



# ANNALES

DE

# L'INSTITUT FOURIER

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**Uniqueness in Rough Almost Complex Structures, and Differential Inequalities**

Tome 60, n° 6 (2010), p. 2261-2273.

[http://aif.cedram.org/item?id=AIF\\_2010\\_\\_60\\_6\\_2261\\_0](http://aif.cedram.org/item?id=AIF_2010__60_6_2261_0)

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# UNIQUENESS IN ROUGH ALMOST COMPLEX STRUCTURES, AND DIFFERENTIAL INEQUALITIES

by Jean-Pierre ROSAY

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ABSTRACT. — The study of  $J$ -holomorphic maps leads to the consideration of the inequations  $|\frac{\partial u}{\partial \bar{z}}| \leq C|u|$ , and  $|\frac{\partial u}{\partial \bar{z}}| \leq \epsilon|\frac{\partial u}{\partial z}|$ . The first inequation is fairly easy to use. The second one, that is relevant to the case of rough structures, is more delicate. The case of  $u$  vector valued is strikingly different from the scalar valued case. Unique continuation and isolated zeroes are the main topics under study. One of the results is that, in almost complex structures of Hölder class  $\frac{1}{2}$ , any  $J$ -holomorphic curve that is constant on a non-empty open set, is constant. This is in contrast with immediate examples of non-uniqueness.

RÉSUMÉ. — L'étude des applications  $J$ -holomorphes conduit à l'étude des équations  $|\frac{\partial u}{\partial \bar{z}}| \leq C|u|$ , et  $|\frac{\partial u}{\partial \bar{z}}| \leq \epsilon|\frac{\partial u}{\partial z}|$ . La première inéquation est facile à utiliser. La seconde, qui intervient naturellement dans les structures non lisses, est plus difficile. De façon intéressante, le cas d'applications vectorielles  $u$  est différent du cas scalaire. Les questions étudiées ont trait à l'unicité de prolongement et aux zéros isolés. Parmi les résultats, il est démontré que, pour les structures presque complexes de classe Hölderienne  $\frac{1}{2}$ , toute courbe  $J$ -holomorphe constante sur un ouvert non vide, est constante. Ceci est en contraste avec des exemples immédiats de non-unicité.

## 1. Introduction

The open unit disc in  $\mathbf{C}$  will be denoted by  $\mathbf{D}$ . Recall that an almost complex structure on  $\mathbf{C}^n$  consists in having for each  $p \in \mathbf{C}^n$  an endomorphism  $J = J(p)$  of the (real) tangent space to  $\mathbf{C}^n$  at  $p$  satisfying  $J^2 = -\mathbf{1}$ , and that a map  $u : \mathbf{D} \rightarrow (\mathbf{C}^n, J)$  is  $J$ -holomorphic if  $\frac{\partial u}{\partial \bar{y}}(z) = [J(u(z))](\frac{\partial u}{\partial x}(z))$ .

The main results in this paper are the following ones:

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*Keywords:*  $J$ -holomorphic curves, differential inequalities, uniqueness.

*Math. classification:* 32Q65, 35R45, 35A02.

PROPOSITION 1.1. — *Let  $J$  be a Hölder continuous  $\mathcal{C}^{\frac{1}{2}}$  almost complex structure defined on  $\mathbf{C}^n$ . Let  $u : \mathbf{D} \rightarrow \mathbf{C}^n$  be a  $J$ -holomorphic map (so  $u$  is of class  $\mathcal{C}^{1, \frac{1}{2}}$ ). If  $u = 0$  on some non empty open subset of  $\mathbf{D}$ , then  $u \equiv 0$ .*

PROPOSITION 1.2. — *There exists a smooth map  $u : \mathbf{D} \rightarrow \mathbf{C}^2$  satisfying  $|\frac{\partial u}{\partial \bar{z}}| \leq \epsilon(z)|\frac{\partial u}{\partial z}|$  with  $\epsilon(z) \rightarrow 0$  as  $z \rightarrow 0$ , and such that:*  
 *$u$  vanishes to infinite order at 0,*  
 *$u$  has a non-isolated zero at 0,*  
*but  $u$  is not identically 0 near 0.*

Note that the phenomena of Proposition 1.2 cannot occur for scalar valued maps  $u$  (i.e., for  $u : \mathbf{D} \rightarrow \mathbf{C}$ ), see the Appendix. So, here we see a difference between vector valued maps and functions.

I shall now explain the motivations for these questions and how they are related.

It is well known and extremely easy to see that for Hölder continuous almost complex structures, that are not Lipschitz, there is not an equivalent of unique analytic continuation. Two distinct  $J$  holomorphic maps  $u$  and  $v$  can agree on a non empty open subset of  $\mathbf{D}$ . Indeed, set  $u(z) = (z, 0)$ , and  $v(z) = (z, 0)$  if  $\text{Im } z \leq 0$ , but  $v(z) = (z, (\text{Im } z)^k)$  if  $\text{Im } z > 0$ . For  $u$  and  $v$  to be both  $J$ -holomorphic maps we simply need that  $[J(z, 0)](1, 0) = (i, 0)$ , and  $[J(x+iy, y^k)](1, 0) = (i, ky^{k-1})$ , if  $y > 0$ . It is immediate that such  $J$ 's of Hölder class  $\mathcal{C}^{\frac{k-1}{k}}$  can be defined. This failure of uniqueness is essentially linked to the failure of uniqueness for O.D.E. such as  $y' = |y|^\alpha$ ,  $\alpha < 1$ , for which the Lipschitz condition is not satisfied. The possibly surprising fact is that uniqueness holds when one of the maps is constant, at least for almost complex structures of class  $\mathcal{C}^{\frac{1}{2}}$ .

If  $J$  is close enough to the standard complex structure  $J_{st}$ ,  $J$ -holomorphic maps are characterized by an equation:

$$(E) \quad \frac{\partial u}{\partial \bar{z}} = \overline{Q(u)} \frac{\partial u}{\partial z}$$

where for  $p \in \mathbf{C}^n$ ,  $Q(p)$  is a  $\mathbf{C}$ -linear map,  $Q(p) = 0$  if  $J(p) = J_{st}$ , and  $Q$  has the same Hölder or  $\mathcal{C}^k$  regularity as  $J$ .

Equation (E) leads to inequalities, in particular:

(1) If  $Q(0) = 0$  and  $Q$  is Lipschitz continuous, (locally at least) one gets:

$$(IN1) \quad \left| \frac{\partial u}{\partial \bar{z}} \right| \leq C|u|$$

where by rescaling the size of  $C$  is irrelevant.

(2) If the operator norm of  $Q(\cdot)$  is  $\leq \epsilon$ ,

$$(IN2) \quad \left| \frac{\partial u}{\partial \bar{z}} \right| \leq \epsilon \left| \frac{\partial u}{\partial z} \right|$$

The size of  $\epsilon$  is important.

In almost complex analysis, it is interesting to know which properties follow from the equations and which one are merely consequences of the above inequalities. (IN1) is very easy to use and there is a summary in the Appendix. (IN2) is useful to get energy estimates (see Lemma 2.4.2 in [14], Remark 2 in 1.d in [9], 2.2 in [4]). Another differential inequality, on the Laplacian, is used in [12] page 44, also to get energy estimates. Contrary to (IN1), (IN2) apparently does not yield good uniqueness results, as Proposition 1.2 illustrates. A much strengthened version of (IN2) will be used for proving Proposition 1.1.

PROPOSITION 1.3. — *Let  $u : \mathbf{D} \rightarrow \mathbf{C}^n$  be a  $\mathcal{C}^1$  map such that for some  $K > 0$*

$$\left| \frac{\partial u}{\partial \bar{z}} \right| \leq K |u|^{\frac{1}{2}} \left| \frac{\partial u}{\partial z} \right|.$$

*If  $u$  vanishes on some non empty open subset of  $\mathbf{D}$ , then  $u \equiv 0$ .*

### Comments.

1) The problems of uniqueness

(a) vanishing on an open set implies vanishing

(b) vanishing to infinite order at a point implies vanishing on a neighborhood

are not strictly related since we are not dealing with smooth maps.

2) The results in this paper are only partial results. More questions are raised than solved, e.g. :

Can Proposition 1.1 be extended to all Hölder continuous almost complex structures?

In Proposition 1.1, can one replace vanishing on an open set by vanishing to infinite order at some point, or having a non-isolated zero? (For non-Hölderian structures, see an example in Remark 4.3).

And vice versa for Proposition 1.2.

The gap is huge between the positive result of Proposition 1.3 and the counterexample of Proposition 1.2, in which (with the notation of Proposition 1.2) we will have  $|\epsilon(2^{-n})| = O(\frac{1}{n})$ .

3) The proof of Proposition 1.1 will follow very closely the first, easier, steps of the proof of Theorem 17.2.1 in [6], and will conclude with the

arguments in 8.5 and 8.6 in [5], mentioned in [6] page 10. A special case of this Theorem, gives the following:

Let  $g$  be a  $C^1$  function defined on  $\mathbf{D}$ . Assume that  $\frac{\partial g}{\partial \bar{z}} + a(z)\frac{\partial g}{\partial z} = 0$ , where  $a$  is a  $C^1$ , with  $|a| < 1$ . If  $g$  vanishes on some non empty open subset of  $\mathbf{D}$ , then  $g \equiv 0$ .

For getting uniqueness for  $J$ -holomorphic discs there are very serious difficulties preventing us from adapting the result:

- The proof makes crucial use of the differentiability of  $a$  that we certainly do not have.
- We would need a generalization to vector valued maps  $g$ , with  $a(z)$  operator valued. The proof in [6] seems to be difficult to adapt if one does not assume  $a(z)$  to be a normal operator ( $a^*a = aa^*$ ). Is such a generalization true?
- And first, and possibly worst: equation (E) is not a  $\mathbf{C}$ -linear equation.

4) Aronszajn type Theorems, such as Theorem 17.2.6 in [6], that prove that vanishing to infinite order implies vanishing, are difficult and the situation is very delicate, as shown both by the fact that these results do not generalize to equations of order  $> 2$  and by the second order counterexamples provided by Alinhac [1]. However, in case of the differential inequality  $|\frac{\partial u}{\partial \bar{z}}| \leq C|u|$  (vector valued case), the proof of uniqueness under the hypothesis of vanishing to infinite order by using Carleman weights simplifies enormously, since one can take advantage of the commutation of  $\frac{\partial}{\partial \bar{z}}$  and multiplication by  $\frac{1}{z^N}$ , for introducing the weights  $\frac{1}{|z|^N}$ , at “no cost” (proving an estimate  $\int_D |\frac{\partial u}{\partial \bar{z}}|^2 \frac{1}{|z|^{2N}} \geq K \int_D |u|^2 \frac{1}{|z|^{2N}}$ , for  $u \in C_0^1(D)$ , with  $K > 0$  independent of  $N$ ), avoiding thus all the difficult part of the proof of Aronszajn’s Theorem.

## 2. Standard $\bar{\partial}$ estimates

*Notation 2.1.* — For the whole paper, we set  $\phi(z) = \phi(x) = x + \frac{x^2}{2}$ ,  $z = x + iy$ . So  $\phi(z) < 0$  if  $-1 \leq x < 0$ ,  $\phi$  is increasing on  $[-1, 1]$ , and  $\phi'' = 1$ .

Integration on  $\mathbf{D}$  will be simply denoted by  $\int$ , standing for  $\int_{\mathbf{D}}$ .

The estimate of Lemma 2.2 is completely standard and its proof is included only for the convenience of the reader. It is more usual to see the dual estimate used for solving  $\bar{\partial}$  requiring the opposite sign of the exponent in the weight function. For generalizations see [6], proof of Theorem 17.2.1. For an elementary introduction, see 4.2 in [7] 4.2 in Chapter IV.

**2.1.**

LEMMA 2.2. — For any  $C^1$  function (or  $\mathbf{C}^n$ -valued map)  $v$  with compact support in  $\mathbf{D}$ , and  $\tau \geq 1$

$$\int \left| \frac{\partial v}{\partial \bar{z}} \right|^2 e^{\tau\phi} \geq \frac{1}{10\tau} \int \left| \frac{\partial v}{\partial z} \right|^2 e^{\tau\phi} + \frac{\tau}{20} \int |v|^2 e^{\tau\phi}.$$

*Proof.* — For short set  $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$  and  $\partial = \frac{\partial}{\partial z}$ . Consider the Hilbert space of measurable functions  $f$  on  $\mathbf{D}$  such that  $\int |f|^2 e^{\tau\phi} < +\infty$ , with the scalar product  $\langle f, g \rangle_\phi = \int f \bar{g} e^{\tau\phi}$ .

Let  $\bar{\partial}^*$  be the adjoint of  $\bar{\partial}$  in that space. Elementary computations show that

$$\bar{\partial}^* v = -(\partial v + \frac{1}{2}\tau\phi'v).$$

One has

$$\begin{aligned} \int |\bar{\partial} v|^2 e^{\tau\phi} &= \langle \bar{\partial} v, \bar{\partial} v \rangle_\phi = \langle v, \bar{\partial}^* \bar{\partial} v \rangle_\phi \\ &= \langle v, \bar{\partial} \bar{\partial}^* v \rangle_\phi + \langle v, [\bar{\partial}^*, \bar{\partial}] v \rangle_\phi \\ &= \langle \bar{\partial}^* v, \bar{\partial}^* v \rangle_\phi + \langle v, [\bar{\partial}^*, \bar{\partial}] v \rangle_\phi. \end{aligned}$$

Another immediate computation gives  $[\bar{\partial}^*, \bar{\partial}]v = \frac{\tau}{4}\phi''v = \frac{\tau}{4}v$ . (This is where positivity comes, from the convexity of  $\phi$ ). Therefore

$$\int |\bar{\partial} v|^2 e^{\tau\phi} = \int |\bar{\partial}^* v|^2 e^{\tau\phi} + \frac{\tau}{4} \int |v|^2 e^{\tau\phi} \geq \frac{1}{5\tau} \int |\bar{\partial}^* v|^2 e^{\tau\phi} + \frac{\tau}{4} \int |v|^2 e^{\tau\phi}.$$

To finish the proof, it is enough to use, in the above inequality, the estimate from below (simply using  $(a+b)^2 \geq \frac{a^2}{2} - b^2$ ):

$$|\bar{\partial}^* v|^2 = |\partial v + \frac{1}{2}\tau\phi'v|^2 \geq \frac{1}{2}|\partial v|^2 - \frac{1}{4}|\tau\phi'v|^2 \geq \frac{1}{2}|\partial v|^2 - \tau^2|v|^2,$$

since  $0 \leq \phi' \leq 2$ . □

**2.2.**

The next Lemma is narrowly tailored to the application in view. The hypotheses are made just ad-hoc and we state the conclusion just as we shall need it, dropping an  $\frac{1}{\tau} \int |\nabla u|^2$  term on the right hand side, that we could keep.

LEMMA 2.3. — *Let  $v$  be a  $C^1$  map from  $\mathbf{D}$  to  $\mathbf{C}^n$ , with compact support in  $\mathbf{D}$ ,  $|v| \leq 1$ . If  $w$  is a measurable function on  $\mathbf{D}$  satisfying  $|w(z)| \leq \theta|v|^{\frac{1}{2}} \text{Min}(|\frac{\partial v}{\partial z}|, 1)$ , with  $\theta = \frac{1}{10}$ , then, for any  $\tau \geq 1$ :*

$$\int \left| \frac{\partial v}{\partial \bar{z}} - w(z) \right|^2 e^{\tau\phi} \geq \frac{\tau}{80} \int |v|^2 e^{\tau\phi} .$$

*Proof.* — We have

$$\begin{aligned} \int \left| \frac{\partial v}{\partial \bar{z}} - w(z) \right|^2 e^{\tau\phi} &\geq \frac{1}{2} \int \left| \frac{\partial v}{\partial \bar{z}} \right|^2 e^{\tau\phi} - \int |w|^2 e^{\tau\phi} \\ &\geq \frac{1}{20\tau} \int \left| \frac{\partial v}{\partial z} \right|^2 e^{\tau\phi} + \frac{\tau}{40} \int |v|^2 e^{\tau\phi} - \theta^2 \int |v| \left[ \text{Min} \left( \left| \frac{\partial v}{\partial z} \right|, 1 \right) \right]^2 e^{\tau\phi} . \end{aligned}$$

The Lemma will be established if we have the (much better than needed) point-wise estimate

$$\theta^2 |v| \left[ \text{Min} \left( \left| \frac{\partial v}{\partial z} \right|, 1 \right) \right]^2 \leq \frac{1}{20\tau} \left| \frac{\partial v}{\partial z} \right|^2 + \frac{\tau}{80} |v|^2 .$$

At points where  $|v| \leq \frac{1}{\tau}$ , the estimate is trivial. At points where  $|v| \geq \frac{1}{\tau}$ , we simply use  $\theta^2 |v| \left[ \text{Min} \left( \left| \frac{\partial v}{\partial z} \right|, 1 \right) \right]^2 \leq \theta^2 |v|$ . We have  $\theta^2 |v| = \theta^2 |v|^{-1} |v|^2 \leq \theta^2 \tau |v|^2 \leq \frac{1}{80} \tau |v|^2$ . □

Remark 2.4. — The proof of Lemma 2.3 ends with a simple pointwise estimate, because we have no regularity assumptions on  $w$  (so, no helpful integration by parts seems to be possible). The Hölder exponent  $\frac{1}{2}$  in Proposition 1.1, will lead below to the consideration of a perturbation term  $w(z)$  satisfying the hypotheses of Lemma 2.3. That exponent  $\frac{1}{2}$  seems to be the limit of what our approach can reach. In Lemma 2.2, a big loss was taken. At some point, we wrote  $\int |\bar{\partial}^* v|^2 e^{\tau\phi} \geq \frac{1}{5\tau} \int |\bar{\partial}^* v|^2 e^{\tau\phi}$ . This is just to say that we could add  $\frac{1}{2} \int |\bar{\partial}^* v|^2 e^{\tau\phi}$ , to the right hand side of the inequality in Lemma 2.2. However, as we shall see, this would not solve the difficulty that we now explain. If we now consider an almost complex structure of class  $C^\alpha$ , we are led to consider a perturbation term  $w$  of the size of  $|v|^\alpha \left| \frac{\partial v}{\partial z} \right|$ . If, for simplicity, we assume  $\left| \frac{\partial v}{\partial z} \right| \leq 1$ , for finishing the proof of Lemma 2.3, under the hypothesis  $|w| \leq \theta |v|^\alpha \left| \frac{\partial v}{\partial \bar{z}} \right|$ , we would need an inequality of the type

$$\left| \frac{\partial v}{\partial z} + \tau \phi' v \right|^2 + \frac{1}{\tau} \left| \frac{\partial v}{\partial z} \right|^2 + \tau |v|^2 \geq C |v|^{2\alpha} \left| \frac{\partial v}{\partial z} \right|^2 ,$$

where  $C > 0$  should be independent of  $\tau$ , and the first term on the left hand side is to try to get advantage from the term previously dropped (that makes the second term superfluous, as it has been seen in the proof of Lemma 2.2). Suppose that at some point  $z$ , with  $|z| < 1$  and  $\text{Re } z \geq -\frac{1}{2}$  (so  $\frac{1}{2} \leq \phi' \leq 2$ ), we have:  $\frac{\partial v}{\partial z} + \tau \phi' v = 0$ , and  $|v(z)| = \frac{\epsilon}{\tau}$ , with  $0 < \epsilon < \frac{1}{2}$

(so  $|\frac{\partial v}{\partial \bar{z}}(z)| \leq 2\epsilon < 1$ ). Then, the left hand side, in the above inequality, is  $\tau(1 + \phi'^2)|v|^2 \leq 5\tau|v|^2 = \frac{5\epsilon^2}{\tau}$ , while the right hand side is  $\geq \frac{C\tau^2}{4}|v|^{2+2\alpha} = \frac{C\epsilon^{2+2\alpha}}{4} \frac{1}{\tau^{2\alpha}}$ . For  $\tau$  large, the inequality will not hold if  $\alpha < \frac{1}{2}$ .

### 3. Proof of Propositions 1.3 and 1.1

Proposition 1.1 is an immediate consequence of Proposition 1.3, since by a linear change of variable in  $\mathbf{R}^{2n}$ , we can assume that  $J(0) = J_{st}$ , thus  $Q(0) = 0$  and  $|Q(p)| \leq K|p|^{\frac{1}{2}}$ . So we now turn to the Proof of Proposition 1.3.

#### 3.1. Reduction to Lemma 3.1

Let  $\omega \subset \mathbf{D}$  be the set of  $z \in \mathbf{D}$  such that  $u \equiv 0$  on a neighborhood of  $z$ . We need to show that its boundary  $b\omega$  is empty. If it is not empty, we can choose  $\zeta_0 \in \omega$  such that  $\text{dist}(\zeta_0, b\omega) = r < 1 - |\zeta_0|$ , and let  $\zeta_1 \in b\omega$  such that  $|\zeta_1 - \zeta_0| = r$ . So, on the disc defined by  $|\zeta - \zeta_0| \leq r$ ,  $u = 0$ . We wish to prove that  $u \equiv 0$  near  $\zeta_1$ , getting thus a contradiction.

Note that the hypotheses of Proposition 1.3 are preserved under holomorphic change of variable for  $z$ . Using a conformal map from a neighborhood of  $\zeta_1$  that maps  $\zeta_1$  to 0 and that maps the intersection of disc  $\{|\zeta - \zeta_0| < r\}$  with a neighborhood of  $\zeta_1$  to a region defined near 0 by  $x > -y^2$ , and by rescaling, the proof of Proposition 1.3 reduces to proving the following:

LEMMA 3.1. — *Let  $A$  and  $\theta > 0$  and let  $u : \mathbf{D} \rightarrow \mathbf{C}^n$  be a  $C^1$  map such that*

$$u(x + iy) = 0, \text{ if } x \geq -Ay^2, \\ \left| \frac{\partial u}{\partial \bar{z}} \right| \leq \theta |u|^{\frac{1}{2}} \left| \frac{\partial u}{\partial z} \right|.$$

*Then  $u \equiv 0$  near 0.*

#### 3.2.

Further reduction. By replacing  $u$  by  $Ku$  for  $K$  large enough, we can assume that  $\theta$  is as small as we wish. In order to apply Lemma 2.3, we take  $\theta = \frac{1}{10}$ . Next (seemingly contradictory to the previous step), we can also assume that  $|u| \leq 1$  and  $|\frac{\partial u}{\partial \bar{z}}| \leq 1$ . This can be achieved by rescaling, replacing the function  $z \mapsto u(z)$ , by the function  $z \mapsto u(\epsilon z)$ , for  $\epsilon > 0$  small enough. This changes the constant  $A$  in the Lemma. We shall therefore use  $\theta = \frac{1}{10}$ ,  $|u| \leq 1$  and  $|\frac{\partial u}{\partial \bar{z}}| \leq 1$ .

### 3.3. Proof of Lemma 3.1

This is the standard game of Carleman's estimates.

In order to apply the  $\bar{\partial}$  estimates of section 2, we need compact support. Let  $\chi \in C^\infty(\mathbf{D})$  be such that  $0 \leq \chi \leq 1$ ,  $\chi(x + iy) = 1$  if  $x > -\alpha$ , and  $\chi(x + iy) = 0$  if  $x < -2\alpha$ , where  $\alpha > 0$  is chosen small enough so that the region defined by  $x < -Ay^2$  and  $x > -2\alpha$  is relatively compact in  $\mathbf{D}$ .

Set  $v = \chi u$ , so  $v$  is a compactly supported map from  $\mathbf{D}$  into  $\mathbf{C}^n$ . We can apply the estimate of Lemma 2.3, with  $w(z) = \frac{\partial u}{\partial \bar{z}}$  if  $x \geq -\alpha$  ( $z = x + iy$ ), and  $w(z) = 0$  if  $x < -\alpha$ . For all  $\tau \geq 1$ :

$$\int \left| \frac{\partial}{\partial \bar{z}}(\chi u)(z) - w(z) \right|^2 e^{\tau\phi} \geq \frac{\tau}{40} \int |\chi u|^2 e^{\tau\phi}.$$

However:

The integrand on the left hand side is zero for  $x > -\alpha$ . So the left hand side is at most  $O(e^{\tau(-\alpha + \frac{\alpha^2}{2})})$ . If  $u(x_0, y) \neq 0$  for some  $x_0 > -\alpha$ , it is immediate to see that (for  $\tau$  large)  $\frac{\tau}{80} \int |\chi u|^2 e^{\tau\phi} > e^{\tau(x_0 + \frac{x_0^2}{2})}$ . Letting  $\tau \rightarrow +\infty$  gives us a contradiction since  $x_0 + \frac{x_0^2}{2} > -\alpha + \frac{\alpha^2}{2}$ .

## 4. Examples (proving Proposition 1.2)

*Example 4.1.* — A smooth map  $z \mapsto u(z) = (u_1(z), u_2(z))$  from a neighborhood of 0 in  $\mathbf{C}$  into  $\mathbf{C}^2$ , such that

$$\frac{|\bar{\partial}u|}{|\partial u|} \rightarrow 0, \text{ as } z \rightarrow 0,$$

$u$  vanishes to infinite order at 0, but is not identically 0 near 0.

Moreover one can adapt the construction so that  $u$  has a non-isolated zero at 0.

### 4.1.

We first give an example without getting a non-isolated zero.

1) If  $n$  is an even positive integer, for  $2^{-n} \leq |z| \leq 2^{-n+1}$ , set

$$u_1(z) = 2^{\frac{n^2}{2}} z^n$$

$$u_2(z) = \chi(z) 2^{\frac{(n-1)^2}{2}} z^{n-1} + (1 - \chi(z)) 2^{\frac{(n+1)^2}{2}} z^{n+1},$$

where  $\chi = 1$  near  $|z| = 2^{-n+1}$ ,  $\chi = 0$  near  $|z| = 2^{-n}$ ,  $|d\chi| = O(2^n)$ , and more generally the  $C^k$  norm of  $\chi$  is  $O(2^{kn})$ .

2) For  $n$  odd, take the same definitions switching  $u_1$  and  $u_2$ .

*Claims.*

- (a)  $u$  is a smooth map vanishing to infinite order at 0,
- (b) For  $2^{-n} \leq |z| \leq 2^{-n+1}$

$$|\bar{\partial}u| \leq \frac{C}{n} |\partial u|,$$

with  $C$  independent on  $n$ .

(a) is straightforward. Note that for  $|z| = K2^{-n}$  (think  $\frac{1}{2} \leq K \leq 2$ ),  $2^{\frac{n^2}{2}} |z^n| = 2^{\frac{-n^2}{2}} K^n \leq 2^{\frac{-n^2}{3}}$ .

We now check (b), for  $n$  even. For  $n$  odd, the checking is the same with  $u_1$  and  $u_2$  switched.

We have  $|\partial u| \geq |\partial u_1| = n2^{\frac{n^2}{2}} |z|^{n-1}$ . The factor  $n$  will be the needed gain.

$\bar{\partial}u_1 = 0$ . So we only have to estimate  $\bar{\partial}u_2$ , in which the non zero terms come from differentiating  $\chi$  whose gradient is of the order of  $2^n$ . One gets (with various constants  $C$ ):

$$\begin{aligned} (*) \quad |\bar{\partial}u_2| &\leq C(2^n 2^{\frac{(n-1)^2}{2}} |z|^{n-1} + 2^n 2^{\frac{(n+1)^2}{2}} |z|^{n+1}) \\ &\leq C |z|^{n-1} (2^{\frac{n^2}{2}} + 2^{\frac{n^2}{2}+2n} |z|^2). \end{aligned}$$

Since  $|z| \simeq 2^{-n}$ , one indeed gets

$$|\bar{\partial}u| \leq C2^{\frac{n^2}{2}} |z|^{n-1} \text{ so } |\bar{\partial}u| \leq \frac{C}{n} |\partial u|.$$

*Remark 4.2.* — The example is not difficult but the matter looks delicate. In particular, the choice of the exponent  $\frac{n^2}{2}$  seems to be somewhat dictated. If in the above definition of  $u_1$  we would set  $u_1(z) = 2^{\frac{n^2}{p}} z^n$  instead of  $u_1(z) = 2^{\frac{n^2}{2}} z^n$ , and do the corresponding change in the definition of  $u_2$ , we should take  $p > 1$  for having decay. Then, for the estimate of the first term on the right hand side in (\*), we would need  $p \leq 2$ . But for the estimate of the second term we would need  $p \geq 2$ .

## 4.2.

We now indicate how to get a non-isolated zero.

Set

$$2^{-n} < r_n = \frac{5}{4}2^{-n} < a_n = \frac{3}{2}2^{-n} < R_n = \frac{7}{4}2^{-n} < 2^{-n+1}.$$

We modify the definition of  $u_1$  and  $u_2$  (resp  $u_2$  and  $u_1$ , if  $n$  is odd) above by setting:

$$u_1(z) = 2^{\frac{n^2}{2}} z^{n-1} (z - a_n)$$

i.e., replacing a factor  $z$  by  $(z - a_n)$ , and accordingly

$$u_2(z) = \chi(z) 2^{\frac{(n-1)^2}{2}} z^{n-2} (z - a_{n-1}) + \psi(z) 2^{\frac{(n+1)^2}{2}} z^n (z - a_{n+1}),$$

where  $\chi = 1$  near  $|z| = 2^{-n+1}$  and  $\chi(z) = 0$  if  $|z| < R_n$ ,  $\psi = 1$  near  $|z| = 2^{-n}$  and  $\psi(z) = 0$  if  $|z| > r_n$ , with estimates on the derivatives, as before. Note that for  $r_n < |z| < R_n$ ,  $u_2(z) = 0$ , (in particular  $u_2(a_n) = 0$ ).

In the region  $r_n < |z| < R_n$ , both  $u_1$  and  $u_2$  are holomorphic. So the differential inequalities have to be checked only in the regions  $2^{-n} < |z| < r_n$  and  $R_n < |z| < 2^{-n+1}$ . In these regions  $|z|$ ,  $|z - a_n|$ ,  $|z - a_{n+1}|$  and  $|z - a_{n-1}|$  all have the same order of magnitude.

The estimate for  $\bar{\partial}u_2$  is basically unchanged: gradient estimates for  $\chi$  and  $\psi$  and point-wise estimates of  $z^{n-2}(z - a_{n-1})$  and  $z^n(z - a_{n+1})$  instead of  $z^{n-1}$  and  $z^{n+1}$ .

Finally one has to estimate  $\partial u_1 = 2^{\frac{n^2}{2}} ((n-1)z^{n-2}(z - a_n) + z^{n-1})$ . For  $n$  large, in the regions under consideration the first term in the parenthesis dominates the second one ( $|z - a_n| \geq \frac{1}{8}|z|$ ), and one has  $|\partial u_1| \geq \frac{n}{10} 2^{\frac{n^2}{2}} |z|^{n-1}$ .

So, as previously

$$|\bar{\partial}u| \leq \frac{C}{n} |\partial u|,$$

with  $C$  independent on  $n$ .

We have  $u_1(a_n) = u_2(a_n) = 0$ .

*Remark 4.3.* — Almost complex structures that are merely continuous and not Hölder continuous are certainly of much less interest. However for these merely continuous almost complex structures, one can still prove the existence of many  $J$ -holomorphic curves, see 5.1 in [8]. The above example, by adding four real dimensions, allows one to find a continuous almost complex structure  $J$  on  $\mathbf{C}^4$ , such that there is a smooth  $J$ -holomorphic map  $U : \mathbf{D} \rightarrow (\mathbf{C}^4, J)$  non identically 0 near 0, but vanishing to infinite order at 0, and with a non-isolated zero. We are still very far from an example with a Hölder continuous almost complex structure, as asked in [10].

We set  $U(z) = (u(z), u_3(z), u_4(z))$ , where  $u = (u_1, u_2)$  is the map of the above example, with  $u(a_n) = 0$ . Looking at the proof, we see that  $\partial u(z) \neq 0$  unless  $z = 0$  or,  $z = \frac{n-1}{n}a_n$ , for some  $n$ , and that  $\bar{\partial}u = 0$  near  $\frac{n-1}{n}a_n$ . For each  $n$ , let  $\psi_n$  be a non negative smooth function with support in a small neighborhood of  $\{2^{-n} \leq |z| \leq 2^{-n+1}\}$ , such that:  $\psi_n(z) = |z - a_n|^2$  near  $a_n$ ,  $\psi_n(z) > 0$  if  $2^{-n} \leq |z| \leq 2^{-n+1}$  and  $z \neq a_n$ ,  $\psi_n$  is constant on a neighborhood of  $\frac{n-1}{n}a_n$ . Note that  $\bar{\partial}\psi_n(a_n) = 0$ , and  $\bar{\partial}\psi_n = 0$  near  $\frac{n-1}{n}a_n$ . For  $\epsilon_n$  small enough, set  $u_3 = \sum_n \epsilon_n \psi_n$ . Then  $u_3$  is a smooth function that vanishes only at the points  $a_n$  and at 0, such that  $\bar{\partial}u_3 = 0$  at  $a_n$  and near  $\frac{n-1}{n}a_n$ , and such that  $\frac{|u_3| + |\nabla u_3|}{|\partial u|} \rightarrow 0$ , as  $z \rightarrow 0$ . Finally one sets  $u_4(z) = zu_3(z)$ . Note the following injectivity property:  $(u_3(z), u_4(z)) = (u_3(z'), u_4(z'))$  if and only if  $z = z'$ , unless  $z = a_n$  for some  $n$  or  $z = 0$ . One can define the almost complex structure on  $\mathbf{C}^4$  by defining the matrix  $Q$  in  $(E)$ . We set  $Q(0) = 0$  and we need  $[Q(u_1(z), u_2(z), u_3(z), u_4(z))](\partial U) = \bar{\partial}U$ . Due to the injectivity property of  $U$  and the vanishing of  $\bar{\partial}U$  near the points  $\frac{n-1}{n}a_n$  where  $\partial u = 0$ , and at the points  $a_n$  where  $U = 0$ , the above requirement on  $Q$  is compatible with the requirement of continuity and  $Q(0) = 0$ , since  $\frac{|\bar{\partial}U|}{|\partial u|}(z) \rightarrow 0$  as  $z \rightarrow 0$ .

## 5. Appendix

### 5.1.

The inequality (IN1) has been used by many authors ([3] and [14] Lemma 3.2.4 – see also [4], [9], [12], [13]). Given a bounded matrix valued function  $z \mapsto A(z)$ , where  $A(z)$  is a  $n \times n$  matrix, one can solve the equation  $\frac{\partial M}{\partial \bar{z}} + AM = 0$  with solution  $M(z)$  an invertible matrix. Locally this is easily obtained by the inverse function Theorem, for a global result see [11] (or the Appendix p. 61 in [13]). If (IN1) is satisfied, there exists a bounded matrix valued map  $z \mapsto A(z)$  (depending on  $u$ ) such that  $\frac{\partial u}{\partial \bar{z}} + A(z)u = 0$ . Define  $v$  by  $u = Mv$ . The above equation yields  $\frac{\partial v}{\partial \bar{z}} = 0$ , so  $v$  is holomorphic, and the zero set of  $u$  is the same as the zero set of the holomorphic vector valued function  $v$ . Globally on  $\mathbf{D}$ , that will give the Blaschke condition for the zero set of  $J$ -holomorphic maps, if  $J$  is  $C^1$ , and if  $|\nabla u|$  is bounded. See more on that topic in [8].

## 5.2.

The scalar case of (IN2) is  $|\frac{\partial u}{\partial \bar{z}}| \leq a |\frac{\partial u}{\partial z}|$  with  $u$  scalar valued, and we should take  $0 \leq a < 1$ . For each function  $u$ , satisfying the inequality, there is a bounded measurable (not continuous) function  $\alpha$ , with  $|\alpha(z)| \leq a < 1$ , such that  $u$  satisfies the Beltrami equation  $\frac{\partial u}{\partial \bar{z}} = \alpha(z) \frac{\partial u}{\partial z}$ . Beltrami equations have been much studied (Bers, Bers-Nirenberg, Morrey, Vekua, ...). A very convenient reference is [2]. It is shown in [2], Theorems 3.1 and 3.2, that, given  $\alpha$  (in our case, not given a priori but associated to  $u$ ), with  $|\alpha| \leq a < 1$ , there exists a Hölder continuous change of variables  $\psi$  such that a function  $v$  satisfies the Beltrami equation  $\frac{\partial v}{\partial \bar{z}} = \alpha(z) \frac{\partial v}{\partial z}$  if and only if  $v \circ \psi$  is holomorphic. So there is again reduction to the holomorphic case, by composition on the right rather than by composition on the left as in 5.1. However, this reduction is only in the scalar case and it is much more difficult. Note that, even if we started from the point of view of  $J$ -holomorphicity  $\frac{\partial u}{\partial \bar{z}} = \beta(u) \frac{\partial u}{\partial z}$ , we switched here to the quasi-conformal point of view  $\frac{\partial v}{\partial \bar{z}} = \alpha(z) \frac{\partial v}{\partial z}$ . This, in the theory of almost complex structures on  $\mathbf{C}$ , corresponds to studying maps from  $(\mathbf{C}, J_{st})$  to  $(\mathbf{C}, J)$ , or vice versa. And it corresponds to the non-linear and linear approaches to the Theorem of Newlander-Nirenberg.

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Manuscrit reçu le 12 octobre 2009,  
accepté le 5 février 2010.

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