpha^2}{2n_\alpha} \geqq \left(\lambda_0 + \frac{\mu_0}{2} \right) \rho^{-2} \right\}.$$

We shall use the following simple fact.

LEMMA 4.2. — For $\lambda \in I_{\lambda_0}$ and all α, $i = 1, \ldots, n_\alpha$

$$r_\alpha^i(\rho, \lambda) \in \mathcal{B}(L^2_{\rho,\lambda_0}(R^n_{p_\alpha}), H^{0,2}(R^n_{p_\alpha})). \tag{4.1}$$

Based on [19] Lemma 1.1, [2] Theorem 4.1, [12] Theorem 1, [8] Lemma 2.8 and [38] Lemma 4.5 the following results are proved in [10], utilizing techniques of [38].

LEMMA 4.3. — For all α

$$H_\pm(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} H(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathcal{B}(H^s(R^{2n}), \tilde{\mathcal{H}}^{-s,2})$$
$$\cap\, \mathcal{B}(H^s(R^n_{k_\alpha}) \otimes L^2_{\rho,\lambda_0}(R^n_{p_\alpha}), \tilde{\mathcal{H}}^{-s,2}) \quad \text{for} \quad \varphi \in (-a, a), \quad \mu \in I_\varphi \tag{4.2}$$

and

$$H_\pm(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} H(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathcal{B}(H^s(R^n_{k_\alpha}) \otimes L^2(R^n_{p_\alpha}), \tilde{\mathcal{H}}^{0,2})$$
$$\text{for} \quad \varphi \neq 0, \quad \mu \in I_\varphi \tag{4.3}$$

LEMMA 4.4. — For $z \in \mathcal{O}$, $\mu \in \mathrm{I}_\varphi$

$$\mathrm{A}_\pm(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{A}(z, \mu e^{2i(\varphi \pm \varepsilon)}) = \lim_{\varepsilon \downarrow 0} \mathrm{A}(\rho e^{i(\varphi \mp \varepsilon)}, \mu e^{2i\varphi}) \in \mathscr{B}(\tilde{\mathscr{H}}^{-s,2}) \quad (4.4)$$

and $\mathrm{A}^2_\pm(z, \mu e^{2i\varphi}) \in \mathscr{C}(\tilde{\mathscr{H}}^{-s,2})$.

For $\varphi \neq 0$ the same holds with $\tilde{\mathscr{H}}^{-s,2}$ replaced by $\tilde{\mathscr{H}}^{0,2}$.

The next result is proved in the same way as [8] Lemma 2.4, details are found in [10].

LEMMA 4.5. — For $\varphi = 0$, $\mu \in \mathrm{R}^+$ and $\varphi \neq 0$, $\mu \in (0, \mu')$

$$\mathscr{N}(1 + \mathrm{A}_{\underset{(-)}{+}}(z, \mu e^{2i\varphi})) \neq \{0\}$$

if and only if

$$\mu e^{2i\varphi} \in \mathscr{R}_0 \cup \sigma_p(\mathrm{H}) \cup \overline{\mathscr{R}}'_0 \qquad (\mathscr{R}'_0 \cup \sigma_p(\mathrm{H}) \cup \overline{\mathscr{R}}_0).$$

Moreover for $0 < \varepsilon \leqq \varepsilon_0$

$$\dim \mathscr{N}(1 + \mathrm{A}_{\underset{(-)}{+}}(z, \mu e^{2i\varphi})) = \dim \mathscr{N}(1 + \mathrm{A}(\rho e^{i(\varphi \mp \varepsilon)}, \mu e^{2i\varphi}).$$

We utilize the following spaces introduced by Yajima [38].

DEFINITION 4.6. —

$$\mathscr{X}_0 = \sum_\alpha (\mathrm{H}^s(\mathrm{R}^n_{k_\alpha}) \otimes \mathrm{L}^2(\mathrm{R}^n_{p_\alpha}))\,; \qquad \mathscr{X}_1 = \mathscr{X}_0 \cap \mathscr{H}_{\rho,\lambda_0}.$$

$$\mathscr{X}_2 = \sum_\alpha (\mathrm{H}^s(\mathrm{R}^n_{k_\alpha}) \otimes \mathrm{L}^2_{\rho,\lambda_0}(\mathrm{R}^n_{p_\alpha}))\,; \qquad \mathscr{X}_3 = \mathscr{X}_2 + \mathrm{H}^s(\mathrm{R}^{2n})$$

$$\mathscr{X}_\alpha = \mathrm{L}^2_{\rho,\lambda_0}(\mathrm{R}^n_{p_\alpha}) + \mathrm{H}^s(\mathrm{R}^n_{p_\alpha})\,; \qquad \tilde{\mathscr{X}}_0 = \mathscr{X}_0 \oplus \sum_\alpha \oplus \sum_{i=1}^{n_\alpha} \mathscr{X}_\alpha$$

$$\tilde{\mathscr{X}}_1 = \mathscr{X}_1 \oplus \sum_\alpha \oplus \sum_{i=1}^{n_\alpha} \oplus \mathscr{X}_\alpha\,; \qquad \tilde{\mathscr{X}}_1^- = \mathscr{X}_1 \oplus \sum_\alpha \oplus \sum_{i=1}^{n_\alpha} \oplus \mathrm{H}^{-s,2}(\mathrm{R}^n_{p_\alpha})$$

$$\tilde{\mathscr{X}}_0^0 = \mathscr{X}_0 \oplus \sum_\alpha \oplus \sum_{i=1}^{n_\alpha} \oplus \mathrm{L}^2(\mathrm{R}^n_{p_\alpha})$$

LEMMA 4.7. — Assume that $\mu \in \mathrm{I}_\varphi$ and

$$\mu e^{2i\varphi} \notin \mathscr{R}_0 \cup \sigma_p(\mathrm{H}) \cup \overline{\mathscr{R}}'_0 \qquad (\mathscr{R}'_0 \cup \sigma_p(\mathrm{H}) \cup \overline{\mathscr{R}}_0).$$

Then

$$\mathrm{Y}_{0\underset{(-)}{+}}(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{Y}_0(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{X}_3, \mathscr{X}_0) \quad (4.5)$$

$$\mathrm{Y}^i_{\alpha\underset{(-)}{+}}(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{Y}^i_\alpha(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{X}_3, \mathscr{X}_\alpha) \quad (4.6)$$

For $\varphi \neq 0$ (4.5) and (4.6) hold with $\mathscr{X}_3$ replaced by $\mathscr{X}_0$.

Proof. — By Lemmas 4.3-4.5

$$\mathrm{K}_\pm(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} \mathrm{K}(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{X}_3, \tilde{\mathscr{H}}^{-s,2}) \quad (4.7)$$

and for $\varphi \neq 0$ (4.7) holds with $\mathscr{X}_3$ replaced by $\mathscr{X}_0$. Then the proof proceeds as in [38], see also [10].

The following limits of operators $Y(z, \zeta)$ for $\varphi \neq 0$ in a topology related to the smoothness technique are established in [10].

LEMMA 4.8. — Let Δ be a Borel set such that $\bar{\Delta}$ is a compact subset of R^+. Then for $\varphi \neq 0$

$$H_\pm(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} H(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{H}, L^2(\Delta, \tilde{\mathscr{H}}^{0,2}).$$

LEMMA 4.9. — Let Δ be a Borel set, such that $\bar{\Delta}$ is a compact subset of I_φ and such that

$$\{ e^{2i\varphi}\bar{\Delta} \} \cap \{ \mathscr{R}_0 \cup \bar{\mathscr{R}}_0' \} = \emptyset \qquad (\{ e^{2i\varphi}\bar{\Delta} \} \cap \{ \mathscr{R}_0' \cup \bar{\mathscr{R}}_0 \} = \emptyset).$$

Then for $\varphi \neq 0$

$$Y_{0\pm}(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} Y_0(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{H}, L^2(\Delta, \mathscr{X}_0)) \tag{4.8}$$

$$Y^i_{\alpha\pm}(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} Y^i_\alpha(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{H}, L^2(\Delta, L^2(R^n_{p_\alpha})) \tag{4.9}$$

We utilize the following limits of the operators $G(z, \zeta)$ and extension of the basic equations of Lemma 2.5 to the boundary values of the operators $G(z, \zeta)$, $Y(z, \zeta)$ and $R_1(z, \zeta)$, established in [10].

LEMMA 4.10. — For $z \in \mathcal{O}$, $\mu \in R^+$

$$G_{0\pm}(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} G_0(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{X}_0) \cap \mathscr{B}(\mathscr{X}_1, \mathscr{X}_2) \tag{4.10}$$

$$G^i_{\alpha\pm}(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} G^i_\alpha(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \mathscr{B}(\mathscr{X}_\alpha, H^s(R^n_{k_\alpha}) \otimes L^2_{\rho,\lambda_0}(R^n_{p_\alpha}) + H^s(R^{2n})) \tag{4.11}$$

and for $\varphi \neq 0$ (4.11) holds in $\mathscr{B}(L^2(R^n_{p_\alpha}), H^s(R^{2n}))$.

From Lemmas 4.7 and 4.10 we obtain the following result, utilizing definition 2.4.

LEMMA 4.11. — For $z \in \mathcal{O}$, $\mu \in R^+$

$$G_\pm(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} G(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \begin{cases} \mathscr{B}(\tilde{\mathscr{X}}_0, \mathscr{X}_0) \cap \mathscr{B}(\tilde{\mathscr{X}}_1, \mathscr{X}_3) & \\ \mathscr{B}(\tilde{\mathscr{X}}^0_0, \mathscr{X}_0) & \text{for } \varphi \neq 0 \end{cases}$$

For $\mu e^{2i\varphi} \notin \mathscr{R}_0 \cup \sigma_p(H) \cup \bar{\mathscr{R}}_0'(\mathscr{R}_0' \cup \sigma_p(H) \cup \bar{\mathscr{R}}_0)$

$$Y_{(\pm)}(z, \mu e^{2i\varphi}) = \lim_{\varepsilon \downarrow 0} Y(z, \mu e^{2i(\varphi \pm \varepsilon)}) \in \begin{cases} \mathscr{B}(\mathscr{X}_3, \tilde{\mathscr{X}}_0) & \\ \mathscr{B}(\mathscr{X}_0, \tilde{\mathscr{X}}_0) & \text{for } \varphi \neq 0 \end{cases}$$

For $z \in \mathcal{O}$, $\mu \in R^+$

$$G_\pm(z, \mu e^{2i\varphi}) = J(z) + W(z)R_{1\pm}(z, \mu e^{2i\varphi}) \tag{4.12}$$

in $\mathscr{B}(\tilde{\mathscr{X}}_0, \mathscr{X}_0)$ and $\mathscr{B}(\tilde{\mathscr{X}}_1, \mathscr{X}_3)$ and for $\varphi \neq 0$ in $\mathscr{B}(\tilde{\mathscr{X}}^0_0, \mathscr{X}_0)$.

For $\mu e^{2i\varphi} \notin \mathscr{R}_0 \cup \sigma_P(\mathrm{H}) \cup \overline{\mathscr{R}}'_0(\mathscr{R}'_0 \cup \sigma_P(\mathrm{H}) \cup \overline{\mathscr{R}}_0)$

$$\mathrm{G}_{(\pm)}(z, \mu e^{2i\varphi})\mathrm{Y}_{(\pm)}(z, \mu e^{2i\varphi}) = \mathrm{I} \tag{4.13}$$

in $\mathscr{B}(\mathscr{X}_3, \mathscr{X}_0)$ and for $\varphi \neq 0$ in $\mathscr{B}(\mathscr{X}_0)$.

Using [*19*] Lemma 1.1 and a convolution representation, it is easy to show that for all α, β, $\mathrm{B}_\alpha(z)\mathrm{R}^0_{\beta(\pm)}(z, z^2/2m)\mathrm{A}_\beta(z)$ is an analytic $\mathscr{B}(\mathscr{H})$-valued function of $z \in \mathscr{O} \setminus \bigcup_{\{\eta | \eta^2/2m \in \mathscr{R}_\beta(\overline{\mathscr{R}}_\beta)\}} \{ \eta t \mid 1 \leq t < \infty \}$. This together with the analyticity and limiting properties of $r^i_\pm(z, z^2/2m)$ yields

LEMMA 4.12 — The $\mathscr{B}(\widetilde{\mathscr{H}}^{-s})$-valued functions $\mathrm{A}_\pm(z, z^2/2m)$ are analytic for

$$z \in \mathscr{O} \setminus \left(\mathrm{R}^+ \cup \bigcup_\alpha \bigcup_{\{\eta | \eta^2/2m \in \mathscr{R}_\alpha(\overline{\mathscr{R}}_\alpha)\}} \{ \eta t \mid 1 \leqq t < \infty \} \right),$$

and for $\rho \in \mathrm{R}^+$

$$\mathrm{A}_\pm(\rho, \rho^2/2m) = \lim_{\varphi \to 0\mp} \mathrm{A}_\pm(z, z^2/2m) \quad \text{in} \quad \mathscr{B}(\widetilde{\mathscr{H}}^{-s}).$$

Now Lemmas 4.3-4.5, 4.7, 4.11 and 4.12 yield

LEMMA 4.13. — For all α, $i = 1, \ldots, n_\alpha$, the $\mathscr{B}(\mathscr{X}_3, \mathscr{X}_\alpha)$-valued functions $\mathrm{Y}^i_{\alpha\underset{(-)}{+}}(z, z^2/2m)$ and the $\mathscr{B}(\mathscr{X}_3, \mathscr{X}_0)$-valued function $\mathrm{Y}_{0\underset{(-)}{+}}(z, z^2/2m)$ are meromorphic for

$$z \in \mathscr{O} \setminus \left(\mathrm{R}^+ \cup \left(\bigcup_\alpha \bigcup_{\{\eta | \eta^2/2m \in \mathscr{R}_\alpha(\overline{\mathscr{R}}_\alpha)\}} \{ \eta t \mid 1 \leqq t < \infty \}\right)\right)$$

with poles at most at points of $\{ z \mid z^2/2m \in \mathscr{R}_0 \cup \overline{\mathscr{R}}'_0(\mathscr{R}'_0 \cup \overline{\mathscr{R}}_0) \}$, and for $\rho^2/2m \in \mathrm{I}_{\lambda_0} \setminus \sigma_p(\mathrm{H})$, $\lambda_0 \in \mathrm{R}^+$

$$\mathrm{Y}^i_{\alpha\pm}(\rho, \rho^2/2m) = \lim_{\varphi \to 0\mp} \mathrm{Y}^i_{\alpha\pm}(z, z^2/2m) \quad \text{in} \quad \mathscr{B}(\mathscr{X}_3, \mathscr{X}_\alpha) \tag{4.14}$$

$$\mathrm{Y}_{0\pm}(\rho, \rho^2/2m) = \lim_{\varphi \to 0\mp} \mathrm{Y}_{0\pm}(z, z^2/2m) \quad \text{in} \quad \mathscr{B}(\mathscr{X}_3, \mathscr{X}_0) \tag{4.15}$$

Proof. — The proof is similar to that of Lemma 3.11 and is based on Lemmas 4.3-4.5, 4.7, 4.11 and 4.12.

5. TRACE OPERATORS

DEFINITION 5.1.

$$\Sigma_\alpha = \{ p_\alpha \in \mathrm{R}^n_{p_\alpha} \mid | p_\alpha | = 1 \}, \qquad \Sigma_0 = \{ p \in \mathrm{R}^{2n} \mid | \bar{p} | = 1 \}, \tag{5.1}$$

$$h^i_\alpha = h_\alpha = \mathrm{L}^2(\Sigma_\alpha), \qquad h_0 = \mathrm{L}^2(\Sigma_0).$$

For $\lambda \in [\lambda_\alpha^i, \lambda_\alpha^{i'})$, let

$$h(\lambda) = \sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} \oplus h_\delta^j \tag{5.2}$$

and for $\lambda \in [0, \infty)$

$$h = h(\lambda) = \left(\sum_\delta \sum_{\lambda_\delta^j \in \sigma_d^r(h_\delta)} \oplus h_\delta^j \right) \oplus h_0 \tag{5.3}$$

Moreover, for $\mu \in \mathrm{R}^+$ we define in accordance with [*28*], p. 44 (see also [*26*] prop. 2.1) and [*14*] prop. 2.2

$$\mathrm{T}_\alpha(\mu) = (2n_\alpha)^{n/4} 2^{-\frac{1}{2}} \mu^{\frac{n-2}{4}} \gamma_\alpha((2n_\alpha\mu)^{\frac{1}{2}}) \in \mathscr{B}(\mathrm{H}^s(\mathrm{R}_{p_\alpha}^n), h_\alpha) \tag{5.4}$$

$$\mathrm{T}_0(\mu) = (2m)^{n/2} 2^{-\frac{1}{2}} \mu^{\frac{n-1}{2}} \gamma_0(2m\mu)^{\frac{1}{2}}) \in \mathscr{B}(\mathscr{X}_0, h_0). \tag{5.5}$$

For $\lambda \in (\lambda_\alpha^i, \lambda_\alpha^{i'})$ we define the operators $\mathrm{T}(\lambda)$ by

$$\mathrm{T}(\lambda) = \sum_\delta \sum_{\{\lambda_\delta^j \leqslant \lambda_\alpha^i\}} \oplus \mathrm{T}_\delta(\lambda - \lambda_\delta^j) \tag{5.6}$$

In case II we define $\mathrm{T}(\lambda)$ also for $\lambda \in (0, \infty)$ by

$$\mathrm{T}(\lambda) = \left(\sum_\delta \sum_{\lambda_\delta^j \in \sigma_d^r(h_\delta)} \oplus \mathrm{T}_\delta(\lambda - \lambda_\delta^j) \right) \oplus \mathrm{T}_0(\lambda). \tag{5.7}$$

By (1.1) we have the following connection between trace operators and dilation operators,

$$\mathrm{T}_\alpha(\mu) = 2^{-\frac{1}{2}} \mu^{-\frac{1}{4}} \gamma_\alpha(1) \mathrm{U}_\alpha^i((2n_\alpha\mu)^{\frac{1}{2}}) \tag{5.8}$$

$$\mathrm{T}_0(\mu) = 2^{-\frac{1}{2}} \mu^{-\frac{1}{4}} \gamma_0(1) \mathrm{U}_0((2m\mu)^{\frac{1}{2}}) \tag{5.9}$$

LEMMA 5.2. — The following identity holds in $\mathscr{B}(\mathrm{H}^s(\mathrm{R}_{p_\alpha}^n), \mathrm{H}^{-s,2}(\mathrm{R}_{p_\alpha}^n))$

$$\mathrm{T}_\alpha^*(\mu)\mathrm{T}_\alpha(\mu) = e^{2i\varphi}(2\pi i)^{-1} \{ r_{\lambda_\alpha+}(\varphi, \lambda_\alpha + \mu e^{2i\varphi}) - r_{\lambda_\alpha-}(\varphi, \lambda_\alpha + \mu e^{2i\varphi}) \} \tag{5.10}$$

and in

$$\mathscr{B}\left(\mathscr{X}_0, \sum_\alpha \mathrm{H}^{-s}(\mathrm{R}_{k_\alpha}^n) \otimes \mathrm{L}^2(\mathrm{R}_{p_\alpha}^n) \right) \qquad \text{for} \quad s > 1$$

$$\mathrm{T}_0^*(\mu)\mathrm{T}_0(\mu) = e^{2i\varphi}(2\pi i)^{-1} \{ \mathrm{R}_{0+}(\varphi, \mu e^{2i\varphi}) - \mathrm{R}_{0-}(\varphi, \mu e^{2i\varphi}) \}. \tag{5.11}$$

Proof. — This follows from (5.4), (5.5) and a well known representation of $(k^2 - \rho^2)_+^{-1} - (k^2 - \rho^2)_-^{-1}$ given in [*26*] 4.4.

The following result is proved in [*10*].

LEMMA 5.3. — Suppose that conditions A I *i)-iii)* are satisfied. Let $u \in \mathscr{H}_1^s$,

$$u = (u_0, \tau_\alpha^1, \ldots, \tau_\alpha^{n_\alpha}, \tau_\beta^1, \ldots, \tau_\beta^{n_\beta}, \tau_\gamma^1, \ldots, \tau_\gamma^{n_\gamma}),$$

and let $\mu \in (0, \lambda_\alpha^{i'} - \lambda_\alpha^i)$ for $\varphi = 0$, $\mu \in \mathrm{R}^+$ for $\varphi \neq 0$.

Assume that for some fixed $\varphi \in (-a, a)$ and $\mu \in \mathrm{R}^+$

$$\mathrm{J}(z)\mathrm{R}_{1\binom{+}{-}}(z, \lambda_\alpha^i + \mu e^{2i\varphi})u \in \mathscr{H}.$$

Then for $\varphi = 0$ and $\lambda_\beta^j \leqq \lambda_\alpha^i$

$$\gamma_\beta(\rho^{-1}(2n_\beta(\lambda_\alpha^i - \lambda_\beta^j + \mu))^{\frac{1}{2}})\tau_\beta^j = 0$$

and for $\rho = 1$, $\mathrm{T}(\lambda_\alpha^i + \mu)u = 0$.

For $\varphi \neq 0$ and $\rho \in \mathrm{R}^+$, $\mathrm{T}_\alpha(\mu/\rho^2)\tau_\alpha^i = 0$.
In particular for $\rho = (2n_\alpha\mu)^{\frac{1}{2}}$, $\gamma_\alpha(1)\tau_\alpha^i = 0$.

LEMMA 5.4. — Suppose that A I *i)-iii)* are satisfied and let $u \in \mathscr{H}_1^s$. For $\varphi = 0$, $\lambda \in (\lambda_\alpha^i, \lambda_\alpha^{i'}) \setminus \sigma_p(\mathrm{H})$

$$\mathrm{T}(\lambda)\mathrm{Y}_\pm(\lambda)\mathrm{G}_\pm(\lambda)u = \mathrm{T}(\lambda)u. \tag{5.12}$$

For $\varphi \neq 0$ and

$$\lambda_\alpha^i + \mu e^{2i\varphi} \in \{ \lambda_\alpha^i + e^{2i\varphi}\mathrm{R}^+ \} \setminus (\mathscr{R}_\alpha^i \cup \overline{\mathscr{R}}_\alpha^{i'})(\{ \lambda_\alpha^i + e^{2i\varphi}\mathrm{R}^+ \setminus (\mathscr{R}_\alpha^{i'} \cup \overline{\mathscr{R}}_\alpha^i)),$$

$$\mathrm{T}_\alpha(\mu/\rho^2)\mathrm{Y}^i_{\alpha\binom{+}{-}}(z, \lambda_\alpha^i + \mu e^{2i\varphi})\mathrm{G}_{\binom{+}{-}}(z, \lambda_\alpha^i + \mu e^{2i\varphi})u = \mathrm{T}_\alpha(\mu/\rho^2)\tau_\alpha^i, \tag{5.13}$$

in particular for $\rho = (2n_\alpha\mu)^{\frac{1}{2}}$

$$\gamma_\alpha(1)\mathrm{Y}^i_{\alpha\binom{+}{-}}(z, \lambda_\alpha^i + z^2/2n_\alpha)\mathrm{G}_{(\pm)}(z, \lambda_\alpha^i + z^2/2n_\alpha)u = \gamma_\alpha(1)\tau_\alpha^i. \tag{5.14}$$

Proof. — By [*2*] Theorem 4.1 and Lemmas 2.5, 3.4 and 3.8

$$\mathrm{J}(z)\mathrm{R}_{1\pm}(z, \lambda)\mathrm{Y}_\pm(z, \lambda)\mathrm{G}_\pm(z, \lambda)u = \mathrm{J}(z)\mathrm{R}_{1\pm}(z, \lambda)u,$$

and the Lemma follows from Lemma 5.3.
The following analogue of Lemma 5.4 is proved in [*10*].

LEMMA 5.5. — Suppose that A II *i)-v)* are satisfied. For $\varphi = 0$ let $u \in \widetilde{\mathscr{X}}_1$, $\mu \in \mathrm{I}_{\lambda_0}$ (Def. 4.1), and for $\varphi \neq 0$ let $u \in \widetilde{\mathscr{X}}_0^0$, $\mu \in \mathrm{R}^+$. Assume that for some fixed φ and μ

$$\mathrm{J}(z)\mathrm{R}_{1(\pm)}(z, \mu e^{2i\varphi})u \in \mathscr{H}.$$

Then for $\varphi = 0$ and $\rho = 1$,

$$\mathrm{T}(\mu)u = 0$$

and for $\varphi \neq 0$,

$$\mathrm{T}_0(\mu/\rho^2)u_0 = 0,$$

in particular for $\rho = (2m\mu)^{\frac{1}{2}}$, $\gamma_0(1)u_0 = 0$.
Then [*2*] Theorem 4.1 and Lemmas 2.5, 4.7 and 4.10 yield

LEMMA 5.6. — Suppose that A II *i)-v)* are satisfied. For $\varphi = 0$, $\mu \in \mathrm{I}_{\lambda_0} \setminus \sigma_p(\mathrm{H})$, $u \in \widetilde{\mathscr{X}}_1$,

$$\mathrm{T}(\mu)\mathrm{Y}_\pm(\mu)\mathrm{G}_\pm(\mu)u = \mathrm{T}(\mu)u. \tag{5.15}$$

For $\varphi \neq 0$, $\mu \in I_\varphi$, $\mu e^{2i\varphi} \notin \mathcal{R}_0 \cup \overline{\mathcal{R}}'_0(\mathcal{R}'_0 \cup \overline{\mathcal{R}}_0)$,

$$T_0(\mu/\rho^2)Y_{0\,\substack{+\\(-)}}(z, \mu e^{2i\varphi})G_{\substack{+\\(-)}}(z, \mu e^{2i\varphi})u = T_0(\mu/\rho^2)u_0\,, \tag{5.16}$$

in particular for $\rho = (2m\mu)^{\frac{1}{2}}$

$$\gamma_0(1)Y_{0\,\substack{+\\(-)}}(z, z^2/2m)G_{\substack{+\\(-)}}(z, z^2/2m)u = \gamma_0(1)u_0\,. \tag{5.17}$$

6. CONSTRUCTION OF WAVE OPERATORS

LEMMA 6.1. — Suppose that A I *i)-iii)* are satisfied. Let $\lambda_\alpha = \lambda_\alpha^i \in \sigma_d(h_\alpha)$, and let Δ be a Borel set such that $\bar{\Delta}$ is a compact subset of R^+ for $\varphi \neq 0$ and of $(0, \lambda_\alpha^{i'} - \lambda_\alpha^i)$ for $\varphi = 0$, and such that in the respective cases

$$\left(\lambda_\alpha^i + \mu e^{2i\varphi}\right) \cap \left(\mathcal{R}_\alpha^i \cup \mathcal{R}_\alpha^{i'} \cup \sigma_p(H) \cup \overline{\mathcal{R}}_\alpha^i \cup \overline{\mathcal{R}}_\alpha^{i'}\right) = \varnothing$$

Then for $f, g \in \mathcal{H}$ and for f and/or g replaced by functions in $L^2(\Delta, H^s(R^{2n}))$ we have for $\varphi \neq 0$

$$e^{-2i\varphi}\int_\Delta (T_\alpha(\mu)Y_{\lambda_\alpha\pm}(\varphi, \lambda_\alpha + \mu e^{2i\varphi})f, T_\alpha(\mu)Y_{\lambda_\alpha\pm}(-\varphi, \bar{\lambda}_\alpha + \mu e^{-2i\varphi})g)_{h_\alpha}d\mu$$
$$= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta(\{R(\varphi, \lambda_\alpha+(\mu+i\varepsilon)e^{2i\varphi}) - R(\varphi, \lambda_\alpha+(\mu-i\varepsilon)e^{2i\varphi})\}f, g)d\mu \tag{6.1}$$

For $\varphi = 0$

$$\sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} \int_\Delta (T_\delta(\lambda_\alpha^i - \lambda_\delta^j + \mu)Y_{\delta\pm}^j(\lambda_\alpha^i + \mu)f, T_\delta(\lambda_\alpha^i - \lambda_\delta^j + \mu)Y_{\delta\pm}^j(\lambda_\alpha^i + \mu)g)_{h_\delta}d\mu$$
$$= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\Delta(\{R(\lambda_\alpha^i + \mu + i\varepsilon) - R(\lambda_\alpha^i + \mu - i\varepsilon)\}f, g)d\mu\,. \tag{6.2}$$

The left hand side of (6.1) or (6.2) defines a bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$.

Under assumptions A II *i)-v)* the same result holds for positive energies, obtained by replacing $\mathcal{R}_\alpha^i$ by $\mathcal{R}_0$ and α and λ_α by 0 in (6.1), (6.2).

Proof. — This is proved for $\varphi = 0$ in [*24*], [*38*]. For $\varphi \neq 0$ the proof is similar, see [*10*].

DEFINITION 6.2. — For $\varphi \neq 0$ and $\lambda_\alpha = \lambda_\alpha^i \in \sigma_d(h_\alpha)$ we denote by $\mathcal{D}_{\lambda_\alpha}(\varphi)$ the set of all Borel sets Δ such that $\bar{\Delta}$ is a compact subset of R^+ and

$$\left(\lambda_\alpha^i + e^{2i\varphi}\bar{\Delta}\right) \cap \left(\mathcal{R}_\alpha^i \cup \mathcal{R}_\alpha^{i'} \cup \overline{\mathcal{R}}_\alpha^i \cup \overline{\mathcal{R}}_\alpha^{i'}\right) = \varnothing\,.$$

Also, $\mathscr{D}_0(\varphi)$ denotes the set of all Borel sets Δ such that $\bar{\Delta}$ is a compact subset of I_φ, and

$$(e^{2i\varphi}\bar{\Delta}) \cap (\mathscr{R}_0 \cup \mathscr{R}'_0 \cup \bar{\mathscr{R}}_0 \cup \bar{\mathscr{R}}'_0) = \varnothing .$$

Under conditions A I *i)-iii)* the operators $E_{\lambda_\alpha}(\varphi, \Delta) \in \mathscr{B}(\mathscr{H})$ are defined for $\lambda_\alpha = \lambda_\alpha^i \in \sigma_d(h_\alpha)$, $\Delta \in \mathscr{D}_\alpha^i(\varphi)$ in accordance with Lemma 6.1 by

$$(E_{\lambda_\alpha}(\varphi, \Delta)f, g) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_\Delta (\{ R(\varphi, \lambda_\alpha + (\mu + i\varepsilon)e^{2i\varphi}) - R(\varphi, \lambda_\alpha + (\mu - i\varepsilon)e^{2i\varphi}) \} f, g) d\mu \qquad f, g \in \mathscr{H}. \quad (6.3)$$

Under assumptions A II *i)-v)* the operators $E_0(\varphi, \Delta) \in \mathscr{B}(\mathscr{H})$ are defined for $\Delta \in \mathscr{D}_0(\varphi)$ in accordance with Lemma 6.1 by

$$(E_0(\varphi, \Delta)f, g) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_\Delta (\{ R(\varphi, (\mu + i\varepsilon)e^{2i\varphi}) - R(\varphi, (\mu - i\varepsilon)e^{2i\varphi}) \} f, g) d\mu , \qquad f, g \in \mathscr{H}. \quad (6.4)$$

LEMMA 6.3. — For fixed $\varphi \neq 0$, the operators $E_{\lambda_\alpha}(\varphi, \Delta)$ satisfy the following conditions:

$$E_{\lambda_\alpha}(\varphi, \Delta_1)E_{\lambda_\beta}(\varphi, \Delta_2) = \delta_{\lambda_\alpha}^{\lambda_\beta} E_{\lambda_\alpha}(\varphi, \Delta_1 \cap \Delta_2) \quad \text{for } \Delta_1 \in \mathscr{D}_{\lambda_\alpha}(\varphi),\ \Delta_2 \in \mathscr{D}_{\lambda_\beta}(\varphi) . \quad (6.5)$$

For any finite set $\{ \Delta_k \}_{k=1}^m$ of pairwise disjoint sets in $\mathscr{D}_{\lambda_\alpha}(\varphi)$

$$E_{\lambda_\alpha}\left(\varphi, \bigcup_{k=1}^m \Delta_k\right) = \sum_{k=1}^m E_{\lambda_\alpha}(\varphi, \Delta_k) \quad (6.6)$$

$$E_{\lambda_\alpha}(\varphi, \Delta)H \subseteqq HE_{\lambda_\alpha}(\varphi, \Delta) \quad \text{for} \quad \Delta \in \mathscr{D}_{\lambda_\alpha}(\varphi) . \quad (6.7)$$

The properties (6.5)-(6.7) also hold with $E_{\lambda_\alpha}(\varphi)$ and/or $E_{\lambda_\beta}(\varphi)$ replaced by $E_0(\varphi)$.

Proof. — The additivity property (6.6) follows from that of the Lebesgue integral in view of Lemma 6.1, and (6.7) follows easily from Definition 6.2. Property (6.5) has been proved in [*10*], utilizing ideas of [*21*].

DEFINITION 6.4. — Under the assumptions of Lemma 6.1 we set for $\varphi \neq 0$

$$\tilde{F}_{\lambda_\alpha\pm}(\varphi, \Delta) = \chi_\Delta(\mu)T_\alpha(\mu)Y_{\lambda_\alpha\pm}(\varphi, \lambda_\alpha + \mu e^{2i\varphi}) \quad (6.8)$$

$$F_{\lambda_\alpha\pm}(\varphi, \Delta) = \tilde{F}_{\lambda_\alpha\pm}(\varphi, \Delta) \mid E_{\lambda_\alpha}(\varphi, \Delta)\mathscr{H} \quad (6.9)$$

For $\varphi = 0$ we define

$$\tilde{F}_{\alpha\pm}^i(\Delta) = \chi_\Delta(\mu) \sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} \oplus\, T_\delta(\lambda_\alpha^i - \lambda_\delta^j + \mu)Y_{\delta\pm}^j(\lambda_\alpha^i + \mu) \quad (6.10)$$

$$F_{\alpha\pm}^i(\Delta) = \tilde{F}_{\alpha\pm}^i \mid E(\lambda_\alpha^i + \Delta)\mathscr{H} \quad (6.11)$$

In case II the operators $\tilde{F}_{0\pm}(\Delta)$, $F_{0\pm}(\Delta)$, $\tilde{F}_{0\pm}(\varphi, \Delta)$ and $F_{0\pm}(\varphi, \Delta)$ are defined similarly, replacing Y_α^i by Y_0 and λ_α^i by 0.

We shall now establish the basic properties of the local inverse wave operators $F_{\lambda_\alpha\pm}(\varphi, \Delta)$ and $F_{0\pm}(\varphi, \Delta)$ associated with each channel for $\varphi \neq 0$ as given by Definition 6.4.

LEMMA 6.5. — The operators $\tilde{F}_{\lambda_\alpha\pm}(\varphi, \Delta)$ are in $\mathscr{B}(\mathscr{H}, L^2(\Delta, h_\alpha))$ and $\tilde{F}_{0\pm}(\varphi, \Delta)$ in $\mathscr{B}(\mathscr{H}, L^2(\Delta, h_0))$.

Proof. — This follows immediately from Lemma 3.6 and [*28*], p. 44 in the case of $\tilde{F}_{\lambda_\alpha\pm}(\varphi, \Delta)$ and from Lemma 4.7, [*14*], prop. 2.2 in the case of $\tilde{F}_{0\pm}(\varphi, \Delta)$.

The following Lemma is proved in [*10*].

LEMMA 6.6. — If $f \in \mathscr{H}$ and

$$\text{if} \quad E_{\lambda_\alpha}(\varphi, \Delta)f = 0, \quad \text{then} \quad \tilde{F}_{\lambda_\alpha}(\varphi, \Delta)f = 0;$$
$$\text{if} \quad E_0(\varphi, \Delta)f = 0, \quad \text{then} \quad \tilde{F}_0(\varphi, \Delta)f = 0$$

THEOREM 6.7. — Under assumptions A I *i)-iii)* the operators $F_{\lambda_\alpha\pm}(\varphi, \Delta)$ are $1-1$ and bicontinuous from $E_{\lambda_\alpha}(\varphi, \Delta)\mathscr{H}$ onto $L^2(\Delta, h_\alpha)$. Moreover, for $\tilde{\Delta} \subseteq \Delta$ and $\hat{u} \in L^2(\Delta, h_\alpha)$ we have

$$F_{\lambda_\alpha\pm}(\varphi, \Delta)E_{\lambda_\alpha}(\varphi, \tilde{\Delta})F_{\lambda_\alpha\pm}^{-1}(\varphi, \Delta)\hat{u} = \chi_{\tilde{\Delta}}(\mu)\hat{u}(\mu). \tag{6.12}$$

Under assumptions A II *i)-v)* the same holds with h_α replaced by h_0 and λ_α replaced by 0.

Proof. — Let $f \in E_{\lambda_\alpha}(\varphi, \Delta)\mathscr{H}$. Then by Lemma 6.1 and (6.8)

$$\begin{aligned} \| f \| &= \sup_{\|g\|=1} |(F_{\lambda_\alpha}(\varphi, \Delta)f, F_{\lambda_\alpha}(-\varphi, \Delta)g)_{L^2(\Delta,h_\alpha)}| \\ &\leq \| F_{\lambda_\alpha}(\varphi, \Delta)f \|_{L^2(\Delta,h_\alpha)} \cdot \| F_{\lambda_\alpha}(-\varphi, \Delta) \|_{\mathscr{B}(\mathscr{H},L^2(\Delta,h_\alpha))} \end{aligned} \tag{6.13}$$

This proves that $F_{\lambda_\alpha}(\varphi, \Delta)$ is $1-1$ and $F_{\lambda_\alpha}^{-1}(\varphi, \Delta)$ is bounded. It then suffices in order to prove that $F_{\lambda_\alpha\pm}(\varphi, \Delta)$ is onto $L^2(\Delta; h_\alpha)$ to show that $\mathscr{R}(F_{\lambda_\alpha\pm}(\varphi, \Delta))$ is dense in this space. This follows in a straightforward way from Lemma 6.6 and (5.13) (cf. [*10*]).

Finally (6.12) is equivalent to

$$F_{\lambda_\alpha\pm}(\varphi, \Delta)E_{\lambda_\alpha}(\varphi, \tilde{\Delta})u = F_{\lambda_\alpha\pm}(\varphi, \tilde{\Delta})u \tag{6.14}$$

on setting $\hat{u} = F_{\lambda_\alpha\pm}(\varphi, \Delta)u$.

By Lemma 6.6 the left hand side of (6.14) equals $F_{\lambda_\alpha\pm}(\varphi, \tilde{\Delta})E_{\lambda_\alpha}(\varphi, \tilde{\Delta})u$, and the Theorem is proved for $F_{\lambda_\alpha\pm}(\varphi, \Delta)$. The proof for $F_{0\pm}(\varphi, \Delta)$ is similar.

For $\varphi = 0$ we have the following result on asymptotic completeness for negative energies.

THEOREM 6.8. — Under assumptions A I *i)-iii)* the operators $F^{i}_{\alpha\pm}(\Delta)$ defined by (6.10) and (6.11) are isometries from $E(\lambda^i_\alpha + \Delta)\mathscr{H}$ onto

$$\sum_{\delta} \sum_{\{\lambda^j_\delta \leq \lambda^i_\alpha\}} \oplus L^2(\lambda^i_\alpha - \lambda^j_\delta + \Delta, h_\delta).$$

There exist unique isometric operators $F_{\pm}(\lambda^i_\alpha, \lambda^{i'}_\alpha)$ from

$$E((\lambda^i_\alpha, \lambda^{i'}_\alpha) \setminus \sigma_p(H))\mathscr{H}$$

onto

$$\sum_{\delta} \sum_{\{\lambda^j_\delta \leq \lambda^i_\alpha\}} \oplus L^2((\lambda^i_\alpha - \lambda^j_\delta, \lambda^{i'}_\alpha - \lambda^j_\delta), h_\delta)$$

such that for all Δ

$$F_{\pm}(\lambda^i_\alpha, \lambda^{i'}_\alpha) \mid E(\lambda^i_\alpha + \Delta)\mathscr{H} = F^i_{\alpha\pm}(\Delta)$$

The operators $F(\lambda^i_\alpha, \lambda^{i'}_\alpha)$ are defined by (6.10) and (6.11) with $\lambda^i_\alpha + \Delta$ replaced by $(\lambda^i_\alpha, \lambda^{i'}_\alpha) \setminus \sigma_p(H)$.

For $\bar{\Delta} \subset (0, \lambda^{i'}_\alpha - \lambda^i_\alpha) \setminus \sigma_p(H)$ and

$$\hat{u} = \sum_{\delta} \sum_{\{\lambda^j_\delta \leq \lambda^i_\alpha\}} \oplus u^j_\delta,\ u^j_\delta \in L^2((\lambda^i_\alpha - \lambda^j_\delta, \lambda^{i'}_\alpha - \lambda^j_\delta), h_\delta),$$

we have

$$F_{\pm}(\lambda^i_\alpha, \lambda^{i'}_\alpha)E(\lambda^i_\alpha + \Delta)F_{\pm}(\lambda^i_\alpha, \lambda^{i'}_\alpha)^{-1}\hat{u} = \sum_{\delta} \sum_{\{\lambda^j_\delta \leq \lambda^i_\alpha\}} \oplus \chi_{\lambda^i_\alpha - \lambda^j_\delta + \Delta}(\mu)u^j_\delta(\mu) \quad (6.15)$$

Moreover

$$F^{-1}_{\pm}(\lambda^i_\alpha, \lambda^{i'}_\alpha)\hat{\mu} = \lim_{t \to \pm\infty} e^{itH}Je^{-itH_1}\hat{u}. \quad (6.16)$$

The same result holds in case II for positive energies, with λ^i_α replaced by 0 and $F^i_{\alpha\pm}$ replaced by $F_{0\pm}$.

Proof. — Since by [5] Lemma 1 $\sigma_p(H)$ accumulates at most at point of $\{0\} \cup \bigcup_{\alpha} \sigma_d(h_\alpha)$, it suffices to prove the first statement for any closed interval Δ. In this case (6.2) shows that $F^i_{\alpha\pm}(\Delta)$ are isometries from $E(\lambda^i_\alpha + \Delta)$ into $\sum_{\delta} \sum_{\{\lambda^j_\delta \leq \lambda^i_\alpha\}} \oplus L^2(\lambda^i_\alpha - \lambda^j_\delta + \Delta, h_\delta)$. It is then proved as in [*38*], using (5.12), that $\mathscr{R}(F^i_{\alpha\pm}(\Delta))$ is dense in and hence equal to this space. The identity (6.16) is proved in [*38*].

7. THE SCATTERING MATRIX

DEFINITION 7.1. — Under assumptions A I *i)-iii)* we define for $\varphi \neq 0$, $\Delta \in \mathscr{D}_{\lambda_\alpha}(\varphi)$ the local scattering operators $S_{\lambda_\alpha}(\varphi, \Delta) \in \mathscr{B}(L^2(\Delta, h_\alpha))$ by

$$S_{\lambda_\alpha}(\varphi, \Delta) = F_{\lambda_\alpha +}(\varphi, \Delta)F^{-1}_{\lambda_\alpha -}(\varphi, \Delta) \quad (7.1)$$

For $\varphi = 0$ we define the scattering operators $S(\lambda_\alpha^i, \lambda_\alpha^{i'})$ by

$$S(\lambda_\alpha^i, \lambda_\alpha^{i'}) = F_+(\lambda_\alpha^i, \lambda_\alpha^{i'})F_-^{-1}(\lambda_\alpha^i, \lambda_\alpha^{i'}) \tag{7.2}$$

Under assumptions A II *i)-v)* we define for $\varphi \neq 0$, $\Delta \in \mathscr{D}_0(\varphi)$ the local scattering operators $S_0(\varphi, \Delta) \in \mathscr{B}(L^2(\Delta, h_0))$ in accordance with Theorem 6.8 by (7.1), where λ_α is replaced by 0. For $\varphi = 0$ we define the scattering operator $S(0, \infty)$ in accordance with Theorem 6.8 by (7.2), where λ_α^i and $\lambda_\alpha^{i'}$ are replaced by 0 and ∞.

It follows from Theorem 6.8 that $S(\lambda_\alpha^i, \lambda_\alpha^{i'})$ and $S(0, \infty)$ are unitary operators on

$$\sum_\delta \sum_{\{\lambda_\delta^j < \lambda_\alpha^i\}} \oplus L^2((\lambda_\alpha^i - \lambda_\delta^j, \lambda_\alpha^{i'} - \lambda_\delta^j), h_\delta)$$

and

$$\left(\sum_\delta \sum_{\lambda_\delta \in \sigma_d(h_\delta)} \oplus L^2((-\lambda_\delta, \infty); h_\delta)\right) \oplus L^2((0, \infty); h_0)$$

respectively.

It follows from Theorem 6.7 that $S_{\lambda_\alpha}(\varphi, \Delta)$ is a bicontinuous isomorphism of $L^2(\Delta, h_\alpha)$ which commutes with $\chi_{\tilde\Delta}(\mu)$ for $\tilde\Delta \subseteq \Delta$, and $S_0(\varphi, \Delta)$ is a bicontinuous isomorphism of $L^2(\Delta; h_0)$ which commutes with $\chi_{\tilde\Delta}(\mu)$ for $\tilde\Delta \subseteq \Delta$.

The following representation of the scattering operator is given in case II for $\varphi = 0$ in [*38*]. For the extension to case I and to $\varphi \neq 0$ we refer to [*10*].

THEOREM 7.2. — Under assumptions A I *i)-iii)* for $\varphi \neq 0$ and $\Delta \in \mathscr{D}_{\lambda_\alpha}(\varphi)$ the local scattering operators $S_{\lambda_\alpha}(\varphi, \Delta)$ and their inverses $S_{\lambda_\alpha}^{-1}(\varphi, \Delta)$ have the following representations for $f \in L^2(\Delta; h_\alpha)$ and a. e. $\mu \in \Delta$,

$$(S_{\lambda_\alpha}(\varphi, \Delta)f)(\mu) = \chi_\Delta(\mu)\mathscr{S}_{\lambda_\alpha}(\varphi, \mu)f(\mu) \tag{7.3}$$

$$(S_{\lambda_\alpha}^{-1}(\varphi, \Delta)f)(\mu) = \chi_\Delta(\mu)\mathscr{S}_{\lambda_\alpha}^{-1}(\varphi, \mu)f(\mu) \tag{7.4}$$

where $\mathscr{S}_{\lambda_\alpha}(\varphi, \mu)$ and its inverse $\mathscr{S}_{\lambda_\alpha}^{-1}(\varphi, \mu)$ are defined by

$$\mathscr{S}_{\lambda_\alpha}(\varphi, \mu) = 1 - 2\pi i e^{-2i\varphi} T_\alpha(\mu)[Y_{\lambda_\alpha -}(-\varphi, \lambda_\alpha + \mu e^{-2i\varphi})W_{\lambda_\alpha}(-\varphi)]^* T_\alpha^*(\mu) \tag{7.5}$$

$$\mathscr{S}_{\lambda_\alpha}^{-1}(\varphi, \mu) = 1 + 2\pi i e^{-2i\varphi} T_\alpha(\mu)[Y_{\lambda_\alpha +}(-\varphi, \lambda_\alpha + \mu e^{-2i\varphi})W_{\lambda_\alpha}(-\varphi)]^* T_\alpha^*(\mu). \tag{7.6}$$

For fixed $\varphi \underset{(>)}{\leq} 0$ the operators $\mathscr{S}_{\lambda_\alpha}(\varphi, \mu)$ form a norm-continuous function of $\mu \in R^+ \setminus \{ \mu \mid \lambda_\alpha + \mu e^{2i\varphi} \in \mathscr{R}_{\lambda_\alpha}(\bar{\mathscr{R}}'_{\lambda_\alpha}) \}$ with values in $\mathscr{B}(h_\alpha)$, and $\mathscr{S}_{\lambda_\alpha}^{-1}$ is a norm-continuous function of $\mu \in R^+ \setminus \{ \mu \mid \lambda_\alpha + \mu e^{2i\varphi} \in \mathscr{R}'_{\lambda_\alpha}(\bar{\mathscr{R}}_{\lambda_\alpha}) \}$ with values in $\mathscr{B}(h_\alpha)$. Moreover

$$1 - \mathscr{S}_{\lambda_\alpha}(\varphi, \mu) \in \mathscr{C}(h_\alpha), \qquad 1 - \mathscr{S}_{\lambda_\alpha}^{-1}(\varphi, \mu) \in \mathscr{C}(h_\alpha). \tag{7.7}$$

For $\varphi = 0$ the scattering operators $S(\lambda_\alpha^i, \lambda_\alpha^{i'})$ and their inverses $S^{-1}(\lambda_\alpha^i, \lambda_\alpha^{i'})$ have the following representations for

$$f \in \sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} \oplus L^2((\lambda_\alpha^i - \lambda_\delta^j, \lambda_\alpha^{i'} - \lambda_\delta^j), h_\delta)$$

and a. e. $\mu \in \Delta$,

$$(S(\lambda_\alpha^i, \lambda_\alpha^{i'})f)(\mu) = \tilde{\chi}_\Delta(\mu)\mathcal{S}_{\lambda_\alpha^i}(\mu)f(\mu) \tag{7.8}$$

$$(S^{-1}(\lambda_\alpha^i, \lambda_\alpha^{i'})f)(\mu) = \tilde{\chi}_\Delta(\mu)\mathcal{S}_{\lambda_\alpha^i}^{-1}(\mu)f(\mu) \tag{7.9}$$

where the scattering matrix $\mathcal{S}_{\lambda_\alpha^i}(\mu)$ and its inverse $\mathcal{S}_{\lambda_\alpha^i}^{-1}(\mu)$ are defined by

$$\mathcal{S}_{\lambda_\alpha^i}(\mu) = 1 - 2\pi i T(\lambda_\alpha^i + \mu)Y_{E+}^S(\lambda_\alpha^i + \mu)W_E T^*(\lambda_\alpha^i + \mu) \tag{7.10}$$

$$\mathcal{S}_{\lambda_\alpha^i}^{-1}(\mu) = 1 + 2\pi i T(\lambda_\alpha^i + \mu)Y_{E-}^S(\lambda_\alpha^i + \mu)W_E T^*(\lambda_\alpha^i + \mu) \tag{7.11}$$

Here

$$W_E(\{\tau_\alpha^1, \ldots, \tau_\alpha^{n_\alpha}, \tau_\beta^1, \ldots, \tau_\beta^{n_\beta}, \tau_\gamma^1, \ldots, \tau_\gamma^{n_\gamma}\}) = \sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} W_\delta^j \tau_\delta^j \tag{7.12}$$

and

$$Y_{E\pm}^S(\lambda) = \{Y_{\alpha\pm}^1(\lambda), \ldots, Y_{\alpha\pm}^{n_\alpha}(\lambda), Y_{\beta\pm}^1(\lambda), \ldots, Y_{\beta\pm}^{n_\beta}(\lambda), Y_{\gamma\pm}^1(\lambda), \ldots, Y_{\gamma\pm}^{n_\gamma}(\lambda)\} \tag{7.13}$$

is the set of components of $Y_\pm(\lambda)$ related to scattering in the interval $(\lambda_\alpha^i, \lambda_\alpha^{i'})$. We recall that $E = \lambda_\alpha^{i'}$ and

$$Y_\pm(\lambda) = \{Y_{E\pm}(\lambda), Y_{E\pm}^S(\lambda)\}.$$

The operator $\mathcal{S}_{\lambda_\alpha^i}(\mu)$ is unitary on $\sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} \oplus h_\delta$ and norm-continuous for $\mu \in (0, \lambda_\alpha^{i'} - \lambda_\alpha^i) \setminus \sigma_p(H)$.

Moreover,

$$1 - \mathcal{S}_{\lambda_\alpha^i}(\mu) \in \mathcal{C}\Big(\sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} \oplus h_\delta\Big) \tag{7.14}$$

and

$$1 - \mathcal{S}_{\lambda_\alpha^i}^{-1}(\mu) \in \mathcal{C}\Big(\sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} \oplus h_\delta\Big) \tag{7.15}$$

Under assumptions A II *i*)-*v*) the scattering operators $S_0(\varphi, \Delta)$ have the representations obtained by replacing α and λ_α by 0 in (7.3)-(7.6), and $S_0(\Delta)$ have representations obtained by replacing $T(\lambda_\alpha^i + \mu)$ by T, Y_E^S by Y and W_E by W in (7.8)-7.11).

We proceed to study the analyticity properties and limits for $\varphi \to 0$ of $\mathcal{S}_{\lambda_\alpha^i}^{-1}(\varphi, \rho^2/2n_\alpha)$ and $\mathcal{S}_{\lambda_\alpha^i}(\varphi, \rho^2/2n_\alpha)$, establishing the connection of these operators with the diagonal elements $[\mathcal{S}_{\lambda_\alpha^i}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)}$ and $[\mathcal{S}_{\lambda_\alpha^i}^{-1}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)}$ of the S-matrix and its inverse in the corresponding energy interval.

DEFINITION 7.3. — For any set $D(\lambda_\alpha^i) \subset C$ we use the notation

$$D(\lambda_\alpha^i)^\sim = \{ z \in \mathcal{O} \mid \lambda_\alpha^i + z^2/2n_\alpha \in D(\lambda_\alpha^i) \}$$

In case II, with λ_α^i replaced by 0,

$$D(0)^\sim = \{ z \in \mathcal{O} \mid z^2/2m \in D(0) \} .$$

THEOREM 7.4. — Under assumptions A I *i)-iii)*, the $\mathcal{B}(h_\alpha)$-valued functions $\mathcal{S}_{\lambda_\alpha^i}(z) = \mathcal{S}_{\lambda_\alpha^i}(\varphi, \rho^2/2n_\alpha)$ and $\mathcal{S}_{\lambda_\alpha^i}^{-1}(z) = \mathcal{S}_{\lambda_\alpha^i}^{-1}(\varphi, \rho^2/2n_\alpha)$ are meromorphic for $z \in \mathcal{O} \setminus R^+$ with poles at most at points of $(\mathcal{R}_\alpha^i \cup \bar{\mathcal{R}}_\alpha^{i'})^\sim$ and $(\mathcal{R}_\alpha^{i'} \cup \bar{\mathcal{R}}_\alpha^i)^\sim$ respectively.

Moreover,

$$\mathcal{S}_{\lambda_\alpha^i}(\rho e^{i\varphi}) \xrightarrow[\varphi \to 0_-]{} ([\mathcal{S}_{\lambda_\alpha^i}^{-1}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)})^* \tag{7.16}$$

and

$$\mathcal{S}_{\lambda_\alpha^i}^{-1}(\rho e^{i\varphi}) \xrightarrow[\varphi \to 0_+]{} ([\mathcal{S}_{\lambda_\alpha^i}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)})^* \tag{7.17}$$

in the uniform operator topology of $\mathcal{B}(h_\alpha)$, uniformly for ρ in any compact subset of $\{ (\lambda_\alpha^i, \lambda_\alpha^{i'}) \setminus \sigma_p(H) \}^\sim$.

There exist closed null sets $N_{\lambda_\alpha^i \pm} \subset (\lambda_\alpha^i, \lambda_\alpha^{i'}) \setminus \sigma_p(H)$, such that

$$[\mathcal{S}_{\lambda_\alpha^i}^{(-1)}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)}$$

is invertible for

$$\rho \in \{ (\lambda_\alpha^i, \lambda_\alpha^{i'}) \setminus (\sigma_p(H) \cup N_{\lambda_\alpha^i {+ \atop (-)}}) \}^\sim ,$$

and

$$\mathcal{S}_{\lambda_\alpha^i}(\rho e^{i\varphi}) \xrightarrow[\varphi \to 0_+]{} ([\mathcal{S}_{\lambda_\alpha^i}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)})^{-1*} \tag{7.18}$$

$$\mathcal{S}_{\lambda_\alpha^i}^{-1}(\rho e^{i\varphi}) \xrightarrow[\varphi \to 0_-]{} ([\mathcal{S}_{\lambda_\alpha^i}^{-1}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)})^{-1*} \tag{7.19}$$

in the uniform operator topology of $\mathcal{B}(h_\alpha)$, uniformly for ρ in any compact subset of $\{ (\lambda_\alpha^i, \lambda_\alpha^{i'}) \setminus (\sigma_p(H) \cup N_{\lambda_\alpha^i {+ \atop (-)}}) \}^\sim$.

For $\lambda_\alpha^i = \lambda_e$ the functions $\mathcal{S}_{\lambda_e}(z)$ and $\mathcal{S}_{\lambda_e}^{-1}(z)$ are meromorphic for $z \in \mathcal{O} \setminus [(2n_\alpha(\lambda_e' - \lambda_e))^{\frac{1}{2}}, \infty)$ with poles at most at points of $\mathcal{R}_{\lambda_e}^\sim$ and $\bar{\mathcal{R}}_{\lambda_e}^\sim$ respectively.

The analogous result holds in case II under assumptions A II *i)-v)*.

Proof. — We consider case I, the proof for case II is similar, except for for the last statement, which requires a different proof (see [*10*]) since we have not proved that $1 - \mathcal{S}_0(z)$ is compact. By (5.4) and the identity

$$\rho^{n/2}\gamma_\alpha(\rho) = \gamma_\alpha(1)U_\alpha(\rho) \tag{7.20}$$

we have, setting $\mu = \rho^2/2n_\alpha$

$$T_\alpha(\mu) = n_\alpha^{\frac{1}{2}}\rho^{-1}\gamma_\alpha(1)U_\alpha(\rho) . \tag{7.21}$$

Introducing (7.21) in (7.5) and (7.6) of Theorem 7.2, we obtain for $\varphi \neq 0$, taking adjoints and replacing φ by $-\varphi$,

$$\begin{aligned} \mathcal{S}_{\lambda_\alpha^i}^*(\bar{z}) &= 1 + 2\pi i z^{-2} n_\alpha \gamma_\alpha(1) U_\alpha(\rho) Y_{\alpha-}^i(\varphi, \lambda_\alpha^i + z^2/2n_\alpha) W_\alpha^i(\varphi) U_\alpha^*(\rho) \gamma_\alpha^*(1) \\ &= 1 + 2\pi i z^{-2} n_\alpha \gamma_\alpha(1) Y_{\alpha-}^i(z, \lambda_\alpha^i + z^2/2n_\alpha) W_\alpha^i(z) \gamma_\alpha^*(1) \end{aligned} \tag{7.22}$$

and similarly

$$\mathcal{S}_{\lambda_\alpha^i}^{-1*}(\bar{z}) = 1 - 2\pi i z^{-2} n_\alpha \gamma_\alpha(1) \mathrm{Y}_{\alpha+}^i(z, \lambda_\alpha^i + z^2/2n_\alpha) \mathrm{W}_\alpha^i(z) \gamma_\alpha^*(1). \quad (7.23)$$

By Lemma 3.11, $\mathcal{S}_{\lambda_\alpha^i}^*(\bar{z})$ and $\mathcal{S}_{\lambda_\alpha^i}^{-1*}(\bar{z})$ are meromorphic for $z \in \mathcal{O} \setminus \mathrm{R}^+$ with poles at most at points of $\mathcal{R}_\alpha^{i'} \cup \overline{\mathcal{R}_\alpha^i}$ and $\mathcal{R}_\alpha^i \cup \overline{\mathcal{R}_\alpha^{i'}}$ respectively, and

$$\lim_{\varphi \to 0_+} \mathcal{S}_{\lambda_\alpha^i}^*(\bar{z}) = 1 + 2\pi i \rho^{-2} n_\alpha \gamma_\alpha(1) \mathrm{Y}_{\alpha-}^i(\rho, \lambda_\alpha^i + \rho^2/2n_\alpha) \mathrm{W}_\alpha^i(\rho) \gamma_\alpha^*(1) \quad (7.24)$$

$$\lim_{\varphi \to 0_-} \mathcal{S}_{\lambda_\alpha^i}^{-1*}(\bar{z}) = 1 - 2\pi i \rho^{-2} n_\alpha \gamma_\alpha(1) \mathrm{Y}_{\alpha+}^i(\rho, \lambda_\alpha^i + \rho^2/2n_\alpha) \mathrm{W}_\alpha^i(\rho) \gamma_\alpha^*(1) \quad (7.25)$$

By (7.10), (7.11) and (7.21), the right hand sides of (7.24) and (7.25) coincide with $[\mathcal{S}_{\lambda_\alpha^i}^{-1}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)}$ and $[\mathcal{S}_{\lambda_\alpha^i}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)}$ respectively, and we have established the analyticity properties of $\mathcal{S}_{\lambda_\alpha^i}(z)$ and $\mathcal{S}_{\lambda_\alpha^i}^{-1}(z)$ and proved (7.16) and (7.17).

The existence of $[\mathcal{S}_{\lambda_\alpha^i}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)}^{-1}$ and $[\mathcal{S}_{\lambda_\alpha^i}^{-1}(\rho^2/2n_\alpha)]_{(\lambda_\alpha^i, \lambda_\alpha^i)}^{-1}$ almost everywhere and (7.18), (7.19) then follows from (7.7) of Theorem 7.2, the analyticity properties and (7.16), (7.17) by a result of Kuroda [*25*].

For $\lambda_\alpha^i = \lambda_e$, the analyticity properties of $\mathcal{S}_{\lambda_e}(z)$ and $\mathcal{S}_{\lambda_e}^{-1}(z)$ follow from those of $\mathrm{Y}_{\alpha\pm}^i(z, \lambda_e + z^2/2n_\alpha)$ given in Lemma 3.11.

Finally, the analyticity properties of $\mathcal{S}_{\lambda_\alpha}(z)$ and $\mathcal{S}_{\lambda_\alpha}^{-1}(z)$ for $\lambda_\alpha \in \mathcal{R}_\alpha \cup \overline{\mathcal{R}}_\alpha$ follow from those of $\mathrm{Y}_{\lambda_\alpha \pm}(z, \lambda_\alpha + z^2/n_\alpha)$ given in Lemma 3.11.

We finally turn to the question of the connection between resolvent resonances and poles of the S-matrix. In view of theorem 7.4 the problem here is, under what conditions a resolvent resonance is a pole of the analytically continued diagonal elements or the S-matrix. The treatment of this problem is more complicated than in the two-body case (see [*8*]), partly because of the possibility of embedded eigenvalues. We shall not give a complete answer here, but prove the following results, which seem to cover most cases.

DEFINITION 7.5. — Assume that conditions A I *i)-iii)* are satisfied and

$$\varphi = 0, \quad \rho = 1, \quad \mu \in (0, \lambda_\alpha^{i'} - \lambda_\alpha^i)$$

or

$$\varphi \neq 0, \quad \rho \in \mathrm{R}^+, \quad \mu \in \mathrm{R}^+.$$

Let $\lambda = \lambda_\alpha^i + \mu e^{2i\varphi}$ and

$$\tilde{\mathcal{N}}(\mathrm{G}_\pm(z, \lambda)) = \mathcal{N}(\mathrm{G}_\pm(z, \lambda)) / \mathcal{N}(\mathrm{J}(z) \mathrm{R}_{1\pm}(z, \lambda)) \quad (7.26)$$

where

$$\mathrm{G}_\pm(z, \lambda) \in \mathcal{B}(\mathcal{H}_1^s, \mathrm{H}^s(\mathrm{R}^{2n})).$$

Let $\mathrm{E}_\alpha^i \Omega = \tau_\alpha^i$ and

$$\bar{\mathrm{E}}_\alpha^i \Omega = \sum_\delta \sum_{\{\lambda_\delta^j \leq \lambda_\alpha^i\}} \oplus \mathrm{E}_\delta^j \Omega, \qquad \Omega \in \mathcal{N}(\mathrm{G}_\pm(z, \lambda)). \quad (7.27)$$

By Lemma 5.3 the operators $\mathrm{T}(\lambda)\bar{\mathrm{E}}_\alpha^i$ for $\varphi = 0$ and $\mathrm{T}_\alpha(\mu\rho^{-2})\mathrm{E}_\alpha^i$ for $\varphi \neq 0$

map $\mathcal{N}(\mathrm{J}(z)\mathrm{R}_{1\pm}(z, \lambda))$ into 0 and thus induce operators denoted by $\tilde{\mathrm{T}}(\lambda)$ and $\tilde{\mathrm{T}}_\alpha(\mu\rho^{-2})$ on $\tilde{\mathcal{N}}(\mathrm{G}_\pm(z, \lambda))$.

For $\varphi = 0$ we set

$$\mathcal{N}_0(\mathrm{G}_\pm(z, \lambda)) = \{ \Omega \in \mathcal{N}(\mathrm{G}_\pm(z, \lambda)) \mid \mathrm{T}(\lambda)\bar{\mathrm{E}}^i_\alpha = 0 \} \tag{7.28}$$

$$\tilde{\mathcal{N}}_0(\mathrm{G}_\pm(z, \lambda)) = \{ \tilde{\Omega} \in \tilde{\mathcal{N}}(\mathrm{G}_\pm(z, \lambda)) \mid \tilde{\mathrm{T}}(\lambda)\tilde{\Omega} = 0 \tag{7.29}$$

and for $\varphi \neq 0$

$$\mathcal{N}_0(\mathrm{G}_\pm(z, \lambda)) = \{ \Omega \in \mathcal{N}(\mathrm{G}_\pm(z, \lambda)) \mid \mathrm{T}_\alpha(\mu\rho^{-2})\mathrm{E}^i_\alpha\Omega = 0 \} \tag{7.30}$$

$$\tilde{\mathcal{N}}_0(\mathrm{G}_\pm(z, \lambda)) = \{ \tilde{\Omega} \in \tilde{\mathcal{N}}(\mathrm{G}_\pm(z, \lambda)) \mid \tilde{\mathrm{T}}_\alpha(\mu\rho^{-2})\tilde{\Omega} = 0 \} . \tag{7.31}$$

We make use of the following result proved in [*10*].

To simplify notation, we consider A_+, G_+, etc.; the same holds for A_-, G_-, etc.

LEMMA 7.6. — Let $\mu \in \mathrm{R}^+$, $\varphi \in (-a, a)$, $\lambda = \lambda^i_\alpha + \mu e^{2i\varphi}$ in case I and $\mu \in \mathrm{I}_\varphi$, $\varphi \in (-a, a)$, $\lambda = \mu e^{2i\varphi}$ in case II, and let

$$\Phi = (u_\alpha, \sigma^1_\alpha, \dots, \sigma^{n_\alpha}_\alpha, u_\beta, \sigma^1_\beta, \dots, \sigma^{n_\beta}_\beta, u_\gamma, \sigma^1_\gamma, \dots, \sigma^{n_\gamma}_\gamma) \in \mathcal{N}(1 + \mathrm{A}_+(z, \lambda)) \tag{7.32}$$

$$\Omega = (u_0, \tau^1_\alpha, \dots, \tau^{n_\alpha}_\alpha, \tau^1_\beta, \dots, \tau^{n_\beta}_\beta, \tau^1_\gamma, \dots, \tau^{n_\gamma}_\gamma) \in \mathcal{N}(\mathrm{G}_+(z, \lambda)) \tag{7.33}$$

where in case I by Lemmas 3.2 and 3.8

$$\mathrm{A}_+(z, \lambda) \in \mathscr{B}(\tilde{\mathscr{H}}^{-s,2}) \qquad \text{and} \qquad \mathrm{G}_+(z, \lambda) \in \mathscr{B}(\mathscr{H}^s_1, \mathrm{H}^s(\mathrm{R}^{2n})) \tag{7.34}$$

and in case II by Lemmas 4.4 and 4.11 for $\varphi = 0$

$$\mathrm{A}_+(z, \lambda) \in \mathscr{B}(\tilde{\mathscr{H}}^{-s,2}) \qquad \text{and} \qquad \mathrm{G}_\pm(z, \lambda) \in \mathscr{B}(\tilde{\mathscr{X}}_0, \mathscr{X}_0) \tag{7.35}$$

and for $\varphi \neq 0$

$$\mathrm{A}_+(z, \lambda) \in \mathscr{B}(\tilde{\mathscr{H}}^{0,2}) \qquad \text{and} \qquad \mathrm{G}_\pm(z, \lambda) \in \mathscr{B}(\tilde{\mathscr{X}}^0_0, \mathscr{X}_0) . \tag{7.36}$$

Let the operators P_λ and Q_λ be defined by $\mathrm{P}_\lambda\Phi = \Omega$, where the components of Ω are given by

$$u_0 = -\sum_\alpha (1 - \mathrm{P}^{\mathrm{E}}_\alpha(z) - \mathrm{V}_\alpha(z)\mathrm{R}^{\mathrm{E}}_{\alpha+}(z, \lambda))\mathrm{A}_\alpha(z)u_\alpha \tag{7.37}$$

$$\tau^i_\delta = - \langle \phi^i_\alpha(\bar{z}) \mid \mathrm{A}_\delta(z)u_\delta \tag{7.38}$$

and $\mathrm{Q}_\lambda\Omega = \Phi$, where the components of Φ are given by

$$u_\delta = \mathrm{B}_\delta(z)(1 + \mathrm{R}_{0+}(z, \lambda)\mathrm{V}_\delta(z))\mathrm{J}(z)\mathrm{R}_{1+}(z, \lambda)\Omega \tag{7.39}$$

$$\sigma^i_\delta = r^i_{\delta+}(z, \lambda) \langle \phi^i_\delta(\bar{z}) \mid \mathrm{V}_\delta(z)(1 + R_{0+}(z, \lambda))\mathrm{J}(z)\mathrm{R}_{1+}(z, \lambda)\Omega . \tag{7.40}$$

Then P_λ is an isomorphism from $\mathcal{N}(1 + \mathrm{A}_+(z, \lambda))$ to $\tilde{\mathcal{N}}(\mathrm{G}_+(z, \lambda))$ and Q_λ induces an isomorphism $\tilde{\mathrm{Q}}_\lambda$ from $\tilde{\mathcal{N}}(\mathrm{G}_+(z, \lambda))$ to $\mathcal{N}(1 + \mathrm{A}_+(z, \lambda))$ such that

$$\tilde{\mathrm{Q}}_\lambda\mathrm{P}_\lambda = \mathrm{I} \qquad \text{and} \qquad \mathrm{P}_\lambda\tilde{\mathrm{Q}}_\lambda = \mathrm{I} .$$

Theorem 7.7. — Assume that A I *i)-iii)* are satisfied.
Let

$$\lambda = \lambda_\alpha + z^2/2n_\alpha \in \mathcal{R}_{\lambda_\alpha} \setminus \mathcal{R}_{\lambda_\alpha'}(\mathcal{R}'_{\lambda_\alpha} \setminus \mathcal{R}_{\lambda_\alpha})$$

for $\varphi < 0$ and

$$\lambda \in \bar{\mathcal{R}}'_{\lambda_\alpha} \setminus \bar{\mathcal{R}}_{\lambda_\alpha}(\bar{\mathcal{R}}_{\lambda_\alpha} \setminus \bar{\mathcal{R}}'_{\lambda_\alpha})$$

for $\varphi > 0$, and define the operators $T_{\lambda(\pm)}$ and $Z_{\lambda(\pm)}$ by

$$\tau = T_{\lambda\overset{+}{(-)}}\Omega = 2\pi i z^{-2}\gamma_\alpha(1)E_{\lambda_\alpha}\Omega\,, \qquad \Omega \in \tilde{\mathcal{N}}(G_{\overset{+}{(-)}}(z,\lambda)) \quad (7.41)$$

$$\Omega = Z_{\lambda\overset{+}{(-)}}\tau = -\,n_\alpha Y_{(\mp)}(z,\lambda)W_{\lambda_\alpha}(z)\gamma_\alpha^*(1)\tau\,, \quad \tau \in \mathcal{N}(\mathcal{S}_{\lambda_\alpha}^{*(-1)}(\bar{z})) \quad (7.42)$$

where $E_{\lambda_\alpha}\Omega = \tau_\alpha$.

The operator $T_{\lambda\overset{+}{(-)}}$ is an isomorphism from $\tilde{\mathcal{N}}(G_{\overset{+}{(-)}}(z,\lambda))$ onto $\mathcal{N}(\mathcal{S}_{\lambda_\alpha}^{*(-1)}(\bar{z}))$ with the inverse $Z_{\lambda\overset{+}{(-)}}$.

The same result holds under assumptions A II *i)-v)* with λ_α replaced by 0.

Proof. — We consider $(T_{\lambda+}, Z_{\lambda+})$ in case I with $\varphi < 0$, the proof for $\varphi > 0$ and for $(T_{\lambda-}, Z_{\lambda-})$ with $\varphi \gtrless 0$ is similar, and case II is analogous. For brevity we set $T_{\lambda+} = T_\lambda$, $Z_{\lambda+} = Z_\lambda$.

1) $W_{\lambda_\alpha}(z)2\pi i n_\alpha z^{-2}\gamma_\alpha^*(1)\gamma_\alpha(1)E_{\lambda_\alpha} = G_+(z,\lambda) - G_-(z,\lambda)$.

By (5.10) of Lemma 5.2

$$\begin{aligned} W_{\lambda_\alpha}(z)2\pi i n_\alpha z^{-2}\gamma_\alpha^*(1)\gamma_\alpha(1)E_{\lambda_\alpha} &= W_{\lambda_\alpha}(z)\{\, r_{\lambda_\alpha+}(z,\lambda) - r_{\lambda_\alpha-}(z,\lambda)\,\}\,E_{\lambda_\alpha} \\ &= W(z)\{\, R_{1+}(z,\lambda) - R_{1-}(z,\lambda) = G_+(z,\lambda) - G_-(z,\lambda)\,. \end{aligned}$$

since $r_{\lambda_\gamma}(z,\zeta)$ for $\lambda_\gamma \neq \lambda_\alpha$ and $R_0(z,\zeta)$ are regular at $\zeta = \lambda$.

2) Z_λ *maps* $\mathcal{N}(\mathcal{S}_{\lambda_\alpha}^*(\bar{z}))$ *one-to-one into* $\tilde{\mathcal{N}}(G_+(z,\lambda))$, *and* $T_\lambda Z_\lambda = 1$.

Let $\tau \in \mathcal{N}(\mathcal{S}_{\lambda_\alpha}^*(\bar{z}))$, i. e.

$$\tau + 2\pi i n_\alpha z^{-2}\gamma_\alpha(1)E_{\lambda_\alpha}Y_-(z,\lambda)W_{\lambda_\alpha}(z)\gamma_\alpha^*(1)\tau = 0 \qquad (7.43)$$

Applying Z_λ to (7.43) we get by 1)

$$\begin{aligned} \Omega - n_\alpha Y_-(z,\lambda)W_{\lambda_\alpha}(z)2\pi i n_\alpha z^{-2}\gamma_\alpha^*(1)\gamma_\alpha(1)E_{\lambda_\alpha}Y_-(z,\lambda)W_{\lambda_\alpha}(z)\gamma_\alpha^*(1)\tau \\ = \Omega + Y_-(z,\lambda)\{\, G_+(z,\lambda) - G_-(z,\lambda)\,\}\,\Omega = 0 \quad (7.44) \end{aligned}$$

Applying $G_-(z,\lambda)$ to (7.44), we get by (3.10)

$$G_+(z,\lambda)\Omega = 0 \qquad (7.45)$$

From (7.41)-(7.43) follows

$$T_\lambda Z_\lambda \tau = \tau\,. \qquad (7.46)$$

Also, if $Z_\lambda\tau \in \mathcal{N}(J(z)R_{1+}(z,\lambda))$, then by Lemma 5.3 $T_\lambda Z_\lambda\tau = 0$ so by (7.46) $\tau = 0$, and 2) is proved.

3) T_λ *maps* $\tilde{\mathcal{N}}(G_+(z,\lambda))$ *into* $\mathcal{N}(\mathcal{S}_{\lambda_\alpha}^*(\bar{z}))$.

Let $\Omega \in \mathcal{N}(G_+(z, \lambda))$ and let τ be given by (7.41). Then by 1) and (5.29)

$$\begin{aligned}\mathcal{S}^*_{\lambda_\alpha}(\bar{z})\tau &= \tau + 2\pi i n_\alpha z^{-2}\gamma_\alpha(1)Y_{\lambda_\alpha -}(z, \lambda)W_{\lambda_\alpha}(z)2\pi i z^{-2}\gamma^*_\alpha(1)\gamma_\alpha(1)E_{\lambda_\alpha}\Omega \\ &= \tau + 2\pi i z^{-2}\gamma_\alpha(1)E_{\lambda_\alpha}Y_-(z, \lambda)\{ G_+(z, \lambda) - G_-(z, \lambda)\}\Omega \\ &= \tau - 2\pi i z^{-2}\gamma_\alpha(1)E_{\lambda_\alpha}Y_-(z, \lambda)G_-(z, \lambda)\Omega \\ &= \tau - 2\pi i z^{-2}\gamma_\alpha(1)\Omega = 0 \end{aligned} \tag{7.47}$$

By (5.21) $T_\lambda\Omega = 0$ for $\Omega \in \mathcal{N}(J(z)R_{1+}(z, \lambda))$, and 3) is proved.

4) $Z_\lambda T_\lambda = Y_-(z, \lambda)G_-(z, \lambda)$.

By 1), for $\Omega \in \mathcal{N}(G_+(z, \lambda))$

$$\begin{aligned} Z_\lambda T_\lambda \Omega &= -n_\alpha Y_-(z, \lambda)W_{\lambda_\alpha}(z)2\pi i z^{-2}\gamma^*_\alpha(1)\gamma_\alpha(1)E_{\lambda_\alpha}\Omega \\ &= -Y_-(z, \lambda)\{ G_+(z, \lambda) - G_-(z, \lambda)\}\Omega = Y_-(z, \lambda)G_-(z, \lambda)\Omega \end{aligned} \tag{7.48}$$

5) T_λ *is one-to-one.*

By 2), 4) and (3.10) for $\Omega \in \mathcal{N}(G_+(z, \lambda))$ the following statements are equivalent,

$$T_\lambda\Omega = 0 \tag{7.49}$$

$$Z_\lambda T_\lambda\Omega = 0 \tag{7.50}$$

$$Y_-(z, \lambda)G_-(z, \lambda)\Omega = 0 \tag{7.51}$$

$$G_-(z, \lambda)\Omega = 0 \tag{7.52}$$

Moreover, (7.49) implies by (5.10)

$$\begin{aligned}\{ J(z)R_{1+}(z, \lambda) - J(z)R_{1-}(z, \lambda)\}\Omega &= |\phi_{\lambda_\alpha}(z)\rangle\{ r_{\lambda_\alpha +}(z, \lambda) - r_{\lambda_\alpha -}(z, \lambda)\}E_{\lambda_\alpha}\Omega \\ &= |\phi_{\lambda_\alpha}(z)\rangle 2\pi i n_\alpha z^{-2}\gamma^*_\alpha(1)\gamma_\alpha(1)E_{\lambda_\alpha}\Omega = 0 \end{aligned} \tag{7.53}$$

Thus, if $\Omega \in \tilde{\mathcal{N}}(G_+(z, \lambda))$ and $T_\lambda\Omega = 0$, then by (7.52) and (7.53) $\Omega \in \tilde{\mathcal{N}}(G_-(z, \lambda))$. From the assumption that $\lambda \notin \mathcal{R}'_{\lambda_\alpha}$ and Lemmas 3.3 and 7.6 it now follows that $\Omega = 0$.

The Theorem now follows from 2), 3) and 5).

We now consider the case, where $\lambda \in \mathcal{R}_{\lambda_\alpha} \cap \mathcal{R}'_{\lambda_\alpha}$ for $\varphi < 0$ ($\lambda \in \bar{\mathcal{R}}_{\lambda_\alpha} \cap \bar{\mathcal{R}}'_{\lambda_\alpha}$ for $\varphi > 0$), but $Y_{(\mp)}\left(\zeta, \lambda_\alpha + \frac{\zeta^2}{2n_\alpha}\right)$ has a simple pole at $\zeta = (2n_\alpha(\lambda - \lambda_\alpha))^{\frac{1}{2}}$ and $\mathcal{N}(G_{(\mp)}(z, \lambda))$ corresponds to an embedded eigenspace $\mathcal{N}(H(z) - \lambda)$. We make use of the following observation.

LEMMA 7.8. — Assume that A I *i)-iii)* are satisfied, and that for $\varphi \underset{(>)}{<} 0$

$$\gamma_\alpha(1)E_{\lambda_\alpha}\Omega = 0 \qquad \text{for all } \Omega \in \mathcal{N}(G_{(\mp)}(z, \lambda)) \tag{7.54}$$

and $Y_{(\mp)}\left(\zeta, \lambda_\alpha + \frac{\zeta^2}{2n_\alpha}\right)$ has a pole of order 1 at $\zeta = z$, i. e. for ζ near z

$$Y_{(\mp)}\left(\zeta, \lambda_\alpha + \frac{\zeta^2}{2n_\alpha}\right) = \frac{A_{(\mp)}}{\zeta - z} + \tilde{Y}_{(\mp)}\left(\zeta, \lambda^i_\alpha + \frac{\zeta^2}{2n_\alpha}\right), \tag{7.55}$$

where $A_{(\mp)} \in \mathcal{B}(H^s(R^{2n}), \mathcal{H}^s_1)$, and $\tilde{Y}_{(\mp)}\left(\zeta, \lambda^i_\alpha + \frac{\zeta^2}{2n_\alpha}\right)$ is regular at $\zeta = z$.

Then $\mathcal{S}^{*(-1)}_{\lambda_\alpha}(\bar{\zeta})$ is regular at $\zeta = z$, and

$$\mathcal{S}^{*(-1)}_{\lambda_\alpha}\left(-\varphi, \frac{\rho^2}{2n_\alpha}\right) = \mathcal{S}^{*(-1)}_{\lambda_\alpha}(\bar{z})$$
$$= 1 \underset{(-)}{+} 2\pi i n_\alpha z^{-2}\gamma_\alpha(1)\mathrm{E}_{\lambda_\alpha}\tilde{\mathrm{Y}}_{(\mp)}(z, \lambda)\mathrm{W}_{\lambda_\alpha}(z)\gamma^*_\alpha(1) \quad (7.56)$$

Moreover,

$$\mathrm{G}_{(\mp)}(z, \lambda)\tilde{\mathrm{Y}}_{(\mp)}(z, \lambda) = \mathrm{I} \quad (7.57)$$

The same holds under assumptions A II *i)-v)* with λ_α replaced by 0.

Proof. — We consider case 1, the proof in case 2 is similar. By (3.10) and (7.55) we have for ζ near z

$$\frac{\mathrm{G}_-\left(\zeta, \lambda_\alpha + \frac{\zeta^2}{2n_\alpha}\right)\mathrm{A}_-}{\zeta - z} + \mathrm{G}_-\left(\zeta, \lambda_\alpha + \frac{\zeta^2}{2n_\alpha}\right)\tilde{\mathrm{Y}}_-\left(\zeta, \lambda_\alpha + \frac{\zeta^2}{2n_\alpha}\right) = \mathrm{I} \quad (7.58)$$

This implies (7.57) and

$$\mathrm{G}_-(z, \lambda)\mathrm{A}_- = 0 \quad (7.59)$$

and hence by (7.54)

$$\gamma_\alpha(1)\mathrm{E}_{\lambda_\alpha}\mathrm{A}_- = 0 \quad (7.60)$$

For ζ near z

$$\mathcal{S}^*_{\lambda_\alpha}(\bar{\zeta}) = 1 + 2\pi i n_\alpha \zeta^{-2}\gamma_\alpha(1)\mathrm{E}_{\lambda_\alpha}\mathrm{Y}_-\left(\zeta, \lambda_\alpha + \frac{\zeta^2}{2n_\alpha}\right)\mathrm{W}_{\lambda_\alpha}(\zeta)\gamma^*_\alpha(1) \quad (7.61)$$

By (7.55) and (7.60) $\mathcal{S}^*_{\lambda_\alpha}(\bar{\zeta})$ is regular at $\zeta = z$, and

$$\mathcal{S}^*_{\lambda_\alpha}(\bar{z}) = 1 + 2\pi i n_\alpha z^{-2}\gamma_\alpha(1)\mathrm{E}_{\lambda_\alpha}\tilde{\mathrm{Y}}_-\left(z, \lambda_\alpha + \frac{z^2}{2n_\alpha}\right)\mathrm{W}_{\lambda_\alpha}(z)\gamma^*_\alpha(1) \quad (7.62)$$

and the Lemma is proved.

THEOREM 7.9. — Assume that A I *i)-iii)* and (7.54), (7.55) are satisfied. Let $\mathrm{T}_{\lambda\underset{(-)}{+}}$ and $\mathrm{Z}_{\lambda\underset{(-)}{+}}$ be defined by (7.41) and (7.42) with $\mathrm{Y}_{(\mp)}(z, \lambda)$ replaced by $\tilde{\mathrm{Y}}_{(\mp)}(z, \lambda)$.

The operator $\mathrm{T}_{\lambda\underset{(-)}{+}}$ induces an isomorphism $\tilde{\mathrm{T}}_{\lambda\underset{(-)}{+}}$ of

$$\tilde{\mathcal{N}}(\mathrm{G}_{(\pm)}(z, \lambda))/\tilde{\mathcal{N}}_0(\mathrm{G}_{(\pm)}(z, \lambda))$$

onto $\mathcal{N}(\mathcal{S}^{*(-1)}_{\lambda_\alpha}(\bar{z}))$ with the inverse $\tilde{\mathrm{Z}}_{\lambda\underset{(-)}{+}}$ induced by $\mathrm{Z}_{\lambda\underset{(-)}{+}}$.

The same results hold under assumptions A II *i)-v)* with λ_α replaced by 0.

Proof. — We consider $(\mathrm{T}_{\lambda+}, \mathrm{Z}_{\lambda+})$ in case I with $\varphi < 0$, the other cases are similar. This is proved as Theorem 7.7 replacing $\mathrm{Y}_-(z, \lambda)$ by $\tilde{\mathrm{Y}}_-(z, \lambda)$, utilizing Lemma 7.8 and making the following modifications in 2), 3) and 5) of the proof.

2′) Z_λ *induces a one-to-one map* $\tilde{\mathrm{Z}}_\lambda$ *of* $\mathcal{N}(\mathcal{S}^*_{\lambda_\alpha}(\bar{z}))$ *into* $\tilde{\mathcal{N}}(\mathrm{G}_+(z, \lambda))/\tilde{\mathcal{N}}_0(\mathrm{G}_+(z, \lambda))$ *and* $\tilde{\mathrm{T}}_\lambda\tilde{\mathrm{Z}}_\lambda = 1$.

3′), 5′) T_λ *induces a one-to-one map* $\tilde{T}_\lambda$ *from* $\tilde{\mathcal{N}}(G_+(z, \lambda))/\tilde{\mathcal{N}}_0(G_+(z, \lambda))$ *into* $\mathcal{N}(\mathcal{S}^*_{\lambda_\alpha}(\bar{z}))$.

REMARK 7.10. — Under the assumptions of Theorem 7.9 we obtain from [11] Lemma 4.6 and Theorem 7.9 a decomposition

$$\tilde{\mathcal{N}}(G_{\underset{(-)}{+}}(z, \lambda)) = \tilde{\mathcal{N}}_0(G_{\underset{(-)}{+}}(z, \lambda) + \tilde{\mathcal{N}}(G_{\underset{(-)}{+}}(z, \lambda))/\tilde{\mathcal{N}}_0(G_{\underset{(-)}{+}}(z, \lambda)) \tag{7.63}$$

such that

$$\tilde{\mathcal{N}}_0(G_{\underset{(-)}{+}}(z, \lambda)) = \tilde{\mathcal{N}}_0(G_{(\mp)}(z, \lambda)) \qquad \text{is isomorphic to } \mathcal{N}(H(z) - \lambda)$$

and

$$\tilde{\mathcal{N}}(G_{\underset{(-)}{+}}(z, \lambda))/\tilde{\mathcal{N}}_0(G_{\underset{(-)}{+}}(z, \lambda)) \qquad \text{is isomorphic to } \mathcal{N}(\mathcal{S}^*_{\lambda_\alpha}(\bar{z})).$$

THEOREM 7.11. — Assume that A I *i)-iii)* are satisfied and that

$$\lambda = \lambda_\alpha + \frac{z^2}{2n_\alpha} \in \mathcal{R}_{\lambda_\alpha} \cup \bar{\mathcal{R}}'_{\lambda_\alpha}(\mathcal{R}'_{\lambda_\alpha} \cup \bar{\mathcal{R}}_{\lambda_\alpha}).$$

Suppose moreover that either of the following conditions is satisfied:

1) $\lambda \notin \mathcal{R}'_{\lambda_\alpha} \cup \bar{\mathcal{R}}_{\lambda_\alpha}(\mathcal{R}_{\lambda_\alpha} \cup \bar{\mathcal{R}}'_{\lambda_\alpha})$.
2) Conditions (7.54) and (7.55) are satisfied, and

$$\tilde{\mathcal{N}}(G_{\underset{(-)}{+}}(z, \lambda))/\tilde{\mathcal{N}}_0(G_{\underset{(-)}{+}}(z, \lambda)) \neq \{0\} \tag{7.64}$$

Then $\mathcal{S}^{(-1)}_{\lambda_\alpha}(\zeta)$ has a pole at $\zeta = z$.
Under assumptions A II *i)-v)* the same result holds with λ_α replaced by 0 and n_α replaced by m.

Proof. — We consider the case of $\mathcal{S}_{\lambda_\alpha}(\zeta)$ for $\varphi < 0$, the cases of $\mathcal{S}_{\lambda_\alpha}(\zeta)$ for $\varphi > 0$ and $\mathcal{S}^{-1}_{\lambda_\alpha}(\zeta)$ are similar.

Assuming 1), we have $\lambda \in \mathcal{R}_{\lambda_\alpha} \setminus \mathcal{R}'_{\lambda_\alpha}$ and hence $\bar{\lambda} \in \bar{\mathcal{R}}_{\lambda_\alpha} \setminus \bar{\mathcal{R}}'_{\lambda_\alpha}$. By Lemmas 3.3 and 7.6 $\bar{\lambda} \in \bar{\mathcal{R}}_{\lambda_\alpha}$ implies that $\tilde{\mathcal{N}}(G_-(\bar{z}. \bar{\lambda})) \neq \{0\}$.
By Theorem 7.7 this together with $\bar{\lambda} \notin \bar{\mathcal{R}}'_{\lambda_\alpha}$ implies that

$$\mathcal{N}(\mathcal{S}^{*-1}_{\lambda_\alpha}(z)) \neq \{0\}.$$

This together with Theorem 7.4 implies that $\mathcal{S}^*_{\lambda_\alpha}(\bar{\zeta})$ is a meromorphic function of $\zeta \in \mathcal{O}_+$ with a pole at $\zeta = \bar{z}$. Hence $\mathcal{S}_{\lambda_\alpha}(\zeta)$ is meromorphic in $\mathcal{O}_-$ with a pole at $\zeta = z$.
Under assumption 2) it follows from Lemma 7.8 and Theorem 7.9 that $\mathcal{S}^{*-1}_{\lambda_\alpha}(\zeta)$ is regular at $\zeta = z$ and

$$\mathcal{N}(\mathcal{S}^{*-1}_{\lambda_\alpha}(z)) \neq \{0\},$$

and we conclude as above that $\mathcal{S}_{\lambda_\alpha}(\zeta)$ has a pole at $\zeta = z$.

The corresponding result for $\mathscr{S}_0(\zeta)$ follows in the same way under assumptions A II *i*)-*v*) from Lemmas 4.5, 7.6 and 7.8 and Theorems 7.4, 7.7, 7.9.

REMARK 7.12. — For $\lambda_\alpha = \lambda_e$ condition 1) is satisfied by [5] Lemma 1.

REFERENCES

[1] S. AGMON, Lower bounds for solutions of Schrödinger equations, *J. d'Anal. Math.*, **23**, 1, 1970, p. 1-25.

[2] S. AGMON, Spectral properties of Schrödinger operators and scattering theory, *Ann. d. Scuola Norm. sup. de Pisa*, Cl. d. Scienze ser. IV, vol. **II**, 2, 1975, p. 151-218.

[3] F. AGUILAR and J. M. COMBES, On a class of analytic perturbations of one-body Schrödinger operators, *Comm. Math. Phys.*, **22**, 1971, p. 269-279.

[4] D. BABBITT and E. BALSLEV, A characterization of dilation-analytic potentials and vectors, *J. Funct. Anal.*, vol. **18**, no. 1, 1975, p. 1-14.

[5] E. BALSLEV and J. M. COMBES, Spectral properties of many-body Schrödinger operators with dilation-analytic interactions, *Comm. Math. Phys.*, t. **22**, 1971, p. 280-299.

[6] E. BALSLEV, Spectral theory of Schrödinger operators of many-body systems with permutation and rotation symmetries, *Ann. Phys.*, t. **73**, 1972, p. 49-107; Schrödinger operators with symmetries I-II, *Reports on Mathematical Physics*, t. **5**, 1974, p. 219-280 et p. 393-413.

[7] E. BALSLEV, Decomposition of many-body Schrödinger operators, *Comm. Math. Phys.*, t. **52**, 1977, p. 127-146.

[8] E. BALSLEV, Analytic scattering theory of 2-body Schrödinger operators, *J. Funct. Anal.*, t. **29**, 3, 1978, p. 375-396.

[9] E. BALSLEV, Scattering theory of dilated 3-body Schrödinger operators, *J. Quantum Chem.*, t. **14**, 1978, p. 361-370.

[10] E. Balslev, *Analytic scattering theory of quantum mechanical three-body systems*, Aarhus University Preprint, no. 26, 1979.

[11] E. BALSLEV, *Analytic scattering theory for many-body systems below the smallest three-body threshold*, Aarhus University Preprint, 1979, to appear, *Comm. Math. Phys.*

[12] J. M. COMBES and L. E. Thomas, Asymptotic behaviour of eigenfunction for multiparticle Schrödinger operators, *Comm. Math. Phys.*, t. **34**, 1973, p. 251-270.

[13] L. D. FADDEEV, *Mathematical aspects of the three-body problem in quantum scattering theory*, Israel program for scientific translations, Jerusalem, 1975.

[14] J. GINIBRE and M. MOULIN, Hilbert space approach to the quantum-mechanical three-body problem, *Ann. Inst. Poincaré*, vol. **XXI**, no. 2, 1974, p. 97-145.

[15] G. A. HAGEDORN, Asymptotic completeness for a class of four particle Schrödinger operators, *Bull. Ann Math. Soc.*, t. **84**, 1, 1978, p. 155-156.

[16] G. A. HAGEDORN, A link between scattering resonances and dilation analytic resonances in few body quantum mechanics, *Comm. Math. Phys.*, *t*. **65**, 2, 1979, p. 181-188.

[17] K. HEPP, On the quantum-mechanical N-body problem, *Helv. Phys. Acta*, t. **42**, 1969, p. 425-458.

[18] J. S. HOWLAND, Abstract stationary theory of multichannel scattering, *J. Funct. Anal.*, t. **22**, 1976, p. 250-282.

[19] R. J. IORIO and M. O'CARROLL, Asymptotic completeness for multiparticle Schrödinger operators with weak potentials, *Comm. math. Phys.*, t. **27**, 1972, p. 137-145.

[20] A. JENSEN, Local decay in time of solutions to Schrödinger's equation with dilation-analytic interaction, *Manuscripta Math.*, t. **25**, 1978, p. 61-77.

[21] T. KAKO and K. YAJIMA, Spectral and scattering theory for a class of non-selfadjoint operators, *Sci. papers College Gen. Ed. Univ. Tokyo*, t. **26**, 1976, p. 73-89.

[22] T. KATO, Wave operators and similarity for some none-selfadjoint operators, *Math. Ann.*, t. **162**, 1966, p. 258-279.
[23] T. KATO, *Perturbation Theory for Linear Operators*, 2 nd. ed. Springer-Verlag, 1976.
[24] T. KATO, Two space scattering, with applications to many-body problems, *J. Fac. Sci. Univ. Tokyo*, Sect. IA, t. **24**, 1977, p. 503-514.
[25] S. T. KURODA, An abstract stationary approach to perturbation of continuous spectra and scattering theory, *J. d'Analyse Math.*, t. **20**, 1967, p. 57-117.
[26] S. T. KURODA, An introduction to scattering theory, *Aarhus University Lecture notes*, Series 51, Feb. 1979.
[27] R. LAVINE, Commutators and scattering theory II, A class of one body problems, *Indiana Univ. Math. J.*, t. **21**, 1972, p. 643-656.
[28] J. L. LIONS and E. MAGENES, *Non-homogeneous boundary value problems and applications*, vol. **I**, Springer-Verlag, 1972.
[29] E. MOURRE, Applications de la méthode de Lavine au problème à trois corps, *Ann. Inst. Poincaré*, Sect. A, t. **26**, 3, 1977, p. 219-262.
[30] M. REED and B. SIMON, *Methods of modern mathematical physics, IV*, Academic Press, 1978.
[31] I. M. SIGAL, Mathematical Foundations of quantum scattering theory for multiparticle systems, *Mem. A. M. S.*, vol. **16**, no. 2, 1978, p. 209.
[32] I. M. SIGAL, On quantum mechanics of many-body systems with dilation analytic potentials, *Bull. A. M. S.*, t. **84**, 1, 1978, p. 152-154.
[33] S. STEINBERG, Meromorphic families of compact operators, *Arch. Rat. Mech. Anal.*, t. **31**, 1968, p. 372-379.
[34] L. E. THOMAS, Asymptotic completeness in two-and three particle quantum mechanical scattering, *Ann. Phys.*, t. **90**, 1975, p. 127-165.
[35] C. VAN WINTER, Complex dynamical variables for multiparticle systems with analytic interactions I-II, *J. Math. Anal. Appl.*, t. **47**, 1974, p. 633-670; t. **48**, 1974, p. 368-399.
[36] C. VAN WINTER, Invariant subspaces of analytic multiparticle Hamiltonians, *J. Math. Anal. Appl.*, t. **49**, 1975, p. 88-123.
[37] C. VAN WINTER, Projection operators associated with analytic multiparticle Hamiltonians Notices, *Am. Math. Soc.*, t. **25**, 1978, A-320.
[38] K. YAJIMA, An abstract stationary approach to three-body scattering, *J. Fac. Sci. Univ. Tokyo*, Sect. IA, t. **25**, 1978, p. 109-132.

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