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ROBERT SCHRADER

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A possible constructive approach to ϕ_4^4. II

by

Robert SCHRADER
Institut für Theoretische Physik,
Freie Universität Berlin, Germany

ABSTRACT. — Normalization conditions for a ϕ_d^4 theory imply regularity properties. We discuss the relevance for the non triviality of ϕ_4^4.

In this note we continue our investigations on a constructive euclidean approach to ϕ_4^4 based on multiplicative renormalization [*9*].

Making a slight modification we present an affirmative answer to the following question raised there: Assume it is possible to normalize all lattice approximations of a would be ϕ_d^4 theory in the same way, may the normalization of the truncated 4-point function be preserved in the limit? This would be the last step in establishing non-triviality of the theory.

Indeed, we will show that this is an easy consequence of certain regularity properties of the truncated 4-point function. These regularity properties in turn will be a consequence of the finiteness of the normalization constants.

Our discussion extends results by Glimm and Jaffe in [*4*] on bounds on the coupling constant in the sense that it does not rely on *a posteriori* structures (such as the euclidean axioms) but allows the application to lattice approximations as well.

Let ϕ be the euclidean field and denote by $\langle \quad \rangle$ the expectation with respect to a translation invariant measure μ on $\mathcal{S}'(\mathbb{R}^\alpha)$. Furthermore $\langle A_1, \ldots, A_n \rangle$ denotes the truncated expectation of the random variables $A_1, \ldots, A_n$.

The following quantities (if they exist) will be called normalization constants for μ:

$$y_1 = \int \langle \phi(x)\,;\,\phi(0) \rangle \, dx$$

$$y_2 = \int g(x) \langle \phi(x)\,;\,\phi(0) \rangle \, dx \tag{1}$$

$$y_3 = \int \langle \phi(x_1)\,;\,\phi(x_2)\,;\,\phi(x_3)\,;\,\phi(0) \rangle \, dx_1 dx_2 dx_3$$

If brief, the constructive approach to ϕ_d^4 in [9] is based on the attempt to try to determine the multiplicative renormalization constants (in any lattice approximation) for fixed normalization constants. In [9] the choice of $g(\rho)$ was ρ^2. Here we will allow g to be any non negative monotonically increasing function of ρ. g will be called an indicator function. From the finiteness of y_i $(i = 1, 2, 3)$ we will derive regularity properties of the truncated two- and four-point function.

We note that the method of estimating quantities in terms of two point functions has become an important tool in euclidean quantum field theory [3] [11] and was initiated by the basic work of Lebowitz [5] [7]. In the language of statistical mechanics, y_1 corresponds to the susceptibility and y_3 to the derivative of the susceptibility with respect to h^2 at $h = 0$, h being the external field.

If $g(\rho) = \rho^\tau(\tau > 0)$, $(y_2 y_1^{-1})^{\frac{1}{\tau}}$ corresponds to a particular definition of the correlation length ξ [2]. Thus a given indicator function g serves to measure the decay of the truncated two-point function. In statistical mechanics the decay rate is known to imply analyticity properties of the thermodynamic functions [6] [1]. We consider our discussion close in spirit. The case

$$g(\rho) = e^{m_0 \rho} \tag{2}$$

will be of particular interest. We will call m_0 the mass gap. This choice of g corresponds to the following definition of the correlation length:

$$\xi = \frac{1}{m_0} \ln \left(\frac{y_2}{y_1} \right) \tag{3}$$

We note that the discussion in [9] may easily be adapted to the cases when $g(\rho)$ is either ρ^τ or of the form (2). These are the cases we shall discuss in more detail.

Now in [9] for each torus $\mathscr{T}$ and parameters α_1, α_2, α_3 we constructed a probability measure $\mu(\alpha_1, \alpha_2, \alpha_3)$ on $\mathbb{R}^{|\mathcal{T}|}$ such that with given lattice spacing a the normalization conditions took the form

$$\begin{aligned} y_1 &= \frac{a^2}{|\mathcal{T}|} \sum_{i,j \in \mathcal{T}} \langle x_i x_j \rangle \\ y_2 &= \frac{a^2}{|\mathcal{T}|} \sum_{i,j \in \mathcal{T}} g(a\,|\,i-j\,|) \langle x_i x_j \rangle \\ y_3 &= -\frac{a^{4+d}}{|\mathcal{T}|} \sum_{i,j,k,l \in \mathcal{T}} \langle x_i\,;\,x_j\,;\,x_k\,;\,x_l \rangle \end{aligned} \tag{1'}$$

where now $\langle\ \ \rangle$ denotes the expectation w. r. t. this measure. Assume now for given g and fixed $y_i > 0$ $(i = 1, 2, 3)$ these normalization conditions may be satisfied for all $\mathcal{T}$ and a. (see conjecture 3 in [9]). This determines $\alpha_i = \alpha_i(\mathcal{T}, a)(i = 1, 2, 3)$. We then define the following functions which are analytic on $\mathbb{C}^{(n-1)d}$ and which when restricted to $\mathbb{R}^{(n-1)d}$ define tempered distributions (the approximate, truncated euclidean Green's functions in momentum space)

$$\tilde{\mathrm{S}}^{\mathrm{T}}_{n,(\mathcal{T},a)}(p_1, \ldots, p_{n-1}) = a^{n-d+\frac{nd}{2}} \sum_{j_l \in \mathcal{T}} e^{ia\sum_{l=1}^{n-1} p_l j_l} \langle x_{j_1}\,;\,\ldots\,;\,x_{j_{n-1}}\,;\,x_0 \rangle \tag{4}$$

From the normalization conditions (1') we obtain

$$\begin{aligned} \tilde{\mathrm{S}}^{\mathrm{T}}_{2,(\mathcal{T},a)}(0) &= y_1 \\ -\,\tilde{\mathrm{S}}^{\mathrm{T}}_{4,(\mathcal{T},a)}(0, 0, 0) &= y_3 \end{aligned} \tag{5}$$

By Griffiths first inequality

$$|\,\tilde{\mathrm{S}}^{\mathrm{T}}_{2,(\mathcal{T},a)}(p)\,| < y_1 (p \in \mathbb{R}^d) \tag{6}$$

and by Lebowitz inequality

$$|\,\tilde{\mathrm{S}}^{\mathrm{T}}_{4,(\mathcal{T},a)}(p_1, p_2, p_3)\,| < y_3 (p_1, p_2, p_3 \in \mathbb{R}^d) \tag{7}$$

Depending on the choice of the indicator function g we obtain more detailed information. With the estimate

$$|\,e^{ipx} - e^{ip'x}\,| < 2\,|\,x\,|^\theta\,|\,p\,|^\theta \qquad (0 \leqslant \theta \leqslant 1)$$

the proof of the first lemma is trivial.

LEMMA 1. — For $g(\rho)$ of the form ρ^τ $(\tau > 0)$ the $\tilde{\mathrm{S}}^{\mathrm{T}}_{2,(\mathcal{T},a)}(\cdot)$ satisfy the uniform estimates

$$\begin{aligned} \left|\frac{\partial^m}{\partial p^m}\tilde{\mathrm{S}}^{\mathrm{T}}_{2,(\mathcal{T},a)}(p)\right| &< \mathrm{C}(\tau)(y_1 + y_2) \\ \left|\frac{\partial^m}{\partial p^m}\tilde{\mathrm{S}}^{\mathrm{T}}_{2,(\mathcal{T},a)}(p) - \frac{\partial^m}{\partial p^m}\tilde{\mathrm{S}}^{\mathrm{T}}_{2,(\mathcal{T},a)}(p')\right| &< \mathrm{C}(\theta, \tau)\,|\,p - p'\,|^\theta (y_1 + y_2) \end{aligned} \tag{8}$$

for any m and $0 \leqslant \theta \leqslant 1$ with $|m| + \theta \leqslant \tau$, all $p, p' \in \mathbb{R}^d$ and $(\mathcal{T}, a)$. In particular they form a bounded family of equicontinuous functions. If g is of the form (2), the $\tilde{S}^{T}_{2,(\mathcal{T},a)}(\cdot)$ are a bounded family of analytic functions in the strip $|\operatorname{Im} p| < m_0$.

The next lemma is more interesting. Let now $p = (p_1, p_2, p_3)$ and let $m = (m_1, m_2, m_3)$ be a multiindex of length $3d$.

LEMMA 2. — If $g(\rho)$ is of the form $\rho^{\tau}(\tau > d)$ the $\tilde{S}^{T}_{4,(\mathcal{T},a)}$ satisfy the uniform estimates

$$\left| \frac{\partial^m}{\partial p^m} \tilde{S}^{T}_{4,(\mathcal{T},a)}(p) \right| < C(\tau, d) y_1^2 \left(1 + \frac{y_2}{y_1}\right)$$

$$\left| \frac{\partial^m}{\partial p^m} \tilde{S}^{T}_{4,(\mathcal{T},a)}(p) - \frac{\partial^m}{\partial p'^m} \tilde{S}^{T}_{4,(\mathcal{T},a)}(p') \right| < C(\tau, \theta, d) \, | p - p' |^{\theta} \, . \, y_1^2 \left(1 + \frac{y^2}{y_1}\right) \tag{9}$$

for all m and all $0 \leqslant \theta \leqslant 1$ with $|m| + \theta < \frac{\tau - d}{3}$. In particular they form a bounded family of equicontinuous functions on $\mathbb{R}^{3d}$. If $g(\rho)$ is of the form (2), they form a bounded family of analytic functions in any strip

$$|\operatorname{Im} p_j| < \frac{m_0}{3} - \varepsilon \qquad (\varepsilon > 0, \ j = 1, 2, 3).$$

Remark. — Analyticity properties of the form given in Lemma 1 and 2 have been known long for the Fourier transforms of the euclidean Green's functions of a Wightman theory [*8*]. Lemma 2 is closely related to theorem 2.2 in [*4*]. We expect that analogous properties will follow for the higher order Green's functions from the conjectured inequalities

$$(-1)^{\frac{n}{2}+1} \langle x_1 ; \ldots ; x_n \rangle \geqslant 0 .$$

For the proof, we extend arguments employed in [*4*] [*10*]. We only prove the second inequality in (9), the first one being easier. Due to Lebowitz inequality, we have

$$0 < - \langle x_{j_1} ; x_{j_2} ; x_{j_3} ; x_0 \rangle < \sum_{q=1}^{8} (A_q)^{\frac{1}{3}} \tag{10}$$

where each A_q is either

$$\langle x_{j_1} x_0 \rangle \langle x_{j_2} x_0 \rangle \langle x_{j_3} x_0 \rangle \langle x_{j_1} x_{j_2} \rangle \langle x_{j_1} x_{j_3} \rangle \langle x_{j_2} x_{j_3} \rangle$$

or a permutation of

$$\langle x_{j_1} x_0 \rangle^2 \langle x_{j_2} x_{j_3} \rangle^2 \langle x_{j_2} x_0 \rangle \langle x_{j_1} x_{j_3} \rangle$$

Let m and θ satisfy the conditions of the lemma, then

$$\left| \frac{\partial^m}{\partial p^m} \tilde{S}^{T}_{4,(\mathcal{T},a)}(p) - \frac{\partial^m}{\partial p^m} \tilde{S}^{T}_{4,(\mathcal{T},a)}(p') \right| < 2^{\theta} \sum_{q=1}^{8} a^{4+d+|m|+\theta} \sum_{j=(j_1,j_2,j_3)\in\mathcal{T}^3} |j|^{m+\theta} (A_q)^{\frac{1}{3}}$$

Choose $\kappa > d$ such that $\kappa + 3(|m| + \theta) < \tau$.

Using the trivial estimates

$$\alpha^n < (1+\alpha)^{n'} \qquad n' \geqslant n,\ \alpha \geqslant 0$$

$$\left(\sum_{i=1}^{l} \alpha_i\right)^{\theta} < l^{\theta} \sum_{i=1}^{l} \alpha^{\theta} < l^{\theta+1} \operatorname*{Max}_i (\alpha_i)$$

$$\alpha_i \geqslant 0\,; \qquad 0 \leqslant \theta < \infty$$

and Hölder's inequality, we obtain

$$\sum_{j\in\mathcal{T}^3} (a\,|\,j\,|)^{|m|+\theta} [\langle x_{j_1}x_0 \rangle \langle x_{j_2}x_0 \rangle \langle x_{j_3}x_0 \rangle \langle x_{j_1}x_{j_2} \rangle \langle x_{j_2}x_{j_3} \rangle \langle x_{j_1}x_{j_3} \rangle]^{\frac{1}{3}}$$

$$\leqslant 3^{|m|+\theta+1} \operatorname*{Max}_{\mu_i\,:\,\sum_{i=1}^{3}\mu_i = |m|+\theta}$$

$$\sum_{j\in\mathcal{T}^3} [(a\,|\,j_1\,|)^{3\mu_1}(1+a\,|\,j_1\,|)^{\kappa}(1+a\,|\,j_3\,|)^{-\kappa} \langle x_{j_1}x_0 \rangle \langle x_{j_2}x_{j_3} \rangle]^{\frac{1}{3}}$$

$$[(a\,|\,j_2\,|)^{3\mu_2}(1+a\,|\,j_2\,|)^{\kappa}(1+a\,|\,j_1\,|)^{-\kappa} \langle x_{j_2}x_0 \rangle \langle x_{j_1}x_{j_3} \rangle]^{\frac{1}{3}}$$

$$[(a\,|\,j_3\,|)^{3\mu_3}(1+a\,|\,j_3\,|)^{\kappa}(1+a\,|\,j_2\,|)^{-\kappa} \langle x_{j_3}x_0 \rangle \langle x_{j_1}x_{j_2} \rangle]^{\frac{1}{3}}$$

$$< 3.6^{\frac{\tau-d}{3}} \left(\sum_{j\in\mathcal{T}} (1+a\,|\,j\,|)^{-\kappa} a^d \right) a^{-4-d} y_1^2 \left(1+\frac{y_2}{y_1}\right)$$

Similarly using

$$|\,j\,| < 3(|\,j_1\,| + |\,j_1 - j_2\,| + |\,j_3 - j_2\,|)$$

we have

$$\sum_{j\in\mathcal{T}^3} (a\,|\,j\,|)^{|m|+\theta} [\langle x_{j_1}x_0 \rangle^2 \langle x_{j_2}x_{j_3} \rangle^2 \langle x_{j_2}x_0 \rangle \langle x_{j_1}x_{j_3} \rangle]^{\frac{1}{3}}$$

$$< 3^{2\left(\frac{\tau-d}{3}\right)+1} \operatorname*{Max}_{\mu_i\ \Sigma\mu_i = |m|+\theta}$$

$$\sum_{j\in\mathcal{T}^3} [(a\,|\,j_1\,|)^{3\mu_1}(1+a\,|\,j_1\,|)^{\kappa}(1+a\,|\,j_1-j_2\,|)^{-\kappa} \langle x_{j_1}x_0 \rangle \langle x_{j_2}x_{j_3} \rangle]^{\frac{1}{3}}$$

$$[(a\,|\,j_1-j_2\,|)^{3\mu_2}(1+a\,|\,j_1-j_2\,|)^{\kappa}(1+a\,|\,j_2-j_3\,|)^{-\kappa} \langle x_{j_1}x_0 \rangle \langle x_{j_1}x_{j_2} \rangle]^{\frac{1}{3}}$$

$$[(a\,|\,j_2-j_3\,|)^{3\mu_3}(1+a\,|\,j_2-j_3\,|)^{\kappa}(1+a\,|\,j_1\,|)^{-\kappa} \langle x_{j_2}x_0 \rangle \langle x_{j_2}x_{j_3} \rangle]^{\frac{1}{3}}$$

$$< 3.(18)^{\frac{\tau-d}{3}} \left(\sum_{j\in\mathcal{T}} (1+a\,|\,j\,|)^{-\kappa} a^d \right) a^{-4-d} y_1^2 \left(1+\frac{y_2}{y_1}\right)$$

The remaining terms are estimated similarly.

This proves (9), since

$$\sum_{j\in\mathcal{T}}(1 + a|j|)^{-\kappa}a^d$$

is bounded by a constant $C(\theta, \tau)$ uniformly in $\mathcal{T}$ and a.

As for the second part, we start with the *a priori* estimate

$$\left|\sum_{j_1,j_2,j_3\in\mathcal{T}} e^{ia\sum_{l=1}^{3}p_l j_l}\langle x_{j_1}; x_{j_2}; x_{j_3}; x_0\rangle\right| < \sum_{q=1}^{8}\sum_{j_1,j_2,j_3\in\mathcal{T}} e^{a\sum_{l=1}^{3}|\mathrm{Im}\, p_l||j_l|}(A_q)^{\frac{1}{3}}$$

and then proceed analogously to the proof of the first part of the lemma. The claimed analyticity with bounds now follows easily.

We are now in a position to discuss the question raised at the beginning. Indeed we have the following.

THEOREM. — Let $(\mathcal{T}_l, a_l)$ be a sequence with $a_l^d|\mathcal{T}_l| \to \infty$, $a_l \to 0$ such that the $\tilde{S}^T_{n,(\mathcal{T}_l,a_l)}$ converge in $\mathbb{R}^{(n-1)d}$ to a distribution $\tilde{S}^T_n$ for all $n > 2$. If $g(\rho)$ is chosen to be of the form $\rho^\kappa(\kappa > d)$ then $\tilde{S}^T_2$ and $\tilde{S}^T_4$ are bounded continuous functions on $\mathbb{R}^d$ and $\mathbb{R}^{3d}$ respectively and the relations (5) are preserved in the limit. The same is true if g is chosen to be of the form (2). Moreover in that case $\tilde{S}^T_2$ and $\tilde{S}^T_4$ have analytic extensions into the strips $|\mathrm{Im}\, p| < m_0$ and $|\mathrm{Im}\, p_j| < \frac{m_0}{3}$ $(j = 1, 2, 3)$ respectively.

Remark. — The existence of such a sequence $(\mathcal{T}_l, a_l)$ is established by the discussion in [*3*]. Also the limiting euclidean Green's functions $\tilde{S}^T_n$ may be obtained from the moments of a measure on $\mathcal{S}'(\mathbb{R}^d)$ which exists by Minlos theorem. Our theorem says in particular that this measure is nontrivial (i. e. non-Gaussian) if $y_3 > 0$. It is instructive to compare this theorem with corollary 2.4 in [*4*].

The proof is easy. Indeed, using Ascoli's lemma (or the fact that a bounded family of analytic functions is normal) we may find convergent subsequences of the $S^T_{n,(\mathcal{T}_l,a_l)}(n = 2$ and $4)$ (in the sense of continuous functions or analytic functions respectively). If there were two subsequences with different limits, they would also differ qua distributions. This, however, contradicts the assumptions and the theorem is proved.

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