ANNALES DE L'I. H. P., SECTION A

I. S. SHIKIN

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Annales de l'I. H. P., section A, tome 11, nº 4 (1969), p. 343-372
<http://www.numdam.org/item?id=AIHPA_1969__11_4_343_0>

Ann. Inst. Henri Poincaré,
Vol. XI, n° 4, 1969, p. 343-372.

Section A :
Physique théorique.

Relativistic effects for magnetohydrodynamic waves

by

I. S. SHIKIN
(Moscow State University, Moscow).

This paper deals with relativistic magnetohydrodynamics (MHD) of ideal conducting medium under assumptions that it has no viscosity and heat conductivity and that its electrical conductivity is infinite. In this case the dissipative processes in regions of continuous motion are absent.

The metrics is used in the form $ds^2 = g_{ik}dx^i dx^k$ with signature $+ - - -$. 4-velocity u^i is normalized by $u^i u_i = 1$. Latin indices run from 0 to 3, Greek indices run from 1 to 3; $x^0 = ct$, t is time, c is the velocity of light, x^α ($\alpha = 1, 2, 3$) are space coordinates.

We shall consider the general system of equations of relativistic MHD, Riemann waves in pseudo-euclidian space-time and relations for strong discontinuities.

It is essential to emphasize that relativistic effects in MHD can become apparent even in the cases when velocities and temperatures of fluid are of nonrelativistic values but the velocities of waves approach to the velocity of light. Such a case can be realized when MHD waves propagate through the medium with sufficiently small density, intense magnetic fields or in the case of propagation at a large angle to the direction of magnetic field.

§ 1. GENERAL EQUATIONS

The system of equations of relativistic MHD consists of the conservation laws for total energy-momentum of fluid and electromagnetic field

$$(T_i^k)^{\rm I}{}_{;k} + (T_i^k)^{\rm II}{}_{;k} = 0 \qquad (1.1a)$$

(covariant differentiation is designated by ;), Maxwell equations

$$\frac{\partial \mathscr{F}_{ik}}{\partial x^l} + \frac{\partial \mathscr{F}_{kl}}{\partial x^i} + \frac{\partial \mathscr{F}_{li}}{\partial x^k} = 0, \tag{1.2}$$

where $\mathscr{F}_{ik}$ is electromagnetic field tensor,
the condition of ideal (infinite) electrical conductivity

$$\mathscr{F}_{ik}u^k = 0. \tag{1.3}$$

In the case of General Relativity the conservation laws (1.1*a*) are consequences of Einstein field equations

$$\mathrm{R}_i^k - \frac{1}{2}\mathrm{R}\delta_i^k = \frac{8\pi k}{c^4}\{(\mathrm{T}_i^k)^{\mathrm{I}} + (\mathrm{T}_i^k)^{\mathrm{II}}\}, \tag{1.1}$$

so that in this case the general equations are (1.1).

Within the framework of special relativity in pseudo-euclidian space-time the equations (1.1*a*) are general.

In (1.1*a*) $(\mathrm{T}_i^k)^{\mathrm{I}}$ is the energy-momentum tensor of ideal fluid

$$(\mathrm{T}_i^k)^{\mathrm{I}} = wu_iu^k - p\delta_i^k, \tag{1.4}$$

where heat function (enthalpy) $w = e + p$, e is internal energy per unit proper volume including the rest energy, p is pressure, u^i is 4-velocity of a fluid element.

$$(\mathrm{T}_i^k)^{\mathrm{II}} = \frac{1}{4\pi}\left(-\mathscr{F}_{il}\mathscr{F}^{kl} + \frac{1}{4}\delta_i^k\mathscr{F}_{lm}\mathscr{F}^{lm}\right)$$

is the energy-momentum tensor for electromagnetic field. 4-divergency of this tensor is equal to

$$(\mathrm{T}_i^k)^{\mathrm{II}}{}_{;k} = -\frac{1}{4\pi}\mathscr{F}_{il}(\mathscr{F}^{ml})_{;m},$$

so that

$$(\mathrm{T}_i^k)^{\mathrm{I}}{}_{;k} = \frac{1}{4\pi}\mathscr{F}_{il}(\mathscr{F}^{ml})_{;m}. \tag{1.5}$$

The equations (1.1*a*) (in the case of general relativity (1.1)), (1.2-5) must be supplemented with thermodynamic state equation of fluid. Two cases must be discerned depending upon the form of this equation:

Case I, when $w = w(p, n)$, where n is density of number of particles (instead of n one may use rest mass density mn, m is particle rest mass), and case II, when $w = w(p)$.

In case II, when $w = w(p)$, the system of equations (1.1-5) is closed system. Such is for instance the case of ultrarelativistic state equation $e = 3p$, $w = 4p$, or the more general case of equations of state

$$e = \frac{p}{\gamma - 1}, \qquad \gamma = \text{const.}$$

In case I, when $w = w(p, n)$, it is still necessary to use the equation of continuity for number of particles

$$(nu^i)_{;i} = 0. \tag{1.6}$$

In case I equation (1.1*a*) by virtue of (1.4) and (1.6) is written

$$nu^k\left(\frac{w}{n}u^i\right)_{;k} - \frac{\partial p}{\partial x^k} = \frac{1}{4\pi}\mathscr{F}_{il}(\mathscr{F}^{ml})_{;m}. \tag{1.7}$$

Projecting (1.7) on the direction of 4-velocity u^i, taking into account relation $u^i(u_{i;k}) = 0$ which is a consequence of $u^i u_i = 1$ and using the condition of ideal conductivity (1.3), one obtains

$$u^k\left\{\frac{\partial}{\partial x^k}\left(\frac{w}{n}\right) - \frac{1}{n}\frac{\partial p}{\partial x^k}\right\} = 0. \tag{1.8}$$

By virtue of thermodynamic identity

$$\mathrm{T}d(\sigma/n) = d(w/n) - dp/n$$

where σ is entropy per unit proper volume and T is temperature equation (1.8) is reduced to

$$u^k\partial(\sigma/n)/\partial x^k = 0. \tag{1.9}$$

It also follows from (1.6) and (1.9) that

$$(\sigma u^i)_{;i} = 0. \tag{1.10}$$

Equations (1.9-10) are true within domains of continuous motion and they demonstrate the conservation of specific entropy (per unit rest mass) along world lines of fluid particles and also of entropy in fluid.

In case I the system of equations (1.1*a*) is equivalent to (1.8) and to any three equations from (1.7).

In case II, when $w = w(p)$, projection of (1.1*a*) on u^i gives

$$\left[\frac{w(p)}{\mathscr{B}(p)}u^k\right]_{;k} = 0, \qquad \mathscr{B}(p) = \exp\int\frac{dp}{w(p)}. \tag{1.11}$$

In view of (1.11) equations (1.1*a*) are written as

$$\frac{wu^k}{\mathscr{B}(p)}[\mathscr{B}(p)u_i]_{;k} - \frac{\partial p}{\partial x^i} = \frac{1}{4\pi}\mathscr{F}_{il}(\mathscr{F}^{ml})_{;m}. \tag{1.12}$$

Thermodynamic identity for quantities per unit volume is

$$\mathrm{T}d\sigma = de - \mu dn = dw - (wd\mathscr{B}/\mathscr{B}) - \mu dn,$$

where μ is chemical potential. In the case when $\mu = 0$ this relation is written as $\mathrm{T}d\sigma = d(w/\mathscr{B})$ and since $\mu = w - \mathrm{T}\sigma$ it yields $dln\sigma = dln(w/\mathscr{B})$, so that in this case the equation (1.1) is reduced to (1.10) which is the equation of continuity for entropy.

In case II the system (1.1*a*) is equivalent to (1.11) and to any three equations from (1.12).

The structure of equations (1.7-8), (1.11-12) shows the expediency of utilization of the following quantities which is being used below along with u^i, w, p and n. These quantities are defined by

in case I

$$\kappa^i = \left(\frac{w}{mnc^2}\right)cu^i, \quad \tilde{\rho} = mn\left(\frac{mnc^2}{w}\right), \quad \mathscr{I} = \frac{c^2}{2}\left(\frac{w}{mnc^2}\right)^2;$$

in case II

$$\kappa^i = \mathscr{B}(p)u^i, \qquad \tilde{\rho} = \frac{w(p)}{\mathscr{B}^2(p)}, \qquad \mathscr{I} = \frac{\mathscr{B}^2(p)}{2}, \quad \mathscr{B}(p) = \exp\int\frac{dp}{w(p)}. \tag{1.13}$$

In case I in nonrelativistic limit both for macroscopic velocities of fluid and for temperatures since $w \approx mnc^2 + \rho\varepsilon + p$ (ε is nonrelativistic internal energy per unit mass) the quantity κ^α becomes equal to fluid velocity, $\tilde{\rho}$ is equal to fluid density and $\mathscr{I}$ becomes equal after subtraction of $\frac{c^2}{2}$ to a nonrelativistic heat content per unit mass $\varepsilon + \frac{p}{\rho}$.

Along with relativistic speed of sound ω which is defined as

$$\omega = c[(\partial p/\partial e)\sigma/n]^{1/2}$$

in case I and as

$$\omega = c(dp/de)^{1/2}$$

in case II, the quantity a^2 is being used which is defined by

$$a^2 = \left(\frac{\partial p}{\partial \tilde{\rho}}\right)\sigma/n \text{ in case I} \qquad \text{and} \qquad a^2 = \frac{dp}{d\tilde{\rho}} \text{ in case II.} \tag{1.14}$$

The quantities a^2 and ω^2 are connected by

$$\text{in case I} \qquad a^2 = \left(\frac{w}{mnc^2}\right)^2 \frac{\omega^2}{1-(\omega/c)^2},$$
$$\text{in case II} \qquad a^2 = \mathscr{B}^2(p)\frac{(\omega/c)^2}{1-(\omega/c)^2}. \tag{1.14a}$$

In the case of MHD of ideally conducting fluid since the relation (1.3) takes place the energy-momentum tensor of electromagnetic field has two eigenvectors; one of then which is the time-like one coincides with 4-velocity u^i and the other is space-like 4-vector h^i, orthogonal to u^i.

h^i is assumed to be normalized by

$$h^i h_i = -|h|^2, \qquad h^i u_i = 0, \tag{1.15}$$

where $\frac{|h|^2}{8\pi}$ is the corresponding eigenvalue of energy-momentum tensor.

In terms of u^i and h^i the electromagnetic field tensor $\mathscr{F}^{ik}$ is written as

$$\mathscr{F}^{ik} = \mathrm{E}^{iklm} h_l u_m, \qquad |h|^2 = \frac{1}{2}\mathscr{F}^{lm}\mathscr{F}_{lm}, \tag{1.16}$$

where $\mathrm{E}^{iklm} = e^{iklm}/(-g)^{1/2}$, e^{iklm} is completely antisymmetric unit pseudo-tensor, $e^{0123} = 1$, $(-g)$ is metric tensor determinant, and the energy-momentum tensor of electromagnetic field is expressed as

$$(\mathrm{T}_i^k)^{\mathrm{II}} = \frac{1}{4\pi}\left\{\left(u_i u^k - \frac{1}{2}\delta_i^k\right)|h|^2 - h_i h^k\right\}. \tag{1.17}$$

Maxwell equations (1.2) in terms of u^i and h^i are written as

$$(u^i h^k - u^k h^i)_{;i} = 0. \tag{1.18}$$

In the case of pseudo-euclidean space-time with Galilean metrics (and also in curved 4-space in a local geodesic system of reference with Galilean metrics in fixed space-time point) the electromagnetic field tensor is expressed in a usual way in terms of electric intensity vector $\vec{\mathrm{E}}$ and magnetic induction vector $\vec{\mathrm{H}}$: $\mathscr{F}^{01} = -\mathrm{E}_x$, $\mathscr{F}^{12} = -\mathrm{H}_z$, etc. Maxwell equations (1.2) in this case are written in usual three-dimensional form

$$\operatorname{div}\vec{\mathrm{H}} = 0, \qquad \operatorname{rot}\vec{\mathrm{E}} = -\frac{1}{c}\frac{\partial\vec{\mathrm{H}}}{\partial t}, \tag{1.19}$$

and ideal conductivity condition (1.3) is written in a well known form

$$\vec{E} = -\frac{1}{c}[\vec{v}, \vec{H}], \qquad \text{i. e. } E_x = \frac{v_z}{c}H_y - \frac{v_y}{c}H_z,$$

$$E_y = \frac{v_x}{c}H_z - \frac{v_z}{c}H_x, \qquad E_z = \frac{v_y}{c}H_x - \frac{v_x}{c}H_y, \tag{1.20}$$

$\vec{v}$ is three-dimensional velocity.

Components of 4-vector h^i are expressed in terms of $\vec{E}$, $\vec{H}$ and $\vec{v}$ by

$$h^0 = -\frac{1}{c}u^0(\vec{H}\,\vec{v}), \qquad h^{1,2,3} = -u^0\left(\vec{H} + \frac{1}{c}[\vec{E}, \vec{v}]\right)_{x,y,z}. \tag{1.21}$$

In accordance with (1.15) $|h|^2$ is expressed by

$$|h|^2 = (H_x^2 + H_y^2 + H_z^2) - (E_x^2 + E_y^2 + E_z^2) \equiv H^2 - E^2.$$

In the proper system of fluid element in which 4-vector u^i is directed along time axis $\vec{v} = 0$ and $\vec{E} = 0$. The values in such a system are designated by * so that magnetic induction vector in the proper system is $\vec{H}^*$. The invariance of $\mathcal{F}_{il}\mathcal{F}^{il}$ gives

$$H^2 - E^2 = H^{*2} \qquad \text{and} \qquad |h|^2 = H^{*2}. \tag{1.22}$$

In the proper system also $h^0 = 0$, $h^{1,2,3} = -H^*_{x,y,z}$.

§ 2. RIEMANN WAVES

In this paragraph the consideration is being done in the framework of special relativity, space-time being pseudo-euclidean with Galelian metrics.

Riemann wave (or *simple wave*) is defined as one-dimensional unsteady motion in which the hydrodynamic and electrodynamic variables in some inertial system of reference depend upon x-coordinate along a direction of propagation and upon t through any combination $\varphi(x, t)$. Therefore all these variables in simple wave can be expressed as functions of one of them, for instance $\tilde{\rho}$.

The relation $\varphi(x, t) = 0$ determines the law of propagation of a fixed phase φ front. This front velocity V is equal to

$$V = -\varphi_t/\varphi_x = \partial x(t, \varphi)/\partial t,$$

being in general a function of φ so that fixed phase fronts propagation is governed by $x = V(\varphi)t + f(\varphi)$. The last relation implicitly determines a function $\varphi(x, t)$.

Unit 4-vector of a normal to hypersurface φ = const is being designated by n^i. It is a space-like 4-vector with non-zero components

$$n^0 = \text{sh}\ \xi, \qquad n^1 = \text{ch}\ \xi; \qquad \text{th}\ \xi = \text{V}/c.$$

Both cases I and II are treated below simultaneously, the difference in results for these two cases being pointed out when necessary.

Since an inertial system of reference is used covariant differentiation is replaced by partial.

The system of equations (1.6), (1.7) with $i = \alpha$, (1.19) and (1.20) for case I and the system of equations (1.11), (1.12) with $i = \alpha$, (1.19) and (1.20) for case II in terms of variables (1.13) yield $\left(\beta \equiv \frac{\text{V}}{c},\ \kappa_{x,y,z} \equiv \kappa^{1,2,3}\right)$:

$$d(\tilde{\rho}\kappa_x) - \beta d(\tilde{\rho}\kappa_0) = 0, \tag{2.1}$$

$$4\pi\tilde{\rho}(\kappa_x - \beta\kappa_0)d\kappa_x = -4\pi dp - d(\text{H}_y^2 + \text{H}_z^2 + \text{E}_y^2 + \text{E}_z^2 - \text{E}_x^2)/2 + \beta d(\text{E}_y\text{H}_z - \text{E}_z\text{H}_y), \tag{2.2}$$

$$4\pi\tilde{\rho}(\kappa_x - \beta\kappa_0)d\kappa_y = d(\text{H}_x\text{H}_y + \text{E}_x\text{E}_y) + \beta d(\text{E}_z\text{H}_x - \text{E}_x\text{H}_z), \tag{2.3}$$

$$4\pi\tilde{\rho}(\kappa_x - \beta\kappa_0)d\kappa_z = d(\text{H}_x\text{H}_z + \text{E}_x\text{E}_z) + \beta d(\text{E}_x\text{H}_y - \text{E}_y\text{H}_x), \tag{2.4}$$

$$d\text{E}_z + \beta d\text{H}_y = 0, \qquad d\text{E}_y - \beta d\text{H}_z = 0, \tag{2.5}$$

$$\text{H}_x = \text{const}; \quad \text{E}_x\kappa_0 = \text{H}_y\kappa_z - \text{H}_z\kappa_y, \quad \text{E}_y\kappa_0 = \text{H}_z\kappa_x - \text{H}_x\kappa_z, \quad \text{E}_z\kappa_0 = \text{H}_x\kappa_y - \text{H}_y\kappa_x. \tag{2.6}$$

In case I this system must be enlarged with

$$(\kappa_x - \beta\kappa_0)(\tilde{\rho}d\mathcal{I} - dp) = 0. \tag{2.7}$$

We shall also use the zero component of equations (1.7) and (1.12):

$$4\pi\tilde{\rho}(\kappa_x - \beta\kappa_0)d\kappa_0 = 4\pi\beta dp + \beta d(\text{E}^2 + \text{H}^2)/2 - d(\text{E}_y\text{H}_z - \text{E}_z\text{H}_y). \tag{2.8}$$

It is found to be convenient to utilize for each phase φ = const the system of reference in which 4-vector n^i is directed along x-axis by making Lorentz rotation in (x, t)-plane [3]. Measured in such a system variables are designated by prime. The transition into primed system means for each phase φ = const the transition into system of reference, in which this phase is in rest (is « frosen »). The relations between primed and unprimed values according to Lorentz transformations are given by

$$\kappa_x = \kappa'_x\ \text{ch}\ \xi + \kappa'_0\ \text{sh}\ \xi, \quad \kappa_y = \kappa'_y, \quad \kappa_z = \kappa'_z, \quad \kappa_0 = \kappa'_0\ \text{ch}\ \xi + \kappa'_x\ \text{sh}\ \xi;$$
$$\text{H}_x = \text{H}'_x, \quad \text{H}_y = \text{H}'_y\ \text{ch}\ \xi - \text{E}'_z\ \text{sh}\ \xi, \quad \text{H}_z = \text{H}'_z\ \text{ch}\ \xi + \text{E}'_y\ \text{sh}\ \xi,$$
$$\text{E}_x = \text{E}'_x, \quad \text{E}_y = \text{E}'_y\ \text{ch}\ \xi + \text{H}'_z\ \text{sh}\ \xi, \quad \text{E}_z = \text{E}'_z\ \text{ch}\ \xi - \text{H}'_y\ \text{sh}\ \xi;$$

$$\text{ch}\ \xi = 1/\sqrt{1 - (\text{V}/c)^2}, \qquad \text{sh}\ \xi = \text{V}/c\sqrt{1 - (\text{V}/c)^2}, \qquad \text{th}\ \xi = \text{V}/c. \tag{2.9}$$

In terms of (2.9) equations (2.1), (2.3-5) become

$$d(\tilde{\rho}\kappa'_x) + \tilde{\rho}\kappa'_0 d\xi = 0, \tag{2.10}$$
$$4\pi\tilde{\rho}\kappa'_x d\kappa'_y = \mathrm{H}_x d\mathrm{H}'_y - \mathrm{H}_x\mathrm{E}'_z d\xi + \mathrm{E}'_y d\mathrm{E}_x, \tag{2.11}$$
$$4\pi\tilde{\rho}\kappa'_x d\kappa'_z = \mathrm{H}_x d\mathrm{H}'_z + \mathrm{H}_x\mathrm{E}'_y d\xi + \mathrm{E}'_z d\mathrm{E}_x, \tag{2.12}$$
$$d\mathrm{E}'_z = \mathrm{H}'_y d\xi, \qquad d\mathrm{E}'_y = -\mathrm{H}'_z d\xi. \tag{2.13}$$

Equations (2.2) and (2.8) by (2.13) become

$$8\pi\tilde{\rho}\kappa'_x(d\kappa'_x + \kappa'_0 d\xi) = -8\pi dp - d(\mathrm{H}^{*2}), \tag{2.14}$$
$$4\pi\tilde{\rho}\kappa'_x(d\kappa'_0 + \kappa'_x d\xi) = \mathrm{E}'_z d\mathrm{H}'_y - \mathrm{E}'_y d\mathrm{H}'_z - (\mathrm{E}'^2_y + \mathrm{E}'^2_z)d\xi, \tag{2.15}$$

where H^{*2} is defined by (1.22).

The equation (2.14) by virtue of (2.10) is written

$$d(\mathrm{H}^{*2}/8\pi) = \kappa'^2_x d\tilde{\rho} - dp. \tag{2.16}$$

The differentiation of $\mathrm{E}'_y\kappa'_0 = \mathrm{H}'_z\kappa'_x - \mathrm{H}_x\kappa'_z$ and $\mathrm{E}'_z\kappa'_0 = \mathrm{H}_x\kappa'_y - \mathrm{H}'_y\kappa'_x$ (condition of ideal conductivity) with use of (2.10-13), (2.15) yields

$$\kappa'^2_x\mathrm{H}'_z d\tilde{\rho} = \tilde{\rho}(\kappa'^2_x - a^2_\mathrm{A})(\mathrm{E}'_y d\xi + d\mathrm{H}'_z), \tag{2.17}$$
$$\kappa'^2_x\mathrm{H}'_y d\tilde{\rho} = \tilde{\rho}(\kappa'^2_x - a^2_\mathrm{A})(-\mathrm{E}'_z d\xi + d\mathrm{H}'_y), \tag{2.18}$$

where

$$a^2_\mathrm{A} = (\mathrm{H}'^2_{//} - \mathrm{E}'^2_\perp)/4\pi\tilde{\rho}; \qquad \mathrm{H}'_{//} \equiv \mathrm{H}_x, \qquad \mathrm{E}'^2_\perp \equiv \mathrm{E}'^2_y + \mathrm{E}'^2_z. \tag{2.19}$$

Equations (2.17) and (2.18) by virtue of $\mathrm{H}_x\mathrm{E}_x + \mathrm{H}'_y\mathrm{E}'_y + \mathrm{H}'_z\mathrm{E}'_z = 0$ (the condition of orthogonality of $\vec{\mathrm{E}}'$ and $\vec{\mathrm{H}}'$ due to (1.21)) yield

$$(\kappa'^2_x - a^2_\mathrm{A})(\mathrm{H}_x\mathrm{E}_x d\xi + \mathrm{H}'_z d\mathrm{H}'_y - \mathrm{H}'_y d\mathrm{H}'_z) = 0. \tag{2.20}$$

The set of equations (2.10), (2.13), (2.16), (2.17-18) (and also (2.15) if necessary) is used below as a main system, electric field $\vec{\mathrm{E}}'$ being considered instead of transversal velocities κ'_y and κ'_z. In case I this system is enlarged with the equation (2.7)

$$\kappa'_x(\tilde{\rho}d\mathscr{I} - dp) = 0. \tag{2.21}$$

We start in analysis of this system with the case when over the region of the wave

$$\kappa'_x = 0. \tag{2.22}$$

A simple wave in which (2.22) takes place does not travel through fluid particles. Due to (2.10) in this case $d\xi = 0$ and $\mathrm{V} = \mathrm{const}$ so that the wave propagates (together with particles of fluid) without distortion of its profile.

As (2.17-18) shows there are two types of simple waves with (2.22) subject to the value of $\mathrm{H}'^2_{//} - \mathrm{E}'^2_\perp$.

If $H_{//}'^2 - E_{\perp}'^2 = 0$ (this situation may be realized only if $H_{//}' = 0$ and then $E_{\perp}' = 0$) the only condition

$$p + \frac{H_{\perp}'^2 - E_{//}'^2}{8\pi} = \text{const}\ (H_{\perp}'^2 = H_y'^2 + H_z'^2,\ E_{//}' = E_x)$$

which is the consequence of (2.16) must be satisfied over the wave while H_y, H_z and v_y, v_z may vary arbitrarily. Such a simple wave is called *tangential.*

If $\kappa_x' = 0$ but $H_{//}'^2 - E_{\perp}'^2 \neq 0$ in the wave $\vec{H} = \text{const}$, $\vec{E} = \text{const}$, $\vec{V} = \text{const}$ and $p = \text{const}$. In case II also $\tilde{\rho} = \text{const}$ so that one simply has a region with constant parameters. In case I due to (2.21) the variable $\tilde{\rho} \equiv (mnc^2)^2/w$ may be altered arbitrarily in the wave as well as other thermodynamic variables (in particular entropy) except pressure. Such a simple wave is called *entropic.*

Let us now consider simple waves in which (2.22) does not take place. In case I it follows from (2.21) that entropy must be constant throughout the region of wave so that p is a function of $\tilde{\rho}$.

As (2.17-20) show two possibilities then arise.

The first is

$$\kappa_x'^2 = a_A^2 \tag{2.23}$$

throughout the wave.

This relation in terms of initial variables is written by virtue of (1.13) and (2.19) as

$$u_x'^2 = U_A^2, \qquad U_A^2 \equiv (H_{//}'^2 - E_{\perp}'^2)/4\pi w. \tag{2.23a}$$

In accordance with (2.16-18) over the region of the wave $\tilde{\rho} = \text{const}$, $p = \text{const}$, $H^{*2} = \text{const}$. Equations (2.10) and (2.13) yield $(h^1)'d\xi = 0$ where h^1 is given by (1.21). The equality $(h^1)' = 0$ which leads to

$$H_{//}'^2 - E_{\perp}'^2 = -|h|^2 u_x'^2$$

in accordance with given below equations (2.27, 2.30) may take place only in degenerate case $H_{//}'^2 - E_{\perp}'^2 = 0$ together with $u_x' = 0$ so that over the region of wave $d\xi = 0$ and also $H_{//}' = \text{const}$, $E_{\perp}' = \text{const}$.

Throughout the region of the wave by virtue of (2.15) and (1.21)

$$(h^1)' = \text{const}.$$

This may be written in covariant form since $(n^i)'$ is directed along x^1-axis as

$$h^i n_i = \text{const}. \tag{2.24}$$

Such a simple wave is *Alfven* wave.

Alfven waves travel through fluid particles without distortion of its profile. Unprimed values (in initial system) depend in this case simply upon $x - Vt$, where $V = \text{const}$. The components of n^i are the same f‹ each phase φ so that primed system of reference is a single inertial system for all phases. In this system the front of Alfven wave is in rest.

Now we consider along with primed system also all other systems in which wave front is in rest. We call each of them *w*-system. It may have an arbitrary velocity in the plane of front, i. e. in (y, z)-plane.

Let us consider a 4-vector [*4*]

$$V^i = (h^k n_k) u^i - (u^k n_k) h^i. \tag{2.25}$$

Components of V^i are expressed by (1.16) in terms of $H'_{//}$ and $\vec{E}'_{\perp}$ as

$$(V^0)' = -H'_x, \qquad (V^1)' = 0, \qquad (V^2)' = E'_z, \qquad (V^3)' = -E'_y, \tag{2.26}$$

so that

$$V^i V_i = H'^2_{//} - E'^2_{\perp}. \tag{2.27}$$

The quantity $H'^2_{//} - E'^2_{\perp}$ has the same value in each *w*-system. As $H'^2_{//} - E'^2_{\perp} > 0$ in considered case of Alfven wave one can choose among *w*-systems such one in which $\vec{E}_{\perp} = 0$ and because of orthogonality of $\vec{E}'$ and $\vec{H}'$ also $E'_{//} = 0$ i. e. electric field $\vec{E}' = 0$. In such *w*-system vectors $\vec{v}'$ and $\vec{H}'$ are collinear and due to (2.24) and (1.21) also

$$|\vec{v}'| = \text{const}, \qquad |\vec{H}'| = \text{const}$$

throughout the wave. The only parameter that may be changed arbitrarily over the wave is in this case an orientation of joint direction of $\vec{v}'$ and $\vec{H}'$. In such a sense Alfven waves are also called *rotational.*

All conditions in Alfven wave can be written in manifestly covariant form.

The formula (2.23) since $u'_x \equiv (u^1)' = -(u^i n_i)$ is written by (2.27) as

$$w(u^i n_i)^2 = V^i V_i / 4\pi. \tag{2.28}$$

In view of (2.26) over the region of Alfven wave

$$V^i = \text{const}. \tag{2.29}$$

According to (2.25) and (1.15) $V^i V_i$ is expressed by

$$V^i V_i = (h^k n_k)^2 - (u^k n_k)^2 |h|^2. \tag{2.30}$$

Substitution of (2.30) into (2.28) yields

$$(u^i n_i)^2 \{ 4\pi w + |h|^2 \} = (h^k n_k)^2, \tag{2.31}$$

so that in Alfven wave

either $(4\pi w+|h|^2)^{1/2}(u^k n_k)=h^k n_k$, or $(4\pi w+|h|^2)^{1/2}(u^k n_k)=-h^k n_k$, (2.31$a$)

that corresponds physically to propagation in opposite directions.

Let 4-vectors $\mathscr{A}^i$ and $\mathscr{B}^i$ be introduced by

$$\mathscr{A}^i=(4\pi w+|h|^2)^{1/2}u^i-h^i,\qquad \mathscr{B}^i=(4\pi w+|h|^2)^{1/2}u^i+h^i \quad (2.32)$$

As (2.31a) shows in Alfven wave either $\mathscr{A}^i n_i=0$ or $\mathscr{B}^i n_i=0$.

In Alfven wave with $\mathscr{A}^i n_i=0$ 4-vector V^i is written due to (2.25) and (2.31a) as $V^i=(u^k n_k)\mathscr{A}^i$ so that by virtue of (2.29) throughout such a wave

$$\mathscr{A}^i n_i=0,\qquad \mathscr{A}^i=\text{const.} \quad (2.33)$$

By analogy in the wave with $\mathscr{B}^i n_i=0$

$$\mathscr{B}^i n_i=0,\qquad \mathscr{B}^i=\text{const.} \quad (2.33a)$$

(2.33-33a), (2.29) and (2.24) together with the condition of invariability of p, $\tilde{p}$ and $|h|^2$ form the set of relations in Alfven simple waves in manifestly covariant form.

Let us now consider the propagation of Alfven wave over medium which is in rest with magnetic field $\vec{H}^*$. The velocity of wave along the normal to its front is being designated by D_A, components of $\vec{H}^*$ along the direction of propagation and at the perpendicular direction being designated accordingly by $H^*_{//}$ and $\vec{H}^*_\perp$.

Together with Lorentz transformation of field (2.23a) yields

$$D_A^2=c^2 H^{*2}_{//}/(4\pi w+H^{*2}). \quad (2.34)$$

The value of $H'^2_{//}-E'^2_\perp$ in Alfven wave is expressed in terms of parameters in rest by (2.23a) and (2.34) as

$$V^i V_i=H'^2_{//}-E'^2_\perp=4\pi w H^{*2}_{//}/(4\pi w+H^{*2}_\perp). \quad (2.35)$$

(2.28) and (2.30) give the covariant relation

$$V^i V_i=4\pi w(h^i n_i)^2/(4\pi w+|h|^2). \quad (2.36)$$

Formulae (2.35) and (2.36) shows that in the case when $H^*_{//}\neq 0$ (i. e. when $(h^i n_i)\neq 0$) in Alfven wave $H'^2_{//}-E'^2_\perp>0$ and 4-vector V^i is time-like one. If $H^*_{//}=0$ then $H'_{//}=E'_\perp=0$, 4-vector V^i reduces to zero,

$$(h^i n_i)=0,\qquad (u^i n_i)=0,\qquad D_A=0.$$

This case is the transitional one between tangential and Alfven waves.

Returning to general analysis of simple waves let us now consider the second from the above-mentioned possibilities when (2.22-23) does not take place. It corresponds to *magnetoacoustic* simple waves.

It is essential that in magnetoacoustic simple wave $d\xi \neq 0$ unlike the cases of tangential, entropic and Alfven waves. Fronts of fixed phase φ travel according to $x = \mathrm{V}(\varphi)t + f(\varphi)$ with different velocities V for different φ so that wave profile is distorted during propagation. As phase φ the variable $\tilde{\rho}$ may be implied for instance. If $f(\varphi) = 0$ the wave is called centered. Lines $\varphi = \mathrm{const}$ in (x, t)-plane form a set of straight lines which has an envelope.

Equations (2.17-18) together with (2.13) yield

$$\tilde{\rho} d(\mathrm{H}_{\perp}'^2 - \mathrm{E}_{\perp}'^2)/d\tilde{\rho} = 2\kappa_x'^2 \mathrm{H}_{\perp}'^2/(\kappa_x'^2 - a_{\mathrm{A}}^2). \tag{2.37}$$

Differentiation of orthogonality condition $\mathrm{H}_x\mathrm{E}_x + \mathrm{H}_y'\mathrm{E}_y' + \mathrm{H}_z'\mathrm{E}_z' = 0$ with assistance of (2.17-18) gives

$$\tilde{\rho} d(\mathrm{E}_x^2)/d\tilde{\rho} = 2\kappa_x'^2 \mathrm{E}_x^2/(\kappa_x'^2 - a_{\mathrm{A}}^2). \tag{2.38}$$

The equation (2.16) due to $dp = a^2 d\tilde{\rho}$ becomes

$$d(\mathrm{H}^{*2}/8\pi) = (\kappa_x'^2 - a^2)d\tilde{\rho}. \tag{2.39}$$

The comparison of (2.37-38) with (2.39) leads to biquadratic equation for $\kappa_x'^2$:

$$(\kappa_x')^2 = (a_{\mathrm{M}})^2; \qquad a_{\mathrm{M}}^4 - a_{\mathrm{M}}^2\left(a^2 + \frac{\mathrm{H}^{*2}}{4\pi\tilde{\rho}}\right) + a^2\frac{\mathrm{H}_{//}'^2 - \mathrm{E}_{\perp}'^2}{4\pi\tilde{\rho}} = 0. \tag{2.40}$$

The equation (2.40) holds also true for small disturbances propagated through medium with constant parametres as well as for weak discontinuities in which discontinuity may occur only in derivatives of hydrodynamic and electrodynamic variables.

Physically the equation (2.40) expresses the fact that fronts of fixed phase travel through fluid particles with the speed of small amplitude magnetoacoustic wave. In (x, t)-plane lines $\varphi = \mathrm{const}$ which form a set of straight lines coincide with characteristics of the corresponding family.

The equation (2.40) in terms of initial variables according to (1.13-14*a*) is written

$$u_x' = \pm \mathrm{U}; \qquad \mathrm{U}^4 - \mathrm{U}^2\left(\frac{\omega^2}{c^2 - \omega^2} + \frac{\mathrm{H}^{*2}}{4\pi w}\right) + \frac{\mathrm{H}_{//}'^2 - \mathrm{E}_{\perp}'^2}{4\pi w}\,\frac{\omega^2}{c^2 - \omega^2} = 0. \tag{2.40a}$$

In covariant form the equation (2.40a) by (1.22) and (2.27) is written as

$$U^4 - U^2\left(\frac{\omega^2}{c^2-\omega^2} + \frac{|h|^2}{4\pi w}\right) + \frac{V^iV_i}{4\pi w}\frac{\omega^2}{c^2-\omega^2} = 0. \tag{2.40b}$$

The roots of (2.40-40a) are

$$\kappa'_x = \pm a_M^\pm, \qquad a_M^\pm = \left\{\frac{1}{2}\left[a^2 + \frac{H^{*2}}{4\pi\tilde{\rho}} \pm\sqrt{\left(a^2 + \frac{(H'^2_\perp - E'^2_{//}) - (H'^2_{//} - E'^2_\perp)}{4\pi\tilde{\rho}}\right)^2 + \frac{(H'^2_{//} - E'^2_\perp)(H'^2_\perp - E'^2_{//})}{4\pi^2\tilde{\rho}^2}}\right]\right\}^{1/2}, \tag{2.41}$$

$$u'_x = \pm U^\pm, \qquad U^\pm = \left\{\frac{1}{2}\left[\frac{\omega^2}{c^2-\omega^2} + \frac{H^{*2}}{4\pi w} \pm\sqrt{\left(\frac{\omega^2}{c^2-\omega^2} + \frac{(H'^2_\perp - E'^2_{//}) - (H'^2_{//} - E'^2_\perp)}{4\pi w}\right)^2 + \frac{(H'^2_{//} - E'^2_\perp)(H'^2_\perp - E'^2_{//})}{4\pi^2 w^2}}\right]\right\}^{1/2}. \tag{2.41a}$$

The upper sign in (2.41-41a) (the values a^+ and U^+) corresponds to *fast* magnetoacoustic waves whereas the lower sign (values a^- and U^-) does to *slow* waves.

According to ideal conductivity condition (1.20)

$$E'^2_{//} = (v'_\perp/c)^2 H'^2_\perp \sin^2\delta$$

where δ is an angle between $\vec{v}'_\perp$ and $\vec{H}'_\perp$ so that values of $H'^2_\perp - E'^2_{//}$ are non-negative:

$$H'^2_\perp - E'^2_{//} \equiv H'^2_\perp\{1 - (v'_\perp/c)^2\sin^2\delta\} \geqslant 0, \tag{2.42}$$

the equality $H'^2_\perp - E'^2_{//} = 0$ taking place only when $H'_\perp = 0$.

In view of (2.42) since (2.40) may be written as

$$(\kappa'^2_x - a^2)(\kappa'^2_x - a_A^2) = \kappa'^2_x(H'^2_\perp - E'^2_{//})/4\pi\tilde{\rho}$$

the following inequalities for magnetoacoustic speeds become apparent (a_A^2 and U_A^2 are defined by (2.19) and (2.23a)):

$$(a_M^+) \geqslant \max(a^2, a_A^2), \qquad (a_M^-)^2 \leqslant \min(a^2, a_A^2);$$
$$(U^+)^2 \geqslant \max\left(\frac{(\omega/c)^2}{1-(\omega/c)^2}, U_A^2\right), \qquad (U^-)^2 \leqslant \min\left(\frac{(\omega/c)^2}{1-(\omega/c)^2}, U_A^2\right), \tag{2.43}$$

where signs of equality may take place with excluding the degenerate case $\kappa'_x = 0$ only when $H'^2_\perp - E'^2_{//} = 0$ i. e. when $H'_\perp = 0$.

For fast magnetoacoustic waves stronger inequality may be written:

$$(a_M^+)^2 \geqslant \max\left(a^2, \frac{H^{*2}}{4\pi\tilde{\rho}}\right), \qquad (U^+)^2 \geqslant \max\left(\frac{(\omega/c)^2}{1-(\omega/c)^2}, \frac{H^{*2}}{4\pi w}\right). \qquad (2.43a)$$

In process of rarefaction in magnetoacoustic simple wave as $\tilde{\rho}$ diminishes the value of H^{*2} in accordance with (2.39) and (2.43) increases in slow waves and decreases in fast ones.

As (2.38) shows there exist solutions with $E_x = 0$ over the region of wave. In the case when $E_x \neq 0$ the diminution of $\tilde{\rho}$ leads to increase of E_x^2 in slow waves and to its decrease in fast waves.

Values of $H_{//}'^2 - E_{\perp}'^2$ in slow magnetoacoustic waves due to inequalities (2.43) may not become negative, $H_{//}'^2 - E_{\perp}'^2$ being equal to zero together with $\kappa_x'^2$.

In contrast with the case of slow waves the value $H_{//}'^2 - E_{\perp}'^2$ in fast waves may become negative.

In order to clear up the conditions under which such a situation may be realized we consider now fast magnetoacoustic wave of small amplitude or fast weak discontinuity and study its propagation with speed D along a normal to wave front through medium in rest with field $\vec{H}^*$, the component of $\vec{H}^*$ along a direction of propagation being again designated by $H_{//}^*$.

In view of

$$H_{//}'^2 - E_{\perp}'^2 = \{H_{//}^{*2} - (D/c)^2 H^{*2}\}/\{1-(D/c)^2\} \qquad (2.44)$$

the equation (2.40a) yields

$$\left(\frac{D_M}{c}\right)^4 (4\pi w + H^{*2}) - \left(\frac{D_M}{c}\right)^2 \left\{\left(\frac{\omega}{c}\right)^2 (4\pi w + H_{//}^{*2}) + H^{*2}\right\} + H_{//}^{*2}\left(\frac{\omega}{c}\right)^2 = 0. \qquad (2.45)$$

Roots of (2.45) are the speeds $D_M^\pm$ of fast and slow weak discontinuities (and also of fast and slow small amplitude wave):

$$D_M^\pm = c\left\{\frac{(\omega/c)^2(4\pi w + H_{//}^{*2}) + H^{*2} \pm \{[(\omega/c)^2(4\pi w + H_{//}^{*2}) - H^{*2}]^2 + 4\pi w(\omega/c)^2 H_{\perp}^{*2}\}^{1/2}}{2(4\pi w + H^{*2})}\right\}^{1/2}. \qquad (2.45a)$$

By analogy with (2.43-43a) the inequalities hold true (D_A^2 is defined by (2.34)):

$$(D_M^+)^2 \geqslant \max\{\omega^2, c^2 H^{*2}/(4\pi w + H^{*2})\}, \qquad (D_M^-)^2 \leqslant \min(\omega^2, D_A^2), \qquad (2.46)$$

where signs of equality takes place only when $H_{\perp}^* = 0$ (for $D_M \neq 0$).

Let us now consider the propagation of weak discontinuities for fixed values of w, ω and H^{*2} and at various angles to the direction of $\vec{H}^*$ (i. e. for various $H^*_{//}$). The calculation of extreme values of $H'^2_{//} - E'^2_{\perp}$ and D^2_M in dependence upon $H^*_{//}$ according to (2.44-45) in view of (2.46) shows that $H'^2_{//} - E'^2_{\perp}$ for both types of discontinuities approachs its maximum value which is equal to H^{*2} when $H^*_{\perp} = 0$ i. e. when the propagation occurs along $\vec{H}^*$ and $H'^2_{//} - E'^2_{\perp}$ approachs its minimum value when $H^*_{//} = 0$ i. e. when the propagation occurs in a perpendicular to $\vec{H}^*$ direction. Minimum value of $H'^2_{//} - E'^2_{\perp}$ is equal to 0 for slow waves and is equal to negative value

$$-[(\omega/c)^2 4\pi w + H^{*2}]H^{*2}/[1-(\omega/c)^2]4\pi w$$

for fast waves.

Therefore for slow weak discontinuity as well as for Alfven one the value

$$H'^2_{//} - E'^2_{\perp} \geqslant 0.$$

It vanishs when $H^*_{//} = 0$, D^-_M being equal to 0 and this case being transitional one between slow magnetoacoustic and tangential discontinuities.

In fast weak discontinuity $H'^2_{//} - E'^2_{\perp}$ is positive according to (2.44) if $(D^+_M)^2 < c^2(H^*_{//}/H^*)^2$, and it becomes zero or negative if

$$(D^+_M)^2 \geqslant c^2(H^*_{//}/H^*)^2. \tag{2.47}$$

The physical meaning of (2.47) becomes apparent after consideration of speed D_{H*} with which the wave front, being propagated at an angle to $\vec{H}^*$, is displaced along the direction of $\vec{H}^*$. Vector $\vec{D}_{H*}$ is received from vector $\vec{D}$ of propagation velocity of discontinuity along a normal to its front by addition with the vector of velocity in the front plane so that to direct the resulting vector along $\vec{H}^*$. The condition (2.47) means that D_{H*} is equal to light velocity c or greater than it (the velocity D_M is naturally less than c).

Substitution of (2.47) into (2.45) gives conditions under which the situation with (2.47) takes place

$$(H^*_{\perp}/H^*)^2 \geqslant 4\pi w[1-(\omega/c)^2]/\{H^{*2}[1-(\omega/c)^2] + 4\pi w\}. \tag{2.48}$$

When H^*, w and ω are given the condition (2.48) is realized for a sufficiently large $(H^*_{\perp}/H^*)^2$ i. e. for propagation of fast discontinuity at a sufficiently large angle to the direction of magnetic field.

We call magnetohydrodynamic weak discontinuities as well as small amplitude waves and also shock waves for which $H'^2_{//} - E'^2_{\perp} \leqslant 0$ by *superfast*.

The covariant description of the above-mentioned situation may be given in term of 4-vector V^i (2.25) in view of (2.27). For slow waves 4-vector V^i is always time-like apart from the transitional case between slow and tangential waves when V^i is equal to zero. For fast waves V^i may be both time-like and isotropic or space-like in the case of superfast waves.

It must be emphasized that the existence of superfast waves is essentially relativistic effect.

Returning to magnetoacoustic Riemannien waves we now consider the behaviour of fast simple wave near the points where $H'_\perp = 0$. In such points $\kappa_x'^2$ in accordance with (2.43) is equal to the greater from values of a^2 and a_A^2: $\kappa_x'^2 = a^2$ when $a^2 > a_A^2$ and $\kappa_x'^2 = a_A^2$ when $a^2 < a_A^2$. If in the the point with $H'_\perp = 0$ $\kappa_x'^2 = a^2$ then is this point according to (2.37) and (2.13) also $dH'_\perp/d\rho = 0$ so that $H'_\perp = 0$ over the region where $\kappa_x'^2 = a^2$. Yet, if in the point with $H'_\perp = 0$, $\kappa_x'^2 = a_A^2$ (this case, naturally, being possible when $H_{//}'^2 - E_\perp'^2 > 0$) then in this point in accordance with (2.37), (2.13) and (2.43) the derivative $d(H_\perp'^2)/d\rho > 0$. This inequality shows that a rarefaction below the value $\tilde{\rho}$ for which $H'_\perp = 0$ would lead to negative values of $H_\perp'^2$ and because of this is impossible, so that the value $\tilde{\rho}$ for which tangential field is switched off utterly is the maximum degree of rarefaction in fast magnetoacoustic wave (an analogous fact is true also in nonrelativistic approximation).

In slow simple wave in points where $H'_\perp = 0$ and $\kappa_x'^2 = a_A^2$ in accordance with (2.37) and (2.43) the derivative $d(H_\perp'^2)/d\tilde{\rho} < 0$ and this leads to switching on of transverse magnetic field $H'_\perp$ with a diminution of $\tilde{\rho}$.

Let us now consider the angle α of $\vec{H}'_\perp$ and the angle β of $\vec{E}'_\perp$ with y-axis. Reducing of the second brackets in (2.20) to zero yields

$$d\alpha/d\xi = H_x E_x / H_\perp'^2. \tag{2.49}$$

Equations (2.13) give

$$d\beta/d\xi = -\ H_x E_x / E_\perp'^2. \tag{2.50}$$

As it was already mentioned above there exist in accordance with (2.38) the solution when $E_x = 0$ over the region of simple wave (such is in particular the case when fluid before the wave is in rest). In this case vectors $\vec{H}'$ and $\vec{E}'_\perp$ are orthogonal (due to $(\vec{E}'_\perp \vec{H}'_\perp) = -E_x H_x$) and in view of (2.49-50) their orientation remains invariable. If $E_x \neq 0$ then vectors $\vec{H}'_\perp$ and $\vec{E}'_\perp$ are rotated in the wave. The angle $(\alpha - \beta)$ between $\vec{H}'_\perp$ and $\vec{E}'_\perp$ monotonously (with ξ) increases or decreases over the wave in dependence upon a sign of $E_x H_x$. In this case, yet, vectors $\vec{H}'_\perp$ and $\vec{E}'_\perp$ may not become orthogonal in any point because it leads to $E_x = 0$ in point where $\vec{H}'_\perp \neq 0$ $\kappa_x'^2 \neq a_A^2$, this being incompatible with (2.38).

In the framework of nonrelativistic approximation the equation (2.49) reduces to $d\alpha = 0$, $\alpha = \text{const}$, so that such polarization effects vanish.

Let us now receive the equation which displays the character of distortion of magnetoacoustic Riemann wave profile. The differentiation of (2.40) with taking into account (2.10), (2.13) and (2.39) yields

$$\kappa_x'^4 - a^2[(H_{//}'^2 - E_{\perp}'^2)/4\pi\tilde{\rho}] - a^2(v_x'/c)[(H_y'E_z' - H_z'E_y')/4\pi\tilde{\rho}]$$
$$= \frac{1}{2}(- v_x' d\tilde{\rho}/c\tilde{\rho}d\xi)\{ 3\kappa_x'^2(\kappa_x'^2 - a^2) + [\kappa_x'^2 - (H_{//}'^2 - E_{\perp}'^2)/4\pi\tilde{\rho}]d[(a\tilde{\rho})^2]/\tilde{\rho}d\tilde{\rho} \}. \tag{2.51}$$

We shall assume the following inequalities for the thermodynamic functions (formulae (1.13) are used):

in case I

$$(\mathcal{I}_{ppp})_{\sigma/n} \equiv [\partial^2(1/\tilde{\rho})/\partial p^2]_{\sigma/n} > 0, \qquad \text{i. e. } \partial^2(w/n^2)/\partial p^2\,|_{\sigma/n} > 0;$$

in case II

$$\mathcal{I}_{ppp} \equiv d^2(1/\tilde{\rho})/dp^2 > 0, \qquad \text{i. e. } \frac{d(\omega w)}{dp} > 2\omega. \tag{2.52}$$

In addition we assume as before that

$$a^2 \equiv \frac{dp}{d\tilde{\rho}} > 0, \qquad \text{i. e. } \mathcal{I}_{pp} < 0, \qquad \text{i. e. } \omega^2 < c^2, \tag{2.53}$$

what means

$$\text{in case I } \left(\frac{\partial p}{\partial e}\right)_{\sigma/n} < 1, \qquad \text{in case II } \frac{dp}{de} < 1. \tag{2.54}$$

In consequence of (2.52) in magnetoacoustic Riemann wave the derivative $d[(a\tilde{\rho})^2]/d\tilde{\rho} > 0$. The expression in braces in the right side of (2.51) according to (2.43) is positive for fast simple waves and is negative for slow ones. The analysis of the sign of the left side of (2.51) with utilization of (2.40) which is complicated by the presence of the third term shows that this sign is positive for fast waves and it must be negative for slow ones, so that in magnetoacoustic simple wave

$$- v_x' d\tilde{\rho}/d\xi > 0. \tag{2.55}$$

The inequality (2.55) shows that the points of magnetoacoustic simple wave profile with greater values of $\tilde{\rho}$ propagate more rapidly than the points with lower values of $\tilde{\rho}$, so that a profile of $\tilde{\rho}$ during wave propagation with increasing time flattens out in the domains of rarefaction and steepens in the domains of compression. This leads to toppling down in the domain of compression and to arising of discontinuity.

§ 3. SHOCK WAVES

We shall consider now *strong discontinuities* at the surface of which an actual jump takes place in the values of hydrodynamic and electrodynamic variables. The results of this paragraph hold true both for flat and for curved space-time. For obvious physical interpretation of the relations being obtained we shall use also for each element of discontinuity surface a local geodesic system of reference in which metrics for this element is reduced to Galilean form and given element of discontinuity surface is in rest. We shall call such a system w-system.

In study of classification of strong discontinuities the results of previous analysis of Riemann waves prove to be available. Among Riemann waves entropic, tangential and Alfven waves as it was already mentioned above propagate without distortion of their profiles with equal for each points velocity and a distribution of parameters in a connected with wave w-system does not depend upon time. In these waves some of hydrodynamic and electrodynamic variables can vary arbitrarily so that there exist a free parameter which for given state ahead a wave admits a certain arbitrariness in a distribution of variables within a wave. In particular a width of wave region may be zero and a distribution of variables may be chosen as a jump. In such a case we obtain correspondingly *contact* (according to entropic wave), *tangential* and *Alfven* strong discontinuities.

Such a reasoning fails for magnetoacoustic Riemann waves which propagate with a distortion of wave profile so that the corresponding strong discontinuities which are called magnetohydrodynamic *shock waves* require special study.

Let a unit 4-vector of a normal to hypersurface which conforms to space-time domain occupied with discontinuity be denoted by $n^i(n^i n_i = -1)$. In w-system n^i is directed along x-axis.

We shall mark the side of discontinuity surface faced to a fluid which flow into it by means of index 1 and the opposite side by index 2 so that a fluid flows from state 1 to state 2. We shall assume that 4-vector n^i has the same direction as a velocity of fluid in w-system.

The difference of values $\mathscr{F}_2$ and $\mathscr{F}_1$ of a variable $\mathscr{F}$ at opposite sides of discontinuity surface will be denoted by $[\mathscr{F}] = \mathscr{F}_2 - \mathscr{F}_1$.

In accordance with energy-momentum conservation laws (1.1a) where energy-momentum tensors for matter and for field are given by (1.4) and (1.7) and according to Maxwell equations (1.18) the following relations must be satisfied at points of discontinuity surface:

$$[W^i]=0, \quad W^i \equiv T^{ik}n_k = \left(w+\frac{|h|^2}{4\pi}\right)u^i(u^k n_k)-\left(p+\frac{|h|^2}{8\pi}\right)n^i-\frac{1}{4\pi}h^i(h^k n_k), \quad (3.1)$$

which expresses the continuity of energy-momentum flux along a normal to discontinuity surface, and

$$[V^i]=0, \qquad V^i \equiv (h^k n_k)u^i-(u^k n_k)h^i. \quad (3.2)$$

The physical meaning of components of 4-vector V^i in w-system is exposed by formulae (2.25-27).

In this paragraph values in w-system are denoted without prime. We also shall denote

$$H_x = H_n, \qquad E_y^2+E_z^2=E_\tau^2, \qquad H_y^2+H_z^2=H_\tau^2, \qquad E_x=E_n,$$

so that in w-system

$$V^iV_i = H_n^2 - E_\tau^2, \qquad [H_n]=0, \qquad [\vec{E}_\tau]=0. \quad (3.3)$$

The set of relations (3.1-2) in case II when an equation of state has a form $w = w(p)$ is closed.

In case I when an equation of state has a form $w = w(p, n)$ the relations (3.1-2) must be enlarged according to (1.6) with the condition of continuity of particles flux

$$[n(u^i n_i)] = 0. \quad (3.4)$$

As it was already mentioned above and as it results from (3.1-2) and (3.4) strong discontinuities with $u^i n_i = 0$ through which flux of matter is absent may be of 2 types.

At *tangential* strong discontinuity

$$V^i = 0, \qquad (u^i n_i)=0, \qquad (h^i n_i)=0, \quad (3.5)$$

and the only relation takes place for a jump of variables:

$$[p+(|h|^2/8\pi)]=0. \quad (3.5a)$$

At *contact* strong discontinuity (in case I)

$$(u^i n_i)=0, \qquad V^i=(h^k n_k)u^i, \qquad [u^i]=0, \qquad [h^i]=0, \qquad [p]=0. \quad (3.6)$$

At such a discontinuity a jump may occur in all thermodynamic variables (in particular in entropy σ/n) apart from pressure and a value of jump may be arbitrary.

In further consideration of relations at discontinuity surface we start with an analysis of relations (3.1-2) without using the condition of continuity of particles flux (3.4). In case II it gives a complete analysis of closed set of relations at discontinuity. In case I the relation (3.4) must then be added.

From 4-vector W^i the components along n^i and V^i may be extracted. It must be noticed that vectors V^i and n^i are orthogonal:

$$V^i n_i = 0. \tag{3.7}$$

Since $u^i u_i = 1$ is follows from the definition of W^i (3.1) that

$$W^i n_i = \left(w + \frac{|h|^2}{4\pi}\right)(u^i n_i)^2 + \left(p + \frac{|h|^2}{8\pi}\right) - \frac{1}{4\pi}(h^i n_i)^2, \tag{3.8}$$

the scalar $(W^i n_i)$ in view of (3.1) being continuous at discontinuity surface:

$$[(W^i n_i)] = 0. \tag{3.9}$$

The relation (3.8) is also written due to (2.30) as

$$W^i n_i = w(u^i n_i)^2 - \frac{V^i V_i}{4\pi} + \left(p + \frac{|h|^2}{8\pi}\right). \tag{3.8a}$$

The projection of W^i onto V^i by virtue of definition of W^i (3.1) and of orthogonality condition $h^i u_i = 0$ yields

$$W^i V_i = w(u^k n_k)(h^k n_k), \tag{3.10}$$

the scalar $W^i V_i$ due to (3.1) and (3.2) also being continuous

$$[(W^i V_i)] = 0. \tag{3.11}$$

Let the component of W^i which is orthogonal to n^i be denoted by $\mathscr{K}^i$, so that W^i in accordance with (3.8a) is expressed as

$$W^i = -\left\{ w(u^i n_i)^2 - \frac{V^i V_i}{4\pi} + \left(p + \frac{|h|^2}{8\pi}\right)\right\} n^i + \mathscr{K}^i. \tag{3.12}$$

Due to (3.1) and (3.9) $\mathscr{K}^i$ is continuous at discontinuity surface

$$[\mathscr{K}^i] = 0. \tag{3.13}$$

The length of 4-vector $\mathscr{K}^i$ due to (3.1) and (3.12) is equal to

$$\mathscr{K}^i \mathscr{K}_i = \left\{ w(u^i n_i)^2 - \frac{V^i V_i}{4\pi} \right\}^2 + \left\{ w(u^i n_i)^2 - \frac{V^i V_i}{4\pi} \right\}\left(w + \frac{|h|^2}{2\pi}\right) + \frac{V^i V_i}{4\pi}\left(w + \frac{|h|^2}{4\pi}\right), \tag{3.14}$$

it also being continuous in accordance with (3.13):

$$[\mathscr{K}^i \mathscr{K}_i] = 0. \tag{3.15}$$

Now let us resolve h^i and u^i into the components along n^i and the components perpendicular to n^i:

$$h^i = -(h^k n_k)n^i + t^i, \quad u^i = -(u^k n_k)n^i + g^i, \quad t^i n_i = g^i n_i = 0. \tag{3.16}$$

According to (3.16), (3.1) and (3.12) $\mathscr{K}^i$ is written as

$$\mathscr{K}^i = \left(w + \frac{|h|^2}{4\pi}\right)(u^k n_k) g^i - \frac{1}{4\pi}(h^k n_k) t^i. \tag{3.17}$$

In terms of g^i and t^i due to (3.2) and (3.16) V^i acquires the form

$$\mathrm{V}^i = (h^k n_k) g^i - (u^k n_k) t^i, \tag{3.18}$$

which explicitly demonstrates the orthogonality of V^i and n^i (3.7).

Let us now extract from $\mathscr{K}^i$ its component along V^i under assumption that $\mathrm{V}^i \mathrm{V}_i \neq 0$ i. e. excluding the case when V^i is an isotropic 4-vector.

The expression (3.17) in view of $\mathscr{K}^i \mathrm{V}_i = \mathrm{W}^i \mathrm{V}_i$ and of (3.10) after calculations yields

$$(\mathrm{V}^k \mathrm{V}_k) \mathscr{K}^i = w(u^l n_l)(h^k n_k) \mathrm{V}^i + \mathscr{F}^i, \qquad \mathscr{F}^i \mathrm{V}_i = 0, \tag{3.19}$$

where

$$\mathscr{F}^i = \left\{ w(u^k n_k)^2 - \frac{\mathrm{V}^k \mathrm{V}_k}{4\pi} \right\} \{ (h^k n_k) t^i - |h|^2 (u^k n_k) g^i \}. \tag{3.20}$$

It is evident due to (3.13), (3.2) and (3.11) that at discontinuity surface

$$[\mathscr{F}^i] = 0, \tag{3.21}$$

and also

$$[\mathscr{F}^i \mathscr{F}_i] = 0, \tag{3.22}$$

where

$$\mathscr{F}^i \mathscr{F}_i = -(\mathrm{V}^i \mathrm{V}_i) \left\{ w(u^k n_k)^2 - \frac{\mathrm{V}^k \mathrm{V}_k}{4\pi} \right\}^2 (|h|^2 - \mathrm{V}^i \mathrm{V}_i). \tag{3.23}$$

Finally, in case $\mathrm{V}^i \mathrm{V}_i \neq 0$ 4-vector W^i is expressed due to (3.12) and (3.19) as

$$\mathrm{W}^i = -\left\{ w(u^i n_i)^2 - \frac{\mathrm{V}^i \mathrm{V}_i}{4\pi} + \left(p + \frac{|h|^2}{8\pi}\right) \right\} n^i + \frac{1}{\mathrm{V}^i \mathrm{V}_i} \{ w(u^k n_k)(h^l u_l) \mathrm{V}^i + \mathscr{F}^i \}, \tag{3.24}$$

where $\mathrm{V}^i n_i = \mathscr{F}^i n_i = \mathscr{F}^i \mathrm{V}_i = 0$, and so the conditions $[\mathrm{W}^i] = 0$ at discontinuity surface may be replaced by (3.8*a*-11) and (3.21) which express the continuity of components of W^i along the invariant directions.

If $V^iV_i = 0$, the relation (3.12) with $\mathscr{K}^i$ given by (3.17) stays true. It is easy to verify that for $V^iV_i = 0$ the relation (3.19) with $\mathscr{F}^i$ given by (3.20) remains valid and 4-vector $\mathscr{F}^i$ becomes equal to $-(W^kV_k)V^i$, so that in the case of isotropic V^i the condition (3.21) does not give anything new as compared with (3.10-11).

Let us consider for clear physical interpretation the corresponding relations in w-system.

The relations (3.8a-9) in w-system in accordance with (3.3) and (1.22) become

$$\left[wu_x^2 - \frac{H_n^2 - E_\tau^2}{4\pi} + \left(p + \frac{H^{*2}}{8\pi}\right)\right] = 0. \tag{3.25}$$

The relations (3.10-11) in view of (1.21) in w-system become

$$[wu_x(H_n u^0 - E_z u_y + E_y u_z)] = 0. \tag{3.26}$$

The relations (3.20-21) according to (3.16) and (1.17) in w-system are written (component $\mathscr{F}^1 = 0$):

$$\begin{aligned}
\mathscr{F}^0 &= \left\{ wu_x^2 - \frac{H_n^2 - E_\tau^2}{4\pi} \right\}(E_zH_y - E_yH_z), \qquad [\mathscr{F}^0] = 0;\\
\mathscr{F}^2 &= \left\{ wu_x^2 - \frac{H_n^2 - E_\tau^2}{4\pi} \right\}(E_xE_y + H_xH_y), \qquad [\mathscr{F}^2] = 0; \qquad (3.27)\\
\mathscr{F}^3 &= \left\{ wu_x^2 - \frac{H_n^2 - E_\tau^2}{4\pi} \right\}(E_xE_z + H_xH_z), \qquad [\mathscr{F}^3] = 0.
\end{aligned}$$

Due to (1.22) and (3.3) in w-system

$$|h|^2 - V^iV_i = H_\tau^2 - E_n^2. \tag{3.28}$$

The relation (2.42) gives

$$H_\tau^2 - E_n^2 \equiv H_\tau^2\{1 - (v_\perp/c)^2 \sin^2\delta\} \geqslant 0, \tag{3.29}$$

the equality to zero taking place only when $H_\tau = 0$.

With keeping in mind (3.28-29) we shall denote

$$|h|^2 - V^iV_i = k^2. \tag{3.30}$$

In view of (3.30) the relations (3.22-23) in case $V^iV_i \neq 0$ and (3.19, 21) in case $V^iV_i = 0$ yield

$$\left[k\left\{ w(u^in_i)^2 - \frac{V^iV_i}{4\pi}\right\}\right] = 0. \tag{3.31}$$

In w-system (3.31) becomes

$$\left[k\left(wu_x^2 - \frac{H_n^2 - E_\tau^2}{4\pi}\right)\right] = 0, \tag{3.32}$$

where

$$k = (H_\tau^2 - E_n^2)^{1/2} = (|h|^2 - V^iV_i)^{1/2}, \tag{3.30a}$$

the choice of sign in formulae (3.30a-3.31) being reasoned below.

Now we shall consider in which cases $\mathscr{F}^i$ would reduce to zero. According to (3.23) this may take place either if $w(u^in_i)^2 = V^iV_i/4\pi$ at both sides of discontinuity surface, or if $k = 0$ at both sides of it, or if $k = 0$ at one side of discontinuity surface and $w(u^in_i)^2 = V^iV_i/4\pi$ at another. The last of these possibilities according to (3.29) corresponds to waves which switchs on or switchs off the transverse magnetic field. The second of these possibilities corresponds as it may easy be shown to usual gasdynamic shock waves which propagate along magnetic field and does not interact with it ($H_\tau = E_n = 0$). The first of these possibilities corresponds to *Alfven* strong discontinuities. Since $\mathscr{F}^i = 0$ the number of relations at Alfven discontinuity decreases so that for given state 1 state 2 is not determined uniquely and there exist a free parameter at wave. The relation at Alfven discontinuity as it was already pointed out above are analogous to those in Alfven simple waves so that the analysis of behaviour of variables in these waves given in § 2 and relations (2.31a-36) remain true also for Alfven strong discontinuities.

For magnetohydrodynamic *shock waves* 4-vector $\mathscr{F}^i$ differs from zero (apart only from the case of shock waves which switch on or switch off the magnetic field).

The further consideration is concerned with MHD shock waves.

The relation (3.21) shows that 4-vector $\mathscr{F}^i$ is the same for states 1 and 2 and together with V^i it determines (in case $V^iV_i \neq 0$) the orthogonal to 4-vector n^i hyperplane which remains invariable while the surface of discontinuity is crossed (if $V^iV_i = 0$ such hyperplane is composed by $\mathscr{K}^i$ and V^i).

In w-system the combination $\mathscr{F}^2E_y + \mathscr{F}^3E_z$ from (3.27) due to invariance of $H_n^2 - E_\tau^2$ yields

$$\left[E_n\left(wu_x^2 - \frac{H_n^2 - E_\tau^2}{4\pi}\right)\right] = 0. \tag{3.33}$$

It also follows from (3.27) due to (3.3) and (3.33) that

$$\left[\vec{H}_\tau\left(wu_x^2 - \frac{H_n^2 - E_\tau^2}{4\pi}\right)\right] = 0, \tag{3.34}$$

so that vectors $(\vec{H}_\tau)_1$ and $(\vec{H}_\tau)_2$ are located at the same plane [1].

Relations (3.33) and (3.34) give

$$E_{n1}\vec{H}_{\tau 2} = E_{n2}\vec{H}_{\tau 1}. \tag{3.35}$$

By virtue of (3.29) and (3.35) one may choose among w-systems such one in which $E_n = 0$ at both sides of discontinuity surface. In such a w-system due to (3.30a) k is equal to absolute value of H_τ.

The solution of relations (3.8a-9) and (3.31) with respect to $w(u^i n_i)^2$ in states 1 and 2 gives

$$\{ w(u^i n_i)^2 \}_1 = \frac{V^i V_i}{4\pi} + k_2 \frac{[p + (|h|^2/8\pi)]}{[k]};$$
$$\{ w(u^i n_i)^2 \}_2 = \frac{V^i V_i}{4\pi} + k_1 \frac{[p + (|h|^2/8\pi)]}{[k]}. \tag{3.36}$$

The substitution of (3.36) into (3.14-15) in view of (3.30) yields

$$\left[p + \frac{k^2}{8\pi}\right]^2 (k_1 + k_2) - \left[p + \frac{k^2}{8\pi}\right]\left\{\frac{k_1 k_2 [k]}{2\pi} + (w_2 k_1 - w_1 k_2)\right\}$$
$$- \frac{V^i V_i}{4\pi}[k]\{(w_2 - w_1) - 2(p_2 - p_1)\} = 0. \tag{3.37}$$

The relation (3.37) binds up thermodynamic quantities w and p and electromagnetic field parameter k in states 1 and 2 at both sides of discontinuity surface.

It is known that reasons of compatibility of shock wave with outgoing small disturbances lead to the existence of two types of MHD shock waves namely of *slow* and *fast shocks* with the following inequalities for fluid velocities ahead and behind a shock front (evolutionary conditions [2]):

$$\begin{aligned} &\text{for slow MHD shocks} \quad U_{A1} > (u_x)_1 \geqslant U_1^-, \quad (u_x)_2 \leqslant U_2^-, \\ &\text{for fast MHD shocks} \quad (u_x)_1 \geqslant U_1^+, \quad U_{A2} < (u_x)_2 \leqslant U_2^+. \end{aligned} \tag{3.38}$$

where the speeds of Alfven and magnetoacoustic waves U_A and $U^\pm$ are defined by (2.23a), (2.40a-41a) (the signs of equality correspond to waves of small amplitude; switch-on and switch-off shocks are not taken into account here).

In a covariant form relations (3.38) are written as

for slow MHD shocks

$$\frac{V^iV_i}{4\pi w_1} > \{(u^in_i)^2\}_1 \geqslant (U_1^-)^2, \qquad \{(u^in_i)^2\}_2 \leqslant (U_2^-)^2,$$

for fast MHD shocks

$$\{(u^in_i)^2\}_1 \geqslant (U_1^+)^2, \qquad \frac{V^iV_i}{4\pi w_2} < \{(u^in_i)^2\}_2 \leqslant (U_2^+)^2. \tag{3.38a}$$

The fact of existence of slow and fast MHD shocks is in agreement with the existence of two types of small amplitude MHD waves and also of weak discontinuities.

The evolutionary conditions (3.38*a*) in particular mean in view of (2.43) that

$$\text{for slow shocks} \quad \{w(u^in_i)^2\}_1 < \frac{V^iV_i}{4\pi}, \quad \{w(u^in_i)^2\}_2 < \frac{V^iV_i}{4\pi},$$

$$\text{for fast shocks} \quad \{w(u^in_i)^2\}_1 > \frac{V^iV_i}{4\pi}, \quad \{w(u^in_i)^2\}_2 > \frac{V^iV_i}{4\pi}. \tag{3.39}$$

It follows from (3.39) that 4-vector V^i for slow MHD shocks is time-like one ($V^iV_i > 0$). In view of (3.37), (3.36) and (2.25) there also exist the limiting possibility when $V^i = 0$, $(u^in_i) = (h^in_i) = 0$, this being a transitional state between slow shock and tangential discontinuity.

For fast shocks, yet, in accordance with (3.38*a*) as well as for fast weak discontinuities 4-vector V^i may be both time-like and also isotropic or space-like in cases of *superfast* MHD shocks. Due to inequalities (3.38) and (2.44) such a situation is realized in particular for shock propagation through medium in rest with parameters which obey (2.48).

For fast MHD shocks with an isotropic 4-vector V^i when $V^iV_i = 0$ the relation (3.37) acquires an especially simple form

$$V^iV_i = 0, \qquad (k_1 + k_2)\left[p + \frac{k^2}{8\pi}\right] - \frac{k_1k_2[k]}{2\pi} - (w_2k_1 - w_1k_2) = 0. \tag{3.37a}$$

If 4-vector V^i at a shock point is time-like then in accordance with (3.3) a *w*-system can be chosen in which $\vec{E}_\tau = 0$. In such a *w*-system due to $(\vec{E}\vec{H}) = 0$ also $E_n = 0$ i. e. at both sides of discontinuity surface the electric field $\vec{E} = 0$, vectors of velocity $\vec{v}$ and of magnetic field $\vec{H}$ being collinear and fluid moving along a direction of magnetic field. In this system 4-vector V^i is directed along time axis. For slow shocks as well as for Alfven discontinuities such a system always may be chosen.

In case of superfast shocks for which $V^iV_i < 0$ in accordance with (3.3) a w-system can be chosen in which $H_n = 0$ i. e. at both sides of a shock front vectors of magnetic field are parallel to shock surface. In such a system V^i and n^i have pure space directions which are orthogonal to each other.

It follows from evolutionary conditions (3.39) that in w-system in view of (3.34) fields $\vec{H}_{\tau 1}$ and $\vec{H}_{\tau 2}$ must have the same direction and in formulae (3.32) in both sides of equality one may consider k as arithmetical value of square root (with sign +), this being assumed in corresponding formulae (in accordance with (3.30a).

In case II when an equation of state is of form $w = w(p)$ (in particular in case of ultrarelativistic state equation $e = 3p$) the relation (3.37) with given values of V^i, p_1, k_1 and of one from p_2 and k_2 determines the value of the other. Then formulae (3.36) determine (u^in_i) in states 1 and 2 and (3.10), (3.20-21), (3.2) and (3.18) determine for given state 1 other values for state 2.

In case of state equation of form $w = w(p)$ which obeys (2.54) the equation (3.37) with given value of V^i in (p, k)-plane determines a curve which passes through the point $p = p_1$, $k = k_1$. Straight lines $p = p_1$ and $k = k_1$ divide the plane (p, k) into 4 quadrants. Due to (3.37) for points of the curve if $p = p_1$ then necessarily $k = k_1$ and *vice versa*, so that each branch of the curve is disposed entirely in one of quadrants. It follows from (3.37) also that the equality $[p + (k^2/8\pi)] = 0$ takes place only together with $[k] = [p] = 0$ so that in each quadrant a sign of $[p + (k^2/8\pi)]$ and a sign of $[p + (k^2/8\pi)]/[k]$ along a corresponding branch of the curve (3.37) remain invariable.

Near $p = p_1$, $k = k_1$ the points of the curve (3.37) correspond to small amplitude waves and (u^in_i) coincides with one of magnetoacoustic speeds $U^{\pm}$. The relations (3.36) in view of (3.30) become in the limit

$$\text{when } p_2 \to p_1 \qquad (wU^2)_1 = \left(\frac{|h|^2}{4\pi}\right)_1 + k_1\frac{[p]}{[k]} = \frac{V^iV_i}{4\pi} + k_1\frac{[p + (k^2/8\pi)]}{[k]}. \tag{3.40}$$

It follows from (3.40) and (2.43-43a) that for weak waves in case $k_1 \neq 0$ the branch of (3.37) with $[p]/[k] > 0$ and also with $[p + (k^2/8\pi)]/[k] > 0$ corresponds to fast waves while the branch of (3.37) with $[p]/[k] < 0$ and also with $[p + (k^2/8\pi)]/[k] < 0$ corresponds to slow waves.

The invariance of a sign of $[p + (k^2/8\pi)]/[k]$ along the curve (3.37) in each quadrant leads then in view of the evolutionary conditions (3.39) to the conclusion that the branch of (3-37) for which $[p]/[k] > 0$ might correspond to fast shocks while the branch of (3.37) for which $[p]/[k] < 0$ might do to slow shocks.

For media with state equation $w = w(p)$ which obeys along with (2.54) the thermodynamic inequality (2.52) apparently the parts of inequalities in evolutionary conditions (3.38) which contain magnetoacoustic speeds lead to the conclusion that the pressure p_2 behind a shock must be greater than the pressure p_1 ahead it i. e. shock waves are compression jumps. Then at fast shock the field k increases ($[k]>0$) whereas at slow shock k decreases ($[k]<0$) and it can be switched-off ($k_2 = 0$). The proof of this statement must use the relations (3.36), (3.37) and (3.40) and be concerned with analysis of curve (3.37).

For *ultrarelativistic* equation of state $e = 3p$, $w = 4p$ which obeys the inequalities (2.52-54) the equation (3.37) is reduced to quadratic equation in $(p_2 - p_1)$:

$$w=4p;\quad (p_2-p_1)^2(3k_1-k_2)-(p_2-p_1)[k]\left(\frac{k_2^2-2k_1k_2-k_1^2}{4\pi}+4p_1-\frac{V^iV_i}{2\pi}\right)$$
$$-\frac{[k^2][k]}{8\pi}\left(\frac{[k]^2}{8\pi}+4p_1\right)=0. \qquad (3.41)$$

A degree of compression in slow MHD shocks is restricted by maximum (finite) value of p_2 when the field k_2 becomes switched-off completely ($k_2 = 0$). In contrast with slow shocks at fast MHD shocks arbitrarily large values of p_2 are admitted. When at fast shock $p_2 \rightarrow \infty$ then

$$k_2 \rightarrow 3k_1, \qquad (v_x)_1 \rightarrow c, \qquad (u_x)_2 \rightarrow 1/8. \qquad (3.42)$$

If a w-system is chosen in which a fluid flows into the shock normally to its front then in such a w-system according to (3.36), (1.20) and (3.26) a fluid must flow out also normally to shock front with the velocity which is equal in virtue of (3.42) to

$$(v_x)_2 \rightarrow c/3, \qquad (3.42a)$$

fields $\vec{E}_\tau$ and $\vec{H}_\tau$ being orthogonal in both the states 1 and 2 and

$$H_{\tau 1} = E_1(E_{n1} = E_{n2} = 0;\ H_{\tau 2} = 3H_{\tau 1}).$$

It must be pointed out for comparison that the relations for velocities (3.42-42*a*) at MHD shocks of maximum intensity coincide with the corresponding relations for shocks in absence of magnetic field, this being connected with the fact that magnetic intensity remains finite while $p_2 \rightarrow \infty$.

In case I when an equation of state has a form $w = w(p, n)$ the relations considered above must be enlarged with the relation (3.4) which expresses continuity of particles flux or of rest mass flux:

$$[mn(u^i n_i)] = 0. \qquad (3.43)$$

Let us denote in accordance with (3.43)

$$j \equiv \{ mnc(u^i n_i) \}_1 = \{ mnc(u^i n_i) \}_2. \tag{3.44}$$

The value of j in nonrelativistic limit turns into ρv_x.

The relations (3.8a-9) and (3.31) because of (3.44) yield

$$j^2 = \frac{V^i V_i}{4\pi} \frac{[k]}{[kw/(mnc)^2]}, \tag{3.45}$$

$$j^2 = -\frac{[p + (k^2/8\pi)]}{[w/(mnc)^2]}. \tag{3.46}$$

Now let us return to the equations (3.14-15). It follows from these equations in view of (3.8a-9) that

$$[\mathscr{K}^i \mathscr{K}_i] = [w^2(u^i n_i)^2] - [p]\{ w(u^i n_i)^2|_1 + w(u^i n_i)^2|_2 \} + \left\{ \frac{1}{2\pi}[w(u^i n_i)^2 k^2] - \left[\frac{k^2}{8\pi}\right](w(u^i n_i)^2|_1 + w(u^i n_i)^2|_2) - \frac{V^i V_i}{4\pi}\left[\frac{k^2}{4\pi}\right]\right\}.$$

The last term in the right side of this relation (in braces) by virtue of (3.31) reduces to $[w(u^i n_i)^2](k_2 - k_1)^2/8\pi$ so that the relation becomes

$$[w^2(u^i n_i)^2] - [p]\{ w(u^i n_i)^2|_1 + w(u^i n_i)^2|_2 \} + \frac{1}{8\pi}[w(u^i n_i)^2][k]^2 = 0. \tag{3.47}$$

The equation (3.47) with use of (3.44) transforms into the equation of *shock adiabat* (*Hugoniot relation*):

$$\left[\left(\frac{w}{mnc}\right)^2\right] - [p]\left\{\left(\frac{w}{(mnc)^2}\right)_1 + \left(\frac{w}{(mnc)^2}\right)_2\right\} + \frac{1}{8\pi}\left[\frac{w}{(mnc)^2}\right](k_2 - k_1)^2 = 0. \tag{3.48}$$

It is essential that the relation (3.48) in terms of variables (1.13) becomes written in the form which formally coincides with the form of shock adiabat in nonrelativistic MHD [5] if variables (1.13) be treated as the corresponding nonrelativistic quantities (with the only distinction that k must be replaced by H_τ). This circumstance permits to apply the known procedure of thermodynamic analysis in nonrelativistic version of MHD to the present case. An analogous fact is valid for shock waves in absence of magnetic field [3].

In connection with this it must be emphasized that though equations (3.45, 36) in terms of (1.13) look also like the corresponding nonrelativistic relations the value $V^i V_i$ in nonrelativistic limit reduces to H_n^2 so that in nonrelativistic version of MHD superfast waves phenomenon vanishes.

In order to ascertain the character of jump in values of thermodynamic variables inequalities for an equation of state must be formulated. We assume that an equation of state obeys the inequalities (2.52-54) as well as the inequality

$$\mathscr{I}_{p,\frac{\sigma}{n}} \equiv \partial(1/\tilde{p})/\partial(\sigma/n)\,|_p > 0. \tag{3.49}$$

In terms of (1.13-14) the set of inequalities (2.52-54), (3.49) is analogous to usual set of thermodynamic inequalities in nonrelativistic theory and under these conditions at shock waves in virtue of increasing of entropy in irreversible processes the pressure p and $\tilde{p}$ as it known must increase. Therefore in present case [4]

$$p_2 > p_1, \qquad (n^2/w)_2 > (n^2/w)_1, \tag{3.50}$$

and also in view of (3.46)

$$[p + (k^2/8\pi)] > 0. \tag{3.50a}$$

Due to the evolutionary conditions (3.39) in view of (3.36) and (3.50a) when shock front is crossed the value of k increases at fast shocks and decreases at slow ones. At slow shocks k_2 may be diminished up to zero, in this case transversal field H_τ being switched-off completely. Using also (3.45) we have therefore

for slow MHD shocks

$$[k] < 0, \qquad [kw/(mnc)^2] < 0; \tag{3.51}$$

for fast MHD shocks

$$[k] > 0, \qquad [kw/(mnc)^2] \begin{cases} > 0 & \text{when} \quad V^iV_i > 0, \\ = 0 & \text{when} \quad V^iV_i = 0, \\ < 0 & \text{when} \quad V^iV_i < 0. \end{cases}$$

Along with (3.50) at shock wave (in view of $\mathscr{I}_2 > \mathscr{I}_1$) also

$$(w/n)_2 > (w/n)_1 \qquad \text{and} \qquad n_2 > n_1. \tag{3.50b}$$

In conclusion let the case be pointed out when at shock wave $(h^in_i) = 0$. According to (3.10-11) at shock front the equality $(h^in_i)_1 = 0$ takes place together with $(h^in_i)_2 = 0$ and *vice versa*. In this case 4-vector V^i in view of (2.25) is space-like and is directed along h^i so that such a shock must be superfast. 4-vector W^i (3.12) in this case is orthogonal to V^i and is represented with use of (3.17) in the form:

$$W^i = -\left\{\left(w + \frac{|h|^2}{4\pi}\right)(u^kn_k)^2 + \left(p + \frac{|h|^2}{8\pi}\right)\right\} n^i + \left(w + \frac{|h|^2}{4\pi}\right)(u^kn_k)g^i, \tag{3.52}$$

the relation (3.2) being written as

$$V^i = -(u^k n_k) h^i, \qquad [V^i] = 0. \tag{3.53}$$

The addition to (3.52-53) of the condition of continuity of particles flux (3.43) yields in view of (3.53)

$$[h^k/n] = 0, \qquad (|h|^2/n^2)_2 = (|h|^2/n^2)_1 = b^2, \tag{3.54}$$

and (3.52) in virtue of (3.54) becomes

$$W^i = -\{ w^*(u^k n_k)^2 + p^* \} n^i + w^*(u^k n_k) g^i; \quad w^* \equiv w + \frac{b^2}{4\pi} n^2, \quad p^* \equiv p + \frac{b^2}{8\pi} n^2. \tag{3.55}$$

The expression (3.55) for W^i formally coincides with the expression for W^i in relativistic hydrodynamics in absence of field for a fluid with heat function w^* and pressure p^*.

In the present case among w-systems such a w-system can be chosen in which $(\vec{v}_\tau)_1 = 0$. Then in view of $[W^i] = 0$ also $(\vec{v}_\tau)_2 = 0$ and according to $(h^i n_i) = 0$ and (1.21) also $H_n = 0$. In this w-system matter flows normally to a direction of magnetic field which is parallel to shock wave front.

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(Manuscrit reçu le 7 juillet 1969).