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Section A :
Physique théorique.

On the possibility of general relativistic oscillations

by

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ABSTRACT. — A group of solutions of Einstein's field equations for a spherically symmetric distribution of matter is investigated. If the mass is at rest at the initial moment and the pressure and density are both zero at the boundary (gas cloud or dust cloud with a density gradient defined everywhere), it is found that oscillations are *not* possible. If the density at the boundary is positive, it is shown that, for some classes of solutions, oscillations are not possible, whereas other classes of solutions satisfy necessary conditions for oscillatory motion. However, these conditions are in general not sufficient.

INTRODUCTION

In a series of papers G. C. MvVittie has investigated the radial motions of a spherically symmetric mass distribution under the influence of gravitation and its pressure gradient (McVittie, 1964, 1966 and 1967). By imposing certain symmetry conditions, instead of assuming a particular equation of state, McVittie was able to solve Einstein's equations, in part at least, for the coefficients of the metric. The analytical solutions of Einstein's equations for the interior of the mass distribution, assuming isotropic

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pressure, can be divided into four different classes (McVittie, 1967; hereafter referred to as Paper I). A special case resulting in expansion or collapse to a singularity, depending on the initial value of the radius of the mass, was investigated by McVittie and Stabell (1967).

The possilility of oscillatory motions among the solutions of Paper I was first pointed out by Nariai (1967) and by Bonnor and Faulkes (1967). An investigation of three of the four classes of solutions showed that necessary (but not sufficient) conditions for oscillations are met in all three cases (McVittie and Stabell, 1968, hereafter referred to as Paper II). The purpose of this paper is to investigate in more detail all four cases with regard to oscillatory motions.

BASIC EQUATIONS

With co-moving coordinates the metric within the mass is written (Paper I):

$$ds^2 = y^2 dt^2 - \frac{R_0^2 S^2 e^\eta}{c^2}(dr^2 + f^2(d\theta^2 + \sin^2\theta d\phi^2)), \tag{1}$$

where R_0 is a constant, S and f are dimensionless functions of t and r, respectively, and y and η are dimensionless functions of the variable z defined by

$$e^z = \frac{Q}{S}, \tag{2}$$

where Q is another function of r. It is also shown in Paper I that

$$y = 1 - \frac{1}{2}\frac{d\eta}{dz}. \tag{3}$$

If the presssure is isotropic, then it is shown that Q, f, and y satisfy three ordinary differential equations of the second order

$$\frac{Q_{rr}}{Q} = \frac{Q_r f_r}{Qf} = a\left(\frac{Q_r}{Q}\right)^2, \tag{4}$$

$$\frac{f_{rr}}{f} - \frac{f_r^2}{f^2} + \frac{1}{f^2} = b\left(\frac{Q_r}{Q}\right)^2, \tag{5}$$

$$y_{zz} + (a - 3 + y)y_z + y\{a + b - 2 - (a - 3)y - y^2\} = 0, \tag{6}$$

where a and b are arbitrary constants and suffixes denote derivatives.

It was shown in Paper I that (4) and (5) are always integrable in terms of elementary functions, but (6) is only so integrable in four special cases, denoted in Paper I by equations (A.26) to (A.29). These four cases, obtained by putting one of the two constants of integration equal to zero, are the group of solutions which will be investigated.

Equations (A.33) and (A.34) of Paper I give, for the density, ρ, and the pressure, p, respectively:

$$8\pi G\rho = 3\left(\frac{S_t}{S}\right)^2 + \frac{c^2e^{-\eta}}{R_0^2S^2}\Big[3(1-f_r^2)f^{-2} - 6(1-y)\frac{Q_rf_r}{Qf} - \{2b - 2y_z + (1-y)(2a-1-y)\}\left(\frac{Q_r}{Q}\right)^2\Big], \quad (7)$$

$$8\pi G\frac{p}{c^2} = \frac{1}{y}\Big[-\frac{2S_{tt}}{S} - (3y-2)\left(\frac{S_t}{S}\right)^2 + \frac{c^2e^{-\eta}}{R_0^2S^2}\Big\{y(1-f_r^2)f^{-2} + 2(y^2-y-y_z)\frac{f_rQ_r}{fQ} + (1-y)(y^2-y-2y_z)\left(\frac{Q_r}{Q}\right)^2\Big\}. \quad (8)$$

To fit the internal solution to an external vacuum Schwarzschild solution it is necessary to put the pressure equal to zero at the boundary:

$$p_b \equiv 0. \quad (9)$$

Henceforth boundary values will be denoted by the suffix b.

In Paper II it was shown that the condition (9) gives the following equation for the scale-function S:

$$2\frac{S_{tt}}{S} + (3y_b - 2)\left(\frac{S_t}{S}\right)^2 + \frac{c^2e^{-\eta_b}}{R_0^2S^2}F(y_b) = 0, \quad (10)$$

where $F(y_b) = B_1y_b + 2(y^2 - y - y_z)_bB_2 + (1 - y_b)(y^2 - y - 2y_z)_bB_3$, and

$$B_1 = \left(\frac{1-f_r^2}{f^2}\right)_b, \quad (11)$$

$$B_2 = \left(\frac{f_rQ_r}{fQ}\right)_b \quad (12)$$

$$B_3 = \left(\frac{Q_r}{Q}\right)_b^2. \quad (13)$$

Inspection of equation (10) shows that it can always be written in the form

$$\frac{d}{dS}(S_t^2) + H(S)S_t^2 + J(S) = 0, \quad (14)$$

where

$$H(S) = \frac{3y_b - 2}{S},$$

and

$$J(S) = \frac{c^2 e^{-\eta_b}}{R_0^2 S} F(y_b).$$

Equation (14) is an ordinary differential equation of first order for S_t^2. When it is solved for S_t^2, we have S_t^2 as a function of S.

In Paper II it was found that without loss of generality one could put:

$$Q_b = 1. \tag{15}$$

As is remarked in Paper II, it is always possible to adjust the constant R_0 in (1) so that S will have the value unity at a pre-assigned instant. It will be assumed that $S = 1$ corresponds to $S_t = 0$, and that the mass is at rest at the initial moment, *i. e.*

$$S_t(0) = 0, \quad S(0) = 1. \tag{16}$$

The four cases, (A.26) to (A.29) of Paper I, will now be dealt with separately.

CASE A. 26

Case A.26 is defined by

$$a \neq 3, b = \frac{-6a^2 + 11a - 4}{25}, \tag{17}$$

and equation (6) is integrable to give the following equation

$$y_z = \frac{a-3}{5} y + \frac{1}{2} y^2. \tag{18}$$

For convenience a new variable, x, is introduced instead of the scale-function by

$$x = S^{\frac{a-3}{5}}.$$

Using condition (16), $S_t = 0$ when $S = 1$, equation (14) can be integrated to give (equ. (2.24) in Paper II):

$$x_t^2 = \frac{a-3}{10}(2n_1 + n_0)\left(\frac{c}{R_0}\right)^2 (1-x)(x-x_1)x^6(1+x)^{-6}, \tag{19}$$

where n_0, n_1 and x_1 are constants defined in Paper II, involving the constants B_1, B_2 and B_3 defined in equations (11) to (13).

Fos oscillatory motions to exist the scale function must vary between unity and a positive value, *i. e.* $x_1 > 0$. The following inequality is then obtained:

$$\frac{-25B_1 + (20a - 10)B_2 + (4a^2 - 4a + 1)B_3}{-75B_1 + (40a + 30)B_2 + (4a^2 + 16a - 9)B_3} < 0.$$

x_t^2 must also be positive during the motion. From equation (19) the following inequality is then obtained:

$$(a - 3)(2n_1 + n_0) > 0.$$

This last inequality can be put into the following form:

$$-75B_1 + (40a + 30)B_2 + (4a^2 + 16a - 9)B_3 > 0.$$

The two constraints: $x_1 > 0$ and $x_t^2 > 0$ then together yields:

$$5B_1 - 2(a + 2)B_2 - (2a - 1)B_3 < 0. \tag{20}$$

It is also required, however, that the density at the boundary, ρ_b, is positive or zero at the initial moment:

$$\rho_b \geq 0 \quad \text{when} \quad S = 1 \tag{21}$$

From equation (7) the following inequality is then easily obtained

$$5B_1 - 2(a + 2)B_2 - (2a - 1)B_3 \geq 0,$$

in contradiction to inequality (20).

We therefore conclude that no oscillatory motion is possible in case A.26.

CASE A. 27

This case is defined by

$$b = 2 - a, \qquad a \neq 3 \tag{22}$$

and equation (6) is integrable to give

$$y_z = -(a - 3)y - y^2. \tag{23}$$

Using condition (16), $S_t = 0$ when $S = 1$, equation (14) can be integrated to give (equ. (2.32) in Paper II):

$$x_t^2 = \frac{3 - a}{8} M_2 \left(\frac{c}{R}\right)^2 (1 - x)(x - x_1)(-x_2)x^{-1}, \tag{24}$$

where M_2, x_1 and x_2 are constants (defined by equs. (2.33) and (2.34) in Paper II) and where now $x = S^{3-a}$.

Inspection of equation (24) shows that oscillations can only exist if the following inequality is satisfied

$$B_1 - (a - 1)B_2 - \frac{1}{4}(a - 1)^2B_3 < 0. \tag{25}$$

However, from equation (7) and the condition (21) for the density at the boundary the following inequality is obtained:

$$B_1 - (a - 1)B_2 - \frac{1}{4}(a - 1)^2B_3 \geq 0.$$

in contradiction to inequality (25).

We therefore conclude that no oscillatory motion is possible in case A.27.

CASE A. 28

This case is defined by

$$a = 3. \tag{26}$$

In the following we discuss various subcases depending on the value of b.

$\underline{b < -1}$:

From equation (6) we get

$$y = \sqrt{-(b+1)}\ \mathrm{tg}\left(\frac{1}{2} z \sqrt{-(b+1)}\right) \tag{27}$$

where a constant of integration is put equal to zero.

From equation (14) is now obtained (after some calculation)

$$S_t^2 = \frac{c^2}{R_0^2} \frac{S^2}{\cos^6 (X/2)} (A \cos X + B \sin X - A) \tag{28}$$

where

$$A = \frac{1}{2} B_1 - B_2 - B_3 - \frac{1}{2} b\, B_3,$$
$$B = \sqrt{-(b+1)}(B_2 + B_3),$$
$$X = \sqrt{-(b+1)} \ln S.$$

Inspection of equation (28) shows that oscillations are possible only if

$$A > 0. \tag{29}$$

However, from equation (7), requiring (21), that the density is positive or zero at the boundary, it is found that

$$A \leq 0,$$

in contradiction to inequality (29).

Hence, no oscillatory motion is possible in this subcase, ($b < -1$).

$\underline{b = -1}$

From equation (6) we now get

$$y = \frac{-2}{z + K}, \tag{30}$$

where K is a constant of integration. If K is put equal to + 1, we obtain from equation (14):

$$S_t^2 = \frac{c^2}{R_0^2} \frac{S^2 \ln S}{(1 - \ln S)^6} D(\ln S - E), \tag{31}$$

where

$$D = -B_1 + 2B_2 + B_3,$$
$$E = 2 \frac{B_1 - 4B_2 - 3B_3}{B_1 - 2B_2 - B_3}.$$

Inspection of equation (31) shows that $S_t = 0$ when $S = 1$, *i. e.* when the motion is to start. Oscillatory motions are possible only if $S_t = 0$ for another positive value of S. S_t^2 must also be positive during the motion. From equation (31) the following inequality is then obtained: $D < 0$, *i. e.*

$$- B_1 + 2B_2 + B_3 < 0. \tag{32}$$

Inspection of (31) also shows that $E < 1$, since $E \geq 1$ would make the denominator equal to zero at some time during the oscillatory motion. From this, the following inequality is obtained:

$$\frac{B_1 - 6B_2 - 5B_3}{B_1 - 2B_2 - B_3} < 0.$$

Hence, using inequality (32), we obtain

$$B_1 - 6B_2 - 5B_3 < 0. \tag{33}$$

From equation (7) with the requirement (21) it is found, however, that

$$B_1 - 6B_2 - 5B_3 \geq 0,$$

is in contradiction to (33). Hence, no oscillatory motions can exist for this subcase.

Without the restriction $K = 1$ we obtain from equation (14):

$$S_t^2 = \frac{c^2}{R_0^2} \frac{S^2 \ln S}{(K - \ln S)^6} (- B_1 + 2B_2 + B_3) \left[\ln S - \frac{2KB_1 - 4(K + 1)B_2 - 2(K + 2)B_3}{B_1 - 2B_2 - B_3}\right]. \tag{34}$$

In this case (with K arbitrary) we make the extra-requirement that $\rho_b = 0$ always.

From equation (7) we then obtain

$$S_t^2 = \frac{c^2}{R_0^2} \frac{S^2}{(K - \ln S)^6} \{ (- B_1 + 2B_2 + B_3)(\ln S)^2 + 2[KB_1 - 2(K + 1)B_2 - (K + 2)B_3] \ln S + K[- KB_1 + 2(K + 2)B_2 + (K + 4)B_3]. \tag{35}$$

The right hand sides of equations (34) and (35) must be identical. Remembering that $(\ln S)^2$ and $\ln S$ and 1 are linearly independent functions and noting that $K = 0$ implies a singularity in the metric, we obtain:

$$K(- B_1 + 2B_2 + B_3) + 4(B_2 + B_3) = 0 \tag{36}$$

With this result it is easily seen that only motions of expansion or contraction are possible in this subcase.

$b > -1$

Using condition (16), $S_t = 0$ when $S = 1$, equation (14) can be integrated to give (equ. (2.29) in Paper II):

$$x_t^2 = \delta n_2 \left(\frac{c}{R_0}\right)^2 (1 - x)(x - x_1)x^4(1 + x)^{-6}, \qquad (37)$$

where δ, n_2 and x_1 are constants defined in Paper II, and $x = S^\delta$.

With the requirements $x_1 > 0$ and $x_t^2 > 0$, necessary for oscillations, the following inequality is obtained from equation (37):

$$B_1 - 2B_2 - (1 + \delta^2)B_3 > 0, \qquad (38)$$

If we require that the density at the boundary is positive at the initial moment, it is easily seen from (7) that we get an inequality equal to (38) and nothing is gained. Using the values given in Table I in Paper II, we find that oscillations are possible provided that

— $1 < b$ using column 1,
— $1 < b < 0$ using columns 2 or 3.

An exhaustive investigation of equation (37) for the function x for all possible values of B_1, B_2 and B_3 is left for a later investigation.

However, if we require that the density at the boundary, ρ_b, is zero at the initial moment,

$$\rho_b = 0 \quad \text{when} \quad S = 1, \qquad (39)$$

it is easily seen from equation (7) that this implies:

$$B_1 - 2B_2 - (1 + \delta^2)B_3 = 0,$$

which is incompatible with (38). Hence, with the requirement (39), no oscillatory motions are possible in this subcase.

The special case $b = 0$

It was shown in Paper II that oscillations are only possible when B_1, B_2 and B_3 take the values of column 1 of Table I in that paper. From column 1 it is seen that

$$\alpha^{-2} > \beta^2 \qquad (40)$$

where α and β are constants of integration defined in Paper II. Inspection of equation (37) shows that oscillatory motions may exist only if

$$\alpha^{-2} < (1 - \beta)^2 \qquad (41)$$

From equation (7) it is found (after some calculations) that the density is positive everywhere in the mass distribution and at any instant if:

$$\frac{3}{x + Q}\{ Q^5[(\beta - 1)^2 - \alpha^{-2}] + x \} > 0 \qquad (42)$$

Since it is also required that $x > 0$ and $Q > 0$, it is seen that inequality (42) is satisfied.

From equation (8) for the pressure it is found (after some lengthy calculations) that

$$8\pi G \frac{p}{c^2} = \frac{x + Q}{x - Q}\left(\frac{c}{R_0}\right)^2 x^2 \frac{Q - 1}{(x + Q)^6(x + 1)^7} R(Q, x), \tag{43}$$

where

$$\begin{aligned} R(Q, x) = {} & [x_1(Q^5 + Q^4 + Q^3 + Q^2 + Q + 7) - (15Q - 27)]x^7 \\ & + [x_1(7Q^5 + 7Q^4 + 7Q^3 + 7Q^2 + 37Q - 5)(8Q^2 - 13Q - 7)]x^6 \\ & + [3x_1Q(7Q^4 + 7Q^3 + 7Q^2 + 27Q - 8) - 5(9Q^3 - 19Q^2 - 7Q - 7)]x^5 \\ & + [5x_1Q^2(7Q^3 + 7Q^2 + 19Q - 9) - 3(8Q^4 - 27Q^3 - 7Q^2 - 7Q - 7)]x^4 \\ & + [5x_1Q^3(7Q^2 + 13Q - 8) - (5Q^5 - 37Q^4 - 7Q^3 - 7Q^2 - 7Q - 7)]x^3 \\ & + [x_1Q^4(27Q - 8) + (7Q^5 + Q^4 + Q^3 + Q^2 + Q + 1)]x^2 \\ & + x_1Q(Q^4 + Q^3 + Q^2 + Q + 1) \end{aligned} \tag{44}$$

From (40) and (41) the following inequality is obtained:

$$\frac{1}{\alpha} < 1.$$

When this is used in the following inequality:

$$\frac{1}{\alpha}\cos r + 1 - \beta < \frac{1}{\alpha} + 1 - \beta$$

(see Table I in Paper II).

One more inequality is obtained:

$$\frac{1}{\alpha}\cos r + 1 - \beta < 2.$$

From the expression for Q given in Table I of Paper II it is then found that:

$$Q > \frac{1}{2}\sqrt{2}.$$

If necessary the co-moving coordinate r is redefined so that $\alpha\beta \leq \cos r \leq 1$. It is then seen that:

$$\left(\frac{1}{\alpha}\cos r + 1 - \beta\right)_{\min} = 1.$$

From the expression for Q given in Table I of Paper II it is then found that:

$$Q_{\max} = 1.$$

Hence, the value of Q is restricted by

$$\frac{1}{2}\sqrt{2} < Q \leq 1. \tag{45}$$

It is then seen that $R(Q, x) > 0$. Since x and Q are independent variables $(x - Q)$ may now take positive values. Hence the pressure p given by equation (43) may take negative values.

Thus, oscillatory motions in this case must be discarded as physically unacceptable.

CASE A. 29

In this case there are no restrictions on the constants a and b.

The investigation is divided into three subcases which are dealt with separately.

$(a-1)^2 + 4b < 0$

From equation (8), using the conditions (9), $p_b \equiv 0$, and (16), $S_t = 0$ when $S = 1$, we obtain

$$S_t^2 = -\frac{c^2}{R_0^2} S^{\frac{3a-5}{2}} \cos^3 u \int S^{\frac{1-a}{2}} \frac{1}{\cos u} \Big\{ \Big(-\sqrt{k}\,\mathrm{tg}\, u - \frac{a-3}{2} \Big) B_1 + 2\Big[2k\,\mathrm{tg}^2 u + (2-a)\sqrt{k}\,\mathrm{tg}\, u + \frac{(a-3)^2}{4} + \frac{a-3}{2} + k \Big] B_2 + \Big(\frac{a-1}{2} - \sqrt{k}\,\mathrm{tg}\, u \Big) \Big[3k\,\mathrm{tg}^2 u + (2-a)\sqrt{k}\,\mathrm{tg}\, u + \frac{(a-3)^2}{4} + \frac{a-3}{2} + 2k \Big] B_3 \Big\} dS, \tag{46}$$

where

$$u = \sqrt{k} \ln S$$

and

$$k = -\frac{(a-1)^2 + 4b}{4}.$$

For this case we require that $\rho_b = 0$ *always*.

From equation (7), again using condition (16), we obtain

$$S_t^2 = -\frac{1}{3}\frac{c^2}{R_0^2} S^{a-1} \cos^2 u \Big\{ 3B_1 - [3(a-1) - 6\sqrt{k}\,\mathrm{tg}\, u] B_2 - \Big[2b + \frac{2k}{\cos^2 u} + \frac{5}{4}(a-1)^2 - 3(a-1)\sqrt{k}\,\mathrm{tg}\, u + k\,\mathrm{tg}^2 u \Big] B_3 \Big\} \tag{47}$$

Combining equations (46) and (47) and differentiating we get (remembering that $1, \frac{1}{\cos u}, \frac{\sin u}{\cos^2 u}, \frac{\sin^2 u}{\cos^3 u}$, and $\frac{\sin^3 u}{\cos^4 u}$ are linearly independent functions):

$$4(a-1)B_1 - 4(a-1)^2 B_2 - (a-1)^3 B_3 = 0$$
$$2(a-1)B_2 + (a-1)^2 B_3 = 0$$
$$(a-1)B_3 = 0$$

These three equations are satisfied if and only if

$$B_1 = B_2 = B_3 = 0 \tag{48}$$

or

$$a = 1, \tag{49}$$

giving either a static solution or motion of expansion or contraction to a singularity.

In the special case $a = 3$ it is sufficient to require $\rho_b = 0$ at the initial moment. From equation (7), using condition (16), $S_t = 0$ when $S = 1$, we obtain

$$B_1 - 2B_2 - B_3 = 0.$$

From equation (8) with the requirement (9), $p_b \equiv 0$, we then obtain

$$S_t^2 = -\frac{c^2}{R_0^2} S^2 \sin u[-kB_3 \sin u + 2\sqrt{k}(B_2 + B_3)\cos u] \tag{50}$$

To avoid singularity in the metric since $y_b = \sqrt{k}\,\mathrm{tg}\,u$, u must take values in the open interval $\left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$. Inspection of equation (50) then shows that for oscillations to exist

$$\lim_{u \to \frac{\pi}{2}} \sin u[-kB_3 \sin u + 2\sqrt{k}(B_2 + B_3)\cos u] > 0.$$

However it is easily seen that this limit is equal to $-k\left(\frac{Q_r}{Q}\right)_b^2 < 0$. Hence, no oscillatory motions are possible in this special case.

$(a - 1)^2 + 4b > 0$

As in the previous case we require $\rho_b = 0$, always. From equation (7), using condition (16) we obtain

$$x_t^2 = -\frac{4}{3}\delta^2 \frac{c^2}{R_0^2} x^{-\frac{a-3}{2\delta}+1}(a_0 + a_1 x + a_2 x^2), \tag{51}$$

where

$$a_0 = 3B_1 - 3(a - 1 + 2\delta)B_2 - 3\left[b + \frac{1}{2}(a-1)^2 + \delta(a-1)\right]B_3 \tag{52}$$

$$a_1 = 6B_1 - (6(a - 1)B_2 + 6bB_3, \tag{53}$$

$$a_2 = 3B_1 - 3(a - 1 - 2\delta)B_2 - 3\left[b + \frac{1}{2}(a-1)^2 - \delta(a-1)\right]B_3, \tag{54}$$

$$x = S^{-2\delta},$$

and

$$\delta^2 = \frac{(a-1)^2 + 4b}{4}.$$

Inspection of equation (51) shows that when $a_2 = 0$ the possible motions are expansion ($a_0 > 0$), contraction ($a_0 < 0$) or static solution ($a_0 = 0$).

In the case $a_2 \neq 0$, using also the condition $x_t = 0$ when $x = 1$ we find that

$$a_0 + a_1 + a_2 = 0. \tag{55}$$

From equations (52), (53) and (54) we get

$$a_0 - a_1 + a_2 = - 12\delta^2 B_3. \tag{56}$$

Adding (55) and (56) yields

$$a_0 + a_2 = - 6\delta^2 B_3. \tag{57}$$

Inspection of equation (51) shows that oscillatory motions are possible only if

$$a_0 > 0 \quad \text{and } a_2 > 0,$$

and this is in contradiction with equation (57).

$(a - 1)^2 + 4b = 0$

From equation (8) for the pressure, using conditions (9), $p_b \equiv 0$ and (16), we obtain:

$$S_t^2 = - \frac{c^2}{R_0^2}(K - \ln S)^3 S^{\frac{3a-5}{2}} \int \frac{S^{\frac{1-a}{2}}}{K - \ln S} F(y_b) dS \tag{58}$$

where $F(y_b)$ is given in equation (10) and K is a constant of integration.

From equation (7), with $\rho_b \equiv 0$ (and imposing condition (16)) we get

$$S_t^2 = - \frac{c^2}{R_0^2}(K - \ln S)^2 S^{a-1} \left\{ \left[B_1 - (a - 1)B_2 - \frac{1}{4}(a - 1)^2 B_3 \right] + [2B_2 + (a - 1)B_3] \frac{1}{K - \ln S} - B_3 \frac{1}{(K - \ln S)^2} \right\}. \tag{59}$$

Combining equations (58) and (59) and differentaiting we get the following three equations (remembering that 1, (K − ln S), $(K - \ln S)^{-2}$ and $(K - \ln S)^{-3}$ are linearly independent functions):

$$- 4(a - 5)B_1 + 4(a - 5)(a - 1)B_2 + (a - 5)(a - 1)^2 B_3 = 0$$
$$2(a - 5)B_2 + (a - 5)(a - 1)B_3 = 0$$
$$(a - 5)B_3 = 0$$

These three equations are satisfied if and only if

$$B_1 = B_2 = B_3 = 0,$$

(giving a static solution), or

$$a = 5.$$

In this latter case we get from equation (59), imposing the condition (16), $S_t = 0$ when $S = 1$:

$$S_t^2 = - \frac{c^2}{R_0^2} S^4 \ln S(T \ln S - U), \tag{60}$$

where

$$T = B_1 - 4B_2 - 4B_3, \tag{61}$$

$$U = 2[KB_1 + (1 - 4K)B_2 + (2 - 4K)B_3], \tag{62}$$

and

$$K = \frac{-B_2 - 2B_3 \pm \sqrt{B_2^2 + B_1B_3}}{T} \tag{63}$$

Inspection of equation (60) shows that oscillatory motions are only possible if

$$T > 0 \tag{64}$$

Following McVittie (see equ. (A.10) in Paper I) we introduce a new radial coordinate q defined by

$$q = \frac{q_b}{Q^4}, \tag{65}$$

By means of McVittie's method of substitution (see equ. (A.23) in Paper I), equation (5), with the new radial coordinate q, can be solved.

We obtain two sets of solutions for the functions f, f_r, Q and Q_r depending on the value of a constant of integration, P. If P = 0 we have:

$$f = \sqrt{2q\,(\ln q - R)}, \tag{66}$$

$$f_r = \ln q + 1 - R, \tag{67}$$

$$Q = \left(\frac{q_b}{q}\right)^{\frac{1}{4}}, \tag{68}$$

$$Q_r = -\frac{q_b^{\frac{1}{4}}\sqrt{2\,(\ln q - R)}}{4q^{\frac{3}{4}}}, \tag{69}$$

where R is still another constant of integration. When P ≠ 0, we have:

$$f = \sqrt{qP(\ln q - R)^2 - \frac{q}{P}}, \tag{70}$$

$$f_r = 1/2\left[P(\ln q - R)^2 - \frac{1}{P}\right] + P(\ln q - R), \tag{71}$$

$$Q = \left(\frac{q_b}{q}\right)^{\frac{1}{4}}, \tag{72}$$

$$Q_r = -\frac{q_b^{\frac{1}{4}}\sqrt{P(\ln q - R)^2 - \frac{1}{P}}}{4q^{\frac{3}{4}}}. \tag{73}$$

We also have from equation (6), using the relation (2)

$$y = \frac{1}{\ln Q - \ln S + K} - 1. \tag{74}$$

Since Q and S are independent variables, inspection of (74) shows that if singularity in the metric is to be avoided, K must be a positive constant. If we choose $P = 0$, then by equations (66) to (69) using the expressions (11), (12) and (13), we obtain:

$$TB_3 = 0 \tag{75}$$

It is easily seen that $B_3 = 0$ implies $B_2 = 0$ and hence $K = 0$, *i. e.* singularity.

Moreover $T = 0$ is in contradiction to (64).

If we choose $P \neq 0$, then by equations (70) to (73) and using the expressions (11), (12) and (13), we obtain

$$T = \frac{1 - W^2}{q_b V}, \tag{76}$$

and

$$K = \frac{(W \pm 1)V}{4(1 - W^2)}, \tag{77}$$

where

$$V = P(\ln q_b - R)^2 - \frac{1}{P}, \tag{78}$$

and

$$W = P(\ln q_b - R). \tag{79}$$

From (76) and (64) it sollows that a necessary condition for oscillations is

$$-1 < W < 1. \tag{80}$$

It is also seen that only the plus sign can be retained in (77) to ensure that K is positive. Inspection of (74) now shows that (80) leads to singularity in the metric. Hence oscillations are not possible in this subcase.

SUMMARY. — The radial motions of a spherically symmetric mass are investigated for a group of solutions of Einstein's equations found by McVittie (Paper I). In particular we have examined the possibilities of oscillations found in Paper II in more detail. Using the expressions for the density and pressure we have checked for inconsistencies or contradictions or for the occurrence of a singularity in the metric. With the general requirements that the pressure is zero at the boundary and the matter configuration is at rest at the initial moment, we have obtained the following results:

a) If the density ot the boundary is zero throughout the motion, *oscillations are not possible.*

b) If the density at the boundary is zero at the initial moment, oscillations are *not* possible in the cases A.26, A.27 and in the subcases A.28 with $b \neq -1$, A.29 with $a = 3$, $b < -1$.

c) If we just require the density to be *positive* at the initial moment, *no* oscillations are possible in either of the cases A.26, A.27 or in the two subcases of A.28: $b < -1$ and $b = 0$.

In addition it was found that in the special case A.28, $b = -1$ with a constant of integration put equal to 1, oscillations are not possible.

d) In case A.28 with $b > -1$ and B_1, B_2, B_3 taken to have values to be found in Table I in Paper II, it is found that *necessary* conditions for oscillations *are satisfied*. But these conditions are *not sufficient*. In one special case ($b = 0$) the pressure is found to take negative values. This solution must therefore be discarded as physically unacceptable.

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