ANNALES DE L'I. H. P., SECTION A

J. GUNSON

Physical states on quantum logics. I

Annales de l'I. H. P., section A, tome 17, nº 4 (1972), p. 295-311
<http://www.numdam.org/item?id=AIHPA_1972__17_4_295_0>

Ann. Inst. Henri Poincaré,
Vol. XVII, n° 4, 1972, p. 295-311.

Section A :
Physique théorique.

Physical States on Quantum Logics. I

by

J. GUNSON

Department of Mathematical Physics,
University of Birmingham, England

ABSTRACT. — We obtain some continuity properties of countably additive measures on the projection lattice of a continuous von Neumann factor. In the hyperfinite case, we prove a generalised form of Gleason's theorem.

RÉSUMÉ. — Nous obtenons quelques propriétés de continuité des mesures dénombrablement additives sur le treillis des projecteurs d'un facteur de von Neumann continu. En le cas hyperfini, nous prouvons une forme généralisée du théorème de Gleason.

1. Introduction

In the propositional calculus approach to the foundations of quantum mechanics pioneered by Birkhoff and von Neumann [1], one starts out with a set $\mathfrak{L}$ of experimentally verifiable propositions, also called " questions " by Mackey [2], which possess a natural ordering induced by a relation of implication. For details, we refer to an earlier paper [3] where the basic ideas are briefly summarised in a set of structure axioms A.1 to A.6. With these, $\mathfrak{L}$ forms an orthomodular partially ordered set, called a generalised quantum logic. A somewhat more restrictive definition is given by Varadarajan [4], who defines a quantum logic to be a σ-complete orthomodular lattice. A basic problem is that of characterising all physical states on a given quantum logic. Depending on just what restrictions are imposed, there are several ways of giving a precise formulation to this problem. Here we choose the following :

1.1. PROBLEM. — *Let $\mathfrak{L}$ be a σ-complete orthomodular lattice. Characterise all countably additive measures on $\mathfrak{L}$.*

For completeness, we give the relevant definitions :

1.2. DEFINITION. — A σ-complete orthomodular lattice is a triple $\{\mathfrak{L}, >, {}^{\perp}\}$ consisting of a set $\mathfrak{L}$, an order relation $>$ making $\mathfrak{L}$ into a σ-complete lattice (i. e. all countable joins and meets exist) with 0 and 1, an orthocomplementation $a \to a^{\perp}$ satisfying $a \vee a^{\perp} = 1$, $a \wedge a^{\perp} = 0$, $a \leq b \Rightarrow a^{\perp} \geq b^{\perp}$, $a^{\perp\perp} = a$ and the orthomodular identity :

$$a \leq b \quad \Rightarrow \quad b = a \vee (b \wedge a^{\perp}).$$

1.3. DEFINITION. — A countably additive measure μ on a σ-complete orthomodular lattice $\mathfrak{L}$ is a map $\mu : \mathfrak{L} \to [0, 1]$ satisfying $\mu(0) = 0$, $\mu(1) = 1$ and if $\{a_n\}$ is a countable set of mutually orthogonal elements of $\mathfrak{L}$, then $\mu\left(\bigvee_n a_n\right) = \sum_n \mu(a_n)$.

This is clearly a generalisation of the well-known concept of a mesure when $\mathfrak{L}$ is a Boolean σ-algebra. The prototype of a quantum logic is the projection lattice $\mathfrak{L}(\mathcal{B}(\mathcal{H}))$ of the von Neumann algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In this case, a complete solution to problem 1.1 was given by Gleason [5].

1.4. THEOREM. — *Let $\mathcal{H}$ be a real or complex separable Hilbert space of dimension ≥ 3. Then every countably additive measure μ on $\mathfrak{L}(\mathcal{B}(\mathcal{H}))$ has the form*

$$\mu(\mathrm{E}) = \mathrm{tr}(\mathrm{TE}), \qquad \mathrm{E} \in \mathfrak{L}(\mathcal{B}(\mathcal{H})) \tag{1.1}$$

where T is a trace-class positive linear operator (depending on μ), satisfying $\mathrm{tr}(\mathrm{T}) = 1$.

The essential point about Gleason's result is that μ is the restriction to $\mathfrak{L}(\mathcal{B}(\mathcal{H}))$ of a normal state on $\mathcal{B}(\mathcal{H})$. In particular, μ extends to a linear functional on the algebra. Consequently, one is led to conjecture that this statement holds in the more general case of an arbitrary von Neumann factor. In this paper, we present some partial results towards a verification of this generalised Gleason theorem, in that we are able to prove some nice continuity properties of these measures which suffice to give a complete proof in the hyperfinite case. Fortunately, this includes many factors of interest to physicists. Although the non-hyperfinite case still remains elusive, it is still very plausible that there is sufficient mobility in the projection lattice given by the automorphism group for the Gleason theorem to hold.

The hyperfinite case has been considered also by Davies [6] and by Aarnes [7] who gave proofs of Gleason's theorem, but only by introducing extra continuity assumptions which make the proofs rather easy. The dimension function of a type II_1 factor is manifestly a countably additive measure. In solving the problem of the " additivity of the trace ", Murray and von Neumann [8] effectively showed that this measure is the restriction of a normal state, the canonical trace. However, their method of proof depends heavily on unitary invariance and apparently does not generalise. A somewhat different approach has been followed by Turner [9], who develops an integration theory for finite factors and, by making rather strong integrability assumptions, proves Gleason's theorem for finite factors.

2. Continuity in Operator Norm

In this section $\mathcal{A}$ denotes a factor acting on a separable Hilbert space $\mathcal{H}$ (or, more generally, a countably decomposable factor). The uppercase letters E, F, G, H are reserved for (orthogonal) projectors in $\mathcal{A}$, whilst U, V, W are reserved for partial isometries in $\mathcal{A}$. The orthogonal complement of any projector E in $\mathcal{A}$ is denoted $E^{\perp}$. The reduced algebra $E \mathcal{A} A$ is denoted $\mathcal{A}_E$.

2.1. Lemma. — *Let* E, F *be projectors in* $\mathcal{A}$ *satisfying* $\| E - F \| < 1$. *Then* (i) $E \wedge F^{\perp} = E^{\perp} \wedge F = 0$ *and* (ii) $E \sim F$.

Proof. — (i) If $x \in E \wedge F^{\perp} \mathcal{H}$, then $\| (E - F) x \| = \| x \|$, whence $x = 0$. A similar argument works for $x \in E^{\perp} \wedge F \mathcal{H}$. (ii) If $\mathcal{A}$ is properly infinite, then the equivalence of E and F is immediate, so we may assume that $\mathcal{A}$ is semifinite. If E and F are not equivalent, then we may take, without loss of generality, $E < F$. This implies that E is finite and moreover that there is a finite projector $G \leq F$ such that $E < G$. The reduced factor $\mathcal{A}_{E \vee G}$ is finite and hence admits a normalised dimension function $\mathcal{D}$. If all orthogonal complements are taken relative to $E \vee G$, then the general additivity of $\mathcal{D}$ gives

$$\mathcal{D}(E \wedge G) < \mathcal{D}(G) + \mathcal{D}(E^{\perp}) = \mathcal{D}(E^{\perp} \vee G) + \mathcal{D}(E^{\perp} \wedge G).$$

But $\mathcal{D}(E^{\perp} \vee G) \leq \mathcal{D}(E \vee G)$. Together, these give $\mathcal{D}(E^{\perp} \wedge G) > 0$, leading to $E^{\perp} \wedge F > 0$, in contradiction with (i).

2.2. Lemma. — *Let* E, F *be equivalent finite projectors in* $\mathcal{A}$. *Then* $E \wedge F^{\perp}$ *is equivalent to* $E^{\perp} \wedge F$.

Proof. — Taking orthogonal complements and dimensions relative to the finite projector $E \vee F$, we get

$$\begin{aligned} 1 &= \mathcal{D}(E \vee F) = \mathcal{D}(F) + \mathcal{D}(F^{\perp}) = \mathcal{D}(E) + \mathcal{D}(F^{\perp}) \\ &= \mathcal{D}(E \vee F^{\perp}) + \mathcal{D}(E \wedge F^{\perp}) = 1 - \mathcal{D}(E^{\perp} \wedge F) + \mathcal{D}(E \wedge F^{\perp}). \end{aligned}$$

Thus $\mathcal{D}(E^{\perp} \wedge F) = \mathcal{D}(E \wedge F^{\perp})$ and the result follows.

2.3. LEMMA. — *Let* E, F *be projectors on* $\mathcal{A}$ *such that* $E \wedge F^{\perp}$ *and* $E^{\perp} \wedge F$ *are equivalent. Then there is a canonical decomposition of* F *with respect to* E *in the form*

(2.1) $$F = X + V(X - X^2)^{1/2} + (X - X^2)^{1/2} V^* + V(I - X) V^*$$

where $X = EFE$ *and* V *is a partial isometry in* $\mathcal{A}$ *with initial projector* $E - E \wedge F$ *and final projector* $E^{\perp} - E^{\perp} \wedge F^{\perp}$.

Proof. — Clearly

$$F = X + E^{\perp} FE + EFE^{\perp} + E^{\perp} FE^{\perp}.$$

Let $V_0 \mid E^{\perp} FE \mid$ be the polar decomposition of $E^{\perp} FE$. Then

$$\mid E^{\perp} FE \mid = (EFE^{\perp} FE)^{1/2} = (EFE - (EFE)^2)^{1/2} = (X - X^2)^{1/2}.$$

Moreover V_0 has initial projector

$$E - E \wedge F^{\perp} - E \wedge F$$

onto $\overline{R}((X - X^2)^{1/2})$ and final projector $E^{\perp} - E^{\perp} \wedge F^{\perp} - E^{\perp} \wedge F$ onto $\overline{R}(E^{\perp} FE)$. By taking the adjoint, we get $EFE^{\perp} = (X - X^2)^{1/2} V_0^*$.

Since, by assumption, $E \wedge F^{\perp} \sim E^{\perp} \wedge F$, there is a partial isometry V_1 with initial projector $E \wedge F^{\perp}$ and final projector $E^{\perp} \wedge F$. Clearly $V_0 V_1^* = V_1^* V_0 = 0$. Hence $V = V_0 + V_1$ is a partial isometry with initial projector $E - E \wedge F$ and final projector $E^{\perp} - E^{\perp} \wedge F^{\perp}$. From $V_1 (X - X^2)^{1/2} = 0$, we get $E^{\perp} FE = V(X - X^2)^{1/2}$, thus giving the second and third terms in the RHS of (2.1). It remains to show that

(2.2) $$E^{\perp} FE^{\perp} = V(I - X) V^*.$$

Consider the three cases :

(*a*) $x \in (E^{\perp} - E^{\perp} \wedge F^{\perp} - E^{\perp} \wedge F)\mathcal{H}$. The identity

$$EFE^{\perp} = (EFE)(EFE^{\perp}) + (EFE^{\perp})(E^{\perp} FE^{\perp})$$

then gives

$$(X - X^2)^{1/2} V^* x = X(X - X^2)^{1/2} V^* x + (X - X^2)^{1/2} V^* (E^{\perp} FE^{\perp}) x.$$

Premultiplying by V and noticing that x lies in the support of $V(X - X^2)^{1/2} V^*$, we have $E^{\perp} FE^{\perp} x = VV^* x - VXV^* x$.

(*b*) $x \in (E^{\perp} \wedge F)\mathcal{H}$. In this case $V^* x = V_1^* x \in (E \wedge F^{\perp}) H$, so that $XV^* x = 0$. Also $VV^* x = x$ and (2.2) is again satisfied.

(*c*) $x \in (E^{\perp} \wedge F^{\perp})\mathcal{H}$. In this case $V^* x = 0$ and also $FE^{\perp} x = 0$, so that (2.2) is trivially satisfied. Combining (*a*), (*b*) and (*c*) we get $E^{\perp} FE^{\perp} = V(I - X) V^*$ on $E^{\perp} \mathcal{H}$, which extends immediately to the whole of $\mathcal{H}$.

2.4. Corollary. — *By setting* $W = X^{1/2} + V(I - X)^{1/2}$ *we get* $WW^* = F$ *and* $W^*W = E$. *Hence* W *is a partial isometry expressing the equivalence of* E *and* F. *From lemmas* 2.1 *and* 2.2 *we see that the conditions of lemma* 2.3 *are satisfied if* E *and* F *are equivalent finite projectors or, alternatively,* $\| E - F \| < 1$.

2.5. Definition. — Two projectors E, F in $\mathcal{A}$ are isoclinic if they are equivalent and there is an angle $\alpha \in \left[0, \frac{\pi}{2}\right]$ with

$$EFE = \cos^2 \alpha\, E, \qquad FEF = \cos^2 \alpha\, F. \tag{2.3}$$

This term is due to Wong [10] and states roughly that the subspaces $E\mathcal{H}$ and $F\mathcal{H}$ are mutually inclined at a constant angle α. Straightforward consequences of this definition are : (*a*) if $\alpha \neq 0$, then $E \wedge F = 0$; (*b*) if $\alpha \neq \frac{\pi}{2}$, then

$$E \wedge F^{\perp} = E^{\perp} \wedge F = 0$$

and

$$F = E \cos^2 \alpha + V \cos \alpha \sin \alpha + V^* \cos \alpha \sin \alpha + VV^* \sin^2 \alpha$$

where V is the isometric part of $E^{\perp} FE$ (*cf.* lemma 2.3); (*c*) if $\alpha \neq \frac{\pi}{2}$, then the equations (2.3) imply that the projectors E and F are equivalent (*cf.* lemma 2.3); (*d*) if E and F are equivalent finite projectors, then each of the equations in (2.3) implies the other one.

2.6. Lemma. — *Let* F, G *be two projectors in* $\mathcal{A}$ *with* $\| F - G \| < 1$ *and* $F \precsim F^{\perp} \wedge G^{\perp}$. *Then there is a projector* H *in* $\mathcal{A}$ *which is isoclinic to both* F *and* G *with angles* α_1 *and* α_2 *respectively and satisfying* $\alpha_1 + \alpha_2 < \frac{\pi}{2}$, *if and only if*

$$| \alpha_1 - \alpha_2 | F \leqq \Theta \leqq (\alpha_1 + \alpha_2) F, \tag{2.4}$$

where

$$\Theta = \sin^{-1} \{ (FG^{\perp} F)^{1/2} \}.$$

Remark. — The requirement (2.4) is a natural one, in that it gives a triangle inequality for all points in the spectrum of Θ. The latter may be regarded as the closure of the set of all (stationary) angles of inclination from $F\mathcal{H}$ to $G\mathcal{H}$. The condition $F \precsim F^{\perp} \wedge G^{\perp}$ ensures that there is enough " room " in $\mathcal{H}$ to encompass all three projectors F, G and H. The example of two-dimensional projectors in a three-dimensional Euclidean space, for which the theorem fails, shows that such a condition is necessary.

Proof. — From lemma 2.3 and its corollary, we may write

$$G = (A^{1/2} + W(F - A)^{1/2})(A^{1/2} + (F - A)^{1/2} W^*), \tag{2.6}$$

where $A = FGF$, $W^* W = F - F \wedge G$, $WW^* = F^\perp - F^\perp \wedge G^\perp$. Setting $(F - A)^{1/2} = (FG^\perp F)^{1/2} = \sin \Theta$ and $\cos \Theta = (I - \sin^2 \Theta)^{1/2}$, we can define the " angle " operator Θ satisfying $0 \leqq \Theta < F \frac{\pi}{2}$. The condition $\| F - G \| < 1$ implies $\| FG^\perp F \| < 1$ and ensures that the spectrum of Θ lies in the half-open interval $\left(0, \frac{\pi}{2}\right)$.

The " if " part of the proof can be reduced to finding a partial isometry V satisfying

(2.7) $$V^* V = F, \qquad VV^* \leqq F^\perp$$

and

(2.8) $$H = (F \cos \alpha_1 + V \sin \alpha_1)(F \cos \alpha_1 + V^* \sin \alpha_1),$$

with

(2.9) $$HGH = \cos^2 \alpha_2 H, \qquad GHG = \cos^2 \alpha_2 G,$$

after selecting the values of α_1 and α_2 in accordance with the condition (2.5) and the hypothesis $\alpha_1 + \alpha_2 < \frac{\pi}{2}$. By substituting (2.6) and (2.8) into the first equation of (2.9), we get, after some cancellations

(2.10) $$(F \cos \alpha_1 + V^* \sin \alpha_1)(F \cos \Theta + W \sin \Theta)(F \cos \Theta + \sin \Theta W^*) \times (F \cos \alpha_1 + V \sin \alpha_1) = F \cos^2 \alpha_2.$$

Setting $U = \dfrac{F \cos \alpha_1 \cos \Theta + V^* W \sin \alpha_1 \sin \Theta}{\cos \alpha_2}$, we can write (2.10) in the form $UU^* = F$. Similarly, from the second equation of (2.9), we get $U^* U = F$. We now proceed to construct a candidate for V in the special case $U = F$.

The condition (2.4) can be written in the form

(2.11) $$\Theta + \alpha_1 F \geqq \alpha_2 F \geqq | \Theta - \alpha_1 F |.$$

Using the monotonicity of $\cos \alpha$ for $0 \leqq \alpha \leqq \pi$, this gives

(2.12) $$\cos (\Theta + \alpha_1 F) \leqq F \cos \alpha_2 \leqq \cos | \Theta - \alpha_1 F |,$$

or

(2.13) $$- \sin \Theta \sin \alpha_1 \leqq F (\cos \alpha_2 - \cos \alpha_1 \cos \Theta) \leqq \sin \Theta \sin \alpha_1.$$

Let B be the self-adjoint operator satisfying

(2.14) $$B \sin \Theta \sin \alpha_1 = F (\cos \alpha_2 - \cos \alpha_1 \cos \Theta)$$

on the support of $\sin \Theta$ ($= F - F \wedge G$) and vanishing on its orthogonal complement. The condition (2.13) tells us that B is a self-adjoint contraction. Moreover, using the hypothesis $F \lesssim F^{\perp} \wedge G^{\perp}$, we can find a partial isometry W′ satisfying $W'^* W' = F$ and $W' W'^* \leq F^{\perp} \wedge G^{\perp}$. We claim that the operator $WB + W'(F - B^2)^{1/2}$ satisfies all the requirements for V, viz. equations (2.7) to (2.9).

Firstly, we remark that since $0 \leq B^2 \leq F$, $(F - B^2)^{1/2}$ is a well-defined non-negative operator in $\mathcal{A}$. Denoting $WB + W'(F - B^2)^{1/2}$ by X, we have

$$X^* X = BW^* WB + (F - B^2)^{1/2} W'^* W' (F - B^2)^{1/2} = B^2 + F - B^2 = F$$

and $XX^* X = XF = X$, so that X is a partial isometry. Moreover, its range projection is $\leq F^{\perp}$ by construction, thus verifying (2.7). Next we define the projector H using equation (2.8) and setting $V = X$. We proceed to verify equation (2.9). From the arguments immediately following (2.9), this is equivalent to showing that the operator

$$U' = \frac{F \cos \alpha_1 \cos \Theta + X^* W \sin \alpha_1 \sin \Theta}{\cos \alpha_2}, \tag{2.15}$$

is isometric on $F\mathcal{H}$. But $X^* W = B$, since $W'^* W = 0$. On using the definition (2.14), the RHS of (2.15) simplifies to the projector F itself. This concludes the construction of a suitable partial isometry V and hence of the required projector H.

The " only if " part is more straightforward. Given the existence of a projector H satisfying (2.8) and (2.9), we obtain the partially isometric operator U defined in the sequel to (2.10) which satisfies $UU^* = U^*U = F$. Let x be an arbitrary normalised element of $F\mathcal{H}$. Then the second equation for U gives

$$\begin{aligned} \cos^2 \alpha_1 (x, \cos^2 \Theta\, x) &+ 2 \sin \alpha_1 \cos \alpha_1 \operatorname{Re} (\cos \Theta\, x, V^* W \sin \Theta\, x) \\ &+ \sin^2 \alpha_1 (V^* W \sin \Theta\, x, V^* W \sin \Theta\, x) = \cos^2 \alpha_2. \end{aligned} \tag{2.16}$$

Using $\| V^* W \| \leq 1$ and the Cauchy-Schwarz inequality on the LHS middle term of (2.16), we obtain the set of inequalities

$$\begin{aligned} \cos^2 \alpha_1 \cos^2 \beta - 2 \sin \alpha_1 \cos \alpha_1 \sin \beta \cos \beta \cos \gamma + \sin^2 \alpha_1 \sin^2 \beta \cos^2 \gamma \\ \leq \cos^2 \alpha_2 \leq \cos^2 \alpha_1 \cos^2 \beta + 2 \sin \alpha_1 \cos \alpha_1 \sin \beta \cos \beta \cos \gamma \\ + \sin^2 \alpha_1 \sin^2 \beta \cos^2 \gamma, \end{aligned} \tag{2.17}$$

where

$$\begin{cases} \| \cos \Theta\, x \| = \cos \beta & \left(0 \leq \beta \leq \frac{\pi}{2}\right), \\ \| V^* W \sin \Theta\, x \| = \sin \beta \cos \gamma & \left(0 \leq \gamma \leq \frac{\pi}{2}\right) \end{cases} \tag{2.18}$$

The second inequality of (2.17) gives

$$\cos \alpha_2 \leq (\cos \alpha_1 \cos \beta + \sin \alpha_1 \sin \beta \cos \gamma) \leq \cos (\alpha_1 - \beta), \tag{2.19}$$

on using the restrictions $0 \leqq \alpha_1, \alpha_2, \beta < \frac{\pi}{2}\cdot$ The first inequality gives

(2.20) $\cos(\alpha_1 + \beta) \leqq (\cos\alpha_1 \cos\beta - \sin\alpha_1 \sin\beta\cos\gamma) \leqq \cos\alpha_2.$

From (2.19) and (2.20), we get the triangle inequality

$$\cos(\alpha_1 + \beta) \leqq \cos\alpha_2 \leqq \cos(\alpha_1 - \beta).$$

This may be put into the equivalent form

(2.21) $$\cos^2(\alpha_1 + \alpha_2) \leqq \cos^2\beta \leqq \cos^2(\alpha_1 - \alpha_2).$$

Since x is an arbitrary element of $F\mathcal{H}$ and $\cos^2\beta = (x, \cos^2\Theta\, x)$, we may again use the monotonicity of the cosine function on $[0, \pi]$ to recover (2.4).

2.7. Definition. — An isoclinic $(n-1)$-sphere in $\mathcal{A}$ is a set of mutually isoclinic projectors constructed as follows : (i) let

$$\{ F_i : i = 1, 2, \ldots, n \}$$

be a set of mutually orthogonal equivalent projectors in $\mathcal{A}$; (ii) let $\{ V_{m1} : m = 1, 2, \ldots, n \}$ be a set of partial isometries with initial projector F_1 and final projectors F_m and with $V_{11} = F_1$; (iii) let $\hat{\mathbf{r}} = \sum_{i=1}^{n} r_i \mathbf{e}_i$ be a unit vector in an n-dimensional Euclidean space with orthonormal basis $\{ \mathbf{e}_i \}$; (iv) let $W(\hat{\mathbf{r}}) = \sum_{i=1}^{n} r_i V_{i1}$. Then the projectors $F(\hat{\mathbf{r}}) = W(\hat{\mathbf{r}})\, W^*(\hat{\mathbf{r}})$ satisfy

(2.22) $$F(\hat{\mathbf{r}}_1)\, F(\hat{\mathbf{r}}_2)\, F(\hat{\mathbf{r}}_1) = (\hat{\mathbf{r}}_1 . \hat{\mathbf{r}}_2)^2\, F(\hat{\mathbf{r}}_1).$$

If we introduce an orientation to distinguish $F(-\hat{\mathbf{r}})$ from $F(\hat{\mathbf{r}})$, then the projectors trace out a manifold analytically homeomorphic to the unit $(n-1)$-sphere in n-dimensional Euclidean space. It is clear that the projectors $F(\hat{\mathbf{r}})$ are the minimal projectors in a type I_n subfactor of the reduced factor $\mathcal{A}_F$ of $\mathcal{A}$, where $F = \sum_i F_i$. The appropriate subfactor is that generated by the set $\{ V_{m1} : m = 1, 2, \ldots, n \}$.

2.8. Lemma (Gleason). — *On an isoclinic $(n-1)$-sphere in $\mathcal{A}$, $n \geqq 3$, a countably additive measure μ is the restriction of a positive quadratic form (regarding the sphere as embedded in a Euclideann-space in the manner described above).*

Proof. — This is a direct consequence of Gleason's theorem for type I_n ($n \geqq 3$) factors, since orthogonality of the projectors F ($\hat{\mathbf{r}}$) corresponds directly to the orthogonality of the vectors $\hat{\mathbf{r}}$ themselves.

2.9. Corollary. — *If* E, F *are isoclinic projectors in* $\mathcal{A}$, *inclined at an angle* $\alpha \leqq \frac{\pi}{3}$ *and* $E \precsim E^{\perp} \wedge F^{\perp}$, *then, for any countably additive measure* μ *on the projection lattice of* $\mathcal{A}$, *we have*

$$(2.23)\quad |\mu(E) - \mu(F)| \leqq 2 \sin \alpha \, [(\mu(E)(\mu(E \vee F) - \mu(E)))^{1/2} + (\mu(F)(\mu(E \vee F) - \mu(F)))^{1/2}].$$

Proof. — From definition (2.5), we have

$$F = (E \cos \alpha + V \sin \alpha)(E \cos \alpha + V^* \sin \alpha).$$

The condition $E \precsim E^{\perp} \wedge F^{\perp}$ implies the existence of a partial isometry W with initial projector E and final projector $G \leqq E^{\perp} \wedge F^{\perp}$. Clearly V and W together generate a type I_3 subfactor of $\mathcal{A}_{E \vee F \vee G}$. The set of projectors

$$\{ H(\beta) : 0 \leqq \beta < 2\pi \}$$

where

$$H(\beta) = (E \cos \beta + V \sin \beta)(E \cos \beta + V^* \sin \beta)$$

forms an isoclinic 1-sphere in this subfactor. From lemma 2.8 we conclude that, on this isoclinic 1-sphere, μ takes the form $A + B \cos^2(\beta + \gamma)$, for certain constants A, B and γ satisfying $2A + B = \mu(E \vee F)$ and $A, B \geqq 0$. We have

$$\mu(E) = A + B \cos^2 \gamma, \qquad \mu(F) = A + B \cos^2(\gamma + \alpha).$$

Hence

$$\begin{aligned} |\mu(E) - \mu(F)| &\leqq B \,|\cos^2 \gamma - \cos^2(\gamma + \alpha)| \\ &\leqq \frac{B\,|\cos 2\gamma - \cos 2(\gamma + \alpha)|}{2} = B \sin \alpha \,|\sin(2\gamma + \alpha)| \\ &\leqq B \sin \alpha \,(|\sin(2\gamma + 2\alpha)| + |\sin 2\gamma|), \qquad \text{for } \alpha \leqq \frac{\pi}{3}. \end{aligned}$$

Using the elementary inequalities $B \cos^2 \gamma \leqq \mu(E)$, $B \cos^2(\gamma + \alpha) \leqq \mu(F)$, $B \sin^2(\gamma + \alpha) \leqq \mu(E \vee F) - \mu(F)$, we get the required inequality (2.23).

2.10. *Remark.* — An inequality simpler than (2.23) which suffices for small values of α can be obtained from the final stages of the above proof by using $|\sin(2\gamma + \alpha)| \leqq 1$ and $B \leqq \mu(E \vee F)$. This gives

$$(2.24)\quad |\mu(E) - \mu(F)| \leqq \mu(E \vee F) \sin \alpha.$$

2.11. Theorem. — *Let the factor $\mathcal{A}$ be continuous and let μ be a countably additive measure on its projection lattice $\mathfrak{L}(\mathcal{A})$. Then μ is continuous in the norm (operator bound) topology.*

Proof. — We will show that for any $\varepsilon \in (0, 1)$ that $\| F - G \| \leqq \varepsilon$ implies $| \mu(F) - \mu(G) | \leqq 8^{1/2} \varepsilon \mu(I)$, where F and G are two projectors in $\mathcal{A}$. Since we always have $\| F - G \| < 1$, we can apply lemmas 2.1 and 2.3 to show that $F \sim G$ and

$$G = (F \cos \Theta + V \sin \Theta)(F \cos \Theta + \sin \Theta \, V^*), \tag{2.25}$$

where $\sin \Theta = (FG^{\perp} F)^{1/2}$. From the identity

$$(F - G)^2 = FG^{\perp} F + F^{\perp} GF^{\perp}$$

and the relation $F^{\perp} GF^{\perp} = V (FG^{\perp} F) V^*$, we obtain

$$\| (F - G)^2 \| = \| F - G \|^2 = \| FG^{\perp} F \|.$$

Hence $\| \sin \Theta \| \leqq \varepsilon$, giving

$$0 \leqq \Theta \leqq F \sin^{-1} \varepsilon. \tag{2.26}$$

We can, without loss of generality, assume that $F \wedge G = 0$. Otherwise, we could always replace F and G by $F - F \wedge G$ and $G - F \wedge G$ without affecting the values of $\| F - G \|$ and $| \mu(F) - \mu(G) |$. We now use the hypothesis that $\mathcal{A}$ is continuous in order to divide F by two, i. e. find two projectors F_1 and F_2 satisfying $F_1 + F_2 = F$, $F_1 \leqq F_2^{\perp}$ and $F_1 \sim F_2$ [11]. Similarly, we can construct a related division of G by setting $G_1 = WF_1 W^*$ and $G_2 = WF_2 W^*$, where

$$W = (F \cos \Theta + V \sin \Theta).$$

The following are straightforward consequences of this definition

$$F_1 \sim G_1 \sim G_2 \sim F_2; \qquad \| F_1 - G_1 \| \text{ and } \| F_2 - G_2 \| \leqq \| F - G \|. \tag{2.27}$$

Moreover, the conditions $F_1 \lesssim F_1^{\perp} \wedge G_1^{\perp}$ and $F_2 \lesssim F_2^{\perp} \wedge G_2^{\perp}$ required in the following application of the corollary to lemma 2.8 are now satisfied. To show, for example, that $F_1 \lesssim F_1^{\perp} \wedge G_1^{\perp}$, we first notice that the projectors $F_1, F_2, E_1 = VF_1 V^*$ and $E_2 = VF_2 V^*$ are all equivalent and mutually orthogonal. Moreover, $F_1 \vee G_1 = F_1 + E_1$. But

$$F_1 < F_1 + E_1 \sim F_2 + E_2 \leqq (F_1 + E_1)^{\perp} = (F_1 \vee G_1) = F_1^{\perp} \wedge G_1^{\perp},$$

whence the result. Proceeding to apply lemma 2.6 to each pair of projectors (F_1, G_1) and (F_2, G_2), we deduce the existence of projectors H_1, H_2 such that H_i is isoclinic to F_i and G_i $(i = 1, 2)$ with the angle of inclination $\frac{\sin^{-1}\varepsilon}{2}$. From corollary 2.9 and remark 2.10, we then obtain the inequalities

$$\text{(2.28)} \quad \| F_i - H_i \| \text{ and } \| G_i - H_i \| \leqq \sin \frac{\sin^{-1}\varepsilon}{2} \qquad (i = 1, 2).$$

Hence

$$\begin{aligned} |\mu(F) - \mu(G)| \leqq \; & |\mu(F_1) - \mu(H_1)| + |\mu(H_1) - \mu(G_1)| \\ & + |\mu(F_2) - \mu(H_2)| + |\mu(H_2) - \mu(G_2)| \\ & \leqq 4 \sin \frac{\sin^{-1}\varepsilon}{2} \mu(I) \leqq 8^{1/2} \varepsilon \mu(I), \end{aligned}$$

as required.

3. Strong Continuity

In this section we extend the given countably additive measure from the projection lattice to the whole hermitian portion of the factor $\mathfrak{A}$ and, in several stages, proceed to demonstrate some further continuity properties of this extended function.

3.1. — Definition. — If μ is a countably additive measure on the projection lattice of the countably decomposable factor $\mathfrak{A}$, then by $\dot{\mu}$, we denote the functional on the set of hermitian elements of $\mathfrak{A}$, defined as follows : (*a*) let T be any such hermitian element and let $T = \int \lambda \, dE_\lambda$ be its spectral resolution, where the spectral family of projectors $\{ E_\lambda : -\infty < \lambda < \infty \}$ is composed of elements of $\mathfrak{A}$ [12]; (*b*) set $\mu_\lambda = \mu(E_\lambda)$, so determining a bounded σ-additive measure on the Borel sets of $\mathbf{R}$. Finally we set

$$\text{(3.1)} \qquad \dot{\mu}(T) = \int \lambda \, d\mu_\lambda.$$

Having already established the continuity of μ in operator norm in theorem 2.11, we now proceed to establish the strong continuity of $\dot{\mu}$. For this purpose, it is convenient to use the τ-norm topology as determined by the norm

$$\text{(3.2)} \qquad T \to (\tau(T^* T))^{1/2},$$

where τ is any faithful normal positive linear functional on $\mathfrak{A}$ (the existence of such functionals is guaranteed by the hypothesis of countable decomposability). In the sequel, we always take $\tau(I) = 1$, so that τ is a faithful normal state.

3.2. Lemma. — *On the unit ball $\mathfrak{A}_1$ of $\mathfrak{A}$, all τ-norm topologies coincide with the strong (and ultrastrong) operator topology.*

Prof. — The support of any faithful normal state is I, so we may use two propositions of Dixmier [13] to conclude that all τ-norm topologies restricted to $\mathcal{A}_1$, coincide with the strong operator topology on $\mathcal{A}_1$.

3.3. Lemma. — *On the hermitian part of the unit ball of $\mathcal{A}$, $\dot{\mu}$ is strongly continuous at the origin.*

Proof. — From lemma 3.2, it is sufficient to prove τ-norm continuity, where τ is any faithful normal state on $\mathcal{A}$. Let $\{ T_n \}$ be a sequence of operators satisfying $0 \leq T_n \leq I$ and converging in τ-norm to the null operator. We claim that $\dot{\mu}(T_n) \to 0$ as $n \to \infty$. For, suppose that $\limsup \dot{\mu}(T_n) > 0$. Then we can find a δ, $\delta > 0$, and an infinite subsequence $\{ S_m \} \subseteq \{ T_n \}$ such that $\dot{\mu}(S_m) \geq \delta$ for all $m = 1, 2, 3, \ldots$. Since, by hypothesis, $\tau(S_m^2) \to 0$ as $m \to \infty$, we can choose a further infinite subsequence $\{ R_n \} \subseteq \{ S_m \}$ with

$$\sum_{n=1}^{\infty} (n+1)^2 \tau(R_n^2) < \infty. \tag{3.3}$$

Forming the left continuous spectral resolution $R_n = \int_0^{1+} \lambda \, dE^{(n)}(\lambda)$ for each R_n, we define

$$E_m^{(n)} = E^{(n)}((n+1)^{-m}) - E^{(n)}((n+1)^{-m-1}) \qquad (m = 0, 1, 2, \ldots), \tag{3.4}$$

whence

$$R_n \leq V_n = \sum_{m=0}^{\infty} (n+1)^{-m} E_m^{(n)} \tag{3.5}$$

and

$$(n+1)^{-1} V_n \leq R_n. \tag{3.6}$$

From (3.3) and (3.6) we get $\sum_{n=1}^{\infty} \tau(V_n^2) < \infty$ and hence, using (3.5) :

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (n+1)^{-2m} \tau(E_m^{(n)}) < \infty. \tag{3.7}$$

If we define

$$X_N = \bigvee_{n \leq N} E_0^{(n)} + (N+1)^{-1} I, \tag{3.8}$$

then the $\{ X_N \}$ form a commuting sequence of elements of A satisfying

$$V_n \leq X_N \qquad (n = N, N+1, N+2, \ldots). \tag{3.9}$$

From the isotony of μ, we have, for $n \geqq N$,

$$(3.10) \qquad \mu(V_n) \leqq \mu(X_N) = \mu(I)(N+1)^{-1} + \mu\Big(\bigvee_{n \geqq N} E_0^{(n)}\Big).$$

The sequence $\bigvee_{n \geqq N} E_0^{(n)}$ of projectors is decreasing as $N \to \infty$ and

$$\tau\Big(\bigvee_{n \geqq N} E_0^{(n)}\Big) \leqq \sum_{n=N}^{\infty} \tau(E_0^{(n)}).$$

However, (3.7) implies that $\sum_{n=1}^{\infty} \tau(E_0^{(n)}) < \infty$. Since the functional τ is faithful, we conclude that $\operatorname{Inf}_N\Big(\bigvee_{n \geqq N} E_0^{(n)}\Big) = 0$. The countable additivity of μ on orthogonal projectors then requires that $\mu\Big(\bigvee_{n \geqq N} E_0^{(n)}\Big) \to 0$ as $N \to \infty$. Finally, from (3.10), we see that $\dot{\mu}(X_N) \to 0$. Since

$$\dot{\mu}(R_N) \leqq \dot{\mu}(V_N) \leqq \dot{\mu}(X_N),$$

this contradicts our supposition that $\dot{\mu}(R_N) \geqq \delta > 0$ for all N.

We next turn to the general case in which $\{T_n\}$ is any sequence of hermitian operators in the unit ball $\mathcal{A}_1$ converging to 0 in τ-norm. If $T_n = T_n^+ - T_n^-$ is the canonical decomposition of T_n into positive and negative parts, then both sequences $\{T_n^+\}$ and $\{T_n^-\}$ converge to 0 in τ-norm [use $\tau(T_n^2) = \tau(T_n^{+2}) + \tau(T_n^{-2})$]. From the first result, both $\dot{\mu}(T_n^+)$ and $\dot{\mu}(T_n^-) \to 0$ as $n \to \infty$. Hence $\dot{\mu}(T_n) = \dot{\mu}(T_n^+) - \dot{\mu}(T_n^-) \to 0$. This completes the proof.

We note that if $\mathcal{A}$ is a finite factor, then the canonical trace determines a positive normal linear functional on $\mathcal{A}$. The corresponding norm is usually denoted $\|\,.\,\|_2$ [14].

3.4. Lemma. — *Let $\{E_n\}$ be a sequence of projectors in $\mathcal{A}$ converging to a projector* F *in the strong operator topology. Then there is a decomposition $E_n = E_{n_0} + E_{n_1}$ and $F = F_{n_0} + F_{n_1}$ into pairs of orthogonal projectors such that $\|E_{n_0} - F_{n_0}\| \to 0$ and the sequences $\{E_{n_1}\}$ and $\{F_{n_1}\}$ both converge strongly to zero.*

Proof. — In the unit ball $\mathcal{A}_1$, lemma 3.2 allows us to use a τ-norm topology. By assumption $\tau((E_n - F)^2) \to 0$. Since

$$(E_n - F)^2 = E_n F^\perp E_n + E_n^\perp F E_n^\perp,$$

we have $\tau(E_n F^\perp E_n) \to 0$. Now $E_n F^\perp E_n$ is a contraction operator with left continuous spectral resolution of the form $\int_0^{1+} \lambda \, dE_n(\lambda)$.

We set $\tau(E_n F^{\perp} E_n) = \varepsilon_n^2$ and temporarily suppress the index n for ease in writing. This gives $\tau(EF^{\perp}E) = \int_0^{1+} \lambda \, d\tau(E(\lambda)) = \varepsilon^2$. If we define $E_1 = E(\infty) - E(\varepsilon)$ and set $E = E_0 + E_1$, this gives

$$\varepsilon^2 \geqq \int_{\varepsilon}^{1+} \lambda \, d\tau(E(\lambda)) \geqq \int_{\varepsilon}^{1+} \varepsilon \, d\tau(E(\lambda)) = \varepsilon\tau(E_1).$$

We thus get

$$(3.11) \qquad \tau(E_1) \leqq \varepsilon, \qquad \| E_0 F^{\perp} E_0 \| \leqq \varepsilon,$$

using

$$(3.12) \qquad \| E_0 F^{\perp} E_0 \| = \int_0^{\varepsilon+} \lambda \, dE(\lambda).$$

Next define

$$(3.13) \qquad F_1 = E_0^{\perp} \wedge F, \qquad F_0 = F - F_1,$$

when

$$(3.14) \qquad \| E_0 F_0^{\perp} E_0 \| \leqq \varepsilon, \qquad E_0^{\perp} \wedge F_0 = 0.$$

From the first equation of (3.14), we obtain $E_0 \wedge F_0^{\perp} = 0$. Hence we can apply lemma 2.3 to the pair E_0, F_0 to show that $E_0 \sim F_0$ and $E_0^{\perp} F_0 E_0^{\perp} = V(E_0 F_0^{\perp} E_0) V^*$ for a suitable partial isometry V. Thus

$$\| (E_0 - F_0) \|^2 = \| (E_0 - F_0)^2 \| \leqq \text{Max} \{ \| E_0 F_0^{\perp} E_0 \|, \| E_0^{\perp} F_0 E_0^{\perp} \| \} \leqq \varepsilon.$$

Also

$$\begin{aligned} \tau(F_1) = \tau(F - F_0) &\leqq \tau(|F - E|) + \tau(E - E_0) + \tau(|E_0 - F_0|) \\ &\leqq (\tau((E - F)^2))^{1/2} + \varepsilon + \| E_0 - F_0 \| \leqq 2\varepsilon + \varepsilon^{1/2}. \end{aligned}$$

Here we have used the inequalities

$$|\tau(A^* B)|^2 \leqq \tau(A^* A)\,\tau(B^* B) \qquad \text{and} \qquad |\tau(AB)| \leqq \tau(|A|)\, \|B\|.$$

Upon reinserting the suffices and using the previous result that $\varepsilon_n \to 0$ as $n \to \infty$, we conclude that $\| E_{n_0} - F_{n_0} \|$, $\tau(E_{n_1})$ and $\tau(F_{n_1})$ all tend to zero as $n \to \infty$. This completes the proof.

3.5. Theorem. — *Let μ be a countably additive measure on the projection lattice of a countably decomposable continuous factor $\mathcal{A}$. Then μ is strongly continuous on the projection lattice.*

Proof. — Using the notation and result of the previous lemma, we take a sequence of projectors $\{ E_n \}$ converging strongly to a projector F.

Since μ is additive on orthogonal projectors, we have

$$\mu(E_n) = \mu(E_{n_0}) + \mu(E_{n_1}) \quad \text{and} \quad \mu(F) = \mu(F_{n_0}) + \mu(F_{n_1})$$

for the decomposition of lemma 3.4. Hence

$$|\mu(E_n) - \mu(F)| \leq |\mu(E_{n_0}) - \mu(F_{n_0})| + \mu(E_{n_1}) + \mu(F_{n_1}),$$

using theorem 2.11 and lemmas 3.3 and 3.4, we see that each term on the RHS of this inequality vanishes as $n \to \infty$. Hence $\mu(E_n) \to \mu(F)$ strongly. Since F is arbitrary, this proves the theorem.

3.6. Theorem. — *Let* $\{T_n\}$ *be a sequence of operators in the hermitian part of* $\mathcal{A}$ *converging strongly to a hermitian operator* T. *Then* $\dot{\mu}(T_n) \to \dot{\mu}(T)$. (Notation as in the previous theorem.)

Proof. — The sequence $\{T_n\}$ is uniformly bounded [15] and so we may take, without loss of generality, $\|T_n\| \leq 1$ for all n. Writing the left continuous spectral resolution of T_n as $\int_{-1}^{1+} \lambda \, dE_n(\lambda)$, we have

$$\dot{\mu}(T_n) = \int_{-1}^{1+} \lambda \, d\mu(E_n(\lambda)). \tag{3.15}$$

Since $\mu(E_n(\lambda))$ is then a left-continuous, monotonic, non-decreasing function of the real variable λ, (3.15) may be interpreted as a Riemann-Stieltjes integral. If λ is not in the point spectrum of T, then the $E_n(\lambda)$ converge strongly, as $n \to \infty$, to the corresponding spectral projection $E(\lambda)$ of T [16]. Consequently, from theorem 3.5, we conclude that $\mu(E_n(\lambda)) \to \mu(E(\lambda))$ except possibly for λ in the point spectrum of T. The latter is a countable subset of $[-1, +1]$. Using an appropriate convergence theorem for sequences of Riemann-Stieltjes integrals [17], we conclude that

$$\int_{-1}^{1+} \lambda \, d\mu(E_n(\lambda)) \to \int_{-1}^{1+} \lambda \, d\mu(E(\lambda)) \quad \text{as} \quad n \to \infty.$$

This completes the proof.

4. Gleason's Theorem for Hyperfinite Factors

The result of theorem 3.6 enables us to prove Gleason's theorem for the hyperfinite case by reducing it to the known type I_n case.

4.1. Definition. — A factor $\mathcal{A}$ is hyperfinite if it is the weak closure of the union of a strictly increasing sequence $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \subset \ldots$ of factors of type $I_{n_1}, I_{n_2}, I_{n_3}, \ldots$ respectively.

The operator norm closure of $\bigcup_i \mathfrak{M}_i$ is a uniformly hyperfinite (UHF) C*-algebra in the sense of Glimm [18]. A hyperfinite factor is continuous and countably decomposable.

4.2. THEOREM. — *If μ is a countably additive measure on the projection lattice of a hyperfinite factor $\mathfrak{A}$, then μ is the restriction of a normal state on $\mathfrak{A}$ to the projection lattice.*

Proof. — As in definition 3.1, we construct the functional $\dot{\mu}$ on the set of hermitian elements of $\mathfrak{A}$. Using the notation of definition 4.1, it is a direct consequence of Gleason's theorem for type I_n $(n \geq 3)$ factors that $\dot{\mu}$ is linear on the subspace of hermitian elements of the *-algebra $\bigcup_i \mathfrak{M}_i$. By Kaplansky's density theorem, the unit ball of the latter subspace is strongly dense in the hermitian part of the unit ball of $\mathfrak{A}$. If S, T and $\rho S + \sigma T$ are hermitian elements in the unit ball of A, then we can use theorem 3.6 to obtain linearity, $\dot{\mu}(\rho S + \sigma T) = \rho \dot{\mu}(S) + \sigma \dot{\mu}(T)$, by continuity. Since $\dot{\mu}$ is positive homogeneous on the hermitian part of $\mathfrak{A}$, it is clear that $\dot{\mu}$ is a positive linear functional which is strongly continuous on the unit ball. By complexifying in the obvious manner,

$$\rho(S + iT) = \dot{\mu}(S) + i\dot{\mu}(T), \tag{4.1}$$

where S and T are hermitian, we construct the required normal state ρ. This completes the proof.

REFERENCES

[1] G. BIRKHOFF and J. VON NEUMANN, *The logic of quantum mechanics* (*Ann. Math.*, t. 37, 1936, p. 823-843).

[2] G. W. MACKEY, *The mathematical foundations of quantum mechanics*, Benjamin, New York, 1963.

[3] J. GUNSON, *On the algebraic structure of quantum mechanics* (*Commun. Math. Phys.*, t. 6, 1967, p. 262-285).

[4] V. S. VARADARAJAN, *Geometry of quantum theory*, vol. I, Van Nostrand, Princeton, 1968.

[5] A. M. GLEASON, *Measures on the closed subspaces of a Hilbert space* (*J. Math. Mech.*, t. 6, 1957, p. 885-894).

[6] E. B. DAVIES, private communication.

[7] J. F. AARNES, *Quasi-states on C*-algebras* (*Trans. Amer. Math. Soc.*, t. 149, 1970, p. 601-625).

[8] F. J. MURRAY and J. VON NEUMANN, *On rings of operators*, II (*Trans. Amer. Math. Soc.*, t. 41, 1937, p. 208-248).

[9] J. E. TURNER, *Ph. D. Thesis*, University of Birmingham, 1968.

[10] Y. C. WONG, *Isoclinic n-planes in Euclidean 2n-space, Clifford parallels in elliptic* $(2n - 1)$*-space and the Hurwitz matrix equations* (*Mem. Amer. Math. Soc.*, No. 41, 1961).

[11] J. DIXMIER, *Les algèbres d'opérateurs dans l'espace hilbertien* (*algèbres de von Neumann*), Gauthier-Villars, Paris, 1957, p. 229.

[12] *See* ref. [11], p. 3.

[13] *See* ref. [11], p. 62.

[14] *See* ref. [11], p. 288.

[15] K. YOSIDA, *Functional analysis*, Springer, Berlin, 1965, p. 69.

[16] T. KATO, *Perturbation theory for linear operators*, Springer, Berlin, 1966, p. 432.

[17] T. H. HILDEBRANDT, *Introduction to the theory of integration*, Academic Press, New York, 1963, ch. II, theorem 15.3.

[18] J. GLIMM, *On a certain class of operator algebras* (*Trans. Amer. Math. Soc.*, t. 95. 1960, p. 318-340).

(*Manuscrit reçu le* 4 *septembre* 1972.)