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A. H. TAUB

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Restricted motions of gravitating spheres (*)

by

A. H. TAUB

Mathematics Department University of California,
Berkeley, California.

ABSTRACT. — Two restricted classes of spherically symmetric time dependent solutions of the Einstein field equations are discussed. For both of these the source of the gravitational field is a perfect fluid which occupies a limited region of space-time. The first class contains solutions discussed by Thompson and Whitrow, by Bondi and by Bonnor and Faulkes. The second class of solutions are those treated by McVittie. It is shown that McVittie's solutions are a subclass of the first class of solutions and that if an equation of state exists then solutions of the first class satisfy McVittie's similarity requirements. Not all of McVittie's solutions satisfy an equation of state. The solutions of the first class treated by the authors listed above are special in that they have uniform energy density. Various methods are described for determining oscillating as well as collapsing and expanding uniform solutions. It is shown that for an oscillating solution, the outer boundary of the matter is always outside the Schwarzschild radius of the total gravitating mass.

1. INTRODUCTION

It is the purpose of this paper to discuss and compare two restricted classes of spherically symmetric time dependent solutions of the Einstein

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field equations in which the source of the gravitational field is a perfect fluid which occupies a limited region of space-time. The first class consists of space-times with metric tensors

$$ds^2 = c^{2\Phi}dt^2 - \frac{1}{c^2} e^{2\Psi}dr^2 - \frac{e^{2\mu}}{c^2} d\Omega^2 \tag{1.1}$$

where Φ, Ψ and μ are functions of r and t and satisfy

$$\Psi_t = \mu_t ; \tag{1.2}$$

we use the notation

$$f_t = \frac{\partial f}{\partial t} \qquad f_r = \frac{\partial f}{\partial r},$$

and

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \tag{1.3}$$

The restriction given by equation (1.2) was introduced by I. H. Thompson and G. J. Whitrow [*1*]. It will be shown below that the Einstein field equations together with the requirement that the coordinate system in which equation (1.1) holds be a co-moving one, implies that equation (1.2) is equivalent to the condition that

$$\mathrm{U}(r, t) \equiv e^{-\Phi}\mathrm{R}_t = \mathrm{R}\frac{\mathfrak{R}_t}{\mathfrak{R}} \tag{1.4}$$

where

$$\log \mathrm{R}(r, t) = \mu \tag{1.5}$$

and

$$\mathfrak{R} = \mathscr{R}(t) \tag{1.6}$$

that is $\mathscr{R}$ is a function of t alone.

When the coordinate system is a co-moving one, the variables r is the analogue of the Lagrangean coordinate of classical hydrodynamics. The function $\mathrm{R}(r, t)$ is the Eulerian coordinate of that theory. That is $\mathrm{R}(r, t)$ is the coordinate position at time t of the fluid particle which at $t = 0$ was at the coordinate position r, if we require that

$$\mathrm{R}(r, 0) = r. \tag{1.7}$$

It is no restriction to impose the condition expressed by equation (1.7) for it may always be satisfied by marking a transformation of the variable r alone.

With this interpretation of $R(r, t)$, it follows that $U(r, t)$ defined by equation (1.4) is the rate of change of R with respect to proper time relative to the observer at $r =$ constant, $\theta =$ constant, $\varphi =$ constant. It plays the role of the fluid velocity relative to the Lagrange coordinates in classical theory. McVittie [2] has called classical motions satisfying equations similar to equation (1.4) linear wave motions.

Since we may normalize the coordinate t by the requirement that

$$\Phi(r_0, t) \equiv 0 \tag{1.8}$$

for some fixed value r_0 of the coordinate r, we find from equation (1.4) that

$$\frac{R_t(r_0, t)}{R(r_0, t)} = \frac{\mathfrak{R}_t}{\mathfrak{R}}.$$

Hence

$$R_0\mathscr{R}(t) = R(r_0, t)$$

where R_0 is a constant. In view of equation (1.7) we have

$$R_0\mathscr{R}(0) = r_0.$$

That is,

$$\mathfrak{R}(t) = \frac{R(r_0, t)}{r_0}\,\mathfrak{R}(0) \tag{1.9}$$

and

$$U(r_0, t) = \frac{r_0}{\mathfrak{R}(0)}\,\mathfrak{R}_t \tag{1.10}$$

Since the r, t coordinate system is a co-moving one the four-velocity of the fluid is given by

$$u^\mu = e^{-\Phi}\delta_4^\mu, \tag{1.11}$$

and satisfies

$$u^\mu u_\mu = 1 \tag{1.12}$$

The stress energy tensor of the fluid is of course given by the equations

$$T^{\mu\nu} = (w + p/c^2)u^\mu u^\nu - \frac{p}{c^2}\,g^{\mu\nu} \tag{1.13}$$

where w is the energy density and p is the pressure.

It should be noted that if the fluid extends to the origin that is occupies the position $r = 0$ for all times, we must have

$$R(0, t) = 0 \tag{1.14}$$

identically in t as a result of the spherical symmetry and the interpretation of $R(r, t)$. Equations (1.7), (1.9) and (1.14) constitute boundary and initial conditions on the function $R(r, t)$.

It is a consequence of equation (1.2) that

$$\mu = \Psi + \log f(r) \tag{1.15}$$

where $f(r)$ is an arbitrary function of r. Hence the line element given by equation (1.1) may be written as

$$ds^2 = e^{2\Phi}dt^2 - \frac{R^2}{c^2 f^2}(dr^2 + f^2 d\Omega^2) \tag{1.16}$$

or as

$$ds^2 = e^{2\Phi}dt^2 - \frac{R^2}{c^2}(d\bar{r}^2 + d\Omega^2) \tag{1.17}$$

where

$$d\bar{r} = \frac{dr}{f(r)}. \tag{1.18}$$

Therefore the proper distance along a curve t = constant, θ = constant, φ = constant is given by

$$r_p = \int \frac{R}{f}\, dr = \int R d\bar{r}. \tag{1.19}$$

The invariant volume element in the hypersurface t = constant is given by

$$dV = \frac{R^3}{f} \sin\theta d\theta d\varphi dr = R^3 \sin\theta d\theta d\varphi d\bar{r} \tag{1.20}$$

The second class of solutions we wish to discuss are those treated by McVittie [3]. For this class the line element is given by equation (1.16), the coordinates r and t are co-moving ones and $\Phi(r, t)$ and $R(r, t)$ are restricted to obey the following conditions

$$\Phi = \Phi(z)$$

$$R = R_0 Q(r) f(r) e^{\eta/2 - z}$$

where R_0 is a constant, Q is a function of r alone and

$$e^z = \frac{Q(r)}{\mathcal{R}(t)},$$

and

$$e^\Phi = 1 - \eta_z/2.$$

That is, McVittie's solutions, which are analogous to the similarity solutions of classical hydrodynamics problems, satisfy equations (1.2), the co-moving condition and additional requirements. They are therefore a subclass of the solutions satisfying equation (1.2) or the equivalent requirement, equation (1.4).

In the next section we shall see that when we impose the condition that an equation of state be satisfied, that is we require that

$$p = p(w) \tag{1.21}$$

then the solutions satisfying equation (1.2) or (1.4) in a co-moving coordinate system must be of the McVittie subclass. However, only special equations of state, that is special function $p(w)$ are allowed.

In subsequent sections we shall show that if equation (1.2) holds, the problem of solving the Einstein field equations is reduced to solving a second order ordinary differential equation. We shall also show the exterior solution may be obtained in an extension of the co-moving coordinate system and discuss the conditions that hold on the boundary between the interior and exterior solutions.

2. THE CONSERVATION LAWS

These laws are expressed by the equations

$$T^{\mu\nu}{}_{;N} = 0 \tag{2.1}$$

where $T^{\mu\nu}$ is given by equation (1.13). When they are evaluated in the co-moving coordinate system for which equations (1.11) and (1.16) hold they reduce to the following equations

$$Rw_t + 3R_t(w + p/c^2) = 0 \tag{2.2}$$

and

$$p_r + (wc^2 + p)\Phi_r = 0 \tag{2.3}$$

The first of these equations represents the conservation of energy and may be written

$$(wdV)_t + p(dV)_t = 0$$

where dV is given by equation (1.19). Thus it states that the rate of change of energy in the proper volume dV is the negative of the pressure times the rate of change of the proper volume. Equation (2.3) is the equation of hydrostatic equilibrium.

Suppose we now assume that an equation of state exists that is the pressure is determined by w alone through an equation of the form of equation (1.20). Then we may define a function $\sigma(w)$ by the equation

$$\frac{d\sigma}{\sigma} = \frac{dw}{w + p/c^2} \tag{2.4}$$

This equation defines σ up to a constant of integration. We shall determine this constant as follows. Let w_0 be determined so that

$$p(w_0) = 0$$

and choose the constant of integration so that

$$\sigma(w_0) = w_0.$$

It is a consequence of equations (2.4) and (2.3) that

$$e^{\Phi(r,t)} = e^{\Phi(r_0,t)} \frac{\sigma}{w + p/c^2} \left(\frac{w + p/c^2}{\sigma}\right)_0$$

where $\left(\frac{w + p}{\sigma}\right)_0$ is evaluated at $r = r_0$. In the problems we wish to consider the fluid is to occupy the region

$$0 \leq r \leq r_0$$

and be bounded by a vacuum for

$$r \geq r_0.$$

Hence at $r = r_0$ we must have

$$p(r_0, t) = 0.$$

In view of the above we must also have

$$w(r_0, t) = w_0,$$

a constant and

$$\sigma(r_0, t) = w_0.$$

Hence,

$$e^{\Phi(r,t)} = e^{\Phi(r_0,t)} \frac{\sigma}{w + p/c^2}.$$

If we further identify r_0 with the value of r used to normalize the coordinate t we then obtain

$$e^{\Phi(r,t)} = \frac{\sigma}{w + p/c^2} = \frac{\mathrm{R}_t}{\mathrm{R}} \frac{\mathfrak{R}}{\mathfrak{R}_t} \tag{2.4'}$$

in view of equation (1.4). Further equation (1.9) holds, that is

$$\mathrm{R}(r_0, t) = r_0 \frac{\mathfrak{R}(t)}{\mathfrak{R}(0)}. \tag{2.5}$$

In view of equation (2.4) it follows from equation (2.2) that

$$\mathrm{R}^3\sigma = h^3(r) = r^3\sigma(r, 0) \tag{2.6}$$

where $h(r)$ is a function of r alone. If the second of equations (2.4′) is used to substitute for R_t in equation (2.2) we obtain

$$w_t = -3\sigma \frac{\mathfrak{R}_t}{\mathfrak{R}}.$$

Since σ is a function of w we conclude that $\mathrm{F}(w)$ defined by

$$\frac{d\mathrm{F}}{dw} = \frac{1}{3\sigma}$$

satisfies the relation

$$\mathrm{F}(w) = \log \frac{\mathrm{Q}(r)}{\mathfrak{R}(t)}.$$

That is

$$w = w(z) \tag{2.7}$$

where

$$e^z = \frac{\mathrm{Q}(r)}{\mathfrak{R}(t)} \tag{2.8}$$

and

$$w_z = 3\sigma. \tag{2.9}$$

It then follows from equation (2.6) that

$$R = h(r)\sigma^{-1/3}(z)$$

and from the first of equations (2.4) that

$$\Phi = \Phi(z).$$

These are just the conditions imposed by McVittie in his discussion of similarity solutions of the Einstein field equations. If we set, as McVittie does,

$$\frac{\eta}{2} - z = -\frac{1}{3}\log \sigma,$$

then we find

$$1 - \eta_z/2 = \frac{1}{3}\frac{\sigma_z}{\sigma} = \frac{1}{3}\frac{w_z}{w + p/c^2} = \frac{\sigma}{w + p/c^2} = e^{\Phi}.$$

This is the condition imposed by McVittie, which insures that the r, t coordinate system is co-moving.

Thus we see that the requirement that an equation of state existe together with the requirement that equation (1.4) (or, as shall we shown, equivalently equation (1.2)) be satisfied implies that McVittie's similarity conditions be satisfied. On the other hand, although all of the solutions satisfying McVittie's similarity conditions satisfy equations (1.4) and (1.2), they do not all imply the existence of an equation of state. This is evident from the results obtained by McVittie in which he gives explicit solutions for which an equation of state does not exist.

It follows from the results given below that the equation of state $p = p(w)$ must satisfy a particular differential equation in order that the field equations be satisfied. Thus only special equations of state are consistent with the Einstein field equations and the assumption that equation (1.2) or (1.4) is satisfied.

Before turning to the discussion of the field equations we consider equations (2.2) and (2.3) when the particle number is conserved. Let us write

$$w = \rho(1 + \varepsilon/c^2)$$

where ρ is the rest particle density and $\varepsilon(p, \rho)$ is the rest specific internal energy. The conservation of particle number is then expressed by the equation

$$(\rho u^{\mu})_{;\mu} = 0.$$

In the co-moving coordinate system in which equation (1.16) holds this equation becomes

$$\rho R^3 = k(r). \tag{2.10}$$

Equation (2.2) then becomes equivalent to

$$S_t = 0$$

or

$$S = S(r) \tag{2.11}$$

where S is the rest specific entropy defined by the equation

$$TdS = d\varepsilon + pd\left(\frac{1}{\rho}\right) \tag{2.12}$$

and T is the temperature.

In case the entropy is a constant independent of r, an equation of state exists and the results given above obtain with

$$\sigma = \rho \tag{2.13}$$

The necessary and sufficient condition for equation (2.13) to hold is that

$$S = S_0 = \text{constant.}$$

In that case equation (2.4) may be written as

$$e^{\Phi(r,t)} = \frac{1}{1 + i/c^2} \tag{2.14}$$

where i is the specific enthalpy and is defined as

$$i = \varepsilon + p/\rho \tag{2.15}$$

3. THE FIELD EQUATIONS

In a previous work [*4*] it has been shown that when the line element is given by equation (1.1) the field equations plus the condition that the coordinate system be co-moving, that is the condition,

$$T^{14} = T^{41} = 0$$

implies that

$$\mu_{rt} - \mu_t\Phi_r - \mu_r\Psi_t + \mu_r\mu_t = 0. \tag{3.1}$$

If we now assume that equation (1.2) obtains, we have

$$\mu_{rt} - \mu_t \Phi_r = 0.$$

When $\mu_t \neq 0$, that is when we are dealing with a time dependent problem it follows that

$$\Phi_r = \frac{\mu_{rt}}{\mu_t} \tag{3.2}$$

or

$$e^{\Phi} = \frac{R_t}{R}\frac{\mathcal{R}}{\mathcal{R}_t}, \tag{3.3}$$

where $\mathcal{R}/\mathcal{R}_t$ is a function of t which enters as a constant of integration. That is, equation (1.4) holds. Conversely if equation (1.4) holds it follows from equation (3.1) that equation (1.2) is satisfied.

It is another consequence of equation (1.2) and the results given in [4] that the condition that the stresses be isotropic, that is the condition that

$$T^1_1 = T^2_2 = T^3_3$$

becomes

$$- c^2 e^{-2\Psi}\mu_r(\mu_r + 2\Phi_r) + c^2 e^{-2\mu} \\ = - c^2 e^{-2\Psi}[\Phi_{rr} + \mu_{rr} + \mu_r^2 + \Phi_r^2 - \Phi_r\Psi_r + \mu_r(\Phi_r - \Psi_r)] = 0$$

or

$$\Phi_{rr} + \mu_{rr} + \Phi_r^2 - \Phi_r\Psi_r - \mu_r(\Phi_r + \mu_r) + e^{-2\mu+2\Psi} = 0.$$

It then follows from equations (1.15) and (1.18) that

$$\Phi_{\bar{r}\bar{r}} + \mu_{\bar{r}\bar{r}} + \Phi_{\bar{r}}^2 - 2\Phi_{\bar{r}}\mu_{\bar{r}} - \mu_{\bar{r}}^2 + 1 = 0$$

where $\bar{r}$ is defined in terms of r by equation (1.18).

On substituting from equation (3.2) into this equation we obtain

$$\left(\mu_{\bar{r}\bar{r}} - \mu_{\bar{r}}^2 + 1\right)_t + \left(\mu_{\bar{r}\bar{r}} - \mu_{\bar{r}}^2 + 1\right)\mu_t = 0.$$

That is,

$$\mu_{\bar{r}\bar{r}} - \mu_{\bar{r}}^2 + 1 = B(\bar{r})e^{-\mu} \tag{3.4}$$

when $B(\bar{r})$ is an arbitrary function of its argument. In terms of the function R introduced via equation (1.5) equation (3.4) may be written as

$$\left(\frac{1}{R}\right)_{\bar{r}\bar{r}} = \frac{1}{R} - \frac{B(\bar{r})}{R^2}. \tag{3.5}$$

When the function $B(\bar{r})$ is specified, equation (3.5) may be integrated to obtain $R(\bar{r}, t)$. This function then determines Φ and Ψ via equations (1.4) and (1.5). The metric is thus determined. The pressure and energy density of the fluid are then calculated from the remaining Einstein field equations, the equations involving T_4^4 and T_1^1. This procedure presupposes that $R_t \neq 0$; for otherwise Φ cannot be determined from equation (1.4). In case $R_t = 0$ and equation (1.2) holds we are essentially dealing with the static case.

The function $R(r, t)$ depends on the variable t because the « constants of integration » of equation (3.5) depend on the time variable.

Instead of calculating p and w from the Einstein field equation directly we may proceed as Bardeen [5] and Misner and Sharp [6] do and calculate these quantities from the derivatives of a function $m(r, t)$ defined as follows:

$$e^{2\Psi} \equiv \frac{R^2}{f^2(r)} = \frac{R_r^2}{1 + \frac{U^2}{c^2} - \frac{2Gm}{c^2 R}} = \frac{R_r^2}{\Gamma^2} \tag{3.6}$$

where U is defined by equation (1.4). Thus in our case

$$\frac{2Gm(r, t)}{c^2} = R\left(1 + \frac{R^2}{c^2}\left(\frac{\mathfrak{R}_t}{\mathfrak{R}}\right)^2 - \left(\frac{R_{\bar{r}}}{R}\right)^2\right) \tag{3.7}$$

and R is given as a solution of equation (3.5).

The Einstein field equations then imply that

$$m_r = 4\pi w R^2 R_r \tag{3.8}$$

and

$$m_t = -\frac{4\pi p}{c^2} R^2 R_t \tag{3.9}$$

The quantity $m(r, t)$ represents the gravitational mass at time t of the fluid particles with Lagrange coordinates less than or equal to r.

If equation (3.7) is differentiated with respect to $\bar{r}$, it follows from equations (3.5) and (3.8) that

$$\frac{c^2 B(\bar{r})}{3G} = m - \frac{4\pi R^3}{3} w \tag{3.10}$$

This equation and the field equations imply equation (1.2). When $R(r, t)$ is given explicitly as a solution of equation (3.5), equation (3.10) together with equation (3.7) serves to determine w as a function of r and t. Equation (3.10) implies equation (2.2) as may be seen by differentiating the former equation with respect to t and using equation (3.9).

On differentiating equation (3.7) with respect to t and using equations (3.9) and (1.4) we obtain

$$-\frac{4\pi Gp}{c^2} = \frac{Gm}{R^3} + \left(\frac{\mathfrak{R}_t}{\mathfrak{R}}\right)^2 + e^{-\Phi}\left(\frac{\mathfrak{R}_{tt}}{\mathfrak{R}} - \frac{\mathfrak{R}_t}{\mathfrak{R}^2}\right) - \frac{R_{\bar{r}}}{R^3}\Phi_{\bar{r}} \tag{3.11}$$

It should be noted that if we consider m, Φ, w and p as functions of R and t then the system of equations consisting of equations (2.4), (3.8) and (3.11) may be written as

$$p_R = (wc^2 + p)\Phi_R \tag{3.12}$$

$$m_R = 4\pi w R^2 \tag{3.13}$$

$$\left[1 + \frac{2}{c^2}\left(\frac{\mathfrak{R}_t}{\mathfrak{R}}\right)^2 - \frac{2Gm}{Rc^2}\right]\Phi_R = \frac{G}{R^2}\left(m + 4\pi\frac{pR^3}{c^2}\right) + R\left[\left(\frac{\mathfrak{R}_t}{\mathfrak{R}}\right)^2 + e^{-\Phi}\left(\frac{\mathfrak{R}_{tt}}{\mathfrak{R}} - \left(\frac{\mathfrak{R}_t}{\mathfrak{R}}\right)^2\right)\right] \tag{3.14}$$

respectively. In case

$$\mathfrak{R}_t = \mathfrak{R}_{tt} = 0$$

these equations are the equations of static equilibrium of a fluid sphere in the Schwarzschild-like coordinates system in which the line element has the form

$$ds^2 = e^{2\Phi}dt^2 - \frac{1}{c^2}\frac{dR^2}{\left(1 - \frac{2Gm}{Rc^2}\right)} - \frac{R^2}{c^2}d\Omega.$$

We observe that the function B when considered as a function of r, not $\bar{r}$, may be evaluated from equation (3.10) at any value of t. If we use $t = 0$ and remember the normalization $R(r, 0) = r$, we have

$$\frac{c^2 B(r)}{3G} = m(r, 0) - \frac{4\pi r^3}{3} w(r, 0).$$

In view of equation (3.8) we may write

$$m(r, 0) = 4\pi\int_{r_i}^{r} w(r, 0)r^2 dr + m(r_i, 0).$$

where r_i is the coordinate of an inner boundary. We may take $r_i = 0$

and in that case require that $m(r_i, 0) = 0$ in order to avoid a singularity in the metric at the origin. We shall assume in all cases that $m(r_i, 0) = 0$. Then we have

$$\frac{c^2 B(r)}{3G} = r\pi \int_{r_i}^{r} w(r, 0) r^2 dr - \frac{4\pi}{3} r^3 w(r, 0).$$

If we define a mean initial density by the equation

$$m(r, 0) = \frac{4\pi}{3} r^3 \bar{w}(r, 0)$$

we have

$$B(r) = \frac{3\pi G}{c^2} r^3 (\bar{w}(r, 0) - w(r, 0)). \tag{3.15}$$

If there is a value of $\bar{r}$ say $\bar{r}_0$ such that

$$p(\bar{r}_0, t) = 0,$$

it follows from equation (2.2) that

$$\frac{4\pi}{3} R^3(\bar{r}_0, t) w(\bar{r}_0, t) = \frac{4\pi}{3} R^3(\bar{r}_0, 0) w(\bar{r}_0, 0).$$

It is then a consequence of equation (3.10) that

$$m(\bar{r}_0, t) = \frac{c^2 B(\bar{r}_0)}{3G} + \frac{4\pi}{3} R^3(\bar{r}_0, 0) w(\bar{r}_0, 0).$$

Therefore

$$m(r_0, t) = 4\pi \int_{r_i}^{r_0} w(r, 0) r^2 dr = \text{constant} = m_e$$

where r_0 is the value of r corresponding to $\bar{r}_0$. The constant m_e is the total gravitational mass of the fluid as measured by an external observer as will be shown below.

4. THE EXTERIOR SOLUTION

We shall now assume that for $r > r_0 (\bar{r} > \bar{r}_0)$ the stress energy tensor vanishes, that is $w = p = 0$ and construct a solution of the field equations in a coordinate system which is an extension of the coordinate system used above. That is we shall assume that equation (1.2) hold and that $R_t \neq 0$. Since w and p are both required to vanish it follows that m must be a constant say m_e. Then in view of equation (3.10) we must have B a constant. In that case equation (3.5) has the first integral

$$\left(\frac{1}{R}\right)_{\bar{r}}^2 = \left(\frac{1}{R}\right)^2 - \frac{2}{3}\frac{B}{R^3} + C(t) = \frac{R_{\bar{r}}^2}{R^4} \tag{4.1}$$

If this equation is solved for $\left(\frac{R_{\bar{r}}}{R}\right)^2$ and the result is substituted into equation (3.7) we find that

$$m = m_e = \frac{c^2 B}{3G} = \text{constant}$$

if and only if

$$C(t) = \frac{1}{c^2}\left(\frac{\mathfrak{R}_t^e}{\mathfrak{R}^e}\right)^2$$

where $\mathscr{R}^e(t)$ is a function of t which is related to the exterior solution. Hence equation (4.1) may be written as

$$\left(\frac{1}{R}\right)_{\bar{r}}^2 = \left(\frac{1}{R}\right)^2 - \frac{2Gm_e}{c^2 R^3} + \frac{1}{c^2}\left(\frac{\mathfrak{R}_t^e}{\mathfrak{R}^e}\right)^2. \tag{4.2}$$

When $R(\bar{r}, t)$ is given as a solution of equation (4.2) and Φ is then determined by the equations

$$e^{\Phi} = \frac{R_t}{R}\frac{\mathfrak{R}^e}{\mathfrak{R}_t^e}, \tag{4.3}$$

the line element

$$ds^2 = e^{2\Phi}dt^2 - \frac{R^2}{c^2}(d\bar{r}^2 + d\Omega^2) \tag{4.4}$$

gives a solution of the Einstein field equations for empty space-time, that is with $T^{\mu\nu} = 0$.

We may again normalize the time coordinate t by the requirement that

$$\Phi(\bar{r}_0, t) = 0.$$

Then we have

$$R_0^e \mathscr{R}^e(t) = R(\bar{r}_0, t) \tag{4.5}$$

where R_0^e is a constant determined from the equation

$$R_0^e \mathfrak{R}^e(0) = R(\bar{r}_0, 0) = r_0(\bar{r}_0),$$

r_0 being the value of r corresponding to $\bar{r} = \bar{r}_0$.
We define

$$X(\bar{r}_0, t) = \frac{1}{R(\bar{r}_0, t)} \tag{4.6}$$

and then may wsite equation (4.2) as

$$\bar{r} - \bar{r}_0 = \int_{X_0(t)}^{X} \frac{dX}{\left(X^2 - \frac{2Gm_e}{c^2} X^3 + \left(\frac{X_{0t}}{X_0}\right)^2\right)^{1/2}} \tag{4.7}$$

where

$$X_0 = X(\bar{r}_0, t) \tag{4.8}$$

and

$$X_{0t} = \frac{d}{dt} X(\bar{r}_0, t) \tag{4.9}$$

We may verify that the constant m_e which enters into equations (4.2) and (4.7) is the gravitational mass as measured by an external observer by showing how the line-element given by equation (4.4) may be transformed into the Schwarzschild one. We first observe that in view of equation (4.2) we may write equation (4.4) as

$$ds^2 = e^{2\Phi} dt^2 - \frac{R_{\bar{r}}^2 d\bar{r}^2}{c^2 \Gamma^2} - \frac{R^2}{c^2} d\Omega^2 \tag{4.10}$$

where

$$\Gamma^2 = 1 + \left(\frac{\mathfrak{R}_t^e R}{c \mathfrak{R}^e}\right)^2 - \frac{2Gm_e}{Rc^2}. \tag{4.11}$$

Since

$$dR = R_{\bar{r}} d\bar{r} + R_t dt = R_{\bar{r}} d\bar{r} + e^{\Phi} R \frac{\mathfrak{R}_t^e}{\mathfrak{R}^e} dt$$

we may write equation (4.10) as

$$ds^2 = \left(1 - \frac{2Gm_e}{Rc^2}\right) d\tau^2 - \frac{dR^2}{c^2\left(1 - \frac{2Gm_e}{Rc^2}\right)} - \frac{R^2}{c^2} d\Omega^2 \tag{4.12}$$

where

$$d\tau = \frac{\Gamma e^{\Phi} dt + \dfrac{R^2 \mathcal{R}_t^e}{c^2 \mathcal{R}^e}\, d\bar{r}}{1 - \dfrac{2Gm_e}{Rc^2}} \tag{4.13}$$

and is a perfect differential in view of equations (4.2), (4.3) and (4.11). Equation (4.12) is the usual form of the Schwarzschild exterior metric in terms of the coordinates τ and R. The first of these coordinates is defined in terms of r and t by equation (4.13) and the second by equation (4.7). We note that if $m_e = 0$, the space-time is flat.

5. BOUNDARY CONDITIONS

It has been shown [4] that the conditions that must be satisfied if the interior solution of the Einstein field equations described in section 3 is to be joined to the exterior solution obtained in section 4 are the following: the boundary separating the two solutions is the hypersurface

$$\bar{r} = \bar{r}_0$$

and on this hypersurface, the pressure must vanish, that is we must have

$$p(\bar{r}_0, t) = 0 \tag{5.1}$$

for the interior solution as well as the exterior one. Note that the density $w(\bar{r}_0, t)$ need not vanish for the interior solution.

Further, the functions Φ and μ in equation (1.1) must be continuous across the hypersurface and their derivatives with respect to t and $\bar{r}$ must also be continuous. The function Ψ and Ψ_t are also required to be continuous at $\bar{r} = \bar{r}_0$ but $\Psi_{\bar{r}}$ need not be continuous there. In our case we have Ψ simply related to μ for both the interior and exterior solutions (cf. equation (1.2)) and the continuity of R and $R_{\bar{r}}$ and R_t ensures the continuity Ψ and its derivatives if the function of integration $f(r)$ is choosen to have a continuous derivative.

Since we have normalized the t coordinate for both the interior and exterior solution by the requirement $\Phi(\bar{r}_0, t) = 0$, we have ensured the continuity of Φ at $\bar{r}_0$. Equations (1.9) and (4.5) and the continuity of $R(\bar{r}, t)$ then require that

$$R_0^e \mathcal{R}^e(t) = R_0 \mathcal{R}(t) \tag{5.2}$$

with

$$R_0^e \mathcal{R}^{(e)}(0) = R_0 \mathcal{R}(0) = r_0(\bar{r}_0)$$

That is, we must have

$$\frac{\mathcal{R}^e(t)}{\mathcal{R}^e(0)} = \frac{\mathcal{R}(t)}{\mathcal{R}(0)} \tag{5.3}$$

identically in t and hence

$$\frac{\mathcal{R}_t^e}{\mathcal{R}^e} = \frac{\mathcal{R}_t}{\mathcal{R}} . \tag{5.4}$$

It then follows from equations (4.2) and (3.7) and the continuity of R and $R_{\bar{r}}$ that

$$m_e = m(\bar{r}_0, t) \tag{5.5}$$

and hence

$$\Gamma_e^2(\bar{r}_0, t) = 1 + \left(\frac{\mathcal{R}_t^e}{c\mathcal{R}^e}\right)^2 R^2 - \frac{2Gm_e}{Rc^2} = \Gamma^2(\bar{r}_0, t) \tag{5.6}$$

where Γ^2 is defined by equation (3.6) and Γ_e^2 is given by equation (4.11).

Since equation (5.6) holds identically in t and since

$$R_{\bar{r}} = R\Gamma$$

it follows from these facts and equation (1.4) that $\Phi_{\bar{r}}$ will be continuous accross $\bar{r} = \bar{r}_0$.

Equation (5.5) enables us to interpret the function $m(\bar{r}, t)$ in terms of the gravitational mass. Equation (5.3) relates arbitrary function $\mathcal{R}(t)$ and $\mathcal{R}^e(t)$. These functions represent the Eulerian displacement of the boundary $\bar{r} = \bar{r}_0$.

6. UNIFORM ENERGY DENSITY

If

$$w_r(r, t) = 0, \tag{6.1}$$

that is if w is a function of t alone, we shall say that the energy density is uniform. This situation arises in a number of cases and has been discussed by Thompson and Whitrow [*1*] by Bondi [*7*] and by Bonnor and Faulkes [*8*].

It follows from equations (3.8) and (3.10) that

$$\frac{c^2 B(\bar{r})}{3G} = m(\bar{r}_i, t) - \int_{r_i}^{\bar{r}} \frac{4\pi R^3}{3} w_{\bar{r}} d\bar{r} \tag{6.2}$$

and hence $B(\bar{r}_i)$ is the mass at $\bar{r} = \bar{r}_i$ a side from constants,

$$B_{\bar{r}} = -\frac{4\pi G w_{\bar{r}} R^3}{c^2} \tag{6.3}$$

Thus, the necessary and sufficient condition that $w_{\bar{r}} = 0$ is that

$$B(\bar{r}) = \text{constant}.$$

If in addition we require that $m(\bar{r}_i, t) = 0$. Then we must have

$$B(\bar{r}) = 0, \tag{6.4}$$

as the criterion for uniform energy density (cf. Thompson and Whitrow [1]).
When equation (6.4) holds equation (3.10) becomes

$$m = \frac{4\pi R^3}{3} w \tag{6.5}$$

and equation (3.5) has the first integral

$$\left(\frac{1}{R}\right)_{\bar{r}}^2 = \left(\frac{1}{R}\right)^2 + C(t) = \frac{R_{\bar{r}}^2}{R^4}. \tag{6.6}$$

On substituting equation (6.6) into equation (3.7) we obtain

$$2Gm(\bar{r}, t) = R^3\left[\left(\frac{\mathfrak{R}_t}{\mathfrak{R}}\right)^2 - c^2 C(t)\right]. \tag{6.7}$$

We may now evaluate $C(t)$ by letting $\bar{r} = \bar{r}_0$ in the above equation. We then have in view of equation (1.9)

$$c^2 C(t) = \left(\frac{\mathfrak{R}_t}{\mathfrak{R}}\right)^2 - \frac{2Gm_e R^3(0)}{r_0^3 \mathfrak{R}^3(t)} \tag{6.8}$$

and

$$m(\bar{r}, t) = \left(\frac{R(\bar{r}, t)}{R(\bar{r}^3, t)}\right)^3 m_e = \left(\frac{R(\bar{r}, t)\mathscr{R}(0)}{r_0 \mathfrak{R}^3(t)}\right)^3 m_e. \tag{6.9}$$

In view of equation (6.5) we have

$$w = w(0)\left(\frac{\mathcal{R}(0)}{\mathrm{R}(t)}\right)^3 \tag{6.10}$$

where

$$\frac{4\pi}{3} r_0^3 w(0) = m_e. \tag{6.11}$$

The last equation is consistent with equation (6.5), the boundary conditions and the fact that m_e is a constant.

It is a consequence of equations (6.9) and (3.9) that

$$\frac{p}{c^2}\left(\frac{\mathcal{R}_t}{\mathcal{R}}\frac{\mathrm{R}}{\mathrm{R}_t} - 1\right)w = (e^{-\Phi} - 1)w \tag{6.12}$$

It follows that $p(\bar{r}_0, t) = 0$.

It is no restriction to set

$$\mathcal{R}(0) = 1$$

for if this condition is not satisfied we may replace $\mathcal{R}(t)$ by $\mathcal{R}(t)/\mathcal{R}(0)$ and for the new $\mathcal{R}(t)$ we shall have $\mathcal{R}(0) = 1$. If we now define

$$\begin{aligned}\Gamma_0^2 &= \Gamma_e^2(\bar{r}_0, t) = \Gamma^2(\bar{r}_0, t) \\ &= 1 + \frac{\mathcal{R}_t^2}{c^2\mathcal{R}^2}\mathrm{R}^2(r_0, t) - \frac{2Gm_e}{c^2\mathrm{R}(\bar{r}_0, t)} \\ &= 1 + \frac{r_0^2}{c^2}\left(\mathcal{R}_t^2 - \frac{8\pi G}{3}\frac{w(0)}{\mathrm{R}(t)}\right),\end{aligned} \tag{6.13}$$

then

$$\mathrm{C}(t) = \frac{1}{\mathcal{R}^2 r_0^2}(\Gamma_0^2 - 1). \tag{6.14}$$

The solution of equation (6.6) is then

$$\mathrm{R}(\bar{r}, t) = \frac{2r_0\mathcal{R}(t)e^{(\bar{r}-\bar{r}_0)}}{\Gamma_0 + 1 - (\Gamma_0 - 1)e^{2(\bar{r}-\bar{r}_0)}} \tag{6.15}$$

where $\mathcal{R}(0) = 1$. If we write

$$r = e^{\bar{r}},$$

that is, choose

$$f(r) = \frac{1}{r}$$

in equation (1.18), then we may write

$$R(r, t) = \frac{2r\mathfrak{R}(t)}{\Gamma_0 + 1 - (\Gamma_0 - 1)(r/r_0)^2}. \tag{6.16}$$

The initial condition

$$R(r, 0) = r$$

can no longer be required for we have normalized the r coordinate by choosing $f(r)$. We note, however, that if

$$\Gamma_0(0) = 1$$

that is, if

$$\mathfrak{R}_t^2(0) = \frac{2Gm_e}{r_0^3} \tag{6.17}$$

then it follows from equation (6.16) that

$$R(r, 0) = r.$$

In this case, it will be seen from the results quoted below that the hypersurface t = constant is a three dimensional flat space.

It follows from equations (6.16) and (1.4) that

$$e^{\Phi(r,t)} = 1 - \mathfrak{R}\{\log [\Gamma_0 + 1 - (\Gamma_0 - 1)(r/r)^2]\}_{\mathfrak{R}} = \frac{1}{1 + p/wc^2} \tag{6.18}$$

where we consider Γ_0 as a function of $\mathfrak{R}$ and the subscript $\mathfrak{R}$ denotes partial differentiation with respect to $\mathfrak{R}$ in which r is kept constant. The function $\mathfrak{R}(t)$ is then determined as a solution of the differential equation (6.13). Equation (6.18) is equivalent to the equation

$$e^{\Phi(r,t)} = 1 - \frac{(r_0^2 - \Gamma^2)\left(\mathfrak{R}\mathfrak{R}_{tt} + \frac{Gm_e}{r_0^3\mathfrak{R}(t)}\right)}{c^2\Gamma_0(\Gamma_0 + 1 - (\Gamma_0 - 1)(r/r_0)^2)}. \tag{6.19}$$

The interior line element is

$$ds^2 = e^{2\Phi}dt^2 - \frac{R^2}{c^2r^2}(dr^2 + r^2d\Omega^2) \tag{6.20}$$

where Φ is given by equation (6.18) or (6.19) and $R(r, t)$ is given by equation (6.16). The energy density is determined by equation (6.10) with $\mathfrak{R}(0) = 1$ and the pressure by equations (6.12) and (6.19) (or (6.18)).

Since the pressure is a function of r and t whereas the energy density is a function of t alone, it is evident that no equation of state exists.

The function $\mathcal{R}(t)$ which enters into the expressions for Φ and R must be such that $\mathcal{R}(0) = 1$ and the pressure is positive but otherwise may be arbitrary. The latter condition may be written as

$$\mathcal{R}\{\log[\Gamma_0 + 1 - (\Gamma_0 - 1)r^2/r_0^2]\}_{\mathcal{R}} \geq 0 \tag{6.21}$$

in which Γ_0 is considered as a function of $\mathcal{R}$. Bonnor and Faulkes have determined classes of functions $\mathcal{R}(t)$ by requiring that the line element given by equation (6.20) be a number of the class given by McVittie [3]. Bondi [7] has determined $\mathcal{R}(t)$ by imposing a condition which relates the central pressure $p(0, t)$ to the energy density.

The central pressure is given in terms of the function $\Gamma_0(\mathcal{R})$ by the equation

$$1 - \mathcal{R}\{\log(\Gamma_0 + 1)\}_{\mathcal{R}} = \frac{1}{1 + p_c/wc^2} \tag{6.22}$$

If we now require

$$\frac{p_c}{w/c^2} = g(w) = g(\mathcal{R}^{-3}) \tag{6.23}$$

we may solve equation (6.22) for $\Gamma_0(\mathcal{R})$. The function $\mathcal{R}(t)$ is then determined by solving equation (6.13), that is the differential equation

$$\mathcal{R}_t^2 = \frac{c^2}{r_0^2}\left[\Gamma_0^2 - 1 + \frac{R_s}{\mathcal{R}(t)}\right] \tag{6.24}$$

where

$$R_s = \frac{2Gm_e}{r_0c^2}$$

and is the ratio of the Schwarzschild radius to the initial boundary radius of the fluid.

It is evident that by choosing any real valued function $\Gamma_0(\mathcal{R})$ which satisfies the inequality (6.21) and the inequality

$$\Gamma_0^2 \geq 1 - \frac{R_s}{\mathcal{R}(t)} \tag{6.25}$$

a physically acceptable solution of the field equations may be obtained. In particular if we choose

$$\Gamma_0 = \text{constant}$$

we will have

$$\Phi = p = 0$$

and the line element given by equation (6.20) reduces to that of a Friedmann universe.

For a $\Gamma_0(\mathfrak{R})$ satisfying the equalities (6.21) and (6.25) the solution of equations (6.24) will lead to the line element of an oscillating sphere of matter if there are real roots to the equation

$$\Gamma_0^2 - 1 + \frac{R_s}{\mathfrak{R}(t)} = 0$$

in the interval

$$0 \leq \mathfrak{R} \leq 1.$$

If $\mathfrak{R}_m$ is such a root we must have

$$\mathfrak{R}_m = \mathfrak{R}(t_m) \geq R_s.$$

since Γ_0 is real. This means that

$$R(r_0, t_m) \geq \frac{2Gm_e}{c^2}.$$

That is, the outer boundary of the fluid is always outside of the Schwarzschild radius of the total mass.

We close this section with the observation that we may choose Γ_0 and $(\Gamma_0)_t$ at $t = 0$ (or at $\mathfrak{R} = 1$) so that the line element given by equations (6.16), (6.19) and (6.20) reduces to that of the Schwarzschild interior solution in isotropic coordinates at $t = 0$. Nevertheless the solution given of the field equations cannot be considered as arising from a static interior solution which has been perturbed at one interior point for the perturbation reaches to all points of the interior instantaneously. The functions $R(r, t)$ and $\Phi(r, t)$ depart from their static values for all values of r if they do for one such value. Thus the perturbation created at one value of r travels throughout the sphere with infinite velocity. This is due to the fact that we are dealing with a physically implausible material, one for which the energy density distribution at any time is uniform but the pressure is a function of radius.

7. SOLUTIONS OBEYING AN EQUATION OF STATE

It was pointed out in section 2 above that such solutions are a subset of McVittie's similarity solutions and have a line element

$$ds^2 = e^{2\Phi}dt^2 - \frac{R^2}{c^2}(d\bar{r}^2 + d\Omega^2) \tag{7.1}$$

where

$$\mathrm{R} = h(\bar{r})\sigma^{-1/3}(z) \tag{7.2}$$

$$e^{\Phi} = \frac{\sigma}{w + p/c^2} = \frac{\mathrm{R}_t}{\mathrm{R}}\frac{\mathfrak{R}}{\mathfrak{R}_t} \tag{7.3}$$

$$z = \log \mathrm{Q}(\bar{r}) - \log \mathfrak{R}(t),$$

σ, w and p are functions of the variable z. In addition, $\mathrm{R}(\bar{r}, t)$ must satisfy equation (3.5), the equation,

$$\left(\frac{1}{\mathrm{R}}\right)_{\bar{r}\bar{r}} = \frac{1}{\mathrm{R}} - \frac{\mathrm{B}(\bar{r})}{\mathrm{R}^2}. \tag{7.5}$$

It should be remembered that in deriving the first of equations (7.3), we have choosen the constant of integration in the definition of σ so that $\sigma(w_0) = w_0$ when w_0 is the value of w such that $p(w_0) = 0$. This requirement implies that

$$\mathrm{R}(\bar{r}_0 t) = r_0(\bar{r}_0)\mathfrak{R}(t) \tag{7.6}$$

where $\bar{r}_0$ is the value of $\bar{r}$ at $r = r_0$, the boundary of the fluid.

We shall write

$$\mathrm{H}(\bar{r}) = 1/h(\bar{r}) \tag{7.7}$$

$$x = \mathrm{Q}(\bar{r})/\mathfrak{R}(t) = e^z \tag{7.8}$$

$$\mathrm{X}(x) = \sigma^{1/3}(z). \tag{7.9}$$

Equation (7.5) then becomes

$$(\mathrm{H}_{\bar{r}\bar{r}} - \mathrm{H})\mathrm{X} + \mathrm{B}\mathrm{H}^2\mathrm{X}^2 + x\mathrm{X}_x\left(2\mathrm{H}_{\bar{r}}\frac{\mathrm{Q}_{\bar{r}}}{\mathrm{Q}} + \frac{\mathrm{H}\mathrm{Q}_{\bar{r}\bar{r}}}{\mathrm{Q}^2}\right) + \mathrm{H}x^2\mathrm{X}_{xx}\frac{\mathrm{Q}_{\bar{r}}^2}{\mathrm{Q}} = 0 \tag{7.10}$$

Since the variables $\bar{r}$ and x are independent we must have

$$\mathrm{H}_{\bar{r}\bar{r}} - \mathrm{H} = \lambda \mathrm{B}\mathrm{H}^2 \tag{7.11}$$

$$2\mathrm{H}_{\bar{r}}\frac{\mathrm{Q}_{\bar{r}}}{\mathrm{Q}} + \frac{\mathrm{H}\mathrm{Q}_{\bar{r}\bar{r}}}{\mathrm{Q}} = \mu \mathrm{B}\mathrm{H}^2 \tag{7.12}$$

$$\frac{\mathrm{H}\mathrm{Q}_{\bar{r}}^2}{\mathrm{Q}^2} = \nu \mathrm{B}\mathrm{H}^2 \tag{7.13}$$

where λ, μ and ν are constants

$$\nu x^2\mathrm{X}_{xx} + \mu x\mathrm{X}_x + \lambda\mathrm{X} + \mathrm{X}^2 = 0. \tag{7.14}$$

It follows from equations (7.12) and (7.13) that

$$\frac{Q_{\bar{r}}}{Q^{\alpha}} = \frac{A}{H^2} \tag{7.15}$$

where A is a constant of integration and

$$\alpha = \frac{\mu}{\nu}. \tag{7.16}$$

Equations (7.11) and (7.13) then lead to a differential equation for the function $Q(\bar{r})$, namely the equation

$$H_{\bar{r}\bar{r}} - H = \beta H \frac{Q_{\bar{r}}^2}{Q^2} \tag{7.17}$$

where

$$\beta = \frac{\lambda}{\nu} \tag{7.18}$$

and H is expressed in terms of Q and $Q_{\bar{r}}$ by means of equation (7.15). The function $B(\bar{r})$ is then determined by equation (7.13).

We now turn to a discussion of equation (7.14) and its consistency with equation (7.3). We may write the former equation as

$$X_{zz} + (\alpha - 1)X_z + \beta X + \gamma X^2 = 0 \tag{7.19}$$

where

$$\gamma = 1/\nu, \tag{7.20}$$

and we have made use of equation (7.8). It follows from equations (2.4), (2.9) and (7.9) that

$$X_z = X\frac{\sigma}{w + p/c^2} = \frac{\sigma^{4/3}}{(w + p/c^2)}$$

Hence

$$e^{\Phi} = Y(z) = \frac{X_z}{X} = \frac{\sigma}{w + p/c^2} = \frac{d\sigma}{dw}. \tag{7.21}$$

Equation (7.19) is a first integral of the equation

$$Y_{zz} + (\alpha - 1 + Y)Y_z - (\beta + (\alpha - 1)Y + Y^2)Y = 0 \tag{7.22}$$

as may be verified by substituting equation (7.21) into equation (7.19) and differentiating with respect to z.

Equation (7.22) is the equation by which McVittie [3] characterizes various solutions. Either it or equation (7.19) may be regarded as a condition on the equation of state, the equation $p = p(w)$ or equivalently $\sigma = \sigma(w)$. Since equation (7.19) may be written as

$$\left(\frac{X_z}{X}\right)_z + \left(\frac{X_z}{X}\right)^2 + (\alpha - 1)\frac{X_z}{X} + \beta + \gamma X = 0$$

It then follows from equations (7.21) and (2.9) that

$$3\sigma \frac{d^2\sigma}{dw^2} + \left(\frac{d\sigma}{dw}\right)^2 + (\alpha - 1)\frac{d\sigma}{dw} + \beta + \sigma\sigma^{1/3} = 0. \tag{7.23}$$

This equation governs the function $\sigma(w)$ and hence determines the equation of state.

Since

$$\frac{d\sigma}{dw} = \frac{\sigma}{w + p/c^2},$$

equation (7.23) may be shown to be equivalent to the equation

$$\frac{\sigma^2}{(w + p/c^2)^2}\left(1 - \frac{3dp}{c^2 dw}\right) + (\alpha - 1)\frac{\sigma}{w + p/c^2} + \beta + \gamma\sigma^{1/3} = 0.$$

In case the flow is isentropic (cf. section 2)

$$\sigma = \rho$$

$$\frac{(w + p/c^2)}{\sigma} = 1 + i/c^2 = f$$

and

$$a^2 = \frac{dp}{c^2 dw} = \rho\frac{d}{d\rho}\log(1 + i/c^2) = \frac{\rho}{f}\frac{df}{d\rho}$$

is the ratio of the velocity of sound to the special relativistic velocity of light. Equation (7.23) then becomes

$$3\rho\frac{df}{d\rho} = f(\alpha - 1)f^2 + (\beta + \gamma\rho^{1/3})f^3. \tag{7.24}$$

Equation (724) must be looked upon as a restriction on the nature of the fluid which allows solutions of the Einstein field equations satisfying equations (1.4) and the condition of constant entropy. In some cases this equation may serve to determine the constants α, β and γ.

The function $\mathcal{R}(t)$ entering in the definition of the variable z via equation (7.3) determines the motion of the boundary of the fluid. If $Q(\bar{r}_0)$ is different from zero the variable z takes on a range of values and equations (7.2) and (7.6) serve to determine $\mathcal{R}(t)$ when $\sigma(z)$ is known. If however as is the case in most solutions $Q(\bar{r}_0) = 0$, we cannot use this method for determining $\mathcal{R}(t)$. We may however determine this function as McVittie does from the requirement that

$$p(\bar{r}_0, t) = 0.$$

This may be done using equation (3.11) in which $m(\bar{r}_0, t) = m_e$, the mass as seen by an external observer.

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