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HODGE–TYPE STRUCTURES AS LINK INVARIANTS

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ABSTRACT. — Based on some analogies with the Hodge theory of isolated hypersurface singularities, we define Hodge–type numerical invariants of any, not necessarily algebraic, link in a three–sphere. We call them *H–numbers*. They contain the same amount of information as the (non degenerate part of the) real Seifert matrix. We study their basic properties, and we express the Tristram–Levine signatures and the higher order Alexander polynomial in terms of them. Motivated by singularity theory, we also introduce the *spectrum* of the link (determined from these *H*–numbers), and we establish some semicontinuity properties for it.

These properties can be related with skein–type relations, although they are not so precise as the classical skein relations.

RÉSUMÉ. — En se fondant sur des analogies avec la théorie de Hodge des singularités isolées des hypersurfaces, nous construisons des invariants numériques de type de Hodge pour un entrelacs quelconque, pas forcément algébrique, dans une sphère de dimension trois. Nous appelons ces invariants les H-nombres. Ils contiennent la même information sur les entrelacs, que la partie non-dégénérée de la matrice de Seifert modulo S-équivalence réelle. Nous étudions leurs propriétés, en particulier, donnons une formule explicite pour les signatures de Tristram et Levine et les polynômes d'Alexander de haut ordre en termes des H-nombres.

De plus, motivés par la théorie des singularités, nous introduisons le spectre d'un entrelacs, qui, lui aussi, peut être exprimé en termes des H-nombres. Nous établissons quelques propriétés de semicontinuité.

Ces propriétés, peuvent être reliées aux relations skein. Cependant, elles ne sont pas aussi précises que les relations skein classiques.

1. Introduction

1.1.

Although a Seifert matrix of a link is not a link invariant itself, it allows to define many link invariants, which are on the one hand very deep,

and easy to compute on the other. These invariants include the Alexander polynomial, the signature and the Tristram–Levine signatures. It might be quite surprising that the signature and the Alexander polynomial, although both come from a Seifert matrix, have completely different properties. For example, the signature detects mirrors, and estimates the four-genus, while the Alexander polynomial estimates the three-genus and does not detect mirrors.

Apparently straying from the knot theory, let us consider a hypersurface singularity in $(\mathbb{C}^{n+1}, 0)$. We can then associate many important objects with it, as the intersection form on the middle homology of the Milnor fiber, the monodromy matrix or the variation operator. These three objects, together with the space they act on, constitute a so–called *variation structure*. If the singularity is isolated, then its variation structure is determined by the variation operator, which is equivalent with the non–degenerate Seifert bilinear form associated with the germ of the singularity. Up to *real* equivalence, each such variation structure is built of some explicitly written indecomposable pieces. The number of times each such piece occurs in a concrete variation structure, is encoded in the so–called *mod 2 equivariant (primitive) Hodge numbers* associated with the singular germ. The name is motivated by the fact that they are, indeed, mod 2 reductions of the equivariant Hodge numbers associated with the mixed Hodge structure of the vanishing cohomology of the singularity, defined as in [20]. In this way one also sees that the information codified in these numbers is equivalent with the real Seifert form [17].

From the mod 2 equivariant Hodge numbers many other invariants can be reread, like the characteristic polynomial of monodromy, or different signature–type invariants. In fact, if $n = 1$, then from them one can even recover completely all the equivariant Hodge numbers of the mixed Hodge structure of the vanishing cohomology.

The algebraic links (case $n = 1$) form a bridge between the singularity theory and the knot theory (but the correspondences can be continued in higher dimensions, too). The Alexander polynomial of an algebraic link is exactly the characteristic polynomial of the monodromy of the corresponding Milnor fibration. The Milnor fiber constitutes a natural Seifert surface of an algebraic link. The corresponding Seifert matrix is the transposed inverse of the variation operator. In other words, the variation structure of (plane curve) singularities is deeply related to link invariants of algebraic links.

Motivated strongly by the case of algebraic links, we associate a variation structure with *any* link in S^3, and we define the analogs of *mod 2 equivariant (primitive) Hodge numbers*. The variation structure is built of the real Seifert matrix of the link and determines the Seifert matrix up to real S-equivalence. In fact, the newly defined numbers codify and determine this structure. Although the present work does not contain any Hodge theoretical discussion, motivated by the above correspondence we still call the introduced numbers *H–numbers*. Actually, the nilpotent part of the suitably defined monodromy operator defines a weight filtration, and also one can define a mod 2–Hodge filtration similarly as in [17], hence a "mod 2 Hodge structure" exists (it would be interesting to extend it to a genuine Hodge structure).

Both the (higher) Alexander polynomial and the Tristram–Levine signatures can be easily expressed in terms of the H–numbers of the link. Their symmetries and their behaviour under taking mirrors, allow us to explain *e.g.* why the Alexander polynomial does not distinguish mirrors, while the signatures do.

In the Hodge theory of hypersurface singularities, the (numerical part) of the mixed Hodge structure was codified by Steenbrink and Varchenko in the so–called spectrum. This codification was motivated by the extremely powerful and mysterious semicontinuity behavior of it under the deformation.

In our present context we also introduce the *mod 2 spectrum* of a link in S^3, and we relate it with the classical link invariants as the higher Alexander polynomials and Tristram–Levine signatures. The relation between the spectrum and Tristram–Levine signatures (see Proposition 4.14) is one of the key ingredients of showing semicontinuity results for spectra by topological methods as in [3]. It also emphasizes the unifying power of the newly introduced invariants, which gather together more conceptually all the classical properties and invariants associated with real Seifert matrices. On the other hand, since we know only the classification of the variation structures over reals, and not over integers, in this discussion we loose some information regarding the integer Seifert matrices, like the determinant.

After the foundations, we try to alloy the two main strategies used in the two theories: the technique of skein relations of classical link theory with the semicontinuity of the spectrum (known in singularity theory). Although at the origin and substance of both sits surgery, for the second case one needs a special surgery with intrinsic monotonicity structure (this, in the singularity theory, is guaranteed by the presence of the deformation). In

our results we will assume the monotonicity of the degree of the Alexander polynomial.

The basic motivation for studying semicontinuity of our structures is the following. First, we believe that the newly introduced H–numbers do not admit so precise skein relations similarly as some of the classical link invariants; or, their form should be packed in a more intelligent way. We believe that this "packing" goes through the spectrum, and the corresponding semicontinuity relations will guide the corresponding surgery formulas. Similarly as in the case of analytic singularities, where the semicontinuity had remarkable applications (see *e.g.* [22]), we expect in the future similar consequences for the newly introduced spectrum too. For more comments see (6.2).

1.2.

The structure of the paper is the following. We begin with a definition and examples of variation structures in Section 2. Then we recall the classification theorem of [17] regarding real, simple (see Definition 2.3) variation structures, and define the H–numbers for links. In Section 4 we relate the classical link invariants (higher Alexander polynomials, rational Nakanishi index, Tristram–Levine signatures) to the H–numbers. In the next section we show some examples. In the last section we gather results about the skein relation, the proofs mostly go through skein relation for the Alexander polynomial or the signatures. Also, we establish some semicontinuity results for the spectrum. By them we wish to draw the attention of the readers to this new phenomenon with the hope that this will bring some deep and powerful instrument in the near future.

2. Variations structures. Definitions and examples

2.1. Definitions

Here we recall some definitions from [18, Section 2]. We begin with fixing some standard notation.

For a finite dimensional complex vector space U, we denote its dual by U^*. The natural isomorphism $\theta \colon U \to U^{**}$ is given by $\theta(u)(\phi) = \phi(u)$. The complex conjugation is denoted by a bar $\bar{\cdot}$; for any $\phi \colon U \to V$ its dual map is denoted by $\phi^* \colon V^* \to U^*$. Let us also recall that, if ϕ is represented by a matrix S in some basis, then ϕ^* is represented by a transpose S^T in the dual basis. For a matrix S, S_{kl} or $S_{k,l}$ denotes the coefficient of S in k-th row and l-th column.

It is convenient to regard hermitian forms in the following way.

DEFINITION 2.1. — *A $\mathbb{C}$–linear endomorphism $b \colon U \to U^*$ with $\overline{b^* \circ \theta} = \varepsilon b$ (where $\varepsilon = \pm 1$) is called ε–hermitian form on U.*

Remark, that we do not assume here that b is non–degenerate. The automorphisms of b consists of isomorphisms $h \colon U \to U$ preserving b, i.e., with $\bar{h}^* \circ b \circ h = b$.

DEFINITION 2.2. — *An ε–hermitian variation structure (abbreviated by HVS) over $\mathbb{C}$ is a quadruple $(U; b, h, V)$, where*

(1) *U is finite dimensional vector space over $\mathbb{C}$;*
(2) *$b \colon U \to U^*$ is an ε–hermitian form on U;*
(3) *$h \colon U \to U$ is a b–orthogonal automorphism of U;*
(4) *$V \colon U^* \to U$ is a $\mathbb{C}$–linear endomorphism such that*

$$\overline{\theta^{-1} \circ V^*} = -\varepsilon V \circ \bar{h}^*$$

$$V \circ b = h - I.$$

Here and afterwards I denotes the identity map. The name of the structure is inherited from the operator V, which usually is a "variation map", cf. (2.4). V, respectively h will be called variation, respectively monodromy operator.

Observe that from (4) it follows immediately that

(2.1) $$b \circ V = \bar{h}^{*-1} - I \quad \text{and} \quad h \circ V \circ \bar{h}^* = V.$$

DEFINITION 2.3. — *The HVS $(U; b, h, V)$ will be called non–degenerate (respectively simple) if b (respectively V) is an isomorphism.*

We will need following lemmas from [17]:

LEMMA 2.4. — *For a triple $(U; b, h)$ satisfying points (1)–(3) from Definition 2.2, if b is non–degenerate, then there exists a unique V, namely $V = (h - I)b^{-1}$, such that $(U; b, h, V)$ constitutes a HVS.*

The non–degenerate triplets $(U; b, h)$ are classified by Milnor [13], see also [19].

LEMMA 2.5. — *For a pair $(U;V)$ with V an isomorphism, there exist unique b and h such that $(U;b,h,V)$ is a HVS. Indeed, $h = -\varepsilon V \bar{V}^{*-1}$ and $b = -V^{-1} - \varepsilon \bar{V}^{*-1}$ satisfy the axioms.*

From the last lemma it follows that the classification of simple HVS is equivalent to the classification of $\mathbb{C}$–linear isomorphisms $V \colon U^* \to U$.

There is a natural notion of an isomorphism of a HVS:

DEFINITION 2.6.
  (a) *Two HVS $(U;b,h,V)$ and $(U';b',h',V')$ are isomorphic, denoted by $\simeq$, if there exists an isomorphism $\phi : U \to U'$ such that $b = \bar{\phi}^* b' \phi$, $h = \phi^{-1} h' \phi$, and $V = \phi^{-1} V' (\bar{\phi}^*)^{-1}$.*
  (b) *If $(U_1;b_1,h_1,V_1)$ and $(U_2;b_2,h_2,V_2)$ are two HVS with the same sign ε, their direct sum is given by $(U_1 \oplus U_2; b_1 \oplus b_2, h_1 \oplus h_2, V_1 \oplus V_2)$. For sum of m copies of $\mathcal{V}$ we write $m \cdot \mathcal{V}$.*
  (c) *The conjugate of $\mathcal{V} = (U;b,h,V)$ is defined as $\bar{\mathcal{V}} = (U;\bar{b},\bar{h},\bar{V})$.*

2.2. Examples and classification of HVS

Here we shall follow closely [17], unless stated otherwise all results in this section are proved in [17]. For $k \geqslant 1$, J_k denotes the $(k \times k)$–Jordan block with eigenvalue 1.

Example 2.7. —   For $\lambda \in \mathbb{C}^* \setminus S^1$ and $k \geqslant 1$, the quadruple

$$\mathcal{V}_\lambda^{2k} = \left(\mathbb{C}^{2k}; \begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix}, \begin{pmatrix} \lambda J_k & 0 \\ 0 & \frac{1}{\bar{\lambda}} J_k^{*-1} \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon(\lambda J_k - I) \\ \frac{1}{\lambda} J_k^{*-1} - I & 0 \end{pmatrix} \right)$$

defines a HVS. Moreover, $\mathcal{V}_\lambda^{2k}$ and $\mathcal{V}_{1/\bar{\lambda}}^{2k}$ are isomorphic.

Before we show the next example we need a computational lemma; here one needs to consider the two square roots of ε. The two canonical sign choices for them are motivated by Hodge theoretical sign–conventions (*cf.* [17], Sections 5 and 6) of the variation structures associated with isolated hypersurface singularities $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$, where $\varepsilon = (-1)^n$.

LEMMA 2.8. — *For any $k > 1$ there are precisely two non-degenerate ϵ–hermitian forms (up to a real positive scaling), denoted by $b_\pm^k$, such that*

$$\bar{b}^* = \varepsilon b \quad \text{and} \quad J_k^* b J_k = b.$$

By convention, the signs are fixed by $(b_\pm^k)_{1,k} = \pm i^{-n^2-k+1}$, where $\epsilon = (-1)^n$. b is also left diagonal, i.e., the entries of b also satisfy: $b_{i,j} = 0$ for $i + j \leqslant k$ and $b_{i,k+1-i} = (-1)^{i+1} b_{1,k}$.

Example 2.9. — Let $\lambda \in S^1$. Up to isomorphism there are two non–degenerate HVS such that $h = \lambda J_k$. These are

$$\mathcal{V}_\lambda^k(\pm 1) = \left(\mathbb{C}^k; b_\pm^k, \lambda J_k, (\lambda J_k - I)(b_\pm^k)^{-1}\right).$$

Notice that these structures are simple unless $\lambda = 1$. In fact, if $\lambda \neq 1$ then any HVS with $h = \lambda J_k$ is both non-degenerate and simple. The case with eigenvalue 1 admits also a pair of degenerate HVS.

LEMMA 2.10. — *For $k \geqslant 2$ there are two degenerate HVS with $h = J_k$. They are*

$$\widetilde{\mathcal{V}}_1^k(\pm 1) = \left(\mathbb{C}^k; \widetilde{b}_\pm, J_k, \widetilde{V}_\pm^k\right),$$

where

$$\widetilde{b}_\pm^k = \begin{pmatrix} 0 & 0 \\ 0 & b_\pm^{k-1} \end{pmatrix}$$

and $\widetilde{V}_\pm^k$ is uniquely determined by b and h (up to an isomorphism). Moreover, $\widetilde{\mathcal{V}}_1^k(\pm 1)$ is simple. In fact, the entries of V^{-1} satisfy: $(V^{-1})_{i,j} = 0$ for $i + j \geqslant k + 2$, $(V^{-1})_{i,k+1-i} = \pm(-1)^{i+1}i^{-n^2-k}$. In order to recognize the isomorphism type, we have to recognize these entries up to a real positive re-scaling.

For $k = 1$ (*i.e.*, $U = \mathbb{C}$), and $h = I$, the structures can be written down more explicitly; there are the following five ε–HVS's with $\varepsilon = (-1)^n$:

$$\mathcal{V}_1^1(\pm 1) = (\mathbb{C}, \pm i^{-n^2}, I, 0)$$
$$\widetilde{\mathcal{V}}_1^1(\pm 1) = (\mathbb{C}, 0, I, \pm i^{n^2+1})$$
$$\mathcal{T} = (\mathbb{C}, 0, I, 0).$$

From all these examples the structures $\mathcal{V}_1^k(\pm 1)$ and $\mathcal{T}$ are non–simple, and $\widetilde{\mathcal{V}}_1^1(\pm 1)$ are simple. Concluding, for any $\lambda \in S^1$ and in each dimension k, there are precisely two non-equivalent simple variation structures with $h = \lambda J_k$. We use the following uniform notation for them:

$$(2.2) \qquad \mathcal{W}_\lambda^k(\pm 1) = \begin{cases} \mathcal{V}_\lambda^k(\pm 1) & \text{if } \lambda \neq 1 \\ \widetilde{\mathcal{V}}_1^k(\pm 1) & \text{if } \lambda = 1. \end{cases}$$

PROPOSITION 2.11. — *A simple HVS is uniquely expressible as a sum of indecomposable ones up to ordering of summands and up to an isomorphism. The indecomposable pieces are*

$$\mathcal{W}_\lambda^k(\pm 1) \text{ for } k \geqslant 1, \, \lambda \in S^1$$
$$\mathcal{V}_\lambda^{2k} \text{ for } k \geqslant 1, \, 0 < |\lambda| < 1.$$

CONVENTION 2.3. — *From now on, all HVS we shall discuss, are assumed to be simple.*

The above proposition allows us to define some invariants of HVS.

DEFINITION 2.12. — *Let $\mathcal{V}$ be a simple HVS $\mathcal{V}$. Let us express it, according to Proposition 2.11, as*

$$(2.4) \qquad \mathcal{V} = \bigoplus_{\substack{0<|\lambda|<1 \\ k \geqslant 1}} q_\lambda^k \cdot \mathcal{V}_\lambda^{2k} \oplus \bigoplus_{\substack{|\lambda|=1 \\ k \geqslant 1,\ u=\pm 1}} p_\lambda^k(u) \cdot \mathcal{W}_\lambda^k(u),$$

where the expression of type $r \cdot \mathcal{V}$ is a shorthand for a sum $\mathcal{V} \oplus \cdots \oplus \mathcal{V}$ (r times), cf. (2.6). Then $\{q_\lambda^k\}_{|\lambda|<1}$ and $\{p_\lambda^k(\pm 1)\}_{\lambda \in S^1}$ are called the H–numbers of $\mathcal{V}$.

Remark 2.13. —   The above classification result is over $\mathbb{C}$ or, equivalently, over $\mathbb{R}$. One can consider HVS's over $\mathbb{Z}$ as well, but then the classification is unknown.

If V is defined over the real numbers, then the above decomposition has some symmetries. Let s be 1 if $\lambda = 1$ and 0 if $\lambda \in S^1 \setminus \{1\}$. Then, with $\varepsilon = (-1)^n$,

$$(2.5) \qquad\qquad\qquad \overline{\mathcal{V}_\lambda^{2k}} = \mathcal{V}_{\bar\lambda}^{2k} \quad \text{for } \lambda \notin S^1, \text{ and}$$

$$(2.6) \qquad\qquad \overline{\mathcal{W}_\lambda^k(\pm 1)} = \mathcal{W}_{\bar\lambda}^k(\pm(-1)^{n+k+1+s}) \quad \text{for } \lambda \in S^1.$$

Therefore we have the following result

LEMMA 2.14. — *If in the HVS $\mathcal{V}$, the matrix V is defined over reals, then*

$$q_\lambda^k = q_{\bar\lambda}^k \ \ (\text{for } |\lambda| < 1) \quad \text{and} \quad p_\lambda^k(\pm 1) = p_{\bar\lambda}^k(\pm(-1)^{n+k+1+s}) \ \ (\text{for } |\lambda| = 1).$$

Moreover, by an easy check of the coefficient $b_{k,1}$ one has

LEMMA 2.15. — *Let V be the variation operator of the simple structure $\mathcal{V}$. Let $-\mathcal{V}$ be the structure corresponding to the variation operator $-V$ (see Lemma 2.5). Then $-\mathcal{W}_\lambda^k(\pm 1) \simeq \mathcal{W}_\lambda^k(\mp 1)$, and $-\mathcal{V}_\lambda^{2k} \simeq \mathcal{V}_\lambda^{2k}$.*

One needs slightly more computations to verify:

LEMMA 2.16. — *Let V be the variation operator of the simple structure $\mathcal{V}$. Let $\mathcal{V}^T$ be the structure determined by the variation operator V^T. Then $\mathcal{V}^T$ and $\mathcal{V}$ are isomorphic.*

Proof. — The statement is clear for $\mathcal{V} = \mathcal{V}_\lambda^{2k}$. Hence, assume that $\mathcal{V} = (U; b, h, V)$ is $\mathcal{W}_\lambda^k(\pm 1)$. Consider the new structure with variation operator $W = V^T$. Since for $\phi = V^T$ one has $\phi^{-1} V^T \overline{\phi}^{*,-1} = \overline{V}^{-1}$, the variation structures associated with W and $\overline{V}^{-1}$ are isomorphic. The monodromy operator of $\overline{V}^{-1}$ is $-\varepsilon \overline{V}^{-1} V^* = h^* = \lambda J_k^T$. Next, consider the anti-diagonal matrix A with $A_{ij} = 1$ if $i + j = k + 1$ and zero otherwise; it satisfies $A = A^{-1} = A^T = \overline{A}$. Base change by A has the effect $A(\lambda J_k^T)A = \lambda J_k$, hence the monodromy operator is twisted to a "normal form", which agrees with h. In particular, it is enough to compare the two variation operators V and $A\overline{V}^{-1}A$. If $\lambda = 1$, by the last sentence of (2.10), it is enough to compare the anti-diagonals of these two operators, which clearly agree. If $\lambda \neq 1$ use (2.8) and the same type of argument. □

2.3. Spectrum and the extended spectrum

One can extract from a variation structure a weaker invariant, whose motivation will be explained in the next subsection when we discuss the spectrum of an isolated hypersurface singularity.

DEFINITION 2.17. — *Let $\mathcal{V}$ be a HVS. The mod 2–spectrum (or, shortly, the spectrum) of $\mathcal{V}$ is a finite set Sp of real numbers from $(0, 2]$ such that any real non-integer number $\alpha \in (0, 2]$ occurs in Sp precisely $s(\alpha)$ times, where*

$$s(\alpha) = \sum_{\substack{k \text{ odd} \\ u = \pm 1}} \frac{k - uv}{2} \cdot p_\lambda^k(u) + \sum_{\substack{k \text{ even} \\ u = \pm 1}} \frac{k}{2} \cdot p_\lambda^k(u),$$

where

$$e^{2\pi i \alpha} = \lambda \text{ and } (-1)^{\lfloor \alpha \rfloor} = v.$$

The H–numbers $p_1^k(\pm 1)$ correspond to elements 1 and 2 in the spectrum, appearing precisely $k/2$ times each if k is even, and $(k \pm 1)/2$ and $(k \mp 1)/2$ times if k is odd.

A consequence of Lemma 2.14 is the following symmetry property

COROLLARY 2.18. — *If V is a real matrix, and $\epsilon = -1$, then $Sp \setminus \mathbb{Z}$ is symmetric with respect to 1.*

Notice that Sp contains no information regarding the blocks with eigenvalues $\lambda \notin S^1$. To enclose the information regarding $\{q_\lambda^k\}_{|\lambda| < 1}$ we define the *extended spectrum*.

Remark that, this construction has no counterpart in the classical Hodge theory.

DEFINITION 2.19. — *The extended spectrum ESp of a HVS $\mathcal{V}$ is a finite subset of complex numbers from $(0,2] \times i\mathbb{R}$ of the form $ESp = Sp \cup ISp$, where $ISp \cap \mathbb{R} = \emptyset$ and any non-real number $z \in (0,2] \times i\mathbb{R}$, $z = x + iy$, occurs in ISp precisely $s(z)$ times, where*

$$s(z) = \begin{cases} \sum k \cdot q_\lambda^k & \text{if } x \leqslant 1,\, y > 0 \text{ and } e^{2\pi i z} = \lambda \\ \sum k \cdot q_\lambda^k & \text{if } x > 1,\, y < 0 \text{ and } e^{2\pi i z} = 1/\bar{\lambda} \\ 0 & \text{if } x \leqslant 1 \text{ and } y < 0, \text{ or } x > 1 \text{ and } y > 0. \end{cases}$$

In other words, a block $\mathcal{V}_\lambda^{2k}$ (where $|\lambda| < 1$) contributes k times to both $x + iy$ and $1 + x - iy$, if $e^{2\pi(-y+ix)} = \lambda$ and $x \in (0,1]$.

We have the following two important properties of ESp.

LEMMA 2.20. — *For any $u \in (0,1)$, let $H_u = (u, u+1) \times i\mathbb{R}$. Then, if $ISp \cap \partial H_u = \emptyset$, we have*

$$\#ISp \cap H_u = \#ISp \setminus H_u.$$

Proof. — This follows directly from a simple observation that out of two numbers $x + iy$ and $1 + x - iy$, one lies in H_u and one does not. □

LEMMA 2.21. — *If $\mathcal{V}$ is real and $\varepsilon = -1$, then $ESp \setminus \mathbb{Z}$ is symmetric (via point–reflection) with respect to 1.*

Proof. — By Corollary 2.18, it is enough to prove that ISp is symmetric. But this follows from the fact that $\lambda = e^{2\pi(-y+ix)}$ yields the points $x + iy$ and $1 + x - iy$ in ISp, while $\bar{\lambda} = e^{2\pi(-y-ix)}$ the points $1 - x + iy$ and $2 - x - iy$. □

2.4. Variations structures of Milnor fibers

The motivation of the definition of HVS comes from the topological invariants of complex isolated hypersurface singularities and their relationship with the mixed Hodge structure on the vanishing cohomology.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ $(n \geqslant 0)$ be an analytic germ such that $f^{-1}(0)$ has an isolated singularity at the origin. Let S^{2n+1} be a small sphere around 0, $K = S^{2n-1} \cap \{f = 0\}$ the link, and

$$\phi \colon S^{2n+1} \setminus K \to S^1, \qquad \phi(z) = f(z)/|f(z)|$$

the Milnor fibration (see [12]) with fiber $F = \phi^{-1}(1)$. Set $F_t = \phi^{-1}(e^{2\pi i t})$ for $t \in [0,1)$ (with $F = F_0$). Then the trivialization of the bundle over $[0,1)$ gives diffeomorphisms (defined up to isotopy) $\gamma_t \colon F \to F_t$ for $t \in [0,1)$, and

extended to $t = 1$, the geometric monodromy $\gamma_1 : F \to F$. They give rise
to a well-defined map

$$\Gamma_t \colon \tilde{H}_n(F_1) \to \tilde{H}_n(F_t)$$

and the *monodromy map*

$$h = \Gamma_1 \colon \tilde{H}_n(F) \to \tilde{H}_n(F).$$

One also defines the intersection form on $b : \tilde{H}_n(F)^{\otimes 2} \to \mathbb{R}$ which is
$(-1)^n$ symmetric. Since γ_1 is chosen such that it is identity on ∂F, one
also defines a *variation* map $V \colon \tilde{H}_n(F, \partial F) \to \tilde{H}_n(F)$ (see [23, Chapter 4.2]
or [1, Chapter 1.2]). Here, by Lefschetz duality one has the identification
$\tilde{H}_n(F, \partial F) \simeq \operatorname{Hom}(\tilde{H}_n(F), \mathbb{R})$. The next fact is well–known (see *e.g.* [17]):

PROPOSITION 2.22. — *The quadruple* $(U = \tilde{H}_n(F, \mathbb{C}),\ b, h, V)$ *form a*
HVS with $\varepsilon = (-1)^n$.

DEFINITION 2.23. — *The variation structure defined above is called the*
variation structure of the singularity f *and it is denoted by* $\mathcal{V}_f$.

Notice that $\mathcal{V}_f$ is defined over $\mathbb{R}$. Additionally, it has some other particu-
lar properties as well. First of all, by the Monodromy Theorem (see *e.g.* [1,
Theorems 3.11 and 3.12] or [23, Chapter 7, §4]), all the eigenvalues of h are
roots of unity. Moreover, the block–decomposition of $\mathcal{V}_f$ is closely related
with the mixed Hodge structure of U.

Recall (see *e.g.* [17] for the facts below) that U carries a mixed Hodge
structure compatible with the monodromy action. Let us denote the cor-
responding equivariant Hodge numbers by $h_\lambda^{a,b}$. The nilpotent part of the
monodromy defines a morphism of Hodge structures of type $(-1, -1)$, let us
denote by $p_\lambda^{a,b}$ the dimensions of the corresponding primitive λ–generalized
eigenspaces, which are, in general, non–trivial for $a + b \geqslant n + s$. Then

$$p_\lambda^{a,b} = h_\lambda^{a,b} - h_\lambda^{a+1,b+1} \quad \text{and} \quad h_\lambda^{a,b} = \sum_{l \geqslant 0} p_\lambda^{a+l,b+l}$$

for any $a + b \geqslant n + s$. Moreover, since $h_\lambda^{a,b} = h_{\bar{\lambda}}^{n+s-a,n+s-b}$, the system of
Hodge numbers $\{h_\lambda^{a,b}\}_{a,b}$ is equivalent with the system of primitive Hodge
numbers $\{p_\lambda^{a,b}\}_{a+b \geqslant n+s}$.

The point is that by [17, Theorem 6.1] one has the following isomorphism
of variation structures:

$$\mathcal{V}_f \simeq \bigoplus_\lambda \bigoplus_{2n \geqslant a+b \geqslant n+s} p_\lambda^{a,b} \cdot \mathcal{W}_\lambda^{a+b+1-n-s}((-1)^b).$$

In particular, for any $k \geqslant 1$ and $u = \pm 1$ one has

$$(2.7) \qquad p_\lambda^k(u) = \sum_{\substack{a+b=k+n+s-1 \\ (-1)^b=u}} p_\lambda^{a,b}.$$

This fact motivates to call the numbers $p_\lambda^k(u)$ the *mod–2 primitive Hodge numbers* of f, or of the corresponding variation structure.

This relation with Hodge theory can be continued. Recall that for any f as above one extracts from the equivariant Hodge numbers the spectrum. Now, if $\mathcal{V}$ is a variation structure associated to an isolated hypersurface singularity, then Sp (defined in (2.17)) is the spectrum of the singularity reduced modulo 2, *i.e.*, if α belongs to the spectrum, then $\alpha \bmod 2 \in (0,2]$ belongs to Sp. In the case of isolated curve singularities Sp is just the spectrum of the singular germ.

Obviously, in general, Sp does not always contain enough information to recover $\mathcal{V}$. However, if all the monodromy eigenvalues are different, then the dimension of all Jordan blocks is one, and $\mathcal{V}$ is determined by Sp. This simple case contains for example all spectra of cuspidal plane curve singularities.

At the end of this subsection we recall the connection of the variation structures with the Seifert form.

DEFINITION 2.24. — *Let us be given two cycles $\alpha, \beta \in \tilde{H}_n(F)$. The Seifert form of the Milnor fibration is defined to be*

$$(2.8) \qquad S(\alpha, \beta) = L(\alpha, \Gamma_{1/2}\beta),$$

where L is the linking number of two n-dimensional cycles in S^{2n+1}.

There is a standard fact that $S(\alpha, \beta) = \langle V^{-1}(\alpha), \beta \rangle$, where $\langle \cdot, \cdot \rangle$ is the Lefschetz pairing $\tilde{H}_n(F, \partial F) \times \tilde{H}_n(F) \to \mathbb{R}$. In particular, in matrix notations, the variation operator is the transposed inverse of the Seifert form.

3. H–numbers for links

3.1. Definitions and first properties

Let us consider S^3 with its standard orientation, and let $L \subset S^3$ be an *oriented link*. Let S be its (integral) Seifert form. By our convention, the Seifert form is $S(\alpha, \beta) = L(\alpha, \beta^+)$, where α, β are cycles on the Seifert surface and β^+ is the push-forward of β in the positive direction. This is

the convention adopted by *e.g.* [4, 11, 15]. Some authors like [6, 7] define $S(\alpha, \beta)$ as $L(\alpha^+, \beta)$. This amounts to transposition of S.

Recall, that two matrices S and S' are *congruent* if there exists an invertible matrix A such that $S' = ASA^T$.

The next results were proved in [8] as a generalization of Theorem 12.2.9 of [7] (we would like to thank H. Trotter for drawing our attention to Keef's paper):

PROPOSITION 3.1.

(a) [8, Proposition 3.1] *Let* $S : V \times V \to \mathbb{R}$ *be a Seifert form of a link. Then, either S is real S–equivalent to the empty matrix or is real S-equivalent to* $\begin{pmatrix} S_0 & 0 \\ 0 & S_{\mathrm{ndeg}} \end{pmatrix}$, *where* $\det S_{\mathrm{ndeg}} \neq 0$ *and S_0 is a zero matrix.*

(b) [8, Theorem 3.5] *Let us be given two matrices S and T, which are S-equivalent. Assume that they are of the form* $S = \begin{pmatrix} S_0 & 0 \\ 0 & S_{\mathrm{ndeg}} \end{pmatrix}$, $T = \begin{pmatrix} T_0 & 0 \\ 0 & T_{\mathrm{ndeg}} \end{pmatrix}$. *Then S_{ndeg} and T_{ndeg} are congruent and* $\dim S_0 = \dim T_0$.

Let us define
$$V := (S_{\mathrm{ndeg}}^T)^{-1}.$$
and take the associated HVS with $\varepsilon = -1$. Its parts are the following: $U = \mathbb{C}^m$, where $m := \mathrm{rank}(V)$, $b = S_{\mathrm{ndeg}} - S_{\mathrm{ndeg}}^T$, and $h = (S_{\mathrm{ndeg}}^T)^{-1} \cdot S_{\mathrm{ndeg}}$.

Observe that taking a congruent of the Seifert matrix results in an isomorphism of HVS. Hence, the whole structure is independent (up to an isomorphism) of the specific choice of the Seifert matrix. Hence it is a link invariant.

DEFINITION 3.2. — *The variation structure $(U; b, h, V)$ defined above is called the variation structure of the link L and is denoted by $\mathcal{V}_L$.*

According to Definition 2.12, we can define the numbers $\{q_\lambda^k\}_{|\lambda|<1}$ and $\{p_\lambda^k(\pm 1)\}_{\lambda \in S^1}$ of the corresponding HVS.

DEFINITION 3.3. — *The numbers $\{q_\lambda^k\}_{|\lambda|<1}$ and $\{p_\lambda^k(\pm 1)\}_{\lambda \in S^1}$ will be called the H–numbers of the link L.*

From basic properties of Seifert matrices we get

Lemma 3.4.

(a) *The H–numbers are symmetric in the sense that for $0 < |\lambda| < 1$ one has $q_\lambda^k = q_{\bar\lambda}^k$, and*

$$p_\lambda^k(\pm 1) = p_{\bar\lambda}^k(\pm(-1)^{k+s}) \qquad \text{for } \lambda \in S^1.$$

(b) *H–numbers are additive with respect to the connected sum of links.*

(c) *If L^{or} is the link L with all its components with opposite orientation, then the H–numbers of L and L^{or} are the same.*

(d) *If L^{mir} is the mirror of L, then the H–numbers are changed as follows: $q_\lambda^k(L^{\mathrm{mir}}) = q_\lambda^k(L)$ for any $|\lambda| < 1$ and*

$$p_\lambda^k(\pm 1)(L^{\mathrm{mir}}) = p_\lambda^k(\mp 1)(L) \qquad \text{for } \lambda \in S^1.$$

Proof. — (a) follows immediately from Lemma 2.14. As for (b), observe that the Seifert matrix of the connected sum is the direct sum of the Seifert matrices of the summands. (c) and (d) follows from the classical facts that the $S(L^{\mathrm{or}}) = S(L)^T$ and $S(L^{\mathrm{mir}}) = -S(L)^T$, *cf.* [15] Propositions (5.4.6) and (5.4.7), combined with (2.14), (2.15) and (2.16). □

If L is an algebraic link, *i.e.*, a link of a plane curve singularity, it has two HVS's: the variation structure of the singularity $\mathcal{V}_f$ (see Definition 2.23) and the variation structure of the oriented link $\mathcal{V}_L$. Obviously, they agree $\mathcal{V}_f \simeq \mathcal{V}_L$, thanks to the relation $V = (S^T)^{-1}$, *cf.* the discussion after Definition 2.24.

One has very strong restrictions for H–numbers of algebraic links: from the classical monodromy theorem (see *e.g.* [1] or [23, Chapter 7, §4]) one reads:

COROLLARY 3.5. — *If L is an algebraic link then $q_\lambda^k = 0$ for any $|\lambda| < 1$. Moreover, $p_\lambda^k(\pm 1) = 0$ if at least one of the following conditions is satisfied*

- λ *is not a root of unity;*
- $\lambda \neq 1$ *and* $k > 2$;
- $\lambda = 1$ *and* $k > 1$.

Corollary 3.5 admits further improvements, see *e.g.* [17, Proposition 6.14].

LEMMA 3.6. — *If L is algebraic link and $\lambda \in S^1 \setminus \{1\}$, then $p_\lambda^2(-1) = 0$ ($p_\lambda^2(+1)$ can be positive) and $p_1^1(-1) = 0$.*

4. Classical link invariants and H–numbers

Having defined the H–numbers, we wish to study their relationship with classical invariants of the link L. Recall that we have the decomposition

$S = S_{\mathrm{ndeg}} \oplus S_0$ and the newly defined numbers are associated with S_{ndeg}, see (3.3).

4.1. Alexander polynomial

Define the polynomial $P(t) \in \mathbb{R}[t]$ by

$$P(t) := \prod_{|\lambda|=1} (t-\lambda)^{\sum_{k,u} k p_\lambda^k(u)} \prod_{0<|\lambda|<1} \left((t-\lambda)(t-1/\bar{\lambda}) \right)^{\sum_k k q_\lambda^k}.$$

LEMMA 4.1. — *The Alexander polynomial $\Delta(t)$ is zero if $S_0 \neq 0$, and it equals $P(t)$ (up to an invertible element of $\mathbb{R}[t, t^{-1}]$) otherwise. In this second case, the degree of $\Delta(t)$ is equal to the cardinality of the extended spectrum ESp.*

Proof. — We have $\Delta(t) = \det(S_{\mathrm{ndeg}} - t S_{\mathrm{ndeg}}^T) = \det S_{\mathrm{ndeg}}^T \cdot \det(h - tI)$. □

As the Alexander polynomial of a knot has no root at $t = 1$, we get:

COROLLARY 4.2. — *If L is a knot then $p_1^k(\pm 1) = 0$.*

The symmetry property of H–numbers (Lemma 2.14) explains (once again) the well-known property of the Alexander polynomial, namely, if we write $\Delta = a_0 + a_1 t + \cdots + a_m t^m$, then $a_n = (-1)^m a_{m-n}$.

4.2. Higher Alexander polynomials

Let us recall briefly the construction of the higher order Alexander polynomials (see [4, Definition 8.10]) via higher order elementary ideals of the matrix $S - tS^T$. We remark that our construction differs slightly from the standard one, because we consider ideals in $\mathbb{R}[t, t^{-1}]$ instead of $\mathbb{Z}[t, t^{-1}]$ (hence we loose some information about $\mathbb{Z}$–torsion elements).

Let ℓ be a positive integer. Consider an $\ell \times \ell$ matrix H over $\mathbb{R}[t, t^{-1}]$. For $0 \leqslant n < \ell$, let E_n be the ideal in $\mathbb{R}[t, t^{-1}]$ generated by the determinants of all $(\ell - n) \times (\ell - n)$ minors of H. As $\mathbb{R}[t, t^{-1}]$ is a principal ideal domain, the ideal E_n is generated by a single element $\Delta_n^H(t) \in \mathbb{R}[t, t^{-1}]$. $\Delta_n^H(t)$ is defined only up to an invertible element in $\mathbb{R}[t, t^{-1}]$, multiplying it by t in the appropriate power we can guarantee that Δ_n^H is in fact a polynomial and, unless it is the zero polynomial, that $\Delta_n^H(0) \neq 0$.

DEFINITION 4.3. — *The polynomial $\Delta_n^H(t)$ for $H = S - tS^T$ is called the n-th Alexander polynomial of the link L and denoted by $\Delta_n(t)$.*

The indexing was chosen so that Δ_0 is the standard Alexander polynomial. If m_0 is the rank of S_0, then $\Delta_n = 0$ for $0 \leqslant n < m_0$ and $\Delta_{m_0} = P$, cf. (4.1).

Our goal now is to express Δ_n in terms of the primitive numbers. Notice that multiplying H by a non-degenerate matrix independent of t or taking its transpose does not change the polynomials Δ_n. Therefore, $\Delta_{n+m_0}^{S-tS^T} = \Delta_n^{h-tI}$. By choosing a suitable basis of U we may also assume that h is in the Jordan form. Moreover, if $H = \lambda J_k - tI$, then $\Delta_0^H = (t-\lambda)^k$ and $\Delta_1^H = 1$. Next, we need to see what happens when we take a direct sum of several matrices.

LEMMA 4.4. — *Let H^1 and H^2 be two square matrices, $H = H^1 \oplus H^2$, and $\Delta_i^{H_1}$, $\Delta_j^{H_2}$, Δ_k^H the corresponding higher Alexander polynomials. For fixed $\lambda \in \mathbb{C}^*$, let a_i (respectively b_j, c_k) be the multiplicity of $(t-\lambda)$ in $\Delta_i^{H_1}$ (respectively $\Delta_j^{H_2}$, $\Delta_k^{H_3}$). Then*

$$c_k = \min\{a_i + b_j \; : \; i + j = k\}.$$

Proof. — It is enough to use the fact that for arbitrary minors A_1, A_2 of H^1 and H^2, $A_1 \oplus A_2$ is a minor of H. Moreover, any minor of H with non-zero determinant arises in this way.   □

Remark 4.5. — Lemma 4.4 works if some Alexander polynomials $\Delta_i^{H_1}$ or $\Delta_j^{H_2}$ are identically zero. We only have to agree that the multiplicity of $(t-\lambda)$ in a zero polynomial is $+\infty$.

Fix $\mu \in \mathbb{C}^*$ and set for each $k \geqslant 1$

$$(4.1) \qquad s_k(\mu) := \begin{cases} p_\mu^k(+1) + p_\mu^k(-1) & \text{if } \mu \in S^1, \\ q_\lambda^k & \text{if } \mu \in \{\lambda, 1/\bar\lambda\}, \quad (|\lambda| < 1). \end{cases}$$

i.e., $s_k(\mu)$ is the number of Jordan blocks of size k with eigenvalue μ. Now let $\Theta := \{\theta_1, \ldots, \theta_r\}$, $\theta_1 \leqslant \cdots \leqslant \theta_r$, be a set of integers, such that each $k \in \mathbb{Z}$ is contained in Θ precisely $s_k(\mu)$ times (hence $r = r(\mu) = \#\Theta = \sum s_k(\mu)$). Define the function

$$I(n) = \begin{cases} \sum_{i=1}^{r(\mu)-n} \theta_i & \text{for } n < r(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

The above facts combined provide:

PROPOSITION 4.6. — *The multiplicity of the root μ in the n–th Alexander polynomial Δ_n^h is $I(n)$.*

In Lemma 4.1 and Proposition 4.6 the exponents of the monomials $(t-\lambda)$ depended on the sums $p_\lambda^k(+1) + p_\lambda^k(-1)$. This, together with Lemma 3.4,

explains in this terminology, why the higher Alexander polynomials of a link and its mirror are the same.

4.3. Rational Nakanishi index

We begin with recalling the definition of the Nakanishi index (see *e.g.* [7, Section 5.4]). Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be a ring of Laurent polynomials with integer coefficient and $\Lambda_{\mathbb{Q}} = \mathbb{Q}[t, t^{-1}]$. For a knot K, set $X = S^3 \setminus K$ and let us consider the Alexander module of K, *i.e.*, the homology group of X

$$H = H_1(X; \Lambda)$$

with coefficients in Λ. This group can be regarded as the homology group of the universal abelian cover of X. It has a natural structure of a Λ module, where t and t^{-1} are deck transformations.

DEFINITION 4.7. — *A square presentation matrix for H is a square matrix A with entries in Λ such that $H \cong \Lambda^n / A\Lambda^n$, where n is the size of A. The Nakanishi index $n(K)$ is the minimal size of a square presentation matrix of the module H.*

Since we are allowed to perform row operations on a square presentation matrix and, independently, column operations, we can always assume that A is diagonal.

It is well known [7, Proposition 5.4.1], that if S is a Seifert matrix of K, then $tS - S^T$ is a square presentation matrix for H. However, in general, its size is not minimal possible. For example, for all torus knots $n(K) = 1$.

We show a relationship between our primitive numbers and the Nakanishi index defined over rational numbers instead of integers.

DEFINITION 4.8. — *The rational Nakanishi index $n_{\mathbb{Q}}(K)$ is a minimal size of a square matrix $A_{\mathbb{Q}}$ with entries in $\Lambda_{\mathbb{Q}}$ such that*

$$H \otimes \mathbb{Q} \cong \Lambda_{\mathbb{Q}}^n / A_{\mathbb{Q}} \Lambda_{\mathbb{Q}}^n.$$

Obviously we have $n(K) \geqslant n_{\mathbb{Q}}(K)$. In [16] it is proved that $n(K)$ is a lower bound for the unknotting number, hence $n_{\mathbb{Q}}(K)$ is a lower bound for it, too. Moreover, $n_{\mathbb{Q}}(K)$ is related to the Alexander polynomials in a following way.

PROPOSITION 4.9. — *If $\Delta_0(K), \ldots, \Delta_n(K)$ are higher order Alexander polynomials with Δ_0 the ordinary Alexander polynomial, then*

$$n_{\mathbb{Q}} = \min\{k \colon \Delta_k(K) \equiv 1\}.$$

In particular $n_\mathbb{Q}$ is the maximal number of Jordan blocks of the monodromy matrix with the same eigenvalue

$$(4.2) \qquad n_\mathbb{Q} = \max_\lambda r(\lambda) = \max \left(\max_{|\lambda|=1} \sum_{k,u} p_\lambda^k(u), \max_{0<|\lambda|<1} \sum_k q_\lambda^k \right).$$

Proof. — First of all observe that given a square presentation matrix $A_\mathbb{Q}$ of size n, $\Delta_l(K)$ is the generator of ideal spanned by all $(n-l) \times (n-l)$ minors of l (see Section 4.2). Hence, if the l−th Alexander polynomial Δ_l is non-trivial, it follows that the size of $A_\mathbb{Q}$ is at least l.

Conversely, if $\Delta_{k-1} \not\equiv 1$ and $\Delta_k \equiv 1$ we may define $A_\mathbb{Q}$ to be a diagonal $k \times k$ matrix with Δ_{l-1}/Δ_l on the (l,l)-th place. Then $\Lambda_\mathbb{Q}^k / A_\mathbb{Q} \Lambda_\mathbb{Q}^k$ is easily seen to be isomorphic as a $\Lambda_\mathbb{Q}$-module to $\Lambda_\mathbb{Q}^n / (tS - S^T)$.

Equation (4.2) follows now from Proposition 4.6. □

4.4. Signatures

Besides Alexander polynomials, the Tristram–Levine signatures can also be computed from the H–numbers. We begin by recalling their definition.

DEFINITION 4.10. — *Let L be a link and S a Seifert matrix of L. The Tristram–Levine signature (or the signature function) is the function associating to each $\zeta \in S^1 \setminus \{1\}$ the signature $\sigma(\zeta)$ of the Hermitian form given by*

$$(4.3) \qquad M_S(\zeta) := (1-\zeta)S + (1-\bar{\zeta})S^T.$$

The nullity $n(\zeta)$ is the nullity of the above form (i.e., $\dim \ker M_S(\zeta)$).

Remark 4.11. — Some authors, like [14, Definition 3.11], define $n(\zeta)$ as the nullity increased by 1. It is merely a matter of convention, we stick to the notation we find more common.

Clearly, in the definition of σ (but not $n(\zeta)$) one can replace S by S_{ndeg}. Hence, in the sequel, for the simplicity of the notations, S will denote S_{ndeg}. Then, $M_S(\zeta)$ equals

$$(4.4) \qquad S((\zeta\bar{\zeta} - \zeta)I + (1-\bar{\zeta})S^{-1}S^T)) = (1-\bar{\zeta})S \cdot (h^{-1} - \zeta I).$$

It is not hard to express these signatures by H–numbers: we compute the signature function associated with each irreducible simple HVS and then we use the additivity of signatures. Notice that for non–real matrices S, $M_S(\zeta)$ in (4.3) should be replaced by $(1-\zeta)S + (1-\bar{\zeta})\bar{S}^T$.

LEMMA 4.12. — *Let V be the variation operator of $\mathcal{V}_\lambda^{2k}$ (see Example 2.7). Let $S = (\bar{V}^T)^{-1}$. Then the signature of $M_S(\zeta)$ is zero and the form is non-degenerate for any ζ.*

Proof. — The non–degeneracy follows from (4.4). For vanishing of the signature notice that $M_S(\zeta)$ has the block form $\begin{pmatrix} 0 & A \\ \bar{A}^T & 0 \end{pmatrix}$ with A non–degenerate. Hence the signature is zero by elementary linear algebra. $\square$

The case of HVS $\mathcal{W}_\lambda^k(u)$ for $|\lambda| = 1$ is slightly more complicated.

LEMMA 4.13. — *Set $S = (\bar{V}^T)^{-1}$, where V is the variation operator of $\mathcal{W}_\lambda^k(u)$. Then the form $M_S(\zeta)$ is non-degenerate for all $\zeta \neq \bar\lambda$. If $\zeta = \bar\lambda$ then it has a one–dimensional kernel. In particular, the nullity of the link L is equal to*

$$n(\zeta) = \sum_{k,u} p_\zeta^k(u) + \dim S_0.$$

Proof. — The first part follows from (4.4). To show the formula for $n(\zeta)$ it is enough to observe that if we decompose $S = S_0 \oplus S_{\mathrm{ndeg}}$, and write M_S and $M_{S_{\mathrm{ndeg}}}$ for corresponding matrices (4.3), then $\dim \ker M_S - \dim \ker M_{S_{\mathrm{ndeg}}} = \dim S_0$. $\square$

The next result is computational. To formulate it we need the next

CONVENTION 4.5. — *Let $\alpha, \beta \in S^1$. We say that*

$$\alpha < \beta$$

if $\alpha = e^{2\pi i x}$, $\beta = e^{2\pi i y}$ with $x, y \in [0,1)$ and $x < y$.

PROPOSITION 4.14. — *Let L be a link, and consider the primitive numbers of the variation structure $\mathcal{V}_L$ associated with its Seifert for as above. Let $\zeta \in S^1 \setminus \{1\}$. Then the Tristram–Levine signature of L is equal to*

$$\sigma(\zeta) = \sigma(\bar\zeta) = -\sum_{\substack{\lambda < \zeta \\ k \text{ odd} \\ u = \pm 1}} u p_\lambda^k(u) + \sum_{\substack{\lambda > \zeta \\ k \text{ odd} \\ u = \pm 1}} u p_\lambda^k(u) + \sum_{\substack{k \text{ even} \\ u = \pm 1}} u p_\zeta^k(u).$$

Sketch of proof. — By additivity of signatures under the direct sum, it is enough to prove the statement if $\mathcal{V}_L = \mathcal{V}_\lambda^k(u)$ for some $\lambda \in S^1$, $k \geqslant 1$ and $u = \pm 1$. Let V be the variation operator corresponding to $\mathcal{V}_\lambda^k(u)$. By Lemma 2.8, V is right diagonal (because b^{-1} is right diagonal) so the corresponding form $M_S(\zeta)$ (see (4.3)) is left diagonal.

Assume that $\lambda \neq \zeta$. Then $M_S(\zeta)$ is non-degenerate. If k is even, we deduce that $M_S(\zeta)$ has a $k/2$-dimensional metabolic subspace, so signature of M_S is zero. If k is odd, the metabolic subspace is $(k-1)/2$ dimensional,

so the signature of M_S is ± 1, more precisely, it is equal to sign $\det M_S(\zeta)$ which can be explicitly computed.

If $\lambda = \zeta$, then $M_S(\zeta)$ is degenerate, we can easily compute that its kernel is one dimensional. If k is odd, it follows that the signature is equal to zero. If k is even, the signature is ± 1. The precise computation of the signature in this case requires much more effort (one can for instance compute V explicitly from the definition) and will not be shown here.   □

As a corollary, if $\zeta \in S^1$ is not an eigenvalue of monodromy, the signature $\sigma(\zeta)$ can be expressed in terms of the (mod 2) spectrum alone.

COROLLARY 4.15.

(a) *Let Sp be the (mod 2)–spectrum of a variation structure $\mathcal{V}_L$ (see Definition 2.17). Let $\zeta = e^{2\pi i x}$, where $x \in (0,1)$. Then*

$$\sigma(\zeta) = -\#Sp \cap (x, x+1) + \#Sp \setminus [x, x+1] + \sum_{\substack{k \text{ even} \\ u=\pm 1}} up^k_\zeta(u).$$

In particular, if ζ is not an eigenvalue of the monodromy h then

$$\sigma(\zeta) = -\#Sp \cap (x, x+1) + \#Sp \setminus (x, x+1).$$

(b) *Let ESp be the extended spectrum of the variation structure $\mathcal{V}_L$ and let $\zeta = e^{2\pi i x}$, $x \in (0,1)$. Let $H_x = (x, x+1) \times i\mathbb{R}$ and assume that $ESp \cap \partial H_x = \emptyset$. Then*

$$\sigma(\zeta) = -\#ESp \cap H_x + \#ESp \setminus H_x.$$

Remark 4.16. — (a) Corollary 4.15(a) can be compared with [10, Proposition 1], where the signatures of the iterated torus knots are computed. In fact, the spectrum of the torus knot (k, l), or equivalently, the spectrum of the singularity $\{x^k + y^l = 0\}$, is

$$Sp = \left\{ \frac{i}{k} + \frac{j}{l}, \ 1 \leqslant i \leqslant k - 1, \ 1 \leqslant j \leqslant l - 1 \right\}.$$

(b) The equivariant signatures of any HVS were computed in [17] and are expressible in terms of H–numbers (primitive equivariant Hodge numbers). However, in general, they are not expressible in terms of the spectrum alone.

For instance, for plane curve singularities, the equivariant signature σ_{-1} cannot be determined, in general, from the spectrum. This is the case with the Tristram–Levine signature $\sigma(-1)$ too. In fact, $\sigma(-1)$ is the signature of $S + S^T$. If S is the Seifert form of the plane curve singularity $f(x, y)$, then $\sigma(S + S^T)$ is the signature of the suspension surface singularity $f(x, y) + z^2$.

For a pair of singularities with the same spectrum but with different $\sigma(-1)$ see *e.g.* [17, (6.10)].

This also shows that H–numbers of a link are not determined by Tristram-Levine signatures and orders of higher Alexander polynomials alone. This can be exemplified by a model situation as follows. Take some $\lambda \in S^1 \setminus \{1\}$ and consider structures such that

$$p_\lambda^3(+1) = p_\lambda^1(-1) = 1 \text{ respectively } p_\lambda^3(-1) = p_\lambda^1(+1) = 1,$$

and all other p^k's for this λ are zero. The two structures are different, but they provide the same contribution to signatures, and orders of zeros of subsequent Alexander polynomials at $t = \lambda$ are in both cases $4, 1, 0, \dots$.

(c) On the other hand, the higher Alexander polynomials with a set of "higher equivariant signatures" determines the set of H–numbers. They are defines as follows, *cf.* [17, (4.4)]. Let $(U; b, h, V)$ be a variation structure, let $U_\lambda \subset U$ be the generalized λ–eigenspace of h, and for each integer $k \geqslant 1$ consider $U_\lambda^{(k)} := \ker((h - \lambda I)^k | U_\lambda)$. On $U_\lambda^{(k)}/U_\lambda^{(k-1)}$ one defines a (± 1)–hermitian form by $B_\lambda^{(k)}(x, y) = B(x, \lambda^{1-k}(h - \lambda I)^{k-1}y)$, where $B(x, y) = b(x)(\bar{y})$. Let $n_\lambda^{(k)}$ be the dimension of $U_\lambda^{(k)}$, while $\sigma_\lambda^{(k)}$ the signature of $B_\lambda^{(k)}$. Then the collection of the integers $\{n_\lambda^{(k)}\}_{k,\lambda}$ is equivalent with the collection of higher order Alexander polynomials, while the collection of pairs of integers $\{n_\lambda^{(k)}, \sigma_\lambda^{(k)}\}_{k,\lambda}$ characterizes completely the variation structure (i.e the real Seifert form).

5. Some examples

5.1.

Let us consider a (right-handed) trefoil with non-degenerate Seifert matrix

$$S = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}.$$

The variation matrix $V = (S^T)^{-1}$ and monodromy matrix $h = V \cdot (V^T)^{-1} = V \cdot S$ are

$$V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad h = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues of h are $\lambda_1 = \frac{1}{2} - \frac{1}{2}i\sqrt{3}$ and $\lambda_2 = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$. We need to diagonalise h. Let us put

$$A = -\frac{1}{i\sqrt[4]{3}} \begin{pmatrix} -\frac{1}{2} - \frac{1}{2}i\sqrt{3} & -1 \\ \frac{1}{2} - \frac{1}{2}i\sqrt{3} & 1 \end{pmatrix}.$$

Then AhA^{-1} is diagonal with diagonal entries λ_1 and λ_2, and

$$AV\bar{A}^T = \frac{1}{\sqrt{3}}\begin{pmatrix} -\frac{3}{2}+i\frac{1}{2}\sqrt{3} & 0 \\ 0 & -\frac{3}{2}-i\frac{1}{2}\sqrt{3} \end{pmatrix} = \begin{pmatrix} -i(\lambda_1-1) & 0 \\ 0 & i(\lambda_2-1) \end{pmatrix}.$$

Thus the HVS of a trefoil is

$$\mathcal{V} = \mathcal{V}^1_{\lambda_1}(-1) \oplus \mathcal{V}^1_{\lambda_2}(+1).$$

The spectrum is $\{\frac{5}{6}, \frac{7}{6}\}$.

5.2.

Let us consider the knot 8_{20}. We have by [5]:

$$S = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ V = (S^T)^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

And

$$h = V \cdot S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}.$$

The monodromy h has eigenvalues $\lambda_1 = \frac{1}{2} - \frac{1}{2}i\sqrt{3}$ and $\lambda_2 = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$. It has two Jordan blocks of size 2. Let A be such matrix that AhA^{-1} is in the Jordan form. *e.g.* :

$$A = \begin{pmatrix} -1 & 2 & -2 & \frac{3}{2}-\frac{3}{2}\sqrt{3}i \\ \sqrt{3}i & \sqrt{3}i & -\frac{3}{2}+\frac{1}{2}\sqrt{3}i & 0 \\ 1 & -2 & 2 & -\frac{3}{2}-\frac{3}{2}\sqrt{3}i \\ \sqrt{3}i & \sqrt{3}i & \frac{3}{2}+\frac{1}{2}\sqrt{3}i & 0 \end{pmatrix}.$$

Then we have

$$W = AV\bar{A}^T = \begin{pmatrix} \frac{3}{2}-\frac{1}{6}i\sqrt{3} & \frac{1}{2}-\frac{1}{2}i\sqrt{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2}+\frac{1}{6}i\sqrt{3} & \frac{1}{2}+\frac{1}{2}i\sqrt{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
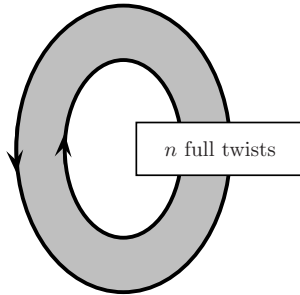
Then $W_{12} = (-1) \cdot (\lambda_1 - 1)$, $W_{34} = (-1) \cdot (\lambda_2 - 1)$. Now the size of each Jordan block is $k = 2$ and $i^k = -1$. Hence both signs in the direct sum decompositions are '+' and

$$\mathcal{V} = \mathcal{V}^2_{\lambda_1}(+1) \oplus \mathcal{V}^2_{\lambda_2}(+1).$$

This is with the agreement with the fact that the Tristram–Levine signature of W is zero, only it is $+1$ at $\zeta = \lambda_{1,2}$. The knot 8_{20} is also reversible.

5.3.

Consider, for any $n \neq 0$, the following link

n full twists

This link represents two unlinks with linking number n. The shaded part between the two strands forms a Seifert surface of genus 1. The Seifert matrix in (n). Hence the variation structure is $\widetilde{\mathcal{V}}_1^1(-1)$ for $n > 0$ and $\widetilde{\mathcal{V}}_1^1(+1)$ for $n < 0$ and apart of that does not depend on n. Therefore we do not see the linking numbers, or the *integral* Seifert form from the H–numbers.

5.4.

According to [5], there are five knots with up to 12 crossings with the Alexander polynomial

$$1 - 4t + 10t^2 - 16t^3 + 19t^4 - 16t^5 + 10t^6 - 4t^7 + t^8 = (t - \mu)^4 (t - \bar{\mu})^4,$$

where $\mu = e^{\pi i/3}$. These are $L_1 = 10_{99}$, $L_2 = 12_{n106}$, $L_3 = 12_{n508}$, $L_4 = 12_{n604}$ and $L_5 = 12_{n666}$. Their monodromy matrices $(S^T)^{-1}S$ are respectively h_1, h_2, h_3, h_4 and h_5. Here we consider h_1 and h_2:

$$
h_1 = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
2 & 1 & 2 & -1 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}
$$

$$h_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

The matrix h_1 has four Jordan blocks of size two, we easily get in this case

$$p_\mu^2(+1) = p_\mu^2(-1) = p_{\bar\mu}^2(+1) = p_{\bar\mu}^2(-1) = 1.$$

The matrix h_2 has a single Jordan block for each eigenvalue. We have

$$p_\mu^4(-1) = p_{\bar\mu}^4(-1) = 1.$$

The matrices h_3 and h_5 have two Jordan blocks of size one and two of size 3. We conclude that

$$p_\mu^1(-1) = p_\mu^3(-1) = p_{\bar\mu}^1(+1) = p_{\bar\mu}^3(+1) = 1.$$

In case of h_4 we have similarly two Jordan blocks of size one and two of size 3. We can compute that

$$p_\mu^1(+1) = p_\mu^3(+1) = p_{\bar\mu}^1(-1) = p_{\bar\mu}^3(-1) = 1.$$

Hence, only the knots 12_{n508} and 12_{n666} are undistinguishable by H–numbers.

Observe that if we take a connected sum of three left–handed trefoils and one right–handed one, then the Alexander polynomial and the signatures for $\zeta \neq \mu, \bar\mu$, shall be the same as in the case of the knots 12_{n508} and 12_{n666}, but the Jordan block structure is different.

6. Skein relations for H–numbers

6.1. Signatures

Although we do not have a precise skein relation for H–numbers, there are several constrains for them, coming mostly from relations for classical invariants.

As usually in skein relation, we consider the three links L_0, L_+ and L_-. Their Seifert matrices S_0, S_+ and S_- can be chosen (see [6, Proof of

Theorem 7.10]) so that S_+ and S_- are $(n+1) \times (n+1)$ matrices, such that

(6.1)
$$S_+ - S_- = \left(\begin{array}{c|c} \underline{\mathbf{0}} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline 0 \ldots 0 & 1 \end{array} \right),$$

where $\underline{\mathbf{0}}$ denotes an $n \times n$ zero matrix. Moreover S_0 arises from S_+ by deleting the $(n+1)$-st row and $(n+1)$-st column.

The following fact is classical (see [7, Lemma 12.3.4] or [14]).

PROPOSITION 6.1. — *For any $\zeta \in S^1 \backslash \{1\}$, we have the following bounds*

$$|\sigma_{L_\pm}(\zeta) - \sigma_{L_0}(\zeta)| + |n_{L_\pm}(\zeta) - n_{L_0}(\zeta)| \leqslant 1.$$

Proof. — We follow the proof of [7]. For fixed ζ, let M_+, M_- and M_0 denote the forms (4.3) for S_+, S_- and S_0, respectively. Let p_+, p_-, p_0, q_+, q_-, q_0 and n_+, n_-, n_0 be the maximal dimension of subspaces on which M_+, M_-, M_0 are, respectively, positive definite, negative definite and zero. As S_0 is submatrix of S_+, M_0 is a restriction of M_+ onto an $n-$dimensional subspace. It follows that

$$p_0 \leqslant p_+ \leqslant p_+ + 1$$
$$q_0 \leqslant q_+ \leqslant q_+ + 1$$
$$n_0 \leqslant n_+ \leqslant n_+ + 1.$$

As $p_0 + q_0 + n_0 + 1 = p_+ + q_+ + n_+$, the statement follows. $\qquad\square$

The above proposition and (4.14) give restriction for possible H–numbers of $L_\pm$ and L_0, when the primitive numbers of one of them are known.

6.2. Semicontinuity of the extended spectrum

The inequality of (6.1) can be used to prove a variant of the semicontinuity of spectra.

Here some comments are in order. The semicontinuity property of (genuine) spectrum of hypersurface singularities says the following: if $\{f_t : t \in (\mathbb{C}, 0)\}$ is a family of isolated singularities, then for any interval $I = [\alpha, \alpha + 1)$ the spectral numbers $\{\mathrm{Spec}(f_t)\}_t$ associated with f_t satisfies: $\# \mathrm{Spec}(f_0) \cap I \geqslant \# \mathrm{Spec}(f_{t\neq 0}) \cap I$, see [21, 22]. The semicontinuity principle is codified in the very geometric substance of the deformation. In

particular, several invariants behave semicontinuously, *e.g.* for the Milnor number $\mu(f_0) \geqslant \mu(f_{t \neq 0})$. If one tries to study this phenomenon in the case of arbitrary links, one needs to assume that the geometric situation mimics in the right way the presence of the deformation. In the next proposition we will assume that $\deg \Delta$ (*i.e.*, the Milnor number in algebraic case) is monotone.

PROPOSITION 6.2. — *Let L_1 and L_2 are two links. Let ESp_1 and ESp_2 be the corresponding extended spectra. Let Δ_{L_i} $(i = 1, 2)$ be the characteristic polynomial of the monodromy operator associated with the non–degenerate part of the Seifert form; in other words, the first non–zero higher order Alexander polynomial. (If the Seifert forms are non–degenerate then Δ_{L_i} is just the Alexander polynomial. See subsections 4.1 and 4.2).*

Assume also that $x \in (0, 1)$ is such that $\partial H_x \cap (ESp_1 \cup ESp_2) = \emptyset$ (where $H_x = (x, x + 1) \times i\mathbb{R}$ as above). Moreover assume that one of the following holds.

  (a) *L_1 arises from L_2 by changing a negative (or left-handed) crossing to positive (or right-handed) crossing (see [11, Exercise 3.2.5] for the necessary definitions) and $\deg \Delta_{L_1} \geqslant \deg \Delta_{L_2}$;*

  (b) *L_1 arises from L_2 by changing one crossing and $\deg \Delta_{L_1} > \deg \Delta_{L_2}$;*

  (c) *L_1 arises from L_2 by a hyperbolic transformation (i.e., L_1 and L_2 can play a role of L_0 and L_∞ at some diagram, see [7, Definition 12.3.3]) and $\deg \Delta_{L_1} > \deg \Delta_{L_2}$.*

Then

$$\# ESp_1 \cap H_x \geqslant \# ESp_2 \cap H_x.$$

Proof. — Let $a_1 = \# ESp_1 \cap H_x$, $a_2 = \# ESp_2 \cap H_x$, $b_1 = \# ESp_1 \setminus H_x$, $b_2 = \# ESp_2 \setminus H_x$. Then by Corollary 4.15(b) we have

$$a_1 + b_1 = \deg \Delta_{L_1} \qquad\qquad a_2 + b_2 = \deg \Delta_{L_2}$$
$$-a_1 + b_1 = \sigma_{L_1}(e^{2\pi i x}) \qquad\qquad -a_2 + b_2 = \sigma_{L_2}(e^{2\pi i x}).$$

Thus

$$a_1 - a_2 = \frac{1}{2}\Big(\deg \Delta_{L_1} - \deg \Delta_{L_2} - \sigma_{L_1}(e^{2\pi i x}) + \sigma_{L_2}(e^{2\pi i x}) \Big).$$

Now, in the case (a), as $\deg \Delta_{L_1} > \deg \Delta_{L_2}$ and L_1 and L_2 have the same number of components, the degrees differ at least by 2. The signatures cannot differ by more than 2 by Proposition 6.1. In the case (b), signature of L_1 is not larger than that of L_2, in case (c), both degree of Alexander and signature cannot differ by more than one. $\qquad\square$

The above results is enough to prove a variant of Theorem 6.7 from [2] with $ESp \cap H_x$ instead of Tristram–Levine signatures. Since for algebraic links the ESp is the same as the ordinary spectrum, we can relate the spectra of singularities of a plane curve with the spectrum of the singularity at infinity. See [3] for details.

Remark 6.3. — In singularity theory the signature is not semicontinuous, see *e.g.* [9]. Hence we do not have the semicontinuity property for each particular Hodge number either. Similar behavior can be observed in the knot theory: the condition $\deg \Delta_{L_1} > \deg \Delta_{L_2}$ (in the notation from Proposition 6.2) alone is not sufficient to determine the sign of $\sigma_{L_1}(\zeta) - \sigma_{L_2}(\zeta)$, so we do not have a strong "semicontinuity property" for signatures of knots. We cannot also expect the semicontinuity property for each H–number of links.

6.3. Higher Alexander polynomials and rational Nakanishi index

Next, we wish to connect the higher order Alexander polynomials associated with $S_\pm$ and S_0, *cf.* (6.1). In order to formulate the result, we need to introduce some additional notation.

CONVENTION 6.4.

(a) *Let us fix $\lambda \in \mathbb{C}$ for this section. For any matrix H with coefficients in $\mathbb{C}[t]$ we define*

$$d(H) = \mathrm{ord}_{t=\lambda} \det H.$$

(b) *For any $m \times m$ matrix K and $1 \leqslant i, j \leqslant m$, we define $K^{i,j}$ as the $(m-1) \times (m-1)$ minor resulting from K by removal of i–th row and j–th column.*

The next easy lemma will be important in the sequel.

LEMMA 6.5. — *For any matrix H of size $m \times m$ and for any $1 \leqslant j \leqslant m$ one has*

$$d(H) \geqslant \min\{d(H^{i,j}) : 1 \leqslant i \leqslant m\}.$$

Proof. — $\det H = \sum_i (-1)^{i+j} H_{i,j} \det H^{i,j}$ by the Laplace expansion of $\det H$ along the j–th column. (Recall that $H_{i,j}$ denotes the element of H at i–th row and j–th column.) It follows that, if all $\det H^{i,j}$ are divisible by $(t - \lambda)^d$, so will be $\det H$. $\qquad\square$

The definition of $d(H)$ is motivated by the higher order Alexander polynomials. Namely, if S is an $m \times m$ Seifert matrix and $H = S - tS^T$, then the multiplicity of a root of the k–th Alexander polynomial (see Definition 4.3) of H at $t = \lambda$ can be expressed as the minimum of $d(K)$, where K runs through all $(m-k) \times (m-k)$ minors of H:

(6.2) $\operatorname{ord}_{t=\lambda} \Delta_k^H = \min\{d(K)\colon K \text{ is an } (m-k) \times (m-k) \text{ minor of } H\}.$

For Seifert matrices S_+, S_- and S_0 of links L_+, L_- and L_0 (notation from Section 6.1) let us define $H_* = S_* - tS_*^T$ ('$*$' is one of '+', '−' or '0') and

$$d_k^* = \operatorname{ord}_{t=\lambda} \Delta_k^{H_*}.$$

An immediate consequence of Lemma 6.5 is that for any k and $* \in \{+, -, 0\}$:

(6.3) $d_k^* \geqslant d_{k+1}^*$

The skein relation gives the following restrictions for values of d_k^*.

PROPOSITION 6.6. — *The integers d_k^+, d_k^- and d_k^0 satisfy the following relations:*

(6.4a) $d_k^0 \geqslant d_{k+1}^{\pm}$

(6.4b) $d_k^{\pm} \geqslant d_{k+1}^0$

(6.4c) $d_k^{\pm} \geqslant d_{k+1}^{\mp}$ *if* $\lambda \neq 1$

(6.4d) $d_k^{\pm} \geqslant \min(d_{k+1}^{\mp} + 1, d_k^{\mp})$ *if* $\lambda = 1$.

Proof. — Let m be the size of H_0. H_0 can be regarded as an $m \times m$ minor of both H_+ and H_- (*cf.* (6.1)). Then any $(m-k) \times (m-k)$ minor of H_0 is also an $((m+1)-(k+1)) \times ((m+1)-(k+1))$ minor of H_+ and of H_-. It follows that $d_{k+1}^{\pm} \leqslant d_k^0$, because in $d_{k+1}^{\pm}$ we take a minimum over larger set. Equation (6.4a) follows.

As for (6.4b), we can divide all possible $(m+1-k) \times (m+1-k)$ minors of H_+ in three categories.

- A_α^k, $\alpha \in \mathcal{A}$, will denote minors lying entirely in H_0;
- B_β^k, $\beta \in \mathcal{B}$, will denote minors containing a part of the last column or row of H_+ but not containing the corner;
- C_γ^k, $\gamma \in \mathcal{C}$, will denote minors containing the element of H_+ lying in $(m+1)$st row and $(m+1)$st column.

Graphically we can present these minors like that
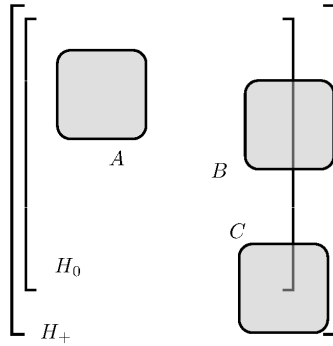
By (6.2) one has (where $d(\alpha) = d(A_\alpha^k)$, and similarly for $d(\beta)$ and $d(\gamma)$):

$$(6.5) \qquad d_k^+ = \min\left(\min_{\alpha \in \mathcal{A}} d(\alpha), \min_{\beta \in \mathcal{B}} d(\beta), \min_{\gamma \in \mathcal{C}} d(\gamma)\right).$$

We need to show that all three minima are greater than d_{k+1}^0.

First of all $\min_{\alpha \in \mathcal{A}} d(\alpha)$ is precisely d_{k-1}^0.

Let $B_\beta^{j,m+1-k}$ be the minor of H_+ resulting by removing the last column and j−th row from B_β (assuming that B_β contains a part of $(m+1)$-st column of H_+ as on the picture, not a part of $(m+1)$st row). Remark that $B_\beta^{j,m+1-k}$ is an $(m-k) \times (m-k)$ minor of H_0. By Lemma 6.5 we have

$$d(B_\beta) \geqslant \min_{j=1,\ldots,m+1-k} d(B_\beta^{j,m+1-k}).$$

Thus

$$\min_{\beta \in \mathcal{B}} d(B_\beta) \geqslant d_k^0.$$

In order to deal with $d(C_\gamma)$ we need to do expand $\det C_\gamma$ first along the last row and then along the last column. Applying Lemma 6.5 twice we get

$$d(C_\gamma) \geqslant \min_{1 \leqslant j,j' \leqslant m+1-k} d\left(C_\gamma^{(j,m+1-k),(j',m+1-k)}\right),$$

where $C_\gamma^{(j,m+1-k),(j',m+1-k)}$ arises from C_γ by deleting j−th and $(m+1-k)$−th column and j−th and $(m+1-k)$−th row. It is thus an $(m-(k+1)) \times (m-(k+1))$ minor of H_+ and also of H_0, because it does not contain neither the last column, nor the last row of H_+. Hence, $d(C_\gamma) \geqslant d_{k+1}^0$. Finally, using (6.2) and (6.5) we obtain $d_k^+ \geqslant \min(d_{k-1}^0, d_k^0, d_{k+1}^0) = d_{k+1}^0$. Hence the inequality (6.4b) is proved.

In order to prove the last two inequalities, let us consider two $(m+1-k) \times (m+1-k)$ minors K_+ and K_- of H_+ and H_- obtained by removing the same columns and the same rows from matrices H_+ and H_-.

As $H_+ - H_-$ is a matrix with $(t-1)$ in the place $(m+1, m+1)$, and zeros everywhere else, $K_+ = K_-$ unless they contain the element at the bottom right corner of H_+ and H_-. If they do not contain,

$$\det K_+ = \det K_-, \text{ so } d(K_+) = d(K_-)$$

If they do,

$$\det K_+ = \det K_- + (t-1)\det K_0,$$

where $K_0 = K_\pm^{m+1-k, m+1-k}$ arises from $K_\pm$ by removing the last column and the last row. In this case we deduce that

$$d(K_+) \geqslant \min(d(K_-), d(K_0) + s),$$

where $s = 1$ if $\lambda = 1$ and 0 otherwise.

This shows in particular that

$$d_k^+ \geqslant \min(d_k^-, s + d_k^0).$$

Now it is enough to observe that by (6.4b) $d_k^0 \geqslant d_{k-1}^-$. $\qquad\square$

In order to apply this skein relation, let us fix λ with $0 < |\lambda| \leqslant 1$ and consider the set Θ (defined after the proof of Lemma 4.4) associated with λ. For any $N \geqslant 1$ set

(6.6) $$P_N = \#\{\theta \in \Theta : \theta \geqslant N\}.$$

P_N can be interpreted as the number of the Jordan blocks of size at least N with eigenvalue λ of the monodromy matrix h; i.e., with the notation of (4.1), one has:

$$P_N = \sum_{k \geqslant N} s_k(\lambda).$$

COROLLARY 6.7. — *Fix some λ. Let P_N^+ and P_N^- be the P_N numbers associated to the links L_+ and L_- respectively as in (6.6). Then for any $N \geqslant 2$ one has*

$$|P_N^+ - P_N^-| \leqslant 2N$$

while for $N = 1$, $|P_1^+ - P_1^-| \leqslant 1$.

Proof. — For $N > 1$, assume that $P_N^+ - P_N^- = a > 0$. By Proposition 4.6 for any $i < P_N^+$ we get

(6.7) $$d_i^+ - d_{i+1}^+ \geqslant N,$$

and for $i \geqslant P_N^-$

(6.8) $$d_i^- - d_{i+1}^- \leqslant N - 1.$$

Therefore, we obtains the next sequence of inequalities:

$$d^+_{P^+_N-1} \overset{(*)}{\geqslant} d^-_{P^+_N} \overset{(**)}{\geqslant} d^-_{P^-_N} - a(N-1) \overset{(*)}{\geqslant}$$

$$\geqslant d^+_{P^-_N+1} - a(N-1) \overset{(***)}{\geqslant} d^+_{P^+_N-1} + (a-2)N - a(N-1).$$

Here the inequalities denoted by $(*)$ follow from (6.4c), $(**)$ from (6.8) and $(***)$ from (6.7). Hence $(a-2)N - a(N-1) \leqslant 0$, or $2N \geqslant a$.

So now assume that $N = 1$. Then $P^{\pm}_1 = \min\{i > 0 : d^{\pm}_i = 0\}$. So let us take k such that $d^-_k > 0 = d^-_{k+1}$ (i.e., $P^-_1 = k + 1$). By (6.4c) we have $d^+_{k-1} \geqslant d^-_k$, hence $d^+_{k-1} > 0$, so $P^+_1 \geqslant k$. The argument follows from symmetry. $\qquad\square$

For knots, the maximum of the values P_1 for all $\lambda \neq 1$ is, by Proposition 4.9, equal to the Nakanishi index $n_{\mathbb{Q}}$. Therefore, Corollary 6.7 implies:

COROLLARY 6.8. — *Let K_+ and K_- be two knots differing by one change of crossing. Then*

$$|n_{\mathbb{Q}}(K_+) - n_{\mathbb{Q}}(K_-)| \leqslant 1.$$

In particular, we reprove a theorem of Nakanishi in a weaker version:

COROLLARY 6.9. — *The rational Nakanishi index of a knot K is bounded from above by the unknotting number of K.*

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