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## A REMARKABLE CONTRACTION OF SEMISIMPLE LIE ALGEBRAS

by Dmitri I. PANYUSHEV & Oksana S. YAKIMOVA

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ABSTRACT. — Recently, E. Feigin introduced a very interesting contraction  $\mathfrak{q}$  of a semisimple Lie algebra  $\mathfrak{g}$  (see arXiv:1007.0646 and arXiv:1101.1898). We prove that these non-reductive Lie algebras retain good invariant-theoretic properties of  $\mathfrak{g}$ . For instance, the algebras of invariants of both adjoint and coadjoint representations of  $\mathfrak{q}$  are free, and also the enveloping algebra of  $\mathfrak{q}$  is a free module over its centre.

RÉSUMÉ. — E. Feigin a introduit la contraction  $\mathfrak{q}$  d'une algèbre de Lie semi-simple  $\mathfrak{g}$  dans arXiv :1007.0646 et arXiv :1101.1898. Nous démontrons que ces algèbres de Lie non-réductives conservent quelque unes des propriétés de  $\mathfrak{g}$ . En particulier, les algèbres des invariants des représentations adjointe et respectivement coadjointe de  $\mathfrak{q}$  sont libres, et l'algèbre enveloppante de  $\mathfrak{q}$  est un module libre sur son centre.

### Introduction

The ground field  $\mathbb{F}$  is algebraically closed and  $\text{char } \mathbb{F} = 0$ . Let  $G$  be a connected semisimple algebraic group of rank  $l$  with Lie algebra  $\mathfrak{g}$ . Recently, E. Feigin introduced a very interesting contraction of  $\mathfrak{g}$  [2]. His motivation came from some problems in Representation Theory [4], and making use of this contraction he also studied certain degenerations of flag varieties [3]. Our goal is to elaborate on invariant-theoretic properties of these contractions of semisimple Lie algebras.

Fix a triangular decomposition  $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$ , where  $\mathfrak{t}$  is a Cartan subalgebra. Then  $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$  is the fixed Borel subalgebra of  $\mathfrak{g}$ . The corresponding subgroups of  $G$  are  $B, U$ , and  $T$ . Using the vector space isomorphism  $\mathfrak{g}/\mathfrak{b} \simeq \mathfrak{u}^-$ , we regard  $\mathfrak{u}^-$  as a  $B$ -module. If  $b \in \mathfrak{b}$  and  $\eta \in \mathfrak{u}^-$ , then

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$(b, \eta) \mapsto b \circ \eta$  stands for the corresponding representation of  $\mathfrak{b}$ . That is, if  $p_- : \mathfrak{g} \rightarrow \mathfrak{u}^-$  is the projection with kernel  $\mathfrak{b}$ , then  $b \circ \eta = p_-([b, \eta])$ .

Following [2, Sect. 2], consider the semi-direct product  $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{g}/\mathfrak{b})^a = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$ , where the superscript ‘a’ means that the  $\mathfrak{b}$ -module  $\mathfrak{u}^-$  is regarded as an abelian ideal in  $\mathfrak{q}$ . We may (and will) identify the vector spaces  $\mathfrak{g}$  and  $\mathfrak{q}$  using the decomposition  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}^-$ . If  $(b, \eta), (b', \eta') \in \mathfrak{q}$ , then the Lie bracket in  $\mathfrak{q}$  is given by

$$(0.1) \quad [(b, \eta), (b', \eta')] = ([b, b'], b \circ \eta' - b' \circ \eta).$$

The corresponding connected algebraic group is  $Q = B \ltimes N$ , where  $N = \exp((\mathfrak{u}^-)^a)$  is an abelian normal unipotent subgroup of  $Q$ . The exponential map  $\exp : (\mathfrak{u}^-)^a \rightarrow N$  is an isomorphism of varieties, and elements of  $Q$  are written as product  $s \cdot \exp(\eta)$ , where  $s \in B$  and  $\eta \in \mathfrak{u}^-$ . If  $(s, \eta) \mapsto s \cdot \eta$  is the representation of  $B$  in  $\mathfrak{u}^-$ , then the adjoint representation of  $Q$  is given by

$$(0.2) \quad \text{Ad}_Q(s \cdot \exp(\eta))(b, \eta') = (\text{Ad}(s)b, s \cdot (\eta' - b \circ \eta)).$$

In this note, we explicitly construct certain polynomials that generate the algebras of invariants  $\mathbb{F}[\mathfrak{q}]^Q$  and  $\mathbb{F}[\mathfrak{q}^*]^Q$ , and thereby prove that these two algebras are free. Furthermore, we also show that these polynomials generate the corresponding fields of invariants,  $\mathbb{F}(\mathfrak{q})^Q$  and  $\mathbb{F}(\mathfrak{q}^*)^Q$ , and that  $\mathbb{F}[\mathfrak{q}]$  is a free  $\mathbb{F}[\mathfrak{q}]^Q$ -module and  $\mathbb{F}[\mathfrak{q}^*]$  is a free  $\mathbb{F}[\mathfrak{q}^*]^Q$ -module. The last assertion implies that the enveloping algebra of  $\mathfrak{q}$ ,  $\mathcal{U}(\mathfrak{q})$ , is a free module over its centre. The Lie algebra  $\mathfrak{q}$  is an *Inönü-Wigner contraction* of  $\mathfrak{g}$  (see [15, Ch. 7 § 2.5]), and we also discuss the corresponding relationship between the invariants of  $G$  and  $Q$ .

Certain classes of non-reductive algebraic Lie algebras  $\mathfrak{q}$  such that  $\mathbb{F}[\mathfrak{q}^*]^Q$  is a polynomial ring have been studied before. They include the centralisers of nilpotent elements in  $\mathfrak{sl}_{l+1}$  and  $\mathfrak{sp}_{2l}$  [9],  $\mathbb{Z}_2$ -contractions of  $\mathfrak{g}$  [10], and the truncated seaweed (biparabolic) subalgebras of  $\mathfrak{sl}_{l+1}$  and  $\mathfrak{sp}_{2l}$  [7]. Our result enlarges this interesting family of Lie algebras.

Let  $\mathfrak{q}_{\text{reg}}^*$  denote the set of regular elements of  $\mathfrak{q}^*$ , i.e.,  $x \in \mathfrak{q}_{\text{reg}}^*$  if and only if  $\dim Q \cdot x$  is maximal. For many problems related to coadjoint representations, it is vital to have that  $\text{codim}(\mathfrak{q}^* \setminus \mathfrak{q}_{\text{reg}}^*) \geq 2$  [10, 9]. However, we prove that if  $\mathfrak{g}$  is simple and not of type  $\mathbf{A}_l$ , then  $\mathfrak{q}^* \setminus \mathfrak{q}_{\text{reg}}^*$  contains a divisor.

*Notation.*

- the centraliser in  $\mathfrak{g}$  of  $x \in \mathfrak{g}$  is denoted by  $\mathfrak{g}^x$ .
- $\kappa$  is the Killing form on  $\mathfrak{g}$ .
- $\mathfrak{g}_{\text{reg}}$  is the set of regular elements of  $\mathfrak{g}$ , i.e.,  $x \in \mathfrak{g}_{\text{reg}}$  if and only if  $\dim \mathfrak{g}^x = l$ .

- If  $X$  is an irreducible variety, then  $\mathbb{F}[X]$  is the algebra of regular functions and  $\mathbb{F}(X)$  is the field of rational functions on  $X$ . If  $X$  is acted upon by an algebraic group  $A$ , then  $\mathbb{F}[X]^A$  and  $\mathbb{F}(X)^A$  denote the subsets of respective  $A$ -invariant functions.
- If  $\mathbb{F}[X]^A$  is finitely generated, then  $X//A := \text{Spec}(\mathbb{F}[X]^A)$  and  $\pi: X \rightarrow X//A$  is determined by the inclusion  $\mathbb{F}[X]^A \hookrightarrow \mathbb{F}[X]$ . If  $\mathbb{F}[X]^A$  is graded polynomial, then the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants*.
- $\mathcal{S}^i(V)$  is the  $i$ -th symmetric power of the vector space  $V$  and  $\mathcal{S}(V) = \bigoplus_{i \geq 0} \mathcal{S}^i(V)$  is the symmetric algebra of  $V$ .

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### 1. On adjoint and coadjoint invariants of Inönü-Wigner contractions

The algebra  $\mathfrak{q} = \mathfrak{h} \ltimes (\mathfrak{u}^-)^\alpha$  is an Inönü-Wigner contraction of  $\mathfrak{g}$ . For this reason, we recall the relevant setting and then describe a general procedure for constructing adjoint and coadjoint invariants of Inönü-Wigner contractions. The  $\mathbb{Z}_2$ -contractions of  $\mathfrak{g}$  (considered in [10, 11]) are special cases of Inönü-Wigner contractions, and for them such a procedure is exposed in [10, Prop. 3.1]. However, the more general situation considered here requires another proof.

For a while, we assume that  $G$  is any connected algebraic group. Let  $H$  be an arbitrary connected subgroup of  $G$  and let  $\mathfrak{m}$  be a complementary subspace to  $\mathfrak{h} = \text{Lie } H$  in  $\mathfrak{g}$ . Using the vector space isomorphism  $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}$ , we regard  $\mathfrak{m}$  as  $H$ -module. Consider the invertible linear map  $c_t: \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $t \in \mathbb{F} \setminus \{0\}$ , such that  $c_t(h + m) = h + tm$  ( $h \in \mathfrak{h}$ ,  $m \in \mathfrak{m}$ ) and define the Lie algebra multiplication  $[\ , \ ]_{(t)}$  on the vector space  $\mathfrak{g}$  by the rule

$$[x, y]_{(t)} := c_t^{-1}([c_t(x), c_t(y)]), \quad x, y \in \mathfrak{g} .$$

Write  $\mathfrak{g}_{(t)}$  for the corresponding Lie algebra. The operator  $(c_t)^{-1} = c_{t^{-1}}: \mathfrak{g} \rightarrow \mathfrak{g}_{(t)}$  yields an isomorphism between the Lie algebras  $\mathfrak{g} = \mathfrak{g}_{(1)}$  and  $\mathfrak{g}_{(t)}$ , hence all algebras  $\mathfrak{g}_{(t)}$  are isomorphic. It is easily seen that  $\lim_{t \rightarrow 0} \mathfrak{g}_{(t)} \simeq \mathfrak{h} \ltimes (\mathfrak{g}/\mathfrak{h})^\alpha = \mathfrak{h} \ltimes \mathfrak{m}^\alpha$ .

The resulting Lie algebra  $\mathfrak{k} := \mathfrak{h} \ltimes \mathfrak{m}^a$  is called an *Inönü-Wigner contraction* of  $\mathfrak{g}$ , cf. Example 7 in [15, Chapter 7, § 2]. The corresponding connected algebraic group is  $K = H \ltimes \exp(\mathfrak{m}^a)$ . We identify the vector spaces  $\mathfrak{g}$  and  $\mathfrak{k}$  using the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ .

*Remark.* — For  $\mathfrak{g}$  semisimple, the contraction  $\mathfrak{g} \rightsquigarrow \mathfrak{b} \ltimes \mathfrak{u}^-$  is presented in a more lengthy way, using structure constants, in [2, Remark 2.3].

**1.1.** To construct invariants of the coadjoint representation of  $\mathfrak{k}$ , we proceed as follows. Let  $f \in \mathcal{S}(\mathfrak{g}) = \mathbb{F}[\mathfrak{g}^*]$  be a homogeneous polynomial of degree  $n$ . Using the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , we consider the bi-homogeneous components of  $f$ :

$$f = \sum_{a \leq i \leq b} f^{(n-i,i)},$$

where  $f^{(n-i,i)} \in \mathcal{S}^{n-i}(\mathfrak{h}) \otimes \mathcal{S}^i(\mathfrak{m}) \subset \mathcal{S}^n(\mathfrak{g})$ , and both  $f^{(n-a,a)}$  and  $f^{(n-b,b)}$  are assumed to be nonzero. In particular,  $f^{(n-b,b)}$  is the bi-homogeneous component having the maximal degree relative to  $\mathfrak{m}$ . Since  $\mathfrak{g}_{(t)}$  and  $\mathfrak{k}$  are just the same vector spaces, we also can regard each  $f^{(n-i,i)}$  as an element of  $\mathcal{S}^n(\mathfrak{g}_{(t)})$  or  $\mathcal{S}^n(\mathfrak{k})$ .

**THEOREM 1.1.** — *If  $f \in \mathcal{S}^n(\mathfrak{g})^G = \mathbb{F}[\mathfrak{g}^*]^G_n$ , then  $f^{(n-b,b)} \in \mathcal{S}^n(\mathfrak{k})^K = \mathbb{F}[\mathfrak{k}^*]^K_n$ .*

*Proof.* — The isomorphism of Lie algebras  $c_{t^{-1}} : \mathfrak{g} \rightarrow \mathfrak{g}_{(t)}$  implies that  $\sum_{a \leq i \leq b} t^{-i} f^{(n-i,i)} \in \mathcal{S}(\mathfrak{g}_{(t)})^{G_{(t)}}$  for all  $t \neq 0$ . It is harmless to replace the last expression with the  $G_{(t)}$ -invariant  $f_{(t)} := \sum_{a \leq i \leq b} t^{n-i} f^{(n-i,i)}$ . Since  $f_{(t)}$  is killed by  $\mathfrak{g}_{(t)}$  for all  $t \neq 0$ , its limit at 0, which is  $f^{(n-b,b)}$ , is killed by  $\lim_{t \rightarrow 0} \mathfrak{g}_{(t)} = \mathfrak{k}$ . Hence  $f^{(n-b,b)}$  is  $K$ -invariant.  $\square$

Let us say that  $f^\bullet := f^{(n-b,b)}$  is the *highest component* of  $f \in \mathbb{F}[\mathfrak{g}^*]^G_n$  (with respect to the contraction  $\mathfrak{g} \rightsquigarrow \mathfrak{k}$ ). Denote by  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$  the linear span of  $\{f^\bullet \mid f \in \mathbb{F}[\mathfrak{g}^*]^G \text{ is homogeneous}\}$ . Clearly, it is a graded algebra, and Theorem 1.1 implies that  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G) \subset \mathbb{F}[\mathfrak{k}^*]^K$ . We say that  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$  is the *algebra of highest components* for  $\mathbb{F}[\mathfrak{g}^*]^G$ .

Invariants of the adjoint representation of  $\mathfrak{k}$  can be constructed in a similar way. Set  $\mathfrak{m}^* := \mathfrak{h}^\perp$ , the annihilator of  $\mathfrak{h}$  in  $\mathfrak{g}^*$ . Likewise,  $\mathfrak{h}^* = \mathfrak{m}^{\perp}$ . Then  $\mathfrak{g}^* = \mathfrak{m}^* \oplus \mathfrak{h}^*$ , and the adjoint operator  $c_t^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  is given by  $c_t^*(m^* + h^*) = t^{-1}m^* + h^*$  ( $m^* \in \mathfrak{m}^*$ ,  $h^* \in \mathfrak{h}^*$ ). Having identified  $\mathfrak{q}^*$  and  $\mathfrak{k}^*$ , we can play the same game with homogeneous elements of  $\mathcal{S}(\mathfrak{g}^*) = \mathbb{F}[\mathfrak{g}^*]$ . If  $\tilde{f} \in \mathcal{S}^n(\mathfrak{g}^*)$ , then  $\tilde{f}^{(i,n-i)}$  denotes its bi-homogeneous component that belongs to  $\mathcal{S}^i(\mathfrak{m}^*) \otimes \mathcal{S}^{n-i}(\mathfrak{h}^*)$ . The resulting assertion is the following:

**THEOREM 1.2.** — For  $\tilde{f} \in \mathcal{S}^n(\mathfrak{g}^*)^G$ , let  $\tilde{f}^{(a,n-a)}$  be the bi-homogeneous component with minimal  $a$ , i.e., having the maximal degree relative to  $\mathfrak{h}^* = \mathfrak{m}^\perp$ . Then  $\tilde{f}^{(a,n-a)} \in \mathcal{S}^n(\mathfrak{k}^*)^K$ .

Likewise, we write  $\tilde{f}^\bullet := \tilde{f}^{(a,n-a)}$  and consider the algebra of highest components,  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$ , which can be regarded as a graded subalgebra of  $\mathbb{F}[\mathfrak{k}]^K$ .

**LEMMA 1.3.** — The graded algebras  $\mathbb{F}[\mathfrak{g}^*]^G$  and  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$  have the same Poincaré series, i.e.,  $\dim \mathbb{F}[\mathfrak{g}^*]^G_n = \dim \mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)_n$  for all  $n \in \mathbb{N}$ ; and likewise for  $\mathbb{F}[\mathfrak{g}]^G$  and  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$ .

*Proof.* — Actually, the assertion concerns vector spaces. Let  $\tilde{V} = \bigoplus_{i \in \mathbb{Z}} \tilde{V}_i$  be a finite-dimensional  $\mathbb{Z}$ -graded vector space and  $V$  an arbitrary subspace of  $\tilde{V}$ . For  $v \in V$ , let  $v^\bullet$  denote the highest component of  $v$  with respect to the  $\mathbb{Z}$ -grading. Set  $\mathcal{L}^\bullet(V) = \text{span}\{v^\bullet \mid v \in V\}$ . We claim that there is a basis for  $V$ , say  $(v_1, \dots, v_m)$ , such that  $(v_1^\bullet, \dots, v_m^\bullet)$  is a basis for  $\mathcal{L}^\bullet(V)$ . (Left to the reader.) In particular,  $\dim V = \dim \mathcal{L}^\bullet(V)$ .

Now, apply this claim to  $\tilde{V} = \mathbb{F}[\mathfrak{g}^*]_n = \bigoplus_i \mathbb{F}[\mathfrak{g}^*]_{(i,n-i)}$  and  $V = \mathbb{F}[\mathfrak{g}^*]^G_n$ . □

It is not always the case that  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G) = \mathbb{F}[\mathfrak{k}^*]^K$  or  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G) = \mathbb{F}[\mathfrak{k}]^K$ . For instance, we will see below that, for  $\mathfrak{g}$  semisimple and  $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{u}^-)^\alpha$ , such an equality holds only for the invariants of the coadjoint representation. By the very construction, the algebras  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}^*]^G)$  and  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G)$  are bi-graded. Moreover, it follows from [10, Theorem 2.7] that the algebras  $\mathbb{F}[\mathfrak{k}^*]^K$  and  $\mathbb{F}[\mathfrak{k}]^K$  are always bi-graded.

**1.2.** If  $\mathfrak{g}$  is semisimple, then we may identify  $\mathfrak{g}$  and  $\mathfrak{g}^*$  (and hence  $\mathcal{S}(\mathfrak{g})$  and  $\mathcal{S}(\mathfrak{g}^*)$ ) using the Killing form  $\varkappa$ . If  $\mathfrak{h}$  is also reductive, then  $\varkappa$  is non-degenerated on  $\mathfrak{h}$  and one can take  $\mathfrak{m}$  to be the orthocomplement of  $\mathfrak{h}$  with respect to  $\varkappa$ . Then  $\mathfrak{h}^\perp \simeq \mathfrak{m}$  and the decompositions of  $\mathfrak{g}$  and  $\mathfrak{g}^*$  considered in the general setting of Inönü-Wigner contractions coincide. Moreover, we can also identify the vector spaces  $\mathfrak{k}$  and  $\mathfrak{k}^*$ . However, to obtain invariants of the adjoint and coadjoint representations of  $\mathfrak{q}$ , one has to take the bi-homogeneous components of maximal degree with respect to *different* summands in the sum  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ . In this situation, Theorems 1.1 and 1.2 admit the following simultaneous formulation:

Suppose that  $f \in \mathbb{F}[\mathfrak{g}]^G_n \simeq \mathcal{S}(\mathfrak{g})^G_n$  and  $f = \sum_{a \leq i \leq b} f^{(n-i,i)}$  is the bi-homogeneous decomposition relative to the sum  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ . (That is,  $\text{deg}_{\mathfrak{h}} f^{(n-i,i)} = n - i$ , etc.) Then, upon identifications of vector spaces  $\mathfrak{g}, \mathfrak{k}$ , and  $\mathfrak{k}^*$ , we have  $f^{(n-a,a)} \in \mathbb{F}[\mathfrak{k}]^K$  and  $f^{(n-b,b)} \in \mathbb{F}[\mathfrak{k}^*]^K$ .

Such a phenomenon was already observed in the case of  $\mathbb{Z}_2$ -contractions of semisimple Lie algebras, *i.e.*, if  $\mathfrak{h}$  is the fixed-point subalgebra of an involution, see [10, Prop. 3.1].

## 2. Invariants of the adjoint representation of $Q$

In this section, we describe the algebra of invariants of the adjoint representation of  $Q$ .

To prove that a certain set of invariants generates the whole algebra of invariants, we use the following lemma of Igusa [6].

LEMMA 2.1 (Igusa). — *Let  $A$  be an algebraic group acting regularly on an irreducible affine variety  $X$ . Suppose that  $S$  is an integrally closed finitely generated subalgebra of  $\mathbb{F}[X]^A$  and the morphism  $\pi: X \rightarrow \text{Spec } S =: Y$  has the properties:*

- (i) *the fibres of  $\pi$  over a dense open subset of  $Y$  contain a dense  $A$ -orbit;*
- (ii)  *$\text{Im } \pi$  contains an open subset  $\Omega$  of  $Y$  such that  $\text{codim}(Y \setminus \Omega) \geq 2$ .*

*Then  $S = \mathbb{F}[X]^A$ . In particular, the algebra of  $A$ -invariants is finitely generated.*

Remark 2.2. — A proof of the Igusa lemma is given, for example, in [11, Lemma 6.1]. This proof shows that the above condition (i) can be replaced with the condition that  $S \subset \mathbb{F}[X]^A$  generates the field  $\mathbb{F}(X)^A$ . (In fact, it is not hard to prove that (i) holds *if and only if*  $S$  separates  $A$ -orbits in a dense open subset of  $X$  *if and only if*  $S$  generates  $\mathbb{F}(X)^A$ .)

LEMMA 2.3. — *If  $t \in \mathfrak{t}$  is regular and  $u \in \mathfrak{u}$  is arbitrary, then (i)  $t + u$  and  $t$  belong to the same  $\text{Ad } U$ -orbit; (ii)  $(t + u) \circ \mathfrak{u}^- = \mathfrak{u}^-$ .*

*Proof.*

(i) Clearly,  $(\text{Ad } U)t \subset t + \mathfrak{u}$  for all  $t \in \mathfrak{t}$ . If  $t$  is regular, then  $\dim(\text{Ad } U)t = \dim \mathfrak{u}$ . It is also known that the orbits of a unipotent group acting on an affine variety are closed. Hence  $(\text{Ad } U)t = t + \mathfrak{u}$ .

(ii) This is obvious if  $u = 0$ . In general, this follows from (i). □

THEOREM 2.4. — *We have  $\mathbb{F}[\mathfrak{q}]^Q \simeq \mathbb{F}[\mathfrak{t}]$ , and the quotient morphism  $\pi_Q: \mathfrak{q} \rightarrow \mathfrak{t}$  is given by  $(t + u, \eta) \mapsto t$ .*

*Proof.* — Clearly,  $\mathbb{F}[\mathfrak{q}]^Q = (\mathbb{F}[\mathfrak{q}]^N)^B$ . We prove that 1)  $\mathbb{F}[\mathfrak{q}]^N \simeq \mathbb{F}[\mathfrak{b}]$  and 2)  $\mathbb{F}[\mathfrak{b}]^B \simeq \mathbb{F}[\mathfrak{t}]$ .

1) Consider the projection  $\pi_N: \mathfrak{q} \rightarrow \mathfrak{q}/(\mathfrak{u}^-)^a \simeq \mathfrak{b}$ . Clearly,  $N$  acts trivially on  $\mathfrak{q}/(\mathfrak{u}^-)^a$  and  $\pi_N$  is a surjective  $N$ -equivariant morphism. Hence  $\mathbb{F}[\mathfrak{b}] \subset \mathbb{F}[\mathfrak{q}]^N$ . By Lemma 2.1, the equality  $\mathbb{F}[\mathfrak{b}] = \mathbb{F}[\mathfrak{q}]^N$  will follow from the fact that general fibres of  $\pi_N$  are  $N$ -orbits.

If  $t \in \mathfrak{t}$  is regular and  $u \in \mathfrak{u}$  is arbitrary, then  $b = t + u$  is a regular semisimple element of  $\mathfrak{g}$ . By (0.2) with  $s = 1$ , we have

$$\text{Ad}_Q(N)(b, \eta) = (b, \eta + b \circ \mathfrak{u}^-).$$

It then follows from Lemma 2.3 that  $\text{Ad}_Q(N)(b, \eta) = (b, \mathfrak{u}^-)$ . On the other hand,  $\pi_N^{-1}(b) = (b, \mathfrak{u}^-)$ , i.e.,  $\pi_N^{-1}(b)$  is a single  $N$ -orbit whenever  $b$  is regular semisimple.

2) Consider the projection  $\pi_B: \mathfrak{b} \rightarrow \mathfrak{b}/\mathfrak{u} \simeq \mathfrak{t}$ . Clearly,  $B$  acts trivially on  $\mathfrak{b}/\mathfrak{u}$  and  $\pi_B$  is a surjective  $B$ -equivariant morphism. Hence  $\mathbb{F}[\mathfrak{t}] \subset \mathbb{F}[\mathfrak{b}]^B$ . By Lemma 2.1, the equality  $\mathbb{F}[\mathfrak{t}] = \mathbb{F}[\mathfrak{b}]^B$  will follow from the fact that general fibres of  $\pi_B$  are  $B$ -orbits. Again, it follows from Lemma 2.3 that if  $t \in \mathfrak{t}$  is regular, then  $(\text{Ad } B)t = t + \mathfrak{u} = \pi_B^{-1}(t)$ . □

*Remark 2.5.* — Theorem 2.4 can be proved in a less informative way. Notice that  $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{u} \times (\mathfrak{u}^-)^a$  and therefore  $\mathbb{F}[\mathfrak{t}] \subset \mathbb{F}[\mathfrak{q}]^Q$ . Let  $x \in \mathfrak{t}$  be regular semisimple. Then  $\mathfrak{q}^x \simeq \mathfrak{g}^x = \mathfrak{t}$ , since  $\mathfrak{g}$  and  $\mathfrak{q}$  are isomorphic as  $T$ -modules. The fibres of the morphism  $\pi_Q: \mathfrak{q} \rightarrow \mathfrak{t}$ , defined in Theorem 2.4, are linear spaces of dimension  $\dim \mathfrak{q} - \dim \mathfrak{t} = \dim(\text{Ad } Q)x$ . Hence a general fibre contains a dense  $Q$ -orbit and Lemma 2.1 applies. We also see that the algebra  $\mathbb{F}[\mathfrak{t}]$  separates  $Q$ -orbits in  $\mathfrak{q}$  in general position and therefore  $\mathbb{F}(\mathfrak{q})^Q = \mathbb{F}(\mathfrak{t})$ .

Comparing with the adjoint representation of  $\mathfrak{g}$ , we see that, for  $\mathfrak{q}$ , the algebra of invariants remains polynomial, but the degrees of basic invariants drastically decrease! All the basic invariants in  $\mathbb{F}[\mathfrak{q}]^Q$  are of degree 1. This clearly means that here  $\mathcal{L}^\bullet(\mathbb{F}[\mathfrak{g}]^G) \subsetneq \mathbb{F}[\mathfrak{q}]^Q$ .

### 3. Invariants of the coadjoint representation of $Q$

In this section, we describe the algebra of invariants of the coadjoint representation of  $Q$ . The coadjoint representation is much more interesting since  $\mathbb{F}[\mathfrak{q}^*] = \mathcal{S}(\mathfrak{q})$  is a Poisson algebra,  $\mathcal{S}(\mathfrak{q})^Q$  is the centre of this Poisson algebra, and  $\mathcal{S}(\mathfrak{q})$  is related to the enveloping algebra of  $\mathfrak{q}$  via the Poincaré-Birkhoff-Witt theorem.

Since  $\mathfrak{q}$  is isomorphic to  $\mathfrak{b} \oplus \mathfrak{g}/\mathfrak{b} \simeq \mathfrak{b} \oplus \mathfrak{u}^-$  as vector space, the dual vector space  $\mathfrak{q}^*$  is isomorphic to  $(\mathfrak{g}/\mathfrak{b})^* \oplus \mathfrak{b}^*$ . Using  $\varkappa$ , we identify  $\mathfrak{b}^*$  with  $\mathfrak{b}^- := \mathfrak{t} \oplus \mathfrak{u}^-$  and  $(\mathfrak{g}/\mathfrak{b})^*$  with  $\mathfrak{u}$ . To stress that  $\mathfrak{q}^*$  is regarded as a  $Q$ -module and  $\mathfrak{b}^-$  appears to be a  $Q$ -stable subspace, we write  $\mathfrak{q}^* = \mathfrak{u} \times \mathfrak{b}^-$ . If  $(b, \eta) \in \mathfrak{q}$  and  $(u, \xi) \in \mathfrak{q}^*$ , i.e.,  $u \in \mathfrak{u}$  and  $\xi \in \mathfrak{b}^-$ , then the coadjoint representation of  $\mathfrak{q}$  is given by the formula:

$$(3.1) \quad (b, \eta) \star (u, \xi) = ([b, u], \phi(u, \eta) + b \star \xi).$$

Here  $(b, \xi) \mapsto b \star \xi$  is the coadjoint representation of  $\mathfrak{b}$ , and

$$\phi: \mathfrak{u} \times \mathfrak{u}^- \simeq \mathfrak{u} \times \mathfrak{u}^* \xrightarrow{\psi} \mathfrak{b}^* \simeq \mathfrak{b}^-,$$

where  $\psi$  is the *moment map* associated with the  $\mathfrak{b}$ -module  $\mathfrak{u}$ . Upon our identifications, the mapping  $\phi$  is directly defined by

$$\varkappa(b, \phi(u, \eta)) := \varkappa([b, u], \eta) = -\varkappa(u, b \circ \eta).$$

Recall some well-known properties of the  $B$ -module  $\mathfrak{u}$ :

- If  $\tilde{e} \in \mathfrak{u}$  is regular nilpotent, then  $\mathfrak{g}^{\tilde{e}} \subset \mathfrak{u}$  [8] and hence  $(\text{Ad } B)\tilde{e}$  is dense in  $\mathfrak{u}$ .
- For any  $e \in \mathfrak{u}$ , the irreducible components of  $(\text{Ad } G)e \cap \mathfrak{u}$  are called *orbital varieties* and each of them has dimension  $\frac{1}{2} \dim(\text{Ad } G)e$  [14, 4.3.11].

Let  $\text{Mor}_G(\mathfrak{g}, \mathfrak{g})$  denote the  $\mathbb{F}[\mathfrak{g}]^G$ -module of polynomial  $G$ -equivariant morphisms  $F: \mathfrak{g} \rightarrow \mathfrak{g}$ . By work of Kostant [8],  $\text{Mor}_G(\mathfrak{g}, \mathfrak{g})$  is a free graded  $\mathbb{F}[\mathfrak{g}]^G$ -module of rank  $l$ . It was noticed by Th. Vust [16, Char. III, § 2] (see also [12]) that a homogeneous basis of this module is obtained as follows. Let  $f_1, \dots, f_l$  be homogeneous algebraically independent generators of  $\mathbb{F}[\mathfrak{g}]^G$ . Each differential  $df_i$  determines a polynomial  $G$ -equivariant morphism (covariant) from  $\mathfrak{g}$  to  $\mathfrak{g}^*$ . Identifying  $\mathfrak{g}$  with  $\mathfrak{g}^*$  via  $\kappa$  yields a homogeneous covariant (or, vector field)  $F_i = \text{grad} f_i: \mathfrak{g} \rightarrow \mathfrak{g}$ . Then  $F_1, \dots, F_l$  form a homogeneous basis for  $\text{Mor}_G(\mathfrak{g}, \mathfrak{g})$ . If  $\deg f_i = d_i$ , then  $\deg F_i = d_i - 1 =: m_i$ . It is customary to say that  $\{m_1, \dots, m_l\}$  are the *exponents* of (the Weyl group of)  $\mathfrak{g}$ . Recall that if  $\mathfrak{g}$  is simple and  $m_1 \leq \dots \leq m_l$ , then  $m_1 = 1$ ,  $m_2 \geq 2$ , and  $m_i + m_{l-i+1}$  is the Coxeter number of  $\mathfrak{g}$ .

The covariants  $F_i$  have the following properties:

- (i)  $F_i(x) \in \mathfrak{g}^x$  for all  $i \in \{1, 2, \dots, l\}$  and  $x \in \mathfrak{g}$ ;
- (ii) The vectors  $F_1(x), \dots, F_l(x) \in \mathfrak{g}$  are linearly independent if and only if  $x \in \mathfrak{g}_{\text{reg}}$  [8, Theorem 9].

It follows that  $(F_1(x), \dots, F_l(x))$  is a basis for  $\mathfrak{g}^x$  if and only if  $x \in \mathfrak{g}_{\text{reg}}$ .

LEMMA 3.1. — *If  $x \in \mathfrak{b}$ , then  $F_i(x) \in \mathfrak{b}$ . If  $y \in \mathfrak{u}$ , then  $F_i(y) \in \mathfrak{u}$ .*

*Proof.* — If  $x \in \mathfrak{b} \cap \mathfrak{g}_{\text{reg}}$ , then  $\mathfrak{g}^x \subset \mathfrak{b}$ . (Indeed,  $[\mathfrak{b}, x] \subset \mathfrak{u}$ , hence  $\dim \mathfrak{b}^x \geq \text{rk } \mathfrak{g}$ . On the other hand,  $\mathfrak{b}^x \subset \mathfrak{g}^x$  and  $\dim \mathfrak{g}^x = \text{rk } \mathfrak{g}$ .) Hence  $F_i(x) \in \mathfrak{g}^x \subset \mathfrak{b}$ . Since  $\mathfrak{b} \cap \mathfrak{g}_{\text{reg}}$  is open and dense in  $\mathfrak{b}$ , the assertion follows.

If  $y \in \mathfrak{u} \cap \mathfrak{g}_{\text{reg}}$ , i.e.,  $y$  is regular nilpotent, then  $\mathfrak{g}^y \subset \mathfrak{u}$  [8]. The rest is the same. □

Consequently, letting  $P_i := F_i|_{\mathfrak{u}}$ , we obtain the covariants  $P_1, \dots, P_l \in \text{Mor}_B(\mathfrak{u}, \mathfrak{u})$ . Actually, we consider the  $P_i$ 's as  $B$ -equivariant morphisms  $P_i: \mathfrak{u} \rightarrow \mathfrak{u} \subset \mathfrak{b}$ . Using these covariants, we define polynomials  $\widehat{P}_i \in \mathbb{F}[\mathfrak{q}^*] = \mathbb{F}[\mathfrak{u} \rtimes \mathfrak{b}^-]$  by the formula

$$(3.2) \quad \widehat{P}_i(u, \xi) = \varkappa(P_i(u), \xi), \quad i = 1, \dots, l,$$

where  $u \in \mathfrak{u}$  and  $\xi \in \mathfrak{b}^-$ .

LEMMA 3.2. — We have  $\widehat{P}_i \in \mathbb{F}[\mathfrak{q}^*]^Q$ .

*Proof.* — Since  $Q = B \times N$ , it suffices to verify that  $\widehat{P}_i$  is both  $B$ - and  $N$ -invariant.

- 1)  $\widehat{P}_i$  is  $B$ -invariant, since  $P_i$  is  $B$ -equivariant.
- 2) For polynomials obtained from covariants  $P_i$  as in (3.2), the invariance with respect to the commutative unipotent group  $N$  is equivalent to that  $[P_i(u), u] = 0, u \in \mathfrak{u}$ . Indeed, for  $\eta \in \mathfrak{u}^-$ , the coadjoint action of  $\exp(\eta) \in N$  is given by  $\exp(\eta) \star (u, \xi) = (u, \xi + \phi(u, \eta))$ . Then

$$\begin{aligned} \widehat{P}_i(\exp(\eta) \cdot (u, \xi)) &= \varkappa(P_i(u), \xi + \phi(u, \eta)) \\ &= \varkappa(P_i(u), \xi) + \varkappa(P_i(u), \phi(u, \eta)) \\ &= \widehat{P}_i(u, \xi) + \varkappa([P_i(u), u], \eta). \end{aligned}$$

Hence  $\widehat{P}_i(\exp(\eta) \cdot (u, \xi)) = \widehat{P}_i(u, \xi)$  for all  $\eta$  if and only if  $[P_i(u), u] = 0$ . The latter follows from the corresponding property (i) for  $F_i$ . □

*Remark.* — We prove below that  $\widehat{P}_i$  is the highest component of  $f_i \in \mathbb{F}[\mathfrak{g}^*]^G$ . In view of Theorem 1.1, this also implies that  $\widehat{P}_i$  is  $Q$ -invariant.

THEOREM 3.3. — The algebra  $\mathbb{F}[\mathfrak{q}^*]^Q$  is freely generated by  $\widehat{P}_1, \dots, \widehat{P}_l$ , and  $\mathbb{F}(\mathfrak{q}^*)^Q$  is the fraction field of  $\mathbb{F}[\mathfrak{q}^*]^Q$ .

*Proof.* — Consider the morphism

$$\pi: \mathfrak{q}^* = \mathfrak{u} \rtimes \mathfrak{b}^- \rightarrow \mathbb{A}^l,$$

given by  $\pi(u, \xi) = (\widehat{P}_1(u, \xi), \dots, \widehat{P}_l(u, \xi))$ . As in Section 2, to prove that  $\pi$  is the quotient by  $Q$ , we are going to apply Lemma 2.1 to  $\pi$ .

If  $e \in \mathfrak{u}$  is regular, then  $P_1(e), \dots, P_l(e)$  are linearly independent and form a basis for  $\mathfrak{g}^e = \mathfrak{u}^e$ . Therefore, (3.2) implies that  $\pi$  is onto, and condition (ii) in Lemma 2.1 is satisfied.

Let us prove that  $\mathbb{F}(\mathfrak{q}^*)^Q = \mathbb{F}(\widehat{P}_1, \dots, \widehat{P}_l)$ . Consider the morphism

$$\tilde{\pi}: \mathfrak{q}^* \rightarrow (\mathfrak{q}^*/\mathfrak{b}^-) \times \mathbb{A}^l = \mathfrak{u} \times \mathbb{A}^l$$

defined by  $\tilde{\pi}(u, \xi) = (u, \widehat{P}_1(u, \xi), \dots, \widehat{P}_l(u, \xi))$ . If  $e \in \mathfrak{u} \cap \mathfrak{g}_{\text{reg}}$ , then Eq. (3.2) shows that  $\tilde{\pi}^{-1}(e, a)$  is an affine subspace of  $\mathfrak{q}^*$  for any  $a \in \mathbb{A}^l$ , and  $\dim \tilde{\pi}^{-1}(e, a) = \dim \mathfrak{b} - l = \dim \mathfrak{u}$ . As in the proof of Theorem 2.4, this implies that  $\tilde{\pi}^{-1}(e, a)$  is a sole  $N$ -orbit. Thus, the coordinate functions on  $\mathfrak{u}$  and  $\widehat{P}_1, \dots, \widehat{P}_l$  separate generic  $N$ -orbits of maximal dimension. By the Rosenlicht theorem [1, 1.6], this implies that all these functions generate the field of  $N$ -invariants on  $\mathfrak{q}^*$ , i.e.,  $\mathbb{F}(\mathfrak{q}^*)^N = \mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l)$ . Since  $B$  has an open orbit in  $\mathfrak{u}$ , we have  $\mathbb{F}(\mathfrak{u})^B = \mathbb{F}$ . Hence

$$\mathbb{F}(\mathfrak{q}^*)^Q = (\mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l))^B = \mathbb{F}(\widehat{P}_1, \dots, \widehat{P}_l).$$

In view of Remark 2.2, this is sufficient for using Lemma 2.1, and we conclude that  $\widehat{P}_1, \dots, \widehat{P}_l$  generate the algebra of  $Q$ -invariants on  $\mathfrak{q}^*$ . □

*Remark 3.4.* — Although we have proved that  $\mathbb{F}(\mathfrak{q}^*)^N = \mathbb{F}(\mathfrak{u})(\widehat{P}_1, \dots, \widehat{P}_l)$ , it is not true that  $\mathbb{F}[\mathfrak{q}^*]^N = \mathbb{F}[\mathfrak{u}][\widehat{P}_1, \dots, \widehat{P}_l]$ . The reason is that the morphism  $\tilde{\pi}$  defined in the previous proof does not satisfy condition (ii) of Lemma 2.1. That is, the closure of the complement of  $\text{Im } \tilde{\pi}$  contains a divisor. One can prove that this divisor is equal to  $D \times \mathbb{A}^l$ , where  $D = \mathfrak{u} \setminus (\text{Ad } B)\tilde{e} = \mathfrak{u} \setminus (\mathfrak{u} \cap \mathfrak{g}_{\text{reg}})$ . Actually, we can explicitly point out a function in  $\mathbb{F}[\mathfrak{q}^*]^N \setminus \mathbb{F}[\mathfrak{u}][\widehat{P}_1, \dots, \widehat{P}_l]$ . Let  $v$  be a non-zero vector in the one-dimensional space  $\mathfrak{b}^U$ . We can regard  $v$  as a linear function on  $\mathfrak{b}^-$  and hence on  $\mathfrak{q}^*$ . Making use of Eq. (0.1), it is not hard to check that the sub-algebra  $(\mathfrak{u}^-)^a \subset \mathfrak{q}$  commutes with  $v$ , i.e.,  $v$  is a required  $N$ -invariant in the symmetric algebra  $\mathcal{S}(\mathfrak{q})$ .

Recall that, for an algebraic group  $A$  with Lie algebra  $\mathfrak{a}$ , the *index* of  $\mathfrak{a}$ ,  $\text{ind } \mathfrak{a}$ , is defined as the minimal codimension of an  $A$ -orbit in the coadjoint representation. By the Rosenlicht theorem, one has  $\text{ind } \mathfrak{a} = \text{trdeg } \mathbb{F}(\mathfrak{a}^*)^A$ . It is easily seen that the index cannot decrease under contractions, hence  $\text{ind } \mathfrak{q} \geq \text{ind } \mathfrak{g} = l$ . The above description of the field of  $Q$ -invariants implies that

**COROLLARY 3.5.** —  $\text{ind } \mathfrak{q} = l$ .

**THEOREM 3.6.** — *The polynomial ring  $\mathbb{F}[\mathfrak{q}^*]$  is a free  $\mathbb{F}[\mathfrak{q}^*]^Q$ -module.*

*Proof.* — Since it is already known that  $\mathbb{F}[\mathfrak{q}^*]^{\mathcal{Q}}$  is a polynomial algebra (of Krull dimension  $l$ ), it suffices to prove that the quotient morphism  $\pi: \mathfrak{q}^* \rightarrow \mathfrak{q}^*/Q \simeq \mathbb{A}^l$  is equidimensional [13, Prop. 17.29]. This, in turn, will follow from the fact that the null-cone  $\mathcal{N} = \pi^{-1}(\pi(0))$  is of dimension  $\dim \mathfrak{q} - l$ . To estimate the dimension of  $\mathcal{N}$ , consider the projection  $p: \mathcal{N} \rightarrow \mathfrak{u}$  and partition  $\mathfrak{u}$  into finitely many orbital varieties (the irreducible components of  $(\text{Ad } G)e_i \cap \mathfrak{u}$ ), where  $\{e_i\}$  runs over a finite set of representatives of all nilpotent  $G$ -orbits. Let  $Z_i$  be an irreducible component of  $(\text{Ad } G)e_i \cap \mathfrak{u}$ . Since  $\pi = (\widehat{P}_1, \dots, \widehat{P}_l)$ , Eq. (3.2) shows that

$$\dim p^{-1}(Z_i) = \dim Z_i + \dim \mathfrak{b} - \dim \text{span}\{P_1(e_i), \dots, P_l(e_i)\}.$$

As  $\dim Z_i = \frac{1}{2} \dim(\text{Ad } G)e_i$ , the condition that  $\dim p^{-1}(Z_i) \leq \dim \mathfrak{q} - l$  can easily be transformed into

$$(3.3) \quad \dim \mathfrak{g}^{e_i} + 2 \dim \text{span}\{P_1(e_i), \dots, P_l(e_i)\} \geq 3l.$$

Recall that  $P_1, \dots, P_l$  are just the restrictions to  $\mathfrak{u}$  of basic covariants  $F_1, \dots, F_l$ , and  $F_j = \text{grad } f_j$ . Consequently,  $\dim \text{span}\{P_1(e_i), \dots, P_l(e_i)\}$  equals the rank of the differential at  $e$  of the quotient morphism  $\pi_{\mathfrak{g}, G}: \mathfrak{g} \rightarrow \mathfrak{g}/G$ . Therefore, (3.3) is precisely the inequality proved in [11, Theorem 10.6].  $\square$

**COROLLARY 3.7.** — *The enveloping algebra  $\mathcal{U}(\mathfrak{q})$  is a free module over its centre  $\mathcal{Z}(\mathfrak{q})$ .*

*Proof.* — This is a standard consequence of the fact that  $\mathbb{F}[\mathfrak{q}^*] = \mathcal{S}(\mathfrak{q})$  is a free module over  $\mathcal{S}(\mathfrak{q})^{\mathcal{Q}}$ ,  $\mathcal{S}(\mathfrak{q})^{\mathcal{Q}}$  is the centre of the Poisson algebra  $\mathcal{S}(\mathfrak{q})$ , and  $\text{gr } \mathcal{Z}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})^{\mathcal{Q}}$ , cf. [8, Theorem 21], [5, Theorem 3.3].  $\square$

*Remark 3.8.* — By Theorem 3.6, the irreducible components of all fibres of  $\pi: \mathfrak{q}^* \rightarrow \mathfrak{q}^*/Q \simeq \mathbb{A}^l$  are of dimension  $\dim \mathfrak{q} - l$ . However, unlike the case of the (co)adjoint representation of  $\mathfrak{g}$ , the zero fibre of  $\pi$  is highly reducible. For, if  $\dim \mathfrak{g}^{e_i} + 2 \dim \text{span}\{P_1(e_i), \dots, P_l(e_i)\} = 3l$ , then every irreducible component of  $(\text{Ad } G)e_i \cap \mathfrak{u}$  gives rise to an irreducible component of  $\pi^{-1}(\pi(0))$ . A complete classification of nilpotent elements of  $\mathfrak{g}$  satisfying this equality is contained in [11, § 10].

**THEOREM 3.9.** — *We have  $\mathcal{L}^\bullet(\mathcal{S}(\mathfrak{g})^G) = \mathcal{S}(\mathfrak{q})^{\mathcal{Q}}$ . The polynomials  $\widehat{P}_1, \dots, \widehat{P}_l \in \mathbb{F}[\mathfrak{q}^*]^{\mathcal{Q}} = \mathcal{S}(\mathfrak{q})^{\mathcal{Q}}$  are the highest components of  $f_1, \dots, f_l \in \mathcal{S}(\mathfrak{g})^G$  in the sense of Subsection 1.1.*

*Proof.*

1) Since  $\deg \widehat{P}_i = \deg f_i$  for all  $i$ , it follows from Lemma 1.3 and Theorem 3.3 that  $\mathcal{L}^\bullet(\mathcal{S}(\mathfrak{g})^G)$  and  $\mathcal{S}(\mathfrak{q})^Q$  have the same Poincaré series. Hence these algebras coincide.

2) Recall that  $\deg f_i = d_i = m_i + 1$ . According to Theorem 1.1, we have to take the decomposition  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}^-$  and pick the bi-homogeneous component of  $f_i$  of maximal degree with respect to  $\mathfrak{u}^-$ .

If the component  $f_i^{(0,d_i)} \in \mathcal{S}^{d_i}(\mathfrak{u}^-)$  were non-trivial, then it would be a  $Q$ -invariant in  $\mathcal{S}(\mathfrak{q})$  and in particular a  $B$ -invariant (Theorem 1.1). Recall that if we work in  $\mathfrak{q}$ , then  $\mathfrak{u}^- \simeq \mathfrak{g}/\mathfrak{b}$  as  $B$ -module. Since  $\mathcal{S}(\mathfrak{g}/\mathfrak{b}) \simeq \mathbb{F}[\mathfrak{u}]$  and  $\mathbb{F}[\mathfrak{u}]^B = \mathbb{F}$ , we get a contradiction. Hence  $f_i^{(0,d_i)} = 0$ .

Then next possible component is  $f_i^{(1,m_i)} \in \mathfrak{b} \otimes \mathcal{S}^{m_i}(\mathfrak{u}^-)$ . Using the identifications  $\mathfrak{b}^* \simeq \mathfrak{b}^-$  and  $\mathfrak{u}^* \simeq \mathfrak{u}^-$ , we have  $f_i^{(1,m_i)} \in \mathbb{F}[\mathfrak{b}^-]_1 \otimes \mathbb{F}[\mathfrak{u}]_{m_i}$ . That is, if considered as a function on  $\mathfrak{g} = \mathfrak{b}^- \oplus \mathfrak{u}$ , it can be written as  $f_i^{(1,m_i)}(\xi, u) = \kappa(\bar{P}_i(u), \xi)$  for some morphism  $\bar{P}_i: \mathfrak{u} \rightarrow \mathfrak{b}$  of degree  $m_i$ . As we have already proved that  $f_i^{(0,d_i)} = 0$ ,  $\bar{P}_i(u)$  is nothing but the value of  $\text{grad } f_i$  at  $u$ . Hence  $\bar{P}_i = P_i$ , and we are done.  $\square$

### 4. Further properties of the coadjoint representation

4.1. For the classical Lie algebras, the basic covariants  $F_i: \mathfrak{g} \rightarrow \mathfrak{g}$  (and hence  $P_i$ ) have a simple description:

- if  $x \in \mathfrak{sl}_{l+1}$ , then  $F_i(x) = x^i, i = 1, 2, \dots, l$ ;
- if  $x \in \mathfrak{sp}_{2l}$  or  $\mathfrak{so}_{2l+1}$ , then  $F_i(x) = x^{2i-1}, i = 1, 2, \dots, l$ ;
- if  $x \in \mathfrak{so}_{2l}$ , then  $F_i(x) = x^{2i-1}, i = 1, 2, \dots, l - 1$ . The covariant  $F_l$  that is related to the pfaffian is described as follows. Let  $x$  be a skew-symmetric matrix of order  $2l$ . For  $i \neq j$ , let  $x_{[ij]}$  be the skew-symmetric sub-matrix of order  $2l - 2$  obtained by deleting  $i$ th and  $j$ th row and column. Set  $a_{ij} = \text{Pf}(x_{[ij]})$  if  $i \neq j$ , and  $a_{ii} = 0$ . Then  $F_l(x) = (a_{ij})_{i,j=1}^{2l}$ . Clearly,  $\deg F_l = l - 1$ , as required.

Results of Sections 2 and 3 explicitly yield the bi-degrees of basic invariants for  $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{u}^-)^a$ . For  $\mathbb{F}[\mathfrak{q}]^Q$ , all the basic invariants have bi-degrees  $(1, 0)$ . For  $\mathbb{F}[\mathfrak{q}^*]^Q$ , the basic invariants have bi-degrees  $(m_i, 1)$ , i.e.,  $\widehat{P}_i \in \mathcal{S}^{m_i}(\mathfrak{u}^-) \otimes \mathfrak{b}$ . In particular, for the coadjoint representation, the total degrees of the basic  $Q$ -invariants remain the same as for  $G$ .

4.2. Hereafter we assume that  $\mathfrak{g}$  is simple and the basic invariants  $f_1, \dots, f_l \in \mathbb{F}[\mathfrak{g}]^G$  are numbered such that  $d_i \leq d_{i+1}$ . Then  $d_l = \mathfrak{h}$  is

the Coxeter number of  $\mathfrak{g}$ . We show that the corresponding  $Q$ -invariant  $\widehat{P}_l$  has a rather simple form. In fact, it turns out to be a product of linear forms.

Let  $\Delta$  be the root system of  $(\mathfrak{g}, \mathfrak{t})$  and  $\Delta^+$  the subset of positive roots corresponding to  $\mathfrak{u}$ . Let  $\Pi = \{\alpha_1, \dots, \alpha_l\}$  (resp.  $\theta$ ) be the set of simple roots (resp. the highest root) in  $\Delta^+$ . Then  $\theta = \sum_{i=1}^l a_i \alpha_i$  and  $\sum_{i=1}^l a_i = \mathfrak{h} - 1$ . For any  $\gamma \in \Delta$ ,  $\mathfrak{g}_\gamma$  denotes the corresponding root subspace, and we fix a nonzero vector  $e_\gamma \in \mathfrak{g}_\gamma$ .

LEMMA 4.1. — *Up to a scalar multiple, we have  $\widehat{P}_l = e_{-\alpha_1}^{a_1} \cdots e_{-\alpha_l}^{a_l} e_\theta \in \mathcal{S}(\mathfrak{q})^Q$ .*

*Proof.* — Recall that  $\mathfrak{q} = \mathfrak{b} \oplus \mathfrak{u}^-$  as vector space, and here  $e_\theta \in \mathfrak{b}$  and  $e_{-\alpha_i} \in \mathfrak{u}^-$ . By the very construction,  $\widehat{P} := e_{-\alpha_1}^{a_1} \cdots e_{-\alpha_l}^{a_l} e_\theta$  is a  $T$ -invariant in  $\mathcal{S}(\mathfrak{q})$ . Then, using Eq. (0.1), one readily verifies that  $\widehat{P}$  is both  $U$ -invariant and  $N$ -invariant. Hence  $\widehat{P}$  is a polynomial in  $\widehat{P}_1, \dots, \widehat{P}_l$ . Since  $\text{bi-deg } \widehat{P} = (\mathfrak{h} - 1, 1)$  and  $m_i < m_l$  for  $i < l$ , the subspace of bi-degree  $(m_l, 1) = (\mathfrak{h} - 1, 1)$  in  $\mathcal{S}(\mathfrak{q})^Q$  is one-dimensional and spanned by  $\widehat{P}_l$ . Hence the assertion.  $\square$

Since  $\dim \mathfrak{q} = \dim \mathfrak{g}$ ,  $\text{ind } \mathfrak{q} = \text{ind } \mathfrak{g}$ , and the (total) degrees of the basic invariants of the coadjoint representations for  $G$  and  $Q$  coincide, we have the equality

$$(4.1) \quad \sum_{i=1}^l \text{deg } \widehat{P}_i = \frac{\dim \mathfrak{q} + \text{ind } \mathfrak{q}}{2},$$

which is very useful in the study of the coadjoint representation, see e.g. [10, Theorem 1.2]. Unfortunately,  $\mathfrak{q}$  does not always possess another important ingredient, the so-called *codim-2* property. Recall that  $x \in \mathfrak{q}^*$  is said to be *regular* if  $\dim Q \cdot x$  is maximal. The set of all regular elements is denoted by  $\mathfrak{q}_{\text{reg}}^*$ . It is an open subset of  $\mathfrak{q}^*$ , and we say that  $\mathfrak{q}$  has the *codim-2* property if  $\text{codim}(\mathfrak{q}^* \setminus \mathfrak{q}_{\text{reg}}^*) \geq 2$ .

THEOREM 4.2. — *The algebra  $\mathfrak{q}$  does not have the codim-2 property if  $\mathfrak{g}$  is not of type  $A_l$ .*

*Proof.* — Suppose that  $\mathfrak{q}$  has the *codim-2* property. Since (4.1) is satisfied, it follows from [10, Theorem 1.2] that the differentials  $(d\widehat{P}_i)_x$ ,  $i = 1, \dots, l$ , are linearly independent if and only if  $x \in \mathfrak{q}_{\text{reg}}^*$ . In particular, any divisor  $\widetilde{D} \subset \mathfrak{q}^*$  contains a point where the differentials of  $\widehat{P}_1, \dots, \widehat{P}_l$  are linearly independent.

On the other hand, Lemma 4.1 shows that if  $a_i \geq 2$  for some  $i$ , then  $d\widehat{P}_l$  vanishes at the hyperplane  $\{e_{-\alpha_i} = 0\}$ , where  $e_{-\alpha_i}$  is regarded as a linear

function on  $\mathfrak{u}$  and hence on  $\mathfrak{q}^*$ . Thus,  $\mathfrak{q}$  cannot have the *codim-2* property unless  $a_i = 1$  for all  $i$ , i.e.,  $\mathfrak{g}$  is of type  $\mathbf{A}_l$ . □

To prove the converse of this theorem, we need some preparations. For  $\alpha_i \in \Pi$ , let  $\mathfrak{u}_i \subset \mathfrak{u}$  denote the kernel of the linear form  $u \mapsto \kappa(e_{-\alpha_i}, u)$ . By [8],  $\mathfrak{u} \setminus \mathfrak{u} \cap \mathfrak{g}_{\text{reg}} = \cup_i \mathfrak{u}_i$ . Set

$$(4.2) \quad \mathcal{Y} = \mathcal{Y}(\mathfrak{q}^*) = \left\{ x \in \mathfrak{q}^* \mid (d\widehat{P}_1)_x, \dots, (d\widehat{P}_l)_x \text{ are linearly independent} \right\}.$$

PROPOSITION 4.3. — *If  $\mathfrak{g} = \mathfrak{sl}_{l+1}$ , then  $\text{codim}(\mathfrak{q}^* \setminus \mathcal{Y}) \geq 2$ .*

*Proof.* — Let  $a = (e, \xi)$  and  $a' = (e', \xi')$  be typical elements of  $\mathfrak{q}^*$ , where  $e, e' \in \mathfrak{u}$  and  $\xi, \xi' \in \mathfrak{b}^-$ . According to formulae of Subsection 4.1,  $\widehat{P}_i(e, \xi) = \kappa(e^i, \xi)$ . Recall that  $(d\widehat{P}_i)_a \in \mathfrak{q}$  and  $\langle (d\widehat{P}_i)_a, a' \rangle$  is the coefficient of  $t$  in the expansion of  $\widehat{P}_i(a + ta')$ . Consequently,

$$\langle (d\widehat{P}_i)_a, a' \rangle = \kappa(e^i, \xi') + \kappa\left(\sum_{k+m=i-1} e^k e' e^m, \xi\right).$$

The vector  $(d\widehat{P}_i)_a$  has the  $\mathfrak{b}$ - and  $\mathfrak{u}^-$ -components, and this equality shows that:

- the  $\mathfrak{b}$ -component of  $(d\widehat{P}_i)_a$  equals  $e^i$ ;
- the  $\mathfrak{u}^-$ -component of  $(d\widehat{P}_i)_a$ , say  $(d\widehat{P}_i)_a\{\mathfrak{u}^-\}$ , is determined by the equation  $\kappa((d\widehat{P}_i)_a\{\mathfrak{u}^-\}, e') = \kappa(\sum_{k+m=i-1} e^k e' e^m, \xi)$ .

Let  $\mathcal{O}^{\text{reg}}$  and  $\mathcal{O}^{\text{sub}}$  denote the regular and subregular nilpotent orbits in  $\mathfrak{sl}_{l+1}$ , respectively. Then  $\overline{\mathcal{O}^{\text{sub}} \cap \mathfrak{u}} = \cup_j \mathfrak{u}_j$ . If  $e \in \mathcal{O}^{\text{reg}} \cap \mathfrak{u}$ , then the  $\mathfrak{b}$ -components of  $(d\widehat{P}_i)_{(e, \xi)}$ ,  $i = 1, \dots, l$ , are linearly independent, regardless of  $\xi$ . Hence  $(\mathcal{O}^{\text{reg}} \cap \mathfrak{u}) \times \mathfrak{b}^- \subset \mathcal{Y}$ .

If  $e \in \mathcal{O}^{\text{sub}} \cap \mathfrak{u}$ , then the  $\mathfrak{b}$ -components of  $(d\widehat{P}_i)_{(e, \xi)}$ ,  $i = 1, \dots, l - 1$ , are still linearly independent for any  $\xi$ , but  $e^l = 0$ . However, if  $e$  is sufficiently general, then the  $\mathfrak{u}^-$ -component of  $(d\widehat{P}_l)_{(e, \xi)}$  appears to be nonzero for all  $\xi$  that belong to a dense open subset of  $\mathfrak{b}^-$ . More precisely, suppose that  $e \in \mathfrak{u}_j$  and  $\kappa(e, e_{-\alpha_i}) \neq 0$  for  $i \neq j$ . Taking  $e' = e_{\alpha_j}$ , one readily computes that  $\sum_{k+m=l-1} e^k e' e^m = e^{j-1} e_{\alpha_j} e^{l-j}$  is a nonzero multiple of  $e_\theta$ . Hence, one can take any  $\xi$  such that  $\kappa(\xi, e_\theta) \neq 0$ .

Thus, there is a dense open subset  $\Omega \subset \cup_i \mathfrak{u}_i \times \mathfrak{b}^-$  such that  $\Omega \subset \mathcal{Y}$ , and the assertion follows. □

It turns out that Proposition 4.3 together with (4.1) is sufficient to conclude that for  $\mathfrak{g} = \mathfrak{sl}_{l+1}$ ,  $\mathfrak{q}$  has the *codim-2* property. This follows from the following general assertion:

THEOREM 4.4. — Let  $R$  be a connected algebraic group with Lie algebra  $\mathfrak{r}$ . Suppose that (i)  $\mathbb{F}[\mathfrak{r}^*]^R = \mathbb{F}[p_1, \dots, p_m]$  is a graded polynomial algebra, (ii)  $\text{ind } \mathfrak{r} = m$ , and (iii)  $\sum_{i=1}^m \deg p_i = (\dim \mathfrak{r} + \text{ind } \mathfrak{r})/2$ . Then the following conditions are equivalent:

- (1)  $\text{codim}(\mathfrak{r}^* \setminus \mathfrak{r}_{\text{reg}}^*) \geq 2$ ;
- (2)  $\text{codim}(\mathfrak{r}^* \setminus \mathcal{Y}(\mathfrak{r}^*)) \geq 2$ , where  $\mathcal{Y}(\mathfrak{r}^*)$  is defined as in (4.2) via the  $p_i$ 's.

If these conditions are satisfied, then actually  $\mathfrak{r}_{\text{reg}}^* = \mathcal{Y}(\mathfrak{r}^*)$ .

Proof. — The implication (1)  $\Rightarrow$  (2) is already proved in [10, Theorem 1.2].

To prove the converse, one can slightly adjust the proof given in [10], see also the proof of Theorem 1.2 in [9]. Set  $n = \dim \mathfrak{r}$ . Let  $T(\mathfrak{r}^*)$  denote the tangent bundle of  $\mathfrak{r}^*$ . The main part of that proof consists in a construction of two homogeneous polynomial sections of  $\wedge^{n-m} T(\mathfrak{r}^*)$ , denoted  $\mathfrak{Y}_1$  and  $\mathfrak{Y}_2$ . Write  $(\mathfrak{Y}_i)_x$  for the value of  $\mathfrak{Y}_i$  at  $x \in \mathfrak{r}^*$ . These sections have the following properties:

- (a) There exist nonzero polynomials  $F_1, F_2 \in \mathbb{F}[\mathfrak{r}^*]$  such that  $F_1 \mathfrak{Y}_1 = F_2 \mathfrak{Y}_2$ ;
- (b)  $(\mathfrak{Y}_1)_x \neq 0$  if and only if  $x \in \mathfrak{r}_{\text{reg}}^*$ ;
- (c)  $(\mathfrak{Y}_2)_x \neq 0$  if and only if  $x \in \mathcal{Y}(\mathfrak{r}^*)$ ;
- (d)  $\deg \mathfrak{Y}_1 = (n - m)/2$  and  $\deg \mathfrak{Y}_2 = \sum_{i=1}^m (\deg p_i - 1)$ .

This only requires assumptions (i) and (ii). If (iii) is also satisfied, then  $\deg \mathfrak{Y}_1 = \deg \mathfrak{Y}_2$ . Therefore either of conditions (1),(2) implies the other. Moreover, properties (a) and (b) imply that if (1) is satisfied, then  $\deg F_2 = 0$ , i.e.,  $F_2 \in \mathbb{F}^\times$ . Likewise, (a) and (c) imply that if (2) is satisfied, then  $\deg F_1 = 0$ . This yields the last assertion.  $\square$

Since  $\mathfrak{q} = \mathfrak{b} \ltimes \mathfrak{u}^-$  does not have the *codim-2* property if  $\mathfrak{g}$  is not of type  $A_l$ , we cannot immediately conclude that in all cases  $x \in \mathfrak{q}_{\text{reg}}^*$  if and only if  $(d\widehat{P}_1)_x, \dots, (d\widehat{P}_l)_x$  are linearly independent. Nevertheless, the fact that  $\widehat{P}_1, \dots, \widehat{P}_l$  are the highest components of the basic  $G$ -invariants  $f_1, \dots, f_l$  allows to circumvent this difficulty. It can be shown in general (see [17]) that the coadjoint representation  $(Q: \mathfrak{q}^*)$  has the following property:

CLAIM 4.5. — For  $x \in \mathfrak{q}^*$  the following conditions are equivalent:

- The orbit  $Q \cdot x$  is of maximal dimension, which is  $\dim \mathfrak{q} - l$  in this situation;
- The differentials  $(d\widehat{P}_i)_x, i = 1, \dots, l$ , are linearly independent.

This generalise a result of Kostant obtained for semisimple Lie algebras [8, Theorem 9].

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