



ANNALES

DE

L'INSTITUT FOURIER

An DE RIJDT & Nikolas VANDER VENNET

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Tome 60, n° 1 (2010), p. 169-216.

http://aif.cedram.org/item?id=AIF_2010__60_1_169_0

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ACTIONS OF MONOIDALLY EQUIVALENT COMPACT QUANTUM GROUPS AND APPLICATIONS TO PROBABILISTIC BOUNDARIES

by An DE RIJDT & Nikolas VANDER VENNET

ABSTRACT. — The notion of monoidal equivalence for compact quantum groups was recently introduced by Bichon, De Rijdt and Vaes. In this paper we prove that there is a natural bijective correspondence between actions of monoidally equivalent quantum groups on unital C^* -algebras or on von Neumann algebras. This correspondence turns out to be very useful to obtain the behavior of Poisson and Martin boundaries under monoidal equivalence of quantum groups. Finally, we apply these results to identify the Poisson boundary for the duals of quantum automorphism groups.

RÉSUMÉ. — La notion de l'équivalence monoïdale pour les groupes quantiques compacts a été introduite récemment par Bichon, De Rijdt et Vaes. Dans cet article, nous montrons : étant donné deux groupes quantiques compacts à équivalence monoïdale, alors il existe une correspondance bijective entre leurs actions. Cette correspondance s'avère être très utile pour obtenir la relation entre les frontières de Poisson et Martin des deux groupes quantiques compacts à équivalence monoïdale. Finalement, nous appliquons ces résultats au calcul des frontières de Poisson des duals associés aux groupes quantiques d'automorphismes.

Introduction

After Woronowicz had introduced the notion of a compact quantum group as a generalization of a compact group, many research topics applying to compact groups were expanded to the general framework of compact quantum groups. One of these topics concerns the study of (ergodic) actions of compact groups on unital C^* -algebras (an action on a C^* -algebra is ergodic if the fixed point algebra reduces to the scalars). We refer to the articles of Høegh-Krohn, Landstad and Størmer [12] and Wasserman

Keywords: Quantum groups, operator algebras, probability theory.

Math. classification: 20G42.

[35, 33, 34] for a deep study of this topic. The abstract theory of *ergodic actions* of compact quantum groups on C^* -algebras was initiated by Boca [8] and Landstad [17]. It turns out that the general theory of (ergodic) actions of compact quantum groups on C^* -algebras is different from the classical theory and in fact much richer. One major difference is that the multiplicity of irreducible representations in an ergodic action can be strictly greater than the dimension of the representation space, which is impossible in the classical case, where the dimension of the representation space is actually an upper bound of this multiplicity. In the quantum case, the upper bound is given by the quantum dimension which is usually larger than the usual dimension.

In [7], Bichon, the first author and Vaes introduced and developed the notion of monoidally equivalent quantum groups. By definition, compact quantum groups are called monoidally equivalent if their representation categories are equivalent as monoidal categories. In their article, they were able to describe certain ergodic actions as unitary fiber functors on the representation category. These ergodic actions are exactly the ergodic actions of full quantum multiplicity. This provides us with a powerful categorical tool for constructing ergodic actions. Moreover, these ergodic actions of full quantum multiplicity provided the first examples of ergodic actions where the multiplicity of the irreducible representations is strictly greater than the dimension of the representation space.

In, [20], Pinzari and Roberts obtained a categorical description of all ergodic actions of a compact quantum group. Inspired by [7], they describe an ergodic action (not necessarily of full quantum multiplicity) of a compact quantum group as a special kind of functor on the representation category. When the ergodic action is of full quantum multiplicity, the corresponding functor is just a unitary fiber functor as in [7]. This categorical description yields a bijective correspondence between ergodic actions of monoidally equivalent quantum groups on unital C^* -algebras.

In this article, we obtain a bijective correspondence between (not necessarily ergodic) actions of monoidally equivalent compact quantum groups on unital C^* -algebras. Moreover, the correspondence is of that kind that it preserves the spectral subspaces of the actions. Restricting this to ergodic actions, this just means that the multiplicities of the irreducible representations are preserved through this correspondence. It should be emphasized that our approach is not categorical. The correspondence is obtained in a concrete, constructive way.

A major application of the bijective correspondence between actions of monoidally equivalent quantum groups is found in the study of Poisson and Martin boundaries for discrete quantum groups. These boundaries find their origin in the study of random walks on discrete groups. A nice survey can be found in [15]. The study of random walks of discrete quantum groups was started by Biane, who considered duals of compact groups and obtained a theory which was parallel to the theory of random walks on discrete abelian groups. Random walks on arbitrary discrete quantum groups, and their Poisson boundaries were introduced by Izumi in [13], whose main motivation came from the study of infinite tensor product actions of compact quantum groups. In [13], Izumi identified the Poisson boundary of the dual of $SU_q(2)$ with the Podleś sphere [21]. Later, Neshveyev and Tuset [19], associated a Martin boundary to a random walk on a discrete quantum group and proved that the Martin boundary of the dual of $SU_q(2)$ is also given by the Podleś sphere. In [14], Izumi, Neshveyev and Tuset identified the Poisson boundary of $SU_q(n)$ but its Martin boundary remains unknown.

Very recently, Tomatsu managed to identify the Poisson boundaries of all amenable discrete quantum groups $\widehat{\mathbb{G}}$ when its underlying compact quantum group \mathbb{G} has commutative fusion rules. This Poisson boundary appears to be the homogeneous space of \mathbb{G} with respect to the maximal closed quantum subgroup of Kac type [24]. This covers the results of Izumi, Neshveyev and Tuset, but is obtained in a completely different way and is also much more general. For Martin boundaries, there are no such results.

Considering the non-amenable case, Vaes and the second author [28] have identified the Poisson and Martin boundary of the class of universal orthogonal quantum groups $A_o(F)$ with higher dimensional Podleś spheres. When $\dim(F) \geq 3$, the dual of $A_o(F)$ is not amenable.

We prove in this article a very general result. We provide a systematic method to relate Poisson and Martin boundaries for the duals of monoidally equivalent quantum groups. The relation goes as follows. Both the Poisson and Martin boundary of a discrete quantum group $\widehat{\mathbb{G}}$, which is the dual of the compact quantum group \mathbb{G} , admit a natural action of \mathbb{G} . If the compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 are monoidally equivalent, the boundaries of their duals $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$ are related through the bijective correspondence we obtained between the actions of \mathbb{G}_1 and \mathbb{G}_2 . This means that if we know the Poisson (Martin) boundary of the dual of a compact quantum group \mathbb{G} , we at the same time know it for the duals of all compact quantum groups which are monoidally equivalent with \mathbb{G} .

Combining our result with Tomatsu's work, we give a concrete identification of the Poisson boundary of a large class of discrete quantum groups. This method makes it also possible to obtain more examples of identifications of Poisson boundaries of non-amenable discrete quantum groups. The main observation is that amenability is not preserved under monoidal equivalence. A first class of examples of this kind are the universal orthogonal quantum groups $A_o(F)$. If the dimension of F is greater than 3, then $A_o(F)$ is not coamenable. The quantum groups $A_o(F)$ and $SU_q(2)$ are monoidally equivalent for the right q . Moreover, the Poisson boundary of $SU_q(2)$ was identified by Izumi and also, in a different way, by Tomatsu ($SU_q(2)$ is coamenable). The correspondence just described gives a concrete identification of the Poisson boundary of $A_o(F)$. As we already saw, this result was already obtained by Vaes and the second author by another method [28]. These were the first examples of identifications of Poisson boundaries of non-amenable discrete quantum groups.

A second and new class of examples of the above type come from quantum automorphism groups $A_{\text{aut}}(D, \omega)$, with D a finite dimensional C^* -algebra. These quantum groups have the fusion rules of $SO(3)$ and are coamenable if and only if the dimension of the C^* -algebra is less than or equal to 4. We prove, in that case that the maximal subgroup of Kac type is the one-dimensional torus \mathbb{T} . In combination with the result of Tomatsu, this provides us with an identification of its Poisson boundary. Using the fact that every quantum automorphism group is monoidally equivalent with a coamenable one, we obtain also an explicit identification of the Poisson boundary of the duals of all such quantum automorphism groups.

Because every $A_o(F)$ is monoidally equivalent with an $SU_q(2)$, the correspondence of Martin boundaries under monoidally equivalent quantum groups gives a direct method to identify the Martin boundary of the duals of the universal compact quantum groups $A_o(F)$. This identification was already obtained by Vaes and the second author in [28] by a different method using a result of [29], allowing to deduce the Martin boundary, in the case of $A_o(F)$, from the Poisson boundary.

Finally, we would like to thank Stefaan Vaes for the numerous remarks and careful reading of the manuscript.

1. Notations

Consider a subset S of a C^* -algebra. We denote by $\langle S \rangle$ the linear span of S and by $[S]$ the closed linear span of S . We use the notation $\omega_{\eta, \xi}(a) =$

$\langle \eta, a\xi \rangle$ and we use inner products that are linear in the second variable. Moreover we denote by $\xi^*: H \rightarrow \mathbb{C}: \eta \mapsto \langle \xi, \eta \rangle$ and denote by \overline{H} the dual Hilbert space of H , i.e. $\overline{H} := \{\xi^* \mid \xi \in H\}$.

The symbol \otimes denotes *tensor products* of Hilbert spaces and *minimal* tensor products of C^* -algebras. We use the symbol \otimes_{alg} for algebraic tensor products of $*$ -algebras and $\overline{\otimes}$ for the tensor product of von Neumann algebras. We also make use of the leg numbering notation in multiple tensor products: if $a \in A \otimes A$, then a_{12}, a_{13}, a_{23} denote the obvious elements in $A \otimes A \otimes A$, e.g. $a_{12} = a \otimes 1$.

The adjointable operators between C^* -modules or bounded operators between Hilbert-spaces H and K are denoted by $\mathcal{L}(H, K)$. We also denote $\mathcal{L}(K, K)$ by $\mathcal{L}(K)$.

Let \mathcal{B} be a unital $*$ -algebra. We call a linear map $\omega: \mathcal{B} \rightarrow \mathbb{C}$ such that $\omega(1) = 1$ a *faithful state* if $\omega(a^*a) \geq 0$ for all $a \in \mathcal{B}$ and $\omega(a^*a) = 0$ if and only if $a = 0$.

2. Preliminaries

Compact quantum groups

We give a quick overview of the theory of compact quantum groups which was developed by Woronowicz in [39]. We refer to [18] for a survey of basic results.

DEFINITION 2.1. — A compact quantum group \mathbb{G} is defined by a pair $(C(\mathbb{G}), \Delta)$, where

- $C(\mathbb{G})$ is a unital C^* -algebra;
- $\Delta: C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$ -homomorphism satisfying the co-associativity relation

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta;$$

- \mathbb{G} satisfies the left and right cancellation property expressed by $\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))$ and $\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)$ are total in $C(\mathbb{G}) \otimes C(\mathbb{G})$.

Remark 2.2. — The notation $C(\mathbb{G})$ suggests the analogy with the basic example given by continuous functions on a compact group. In the quantum case however, there is no underlying space \mathbb{G} and $C(\mathbb{G})$ is a non-abelian C^* -algebra.

A fundamental result in the theory of compact quantum groups is the existence of a unique Haar state.

THEOREM 2.3 (Woronowicz, [36]). — *Let \mathbb{G} be a compact quantum group. There exists a unique state h on $C(\mathbb{G})$ which satisfies $(\text{id} \otimes h)\Delta(a) = h(a)1 = (h \otimes \text{id})\Delta(a)$ for all $a \in C(\mathbb{G})$. The state h is called the Haar state of \mathbb{G} .*

Another crucial set of results in the framework of compact quantum groups is the Peter-Weyl representation theory.

DEFINITION 2.4. — *A unitary representation U of a compact quantum group \mathbb{G} on a Hilbert space H is a unitary element $U \in \mathcal{L}(H \otimes C(\mathbb{G}))$ satisfying*

$$(2.1) \quad (\text{id} \otimes \Delta)(U) = U_{12}U_{13}.$$

Whenever U^1 and U^2 are unitary representations of \mathbb{G} on the respective Hilbert spaces H_1 and H_2 , we define

$$\text{Mor}(U^1, U^2) := \{T \in \mathcal{L}(H_2, H_1) \mid U_1(T \otimes 1) = (T \otimes 1)U_2\}.$$

The elements of $\text{Mor}(U^1, U^2)$ are called intertwiners. We use the notation $\text{End}(U) := \text{Mor}(U, U)$. A unitary representation U is said to be irreducible if $\text{End}(U) = \mathbb{C}1$. If $\text{Mor}(U^1, U^2)$ contains a unitary operator, the representations U^1 and U^2 are said to be unitarily equivalent.

We have the following essential result.

THEOREM 2.5. — *Every irreducible representation of a compact quantum group is finite-dimensional. Every unitary representation is unitarily equivalent to a direct sum of irreducibles.*

Because of this theorem, we almost exclusively deal with finite-dimensional representations. By choosing an orthonormal basis of the Hilbert space H , a finite-dimensional unitary representation of \mathbb{G} can be considered as a unitary matrix (U_{ij}) with entries in $C(\mathbb{G})$ and (2.1) becomes

$$\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}.$$

The product in the C^* -algebra $C(\mathbb{G})$ yields a tensor product on the level of unitary representations.

DEFINITION 2.6. — *Let U^1 and U^2 be unitary representations of \mathbb{G} on the respective Hilbert spaces H_1 and H_2 . We define the tensor product*

$$U^1 \otimes U^2 := U_{13}^1 U_{23}^2 \in \mathcal{L}(H_1 \otimes H_2 \otimes C(\mathbb{G})).$$

Notation 2.7. — Let \mathbb{G} be a compact quantum group. We denote by $\text{Irred}(\mathbb{G})$ the set of equivalence classes of irreducible unitary representations. We choose representatives U^x on the Hilbert space H_x for every $x \in \text{Irred}(\mathbb{G})$. Whenever $x, y \in \text{Irred}(\mathbb{G})$, we use $x \otimes y$ to denote the unitary representation $U^x \otimes U^y$. The class of the trivial unitary representation is denoted by ε . We define the natural numbers $\text{mult}(z, x \otimes y)$ such that

$$x \otimes y \cong \bigoplus_{z \in \text{Irred}(\mathbb{G})} \text{mult}(z, x \otimes y) \cdot U^z.$$

The collection of natural numbers $\text{mult}(z, x \otimes y)$ are called the *fusion rules* of \mathbb{G} .

The set $\text{Irred}(\mathbb{G})$ is equipped with a natural involution $x \mapsto \bar{x}$ such that $U^{\bar{x}}$ is the unique (up to unitary equivalence) irreducible unitary representation satisfying

$$\text{Mor}(x \otimes \bar{x}, \varepsilon) \neq \{0\} \neq \text{Mor}(\bar{x} \otimes x, \varepsilon).$$

The unitary representation $U^{\bar{x}}$ is called the *contragredient* of U^x .

For every $x \in \text{Irred}(\mathbb{G})$, we take non-zero elements $t_x \in \text{Mor}(x \otimes \bar{x}, \varepsilon)$ and $s_x \in \text{Mor}(\bar{x} \otimes x, \varepsilon)$ satisfying $(t_x^* \otimes 1)(1 \otimes s_x) = 1$. Write the antilinear map

$$(2.2) \quad j_x : H_x \longrightarrow H_{\bar{x}} : \xi \longmapsto (\xi^* \otimes 1)t_x$$

and define $Q_x := j_x^* j_x$. We normalize t_x in such a way that $\text{Tr}(Q_x) = \text{Tr}(Q_x^{-1})$. This uniquely determines Q_x and fixes t_x, s_x up to a number of modulus 1. Note that $t_x^* t_x = \text{Tr}(Q_x)$.

DEFINITION 2.8. — For $x \in \text{Irred}(\mathbb{G})$, the value $\text{Tr}(Q_x)$ is called the quantum dimension of x and denoted by $\text{dim}_q(x)$. Note that $\text{dim}_q(x) \geq \text{dim}(x)$, with equality holding if and only if $Q_x = 1$.

The irreducible representations of \mathbb{G} and the Haar state h are connected by the *orthogonality relations*.

$$(2.3) \quad \begin{aligned} (\text{id} \otimes h)(U^x(\xi \eta^* \otimes 1)(U^y)^*) &= \frac{\delta_{x,y} 1}{\text{dim}_q(x)} \langle \eta, Q_x \xi \rangle, \\ (\text{id} \otimes h)((U^x)^*(\xi \eta^* \otimes 1)U^y) &= \frac{\delta_{x,y} 1}{\text{dim}_q(x)} \langle \eta, Q_x^{-1} \xi \rangle \end{aligned}$$

for $\xi \in H_x$ and $\eta \in H_y$.

Notation 2.9. — Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group. We denote by $\mathcal{C}(\mathbb{G})$ the set of coefficients of finite dimensional representations

of \mathbb{G} . Hence,

$$\mathcal{C}(\mathbb{G}) = \langle (\omega_{\xi, \eta} \otimes \text{id})(U^x) \mid x \in \text{Irred}(\mathbb{G}), \xi, \eta \in H_x \rangle$$

Then, $\mathcal{C}(\mathbb{G})$ is a unital dense $*$ -subalgebra of $C(\mathbb{G})$. Restricting Δ to $\mathcal{C}(\mathbb{G})$, $\mathcal{C}(\mathbb{G})$ becomes a Hopf $*$ -algebra.

Also, for $x \in \text{Irred}(\mathbb{G})$, denote by

$$\mathcal{C}(\mathbb{G})_x = \langle (\omega_{\xi, \eta} \otimes \text{id})(U^x) \mid \xi, \eta \in H_x \rangle$$

Note that $\Delta: \mathcal{C}(\mathbb{G})_x \rightarrow \mathcal{C}(\mathbb{G})_x \otimes \mathcal{C}(\mathbb{G})_x$ and that $\mathcal{C}(\mathbb{G})_x^* = \mathcal{C}(\mathbb{G})_{\bar{x}}$.

DEFINITION 2.10. — *The reduced C^* -algebra $C_r(\mathbb{G})$ is defined as the norm closure of $\mathcal{C}(\mathbb{G})$ in the GNS-representation with respect to h . The universal C^* -algebra $C_u(\mathbb{G})$ is defined as the enveloping C^* -algebra of $\mathcal{C}(\mathbb{G})$. The von Neumann algebra $L^\infty(\mathbb{G})$ is defined as the von Neumann algebra generated by $C_r(\mathbb{G})$. Note that if \mathbb{G} is the dual of a discrete group Γ , we have $C_r(\mathbb{G}) = C_r^*(\Gamma)$ and $C_u(\mathbb{G}) = C^*(\Gamma)$ and $L^\infty(\mathbb{G}) = \mathcal{L}(\Gamma)$.*

Remark 2.11. — Given an arbitrary compact quantum group \mathbb{G} , we have surjective homomorphisms $C_u(\mathbb{G}) \rightarrow C(\mathbb{G}) \rightarrow C_r(\mathbb{G})$, but most of the time we are only interested in $C_r(\mathbb{G})$ and $C_u(\mathbb{G})$. So, given the underlying Hopf $*$ -algebra, there exists different C^* -versions. From this point of view, we only consider two quantum groups different if the underlying Hopf $*$ -algebras are different.

DEFINITION 2.12. — *A compact quantum group \mathbb{G} is said to be coamenable if the homomorphism $C_u(\mathbb{G}) \rightarrow C_r(\mathbb{G})$ is an isomorphism.*

PROPOSITION 2.13. — *The Haar state h is a KMS-state on both $C_r(\mathbb{G})$ and $C_u(\mathbb{G})$ and the modular group is determined by*

$$(2.4) \quad (\text{id} \otimes \sigma_t^h)(U^x) = (Q_x^{it} \otimes 1)U^x(Q_x^{it} \otimes 1)$$

for every $x \in \text{Irred}(\mathbb{G})$.

Discrete quantum groups and duality

Following Van Daele ([30]), a discrete quantum group is a multiplier Hopf $*$ -algebra whose underlying $*$ -algebra is a direct sum of matrix algebras. The dual of a compact quantum group is such a discrete quantum group and is defined as follows.

DEFINITION 2.14. — Let \mathbb{G} be a compact quantum group. We define the dual (discrete) quantum group $\widehat{\mathbb{G}}$ as follows.

$$c_0(\widehat{\mathbb{G}}) = \bigoplus_{x \in \text{Irred}(\mathbb{G})} \mathcal{L}(H_x), \quad \ell^\infty(\widehat{\mathbb{G}}) = \prod_{x \in \text{Irred}(\mathbb{G})} \mathcal{L}(H_x).$$

We denote the minimal central projections of $\ell^\infty(\widehat{\mathbb{G}})$ by p_x , $x \in \text{Irred}(\mathbb{G})$. We have a natural unitary $\mathbb{V} \in M(c_0(\widehat{\mathbb{G}}) \otimes C(\mathbb{G}))$ given by

$$(2.5) \quad \mathbb{V} = \bigoplus_{x \in \text{Irred}(\mathbb{G})} U^x.$$

This unitary \mathbb{V} implements the duality between \mathbb{G} and $\widehat{\mathbb{G}}$. We have a natural comultiplication

$$\widehat{\Delta}: \ell^\infty(\widehat{\mathbb{G}}) \longrightarrow \ell^\infty(\widehat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\widehat{\mathbb{G}}): (\widehat{\Delta} \otimes \text{id})(\mathbb{V}) = \mathbb{V}_{13} \mathbb{V}_{23}.$$

One can deduce from this the following equivalent way to define the coproduct structure on $\ell^\infty(\widehat{\mathbb{G}})$.

$$\widehat{\Delta}(a)S = Sa \quad \text{for all } a \in \ell^\infty(\widehat{\mathbb{G}}) \text{ and } S \in \text{Mor}(y \otimes z, x).$$

The notation introduced above is aimed to suggest the basic example where \mathbb{G} is the dual of a discrete group Γ , given by $C(\mathbb{G}) = C^*(\Gamma)$ and $\Delta(\lambda_x) = \lambda_x \otimes \lambda_x$ for all $x \in \Gamma$. The map $x \mapsto \lambda_x$ yields an identification of Γ and $\text{Irred}(\mathbb{G})$ and then, $\ell^\infty(\widehat{\mathbb{G}}) = \ell^\infty(\Gamma)$.

The discrete quantum group $\ell^\infty(\widehat{\mathbb{G}})$ comes equipped with a natural modular structure.

Notation 2.15. — We have canonically defined states φ_x and ψ_x on $\mathcal{L}(H_x)$ related to (2.3) as follows.

$$(2.6) \quad \begin{aligned} \psi_x(A)1 &= \frac{1}{\dim_q(x)} t_x^*(A \otimes 1)t_x = \frac{\text{Tr}(Q_x A)}{\text{Tr}(Q_x)} 1 \\ &= (\text{id} \otimes h)(U^x(A \otimes 1)(U^x)^*) \quad \text{and} \\ \varphi_x(A)1 &= \frac{1}{\dim_q(x)} t_{\bar{x}}^*(1 \otimes A)t_{\bar{x}} = \frac{\text{Tr}(Q_x^{-1} A)}{\text{Tr}(Q_x^{-1})} 1 \\ &= (\text{id} \otimes h)((U^x)^*(A \otimes 1)U^x), \end{aligned}$$

for all $A \in \mathcal{L}(H_x)$.

Remark 2.16. — The states φ_x and ψ_x are significant, since they provide a formula for the invariant weights on $\ell^\infty(\widehat{\mathbb{G}})$. The left invariant weight is given by $\sum_{x \in \text{Irred}(\mathbb{G})} \dim_q(x)^2 \psi_x$, and the right invariant weight is given by $\sum_{x \in \text{Irred}(\mathbb{G})} \dim_q(x)^2 \varphi_x$.

DEFINITION 2.17. — A discrete quantum group $\widehat{\mathbb{G}}$ is amenable if there exists a left invariant mean on $\ell^\infty(\widehat{\mathbb{G}})$, i.e. a state $m \in \ell^\infty(\widehat{\mathbb{G}})^*$ s.t.

$$m((\omega \otimes \text{id})\widehat{\Delta}(x)) = m(x)\omega(1)$$

for all $\omega \in \ell^\infty(\widehat{\mathbb{G}})_*$ and $x \in \ell^\infty(\widehat{\mathbb{G}})$.

Remark 2.18. — It was proven [23] that $\widehat{\mathbb{G}}$ is amenable if and only if \mathbb{G} is coamenable.

Examples: the universal orthogonal compact quantum groups

We consider a class of compact quantum groups which was introduced by Wang and Van Daele in [31]. These compact quantum groups can in general not be obtained as deformations of classical objects.

DEFINITION 2.19. — Let $F \in \text{GL}(n, \mathbb{C})$ satisfying $F\overline{F} = \pm 1$. We define the compact quantum group $\mathbb{G} = A_o(F)$ as follows.

- $C_u(\mathbb{G})$ is the universal C^* -algebra with generators (U_{ij}) and relations making $U = (U_{ij})$ a unitary element of $M_n(\mathbb{C}) \otimes C(\mathbb{G})$ and $U = F\overline{U}F^{-1}$, where $(\overline{U})_{ij} = (U_{ij})^*$.
- $\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}$.

In these examples, the unitary matrix U is a representation, called the *fundamental representation*. The definition of $\mathbb{G} = A_o(F)$ makes sense without the requirement $F\overline{F} = \pm 1$, but the fundamental representation is irreducible if and only if $F\overline{F} \in \mathbb{R}1$. We then normalize such that $F\overline{F} = \pm 1$.

Remark 2.20. — It is easy to classify the quantum groups $A_o(F)$. For $F_1, F_2 \in \text{GL}(n, \mathbb{C})$ with $F_i\overline{F}_i = \pm 1$, we write $F_1 \sim F_2$ if there exists a unitary matrix v such that $F_1 = vF_2v^t$, where v^t is the transpose of v . Then, $A_o(F_1) \cong A_o(F_2)$ if and only if $F_1 \sim F_2$. It follows that the $A_o(F)$ are classified up to isomorphism by n , the sign of $F\overline{F}$ and the eigenvalue list of F^*F (see e.g. Section 5 of [7] where an explicit fundamental domain for the relation \sim is described).

If $F \in \text{GL}(2, \mathbb{C})$, we get up to equivalence, the matrices

$$(2.7) \quad F_q = \begin{pmatrix} 0 & |q|^{1/2} \\ -(\text{sgn } q)|q|^{-1/2} & 0 \end{pmatrix}$$

for $q \in [-1, 1]$, $q \neq 0$, with corresponding quantum groups $A_o(F_q) \cong \text{SU}_q(2)$, see [37]. In this case the quantum dimension of the fundamental representation equals $\text{Tr}(F_q^*F_q) = |q + 1/q|$.

The following result has been proven by Banica [1]. It tells us that the compact quantum groups $A_o(F)$ have the same fusion rules as the group $SU(2)$.

THEOREM 2.21. — *Let $F \in GL(n, \mathbb{C})$ and $F\bar{F} = \pm 1$. Let $\mathbb{G} = A_o(F)$. Then $\text{Irred}(\mathbb{G})$ can be identified with \mathbb{N} in such a way that*

$$x \otimes y \cong |x - y| \oplus (|x - y| + 2) \oplus \cdots \oplus (x + y),$$

for all $x, y \in \mathbb{N}$.

Further on, we will introduce another class of compact quantum groups that we need in this article, namely quantum automorphism groups, but therefore we need the notion of an action.

3. Actions of quantum groups

Actions and spectral subspaces

DEFINITION 3.1. — *Let B be a unital C^* -algebra and \mathbb{G} be a compact quantum group. A (right) action of \mathbb{G} on B is a unital $*$ -homomorphism $\delta: B \rightarrow B \otimes C(\mathbb{G})$ satisfying*

$$(\delta \otimes \text{id})\delta = (\text{id} \otimes \Delta)\delta \quad \text{and} \quad [\delta(B)(1 \otimes C(\mathbb{G}))] = B \otimes C(\mathbb{G}).$$

The action δ is said to be ergodic if the fixed point algebra $B^\delta := \{x \in B \mid \delta(x) = x \otimes 1\}$ equals $\mathbb{C}1$. In that case, B admits a unique invariant state ω given by $\omega(b)1 = (\text{id} \otimes h)\delta(b)$.

DEFINITION 3.2. — *Let $\delta: B \rightarrow B \otimes C(\mathbb{G})$ be an action of the compact quantum group \mathbb{G} on the unital C^* -algebra B . For every $x \in \text{Irred}(\mathbb{G})$, we define the spectral subspace associated with x by*

$$K_x = \{X \in \overline{H}_x \otimes B \mid (\text{id} \otimes \delta)(X) = X_{12}U_{13}^x\}.$$

Defining $\text{Hom}(H_x, B) = \{S: H_x \rightarrow B \mid S \text{ linear and } \delta(S\xi) = (S \otimes \text{id})(U^x(\xi \otimes 1))\}$, we have $K_x \cong \text{Hom}(H_x, B)$, associating to every $X \in K_x$ the operator $S_X: H_x \rightarrow B: \xi \mapsto X(\xi \otimes 1)$.

Remark 3.3. — For each $x \in \text{Irred}(\mathbb{G})$, K_x is a bimodule over the fixed point algebra B^δ in a natural way. Indeed, for $a \in B^\delta$ and $X \in K_x$, $a \bullet X := (1 \otimes a)X$ and $X \bullet a = X(1 \otimes a)$ turns K_x into a B^δ -bimodule. Moreover, one can check easily that

$$(3.1) \quad \langle \cdot, \cdot \rangle: K_x \times K_x \longrightarrow B^\delta: \langle X, Y \rangle = XY^*.$$

gives an inner product, turning K_x in a left Hilbert C^* -module over the fixed point algebra. We refer to [16] for the theory of Hilbert C^* -modules.

We can also turn K_x in a right Hilbert C^* -module. Denote by $E: B \rightarrow B^\delta: x \mapsto (\text{id} \otimes h)\delta$ the conditional expectation onto the fixed point algebra. For $X, Y \in K_x$, one can check, using the fact that $(E \otimes \text{id})\delta(x) = E(x) \otimes 1$, that for each state ω on B^δ , $(\text{id} \otimes \omega E)(X^*Y)$ is an intertwiner for U^x and hence scalar. This means that we can define

$$(3.2) \quad \langle \cdot, \cdot \rangle_\sim: K_x \times K_x \longrightarrow B^\delta \quad \text{by} \quad 1 \otimes \langle X, Y \rangle_\sim := (\text{id} \otimes E)(X^*Y),$$

which makes K_x a right Hilbert C^* -module over B^δ .

In the case where δ is ergodic with invariant state ω , K_x can be turned in a Hilbert space because $B^\delta = \mathbb{C}$, with scalar product defined by $\langle X, Y \rangle_{l1} = YX^*$ and $\langle X, Y \rangle_{r1} = (\text{id} \otimes \omega)(X^*Y)$. Remark that we switched orders in the first scalar product to have conjugate linearity in the first variable.

DEFINITION 3.4. — We define \mathcal{B} as the subspace of B generated by the spectral subspaces, i.e.

$$\mathcal{B} := \langle X(\xi \otimes 1) \mid x \in \text{Irred}(\mathbb{G}), X \in K_x, \xi \in H_x \rangle.$$

Also, we define

$$\mathcal{B}_x := \langle X(\xi \otimes 1) \mid X \in K_x, \xi \in H_x \rangle.$$

Note that $\delta: \mathcal{B}_x \rightarrow \mathcal{B}_x \otimes_{\text{alg}} \mathcal{C}(\mathbb{G})_x$ and that $\mathcal{B}_x^* = \mathcal{B}_{\bar{x}}$.

Observe that \mathcal{B} is a dense unital $*$ -subalgebra of B and that the restriction $\delta: \mathcal{B} \rightarrow \mathcal{B} \otimes_{\text{alg}} \mathcal{C}(\mathbb{G})$ defines an action of the Hopf $*$ -algebra $(\mathcal{C}(\mathbb{G}), \Delta)$ on \mathcal{B} .

Remark 3.5. — If δ is ergodic, \mathcal{B}_x is finite dimensional and its dimension is of the form $\dim H_x \cdot \text{mult}(\delta, x)$, where $\text{mult}(\delta, x)$ is called the multiplicity of x in δ . Note that as a vector space $B_x \simeq H_x \otimes K_x$, so $\text{mult}(\delta, x) = \dim K_x$.

Suppose now that $\delta: B \rightarrow B \otimes C(\mathbb{G})$ is an ergodic action. Let $x \in \text{Irred}(\mathbb{G})$. Take $t \in \text{Mor}(\bar{x} \otimes x, \varepsilon)$, normalized in such a way that $t^*t = \dim_q(x)$. Define the antilinear map

$$(3.3) \quad R_x: K_x \longrightarrow K_{\bar{x}}: R_x(v) = (t^* \otimes 1)(1 \otimes v^*).$$

Since t is fixed up to a number of modulus one, $L_x := R_x^*R_x$ is a well defined positive element of $\mathcal{L}(K_x)$.

DEFINITION 3.6. — We put $\text{mult}_q(x) := \sqrt{\text{Tr}(L_x)\text{Tr}(L_{\bar{x}})}$ and we call $\text{mult}_q(x)$ the quantum multiplicity of x in δ .

Remark 3.7. — It can be proven, for example in [7], that $\text{mult}_q(x) \leq \dim_q(x)$ for all $x \in \text{Irred}(\mathbb{G})$. If equality holds for all $x \in \widehat{\mathbb{G}}$, we say that δ is of *full quantum multiplicity*.

Terminology 3.8. — An action $\delta: B \rightarrow B \otimes C(\mathbb{G})$ of \mathbb{G} on B is said to be *universal* if B is the universal enveloping C^* -algebra of \mathcal{B} . It is said to be *reduced* if the conditional expectation $(\text{id} \otimes h)\delta$ of B on the fixed point algebra B^δ is faithful.

Remark 3.9. — From remark 2.11, we saw that a compact quantum group $(C(\mathbb{G}), \Delta)$ has many C^* -versions, while the underlying Hopf*-algebra is the same. The same remark applies to actions. We have that $B_u \rightarrow B \rightarrow B_r$ for an action $\delta: B \rightarrow B \otimes C(\mathbb{G})$. So again, we only consider two actions to be different if the underlying Hopf *-algebra actions are different. We make extensively use of this fact.

Actions on von Neumann algebras are defined as follows.

DEFINITION 3.10. — *A right action of a compact (resp. discrete) quantum group \mathbb{G} (resp. $\widehat{\mathbb{G}}$) on a von Neumann algebra N is an injective normal unital *-homomorphism*

$$\delta: N \longrightarrow N \overline{\otimes} L^\infty(\mathbb{G}) \quad \text{resp.} \quad \delta: N \longrightarrow N \overline{\otimes} \ell^\infty(\widehat{\mathbb{G}})$$

satisfying $(\delta \otimes \text{id})\delta = (\text{id} \otimes \Delta)\delta$, resp. $(\delta \otimes \text{id})\delta = (\text{id} \otimes \widehat{\Delta})\delta$.

Remark 3.11. — In the case of an action of a compact quantum group on a von Neumann algebra, we do not require the density condition like for C^* -algebraic actions. The reason is that this is automatically fulfilled for von Neumann algebras. This is a quite deep result and we refer to [26], theorem 2.6 for a proof. This implies that the spectral subalgebra as defined in 3.4 remains (weakly) dense in N .

Remark 3.12. — Because every action $\delta: B \rightarrow B \otimes C(\mathbb{G})$ has a unitary implementation, it can be extended to a von Neumann algebraic action.

Quantum subgroups and homogeneous spaces

DEFINITION 3.13 (Tomatsu, [24]). — *Let $(\mathbb{G}, \Delta_{\mathbb{G}})$ and $(\mathbb{H}, \Delta_{\mathbb{H}})$ be compact quantum groups. We call \mathbb{H} a closed (algebraic) quantum subgroup of \mathbb{G} whenever there is given a surjective *-homomorphism $r_{\mathbb{H}}: \mathcal{C}(\mathbb{G}) \rightarrow \mathcal{C}(\mathbb{H})$ satisfying $\Delta_{\mathbb{H}} \circ r_{\mathbb{H}} = (r_{\mathbb{H}} \otimes r_{\mathbb{H}})\Delta_{\mathbb{G}}$.*

Remark 3.14 (Tomatsu, [24]). — In general, a quantum subgroup is a quotient of $C(\mathbb{G})$. It is clear that a quantum subgroup is always an algebraic quantum subgroup. When the quantum group \mathbb{G} is co-amenable, any algebraic quantum subgroup is naturally regarded as a quantum subgroup.

DEFINITION 3.15. — Let $(\mathbb{G}, \Delta_{\mathbb{G}})$ be a compact quantum group with quantum subgroup $(\mathbb{H}, \Delta_{\mathbb{H}})$. Define the Hopf*-algebra action $\gamma_{\mathbb{H}}: C(\mathbb{G}) \rightarrow C(\mathbb{H}) \otimes_{\text{alg}} C(\mathbb{G}): x \mapsto (r_{\mathbb{H}} \otimes \text{id})\Delta_{\mathbb{G}}(x)$. Define the homogeneous space $\mathcal{C}(\mathbb{H} \setminus \mathbb{G})$ as the fixed point subalgebra of $C(\mathbb{G})$ under $\gamma_{\mathbb{H}}$.

Remark 3.16. — The restriction of the comultiplication to $\mathcal{C}(\mathbb{H} \setminus \mathbb{G})$ gives a Hopf*-action

$$(3.4) \quad \Delta_{\mathbb{H} \setminus \mathbb{G}}: \mathcal{C}(\mathbb{H} \setminus \mathbb{G}) \longrightarrow \mathcal{C}(\mathbb{H} \setminus \mathbb{G}) \otimes_{\text{alg}} \mathcal{C}(\mathbb{G}).$$

Since the action $\gamma_{\mathbb{H}}$ is invariant under the Haar measure of \mathbb{G} , we can extend it to $C_r(\mathbb{G})$ and $L^\infty(\mathbb{G})$ and hence define $C_r(\mathbb{H} \setminus \mathbb{G})$ and $L^\infty(\mathbb{H} \setminus \mathbb{G})$. By universality, $\gamma_{\mathbb{H}}$ is also extendable to $C_u(\mathbb{G})$, which gives us $C_u(\mathbb{H} \setminus \mathbb{G})$.

The restriction of the comultiplication to $C_r(\mathbb{H} \setminus \mathbb{G})$, respectively $C_u(\mathbb{H} \setminus \mathbb{G})$, or $L^\infty(\mathbb{H} \setminus \mathbb{G})$ gives again an action as in formula (3.4).

LEMMA 3.17. — The restriction of $r_{\mathbb{H}}$ to the quotient $\mathcal{C}(\mathbb{H} \setminus \mathbb{G})$ is the map $a \mapsto \epsilon_{\mathbb{G}}(a)1$.

Proof. — For $a \in \mathcal{C}(\mathbb{H} \setminus \mathbb{G})$,

$$\Delta_{\mathbb{H}}(r_{\mathbb{H}}(a)) = (r_{\mathbb{H}} \otimes r_{\mathbb{H}})\Delta_{\mathbb{G}}(a) = 1 \otimes r_{\mathbb{H}}(a),$$

We now apply $(\text{id} \otimes \epsilon_{\mathbb{H}})$ to both sides of the equation and use the fact that $\epsilon_{\mathbb{H}}r_{\mathbb{H}} = \epsilon_{\mathbb{G}}$ ([24]). Then

$$r_{\mathbb{H}}(a) = (\text{id} \otimes \epsilon_{\mathbb{H}})\Delta_{\mathbb{H}}(r_{\mathbb{H}}(a)) = \epsilon_{\mathbb{G}}(a)1,$$

which ends the proof. □

In the last chapter, we will need a special kind of subgroup.

DEFINITION 3.18. — Consider a compact quantum group $(\mathbb{G}, \Delta_{\mathbb{G}})$. We call a quantum subgroup $(\mathbb{H}, \Delta_{\mathbb{H}})$ of Kac type maximal, if for any quantum subgroup \mathbb{K} of Kac type, $L^\infty(\mathbb{H} \setminus \mathbb{G}) \subset L^\infty(\mathbb{K} \setminus \mathbb{G})$.

Every compact quantum group has a unique maximal quantum subgroup of Kac type (see [22]). We call it the *canonical Kac subgroup* of the quantum group.

Invariant subalgebras

A more general notion is that of an invariant subalgebra.

DEFINITION 3.19. — Consider a compact quantum group \mathbb{G} with comultiplication Δ . A right invariant subalgebra of \mathbb{G} is a unital C^* -algebra $B \subset C(\mathbb{G})$ such that $\Delta(B) \subset B \otimes C(\mathbb{G})$.

We can define an ergodic action δ of \mathbb{G} on B by just restricting Δ to B . We get the following easy proposition.

PROPOSITION 3.20. — Consider a compact quantum group \mathbb{G} and a right invariant subalgebra B of $C(\mathbb{G})$. Denote the action of \mathbb{G} on B by δ . For all $x \in \text{Irred}(\mathbb{G})$, $\text{mult}(\delta, x) \leq \dim(x)$ and equality in all x is only reached when $B = C(\mathbb{G})$.

Proof. — Let $x \in \text{Irred}(\mathbb{G})$. From the definition of a spectral subspace, we get

$$K_x = \{X \in \overline{H}_x \otimes B \mid (\text{id} \otimes \Delta)(X) = X_{12}U_{13}^x\}.$$

It is clear that

$$K_x \subset \mathbb{K}_x := \{X \in \overline{H}_x \otimes C(\mathbb{G}) \mid (\text{id} \otimes \Delta)(X) = X_{12}U_{13}^x\}$$

with \mathbb{K}_x the spectral subspace of the comultiplication Δ . Now $\mathbb{K}_x \cong \overline{H}_x$ where the bijection is given by $\overline{H}_x \rightarrow K_x: \bar{\xi} \mapsto (\xi^* \otimes 1)U^x$. Then

$$\text{mult}(\delta, x) = \dim(K_x) \leq \dim(H_x) = \dim(x).$$

Equality for all $x \in \text{Irred}(\mathbb{G})$ means that $K_x = \mathbb{K}_x$, so $B_x = C(\mathbb{G})_x$ and hence $B = C(\mathbb{G})$. □

Quantum automorphism groups

In this section we consider a class of universal quantum groups, namely the quantum automorphism groups as studied by Wang in [32] and Banica in [6, 4]. We only consider C^* -algebras with a special kind of states.

DEFINITION 3.21. — Let (D, ω) be a finite dimensional C^* -algebra of dimension ≥ 4 with a state. Denote by $\mu: D \otimes D \rightarrow D$ the multiplication. Take $\delta > 0$. If for the inner product implemented by ω , $\mu\mu^* = \delta^2 1$, we call ω a δ -form.

If D is a matrix-algebra, every state is of the form $\text{Tr}(F \cdot)$ and a δ -form with $\delta^2 = \text{Tr}(F^{-1})$. This can easily be checked by writing out $\mu\mu^*$ in terms of the orthonormal basis $(e_{ij}F^{-\frac{1}{2}})_{i,j=1,\dots,n}$ of D .

We can now give the definition of a quantum automorphism group:

DEFINITION 3.22 ([4]). — *Let (D, ω) be a finite-dimensional C^* -algebra with a δ -form. We define the compact quantum group $\mathbb{G} = A_{\text{aut}}(D, \omega)$ as follows. \mathbb{G} is defined by an action $\alpha: D \rightarrow D \otimes C_u(\mathbb{G})$ with the following properties:*

- $C_u(\mathbb{G})$ is defined as the universal C^* -algebra generated by

$$\{(\omega \otimes \text{id})\alpha(a) \mid \omega \in D^*, a \in D\}.$$

- Whenever $\beta: D \rightarrow D \otimes C_u(\mathbb{G}_1)$ is an action of a compact quantum group \mathbb{G}_1 , there exists a unique $*$ -homomorphism $\pi: C_u(\mathbb{G}) \rightarrow C_u(\mathbb{G}_1)$ satisfying $\beta = (\text{id} \otimes \pi)\alpha$.

Remark 3.23. — Let $n = \dim(D)$. In this article, we consider only the cases where $n \geq 4$. In the cases $n = 1, 2, 3$, we just get the permutation group S_n .

Representation Theory

In [6], Banica has determined the irreducible representations and their fusion rules for all quantum automorphism groups.

If B and ω are as above, the fusion rules of $A_{\text{aut}}(D, \omega)$ are those of $\text{SO}(3)$. This means that the irreducible representations are labeled by \mathbb{N} . We choose $U_i \in \mathcal{L}(H_i) \otimes \mathcal{C}(A_{\text{aut}}(D, \omega))$ the representative of the irreducible representation with label i in such a way that U_0 is the trivial representation ε and that $U = U_0 \oplus U_1 \in \mathcal{L}(D) \otimes \mathcal{C}(A_{\text{aut}}(D, \omega))$ is the fundamental representation. The fusion rules are given by:

$$U_i \otimes U_j = U_{|i-j|} + U_{|i-j|+1} + \dots + U_{i+j}.$$

4. Monoidal equivalence

General theory

The notion of monoidal equivalence was introduced in [7]. In this section, we give an overview of the results we will need.

DEFINITION 4.1 (Def. 3.1 in [7]). — Two compact quantum groups $\mathbb{G}_1 = (C(\mathbb{G}_1), \Delta_1)$ and $\mathbb{G}_2 = (C(\mathbb{G}_2), \Delta_2)$ are said to be monoidally equivalent if there exists a bijection $\varphi: \text{Irred}(\mathbb{G}_1) \rightarrow \text{Irred}(\mathbb{G}_2)$ satisfying $\varphi(\varepsilon) = \varepsilon$, together with linear isomorphisms

$$\begin{aligned} \varphi: \text{Mor}(x_1 \otimes \cdots \otimes x_r, y_1 \otimes \cdots \otimes y_k) \\ \longrightarrow \text{Mor}(\varphi(x_1) \otimes \cdots \otimes \varphi(x_r), \varphi(y_1) \otimes \cdots \otimes \varphi(y_k)) \end{aligned}$$

satisfying the following conditions:

$$\begin{aligned} \varphi(1) = 1, \quad \varphi(S \otimes T) = \varphi(S) \otimes \varphi(T), \\ \varphi(S^*) = \varphi(S)^*, \quad \varphi(ST) = \varphi(S)\varphi(T) \end{aligned}$$

whenever the formulas make sense. In the first formula, we consider $1 \in \text{Mor}(x, x) = \text{Mor}(x \otimes \varepsilon, x) = \text{Mor}(\varepsilon \otimes x, x)$. Such a collection of maps φ is called a monoidal equivalence between \mathbb{G}_1 and \mathbb{G}_2 .

By Theorem 3.9 and Proposition 3.13 of [7], we have the following fundamental result.

THEOREM 4.2. — Let φ be a monoidal equivalence between compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 .

- There exists, up to *-isomorphism, a unique unital *-algebra \mathcal{B} equipped with a faithful state ω and unitary elements $X^x \in \mathcal{B}(H_x, H_{\varphi(x)}) \otimes \mathcal{B}$ for all $x \in \text{Irred}(\mathbb{G}_1)$, satisfying
 1. $X_{13}^y X_{23}^z (S \otimes 1) = (\varphi(S) \otimes 1) X^x$ for all $S \in \text{Mor}(y \otimes z, x)$,
 2. the matrix coefficients of the X^x form a linear basis of \mathcal{B} ,
 3. $(\text{id} \otimes \omega)(X^x) = 0$ if $x \neq \varepsilon$.
- There exist unique commuting ergodic actions $\delta_1: \mathcal{B} \rightarrow \mathcal{B} \otimes_{\text{alg}} \mathcal{C}(\mathbb{G}_1)$ and $\delta_2: \mathcal{B} \rightarrow \mathcal{C}(\mathbb{G}_2) \otimes_{\text{alg}} \mathcal{B}$ satisfying

$$(\text{id} \otimes \delta_1)(X^x) = X_{12}^x U_{13}^x \quad \text{and} \quad (\text{id} \otimes \delta_2)(X^x) = U_{12}^{\varphi(x)} X_{13}^x$$

for all $x \in \text{Irred}(\mathbb{G})$.

- The state ω is invariant under δ_1 and δ_2 . Denoting by B_r the C^* -algebra generated by \mathcal{B} in the GNS-representation associated with ω and denoting by B_u the universal enveloping C^* -algebra of \mathcal{B} , the actions δ_1, δ_2 admit unique extensions to actions on B_r and B_u .

This algebra \mathcal{B} is called the link algebra of \mathbb{G}_1 and \mathbb{G}_2 under the monoidal equivalence φ .

Note that in the case $\mathbb{G} = \mathbb{G}_1 = \mathbb{G}_2$ and φ the identity map, we have $\mathcal{B} = \mathcal{C}(\mathbb{G})$ and $X^x = U^x$ for every $x \in \text{Irred}(\mathbb{G})$. The following unitary

operator generalizes (2.5).

$$(4.1) \quad \mathbb{X} := \bigoplus_{x \in \text{Irred}(\mathbb{G})} X^x \quad \text{where} \quad \mathbb{X} \in \prod_{x \in \text{Irred}(\mathbb{G})} (\mathcal{L}(H_x, H_{\varphi(x)}) \otimes B).$$

PROPOSITION 4.3. — *The invariant state ω is a KMS state on B_r and B_u and its modular group is determined by*

$$(4.2) \quad (\text{id} \otimes \sigma_t^\omega)(X^x) = (Q_{\varphi(x)}^{it} \otimes 1)X^x(Q_x^{it} \otimes 1)$$

for every $x \in \text{Irred}(\mathbb{G}_1)$.

Remark 4.4. — Define $\mathcal{B}_x := \langle (\omega_{\xi, \eta} \otimes \text{id})(X^x) \mid \xi \in H_{\varphi(x)}, \eta \in H_x \rangle$. Then, as a vector space

$$\mathcal{B} = \bigoplus_{x \in \text{Irred}(\mathbb{G})} \mathcal{B}_x.$$

Moreover, the \mathcal{B}_x are exactly the spaces \mathcal{B}_x in definition 3.4 coming from the spectral subspaces of δ_1 and δ_2 , while \mathcal{B} is exactly the dense $*$ -algebra given in Definition 3.4.

The orthogonality relations (2.3) generalize and take the following form.

$$(4.3) \quad \begin{aligned} (\text{id} \otimes \omega)(X^x(\xi_1 \eta_1^* \otimes 1)(X^y)^*) &= \frac{\delta_{x,y} 1}{\dim_q(x)} \langle \eta_1, Q_x \xi_1 \rangle, \\ (\text{id} \otimes \omega)((X^x)^*((\xi_2 \eta_2^* \otimes 1)X^y)) &= \frac{\delta_{x,y} 1}{\dim_q(x)} \langle \eta_2, Q_{\varphi(x)}^{-1} \xi_2 \rangle, \end{aligned}$$

for $\xi_1 \in H_x, \eta_1 \in H_y, \xi_2 \in H_{\varphi(x)}$ and $\eta_2 \in H_{\varphi(y)}$.

Concrete examples

In this section, we investigate in a closer way monoidal equivalence for specific quantum groups, namely the universal quantum groups $A_o(F)$ and the quantum automorphism groups. The case of the quantum groups $A_o(F)$ was already studied in detail in [7]. If $\mathbb{G}_1 = A_o(F_1)$ and $\mathbb{G}_2 = A_o(F_2)$, the following theorem gives a concrete expression of their link algebra.

THEOREM 4.5 (Thms. 5.3 and 5.4 in [7]). — *Let $F_1 \in M_{n_1}(\mathbb{C})$ and $F_2 \in M_{n_2}(\mathbb{C})$ such that $F_1 \bar{F}_1 = \pm 1$ and $F_2 \bar{F}_2 = \pm 1$.*

- *The compact quantum groups $A_o(F_1)$ and $A_o(F_2)$ are monoidally equivalent iff $F_1 \bar{F}_1$ and $F_2 \bar{F}_2$ have the same sign and $\text{Tr}(F_1^* F_1) = \text{Tr}(F_2^* F_2)$.*

- Assume that $A_o(F_1)$ and $A_o(F_2)$ are monoidally equivalent. Denote by $C_u(A_o(F_1, F_2))$ the universal unital C^* -algebra generated by the coefficients of $Y \in M_{n_2, n_1}(\mathbb{C}) \otimes C_u(A_o(F_1, F_2))$ with relations Y unitary and

$$Y = (F_2 \otimes 1)\bar{Y}(F_1^{-1} \otimes 1).$$

Then, $C_u(A_o(F_1, F_2)) \neq 0$ and there exists a unique pair of commuting universal ergodic actions, α_1 of $A_o(F_1)$ and α_2 of $A_o(F_2)$, such that

$$(\text{id} \otimes \alpha_1)(Y) = Y_{12}(U_1)_{13} \quad \text{and} \quad (\text{id} \otimes \alpha_2)(Y) = (U_2)_{12}Y_{13}.$$

Here, U_i denotes the fundamental representation of $A_o(F_i)$.

- $(C_u(A_o(F_1, F_2)), \alpha_1, \alpha_2)$ is isomorphic with the C^* -algebra B_u and the actions thereon given by theorem 4.2.

Remark 4.6. — It is also true that any compact quantum group which is monoidally equivalent with $A_o(F)$ where $F \in \text{GL}(n, \mathbb{C})$ and $F\bar{F} = \pm 1$ is itself of the form of $A_o(F_1)$ where $F_1 \in \text{GL}(n_1, \mathbb{C})$ and $F_1\bar{F}_1 = \pm 1$. Even more holds, Banica [1] showed that any quantum group with fusion rules of $SU(2)$ is of the form $A_o(F)$ where $F \in \text{GL}(n, \mathbb{C})$ and $F\bar{F} = \pm 1$.

Next, we obtain a concrete expression of the link algebra in the case that $\mathbb{G}_1 = A_{\text{aut}}(D_1, \omega_1)$ and $\mathbb{G}_2 = A_{\text{aut}}(D_2, \omega_2)$. We prove the following theorem.

THEOREM 4.7. — *Let D_1 and D_2 be finite dimensional C^* -algebras and ω_1 and ω_2 respectively a δ_1 -form and a δ_2 -form on D_1 , respectively D_2 .*

- The compact quantum groups

$$\mathbb{G}_1 = A_{\text{aut}}(D_1, \omega_1) \quad \text{and} \quad \mathbb{G}_2 = A_{\text{aut}}(D_2, \omega_2)$$

are monoidally equivalent if and only if $\delta_1 = \delta_2$.

- Suppose that $A_{\text{aut}}(D_1, \omega_1)$ and $A_{\text{aut}}(D_2, \omega_2)$ are monoidally equivalent. Denote by $C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2)))$ the universal C^* -algebra generated by the matrix elements of a unital $*$ -homomorphism

$$\gamma: D_1 \longrightarrow D_2 \otimes C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2)))$$

with relations $(\omega_2 \otimes \text{id})\gamma(x) = \omega_1(x)1$ for all $x \in D_1$.

Then $C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2))) \neq 0$ and there exists a unique pair of commuting ergodic actions of full quantum multiplicity α_1 of $A_{\text{aut}}(D_1, \omega_1)$ and α_2 of $A_{\text{aut}}(D_2, \omega_2)$, such that

$$(\text{id} \otimes \alpha_1)\gamma = (\gamma \otimes \text{id})\beta_1 \quad \text{and} \quad (\text{id} \otimes \alpha_2)\gamma = (\beta_2 \otimes \text{id})\gamma,$$

where $\beta_1: D_1 \rightarrow D_1 \otimes C_u(A_{\text{aut}}(D_1, \omega_1))$ and $\beta_2: D_2 \rightarrow D_2 \otimes C_u(A_{\text{aut}}(D_2, \omega_2))$ are the actions of the quantum automorphism groups.

- $(C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2))), \alpha_1, \alpha_2)$ is isomorphic with the C^* -algebra B_u and the actions thereon given by proposition 4.2.

Proof. — Denote by μ_1, μ_2 and η_1, η_2 the multiplication and unital map of respectively D_1 and D_2 . The proof of the first point goes as follows. First suppose that $\delta_1 = \delta_2$. Take now U , respectively V the fundamental representation of $A_{\text{aut}}(D_1, \omega_1)$, respectively $A_{\text{aut}}(D_2, \omega_2)$ corresponding to the actions of this quantum groups. Consider the graded C^* -algebras $(\text{Mor}(U^m, U^n))_{n,m}$ and $(\text{Mor}(V^m, V^n))_{n,m}$. We know from [6] that there is an isomorphism $\pi: (\text{Mor}(U^n, U^m))_{n,m} \rightarrow (\text{Mor}(V^n, V^m))_{n,m}$ which satisfies $\pi(\mu_1) = \mu_2$ and $\pi(\eta_1) = \eta_2$. We now can work analogously to the case of $A_o(F)$ that was covered in [7].

We now set $\text{Irred}(\mathbb{G}_1) = \mathbb{N}$ and $P_n \in \text{Mor}(U^n, U^n)$ the unique projection for which $P_n T = 0$ for all $r < n$ and all $T \in \text{Mor}(U^r, U^n)$. We define U_n as the restriction of U^n to the image of P_n and identify

$$\begin{aligned} & \text{Mor}(n_1 \otimes \dots \otimes n_r, m_1 \otimes \dots \otimes m_k) \\ &= (P_{m_1} \otimes \dots \otimes P_{m_k}) \text{Mor}(U^{n_1+\dots+n_r}, U^{m_1+\dots+m_k})(P_{n_1} \otimes \dots \otimes P_{n_r}). \end{aligned}$$

Define now $H_{\psi(n)} := \pi(P_n)D_1^n$ and define for $S \in \text{Mor}(n_1 \otimes \dots \otimes n_r, m_1 \otimes \dots \otimes m_k)$, $\psi(S)$ by the restriction of π to $\text{Mor}(n_1 \otimes \dots \otimes n_r, m_1 \otimes \dots \otimes m_k)$. Then ψ is a unitary fiber functor which gives a monoidal equivalence between \mathbb{G}_1 and \mathbb{G}_2 .

Conversely, suppose that $A_{\text{aut}}(D_1, \omega_1) \underset{\text{mon}}{\sim} A_{\text{aut}}(D_2, \omega_2)$. Denote by u_1 , respectively v_1 the irreducible representation with label 1 of $A_{\text{aut}}(D_1, \omega_1)$ and $A_{\text{aut}}(D_2, \omega_2)$. Then $\dim_q(u_1)1 = \omega_1 \mu_1 \mu_1^* \omega_1^* = \delta_1^2 1$ and because monoidal equivalence preserves the quantum dimension, δ_1 and δ_2 must be equal. This proves the first part of the theorem.

For the proof of the other parts of the theorem, we first make the following observation. Consider two finite dimensional C^* -algebras (D_1, ω_1) and (D_2, ω_2) with δ -forms and their quantum automorphism groups $A_{\text{aut}}(D_1, \omega_1) := \mathbb{G}_1$ and $A_{\text{aut}}(D_2, \omega_2) := \mathbb{G}_2$. Denote now by $H_1^i = D_i \oplus \mathbb{C}$ and $U_i^1 \in \mathcal{L}(H_1^i) \otimes \mathcal{C}(\mathbb{G}_i)$ for $i = 1, 2$ the representative of the irreducible representation with label 1. Denote by $\theta_i \in \text{Mor}((U_i^1), (U_i^1)^2)$ and $\gamma_i \in \text{Mor}(U_i^0, (U_i^1)^2)$ the obvious "components" of the multiplication. From the construction in the first part of the theorem, it follows that there is a monoidal equivalence φ between \mathbb{G}_1 and \mathbb{G}_2 which sends θ_1 and γ_1 , to θ_2

and γ_2 . If we further below talk about the monoidal equivalence between $A_{\text{aut}}(D_1, \omega_1)$ and $A_{\text{aut}}(D_2, \omega_2)$, we will always mean this one.

We first remark that if $C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2))) \neq 0$, the actions are given by universality. Indeed,

$(\gamma \otimes \text{id})\beta_1: D_1 \longrightarrow D_2 \otimes C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2))) \otimes C_u(A_{\text{aut}}(D_1, \omega_1))$
 is a *-homomorphism which satisfies

$$(\omega_2 \otimes \text{id} \otimes \text{id})(\gamma \otimes \text{id})\beta_1(x) = \omega_1(x)1$$

for $x \in D_1$. So by universality, there exists a *-homomorphism

$$\begin{aligned} \alpha_1: C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2))) \\ \longrightarrow C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2))) \otimes C_u(A_{\text{aut}}(D_1, \omega_1)) \end{aligned}$$

satisfying $(\text{id} \otimes \alpha_1)\gamma = (\gamma \otimes \text{id})\beta_1$. Because β_1 is an action and the coefficients of γ generate $C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2)))$, it follows that α_1 is an action. We define α_2 in an analogous way.

Consider now the C^* -algebra B_u we get from the monoidal equivalence. Denote by θ_i, γ_i the components of the multiplication of $D_i, i = 1, 2$. As we said above, we may suppose that the monoidal equivalence sends θ_1 and γ_1 respectively to θ_2 and γ_2 . Denote by U_i the irreducible representation of $A_{\text{aut}}(D_i, \omega_i)$ with label 1. Because every irreducible representation is contained in a tensor power of the one with label 1, the matrix coefficients of $X^1 \in \mathcal{L}(D_1 \ominus \mathbb{C}, D_2 \ominus \mathbb{C}) \otimes B_u$ generate B_u as a C^* -algebra. By identification, X^1 provides us with a linear map

$$\Gamma: D_1 \ominus \mathbb{C} \longrightarrow (D_2 \ominus \mathbb{C}) \otimes B_u$$

which we can easily extend to D_1 by setting $\Gamma(1) = 1$. Because

$$X^1(\theta_1 \otimes 1) = (\theta_2 \otimes 1)X_{13}^1 X_{23}^1 \text{ and } (\gamma_1 \otimes 1) = (\gamma_2 \otimes 1)X_{13}^1 X_{23}^1,$$

Γ is multiplicative, obviously unital and $\omega_1(x)1 = (\omega_2 \otimes \text{id})\Gamma(x)$. It also preserves the involution because $X_{23}(\gamma_1^* \otimes 1) = X_{13}^*(\gamma_2^* \otimes 1)$ and γ_1 and γ_2 implement the involution on respectively D_1 and D_2 . By universality there exists now a unital *-homomorphism

$$\rho: C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2))) \longrightarrow B_u$$

such that $\Gamma = (\text{id} \otimes \rho)\gamma$. It is now left to show that ρ is an isomorphism.

Because γ satisfies the equation $(\omega_2 \otimes \text{id})\gamma(x) = \omega_1(x)1$, we can look at the restriction of γ given by

$$\gamma: D_1 \ominus \mathbb{C} \longrightarrow (D_2 \ominus \mathbb{C}) \otimes C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2))).$$

Denote by $Y \in \mathcal{L}((D_1 \ominus \mathbb{C}), (D_2 \ominus \mathbb{C})) \otimes C_u(A_{\text{aut}}((D_1, \omega_1), (D_2, \omega_2)))$ the element corresponding to this restricted *-homomorphism. This element satisfies the equations

$$Y(\theta_1 \otimes 1) = (\theta_2 \otimes 1)Y_{13}Y_{23} \quad \text{and} \quad \gamma_1 \otimes 1 = (\gamma_2 \otimes 1)Y_{13}Y_{23}$$

because γ is a unital homomorphism. Remark that Y is unitary because γ also preserves the involution. Because the multiplication and the unital map generate all the intertwiners of $A_{\text{aut}}(D_i, \omega_i)$, $i = 1, 2$, and so is also true for θ_i and γ_i , it holds that

$$Y^{\otimes n}(P_n \otimes 1) = (Q_n \otimes 1)Y^{\otimes n}$$

where P_n and Q_n are the unique projections in respectively $\text{Mor}(U_1^n, U_1^n)$ and $\text{Mor}(U_2^n, U_2^n)$ on the irreducible representation with label n . Defining σ such that $(\text{id} \otimes \sigma)(X^n) = Y^{\otimes n}(P_n \otimes 1)$, gives a unital *-homomorphism with $\sigma\rho = \rho\sigma = \text{id}$. □

Remark 4.8. — We can also prove that every compact quantum group \mathbb{G} which is monoidally equivalent to a quantum automorphism group $A_{\text{aut}}(D, \omega)$ is isomorphic to another quantum automorphism group $A_{\text{aut}}(D_1, \omega_1)$. It is to our best knowledge not clear if every compact quantum group with the fusion rules of $\text{SO}(3)$ is a quantum automorphism group.

5. The Poisson boundary of a discrete quantum group

We give a brief survey of Izumi’s theory of Poisson boundaries for discrete quantum groups.

Fix a discrete quantum group $\widehat{\mathbb{G}}$.

Notation 5.1. — For every normal state $\phi \in \ell^\infty(\widehat{\mathbb{G}})_*$, we define the convolution operator

$$P_\phi: \ell^\infty(\widehat{\mathbb{G}}) \longrightarrow \ell^\infty(\widehat{\mathbb{G}}): P_\phi(a) = (\text{id} \otimes \phi)\widehat{\Delta}(a).$$

We are only interested in special states $\phi \in \ell^\infty(\widehat{\mathbb{G}})$, motivated by the following straightforward proposition. For every probability measure μ on $\text{Irred}(\mathbb{G})$, we set

$$\psi_\mu = \sum_{x \in \text{Irred}(\mathbb{G})} \mu(x)\psi_x \quad \text{and} \quad P_\mu := P_{\psi_\mu}.$$

Recall that the states ψ_x are defined in notation 2.15. Note that we have a convolution product $\mu * \nu$ on the measures on $\text{Irred}(\mathbb{G})$, such that $\psi_{\mu*\nu} = (\psi_\mu \otimes \psi_\nu)\hat{\Delta}$.

PROPOSITION 5.2. — *Let ϕ be a normal state on $\ell^\infty(\widehat{\mathbb{G}})$. Then the following conditions are equivalent.*

- ϕ has the form ψ_μ from some probability measure μ on $\text{Irred}(\mathbb{G})$.
- The Markov operator P_ϕ preserves the center of $\ell^\infty(\widehat{\mathbb{G}})$.
- ϕ is invariant under the adjoint action of \mathbb{G} on $\ell^\infty(\widehat{\mathbb{G}})$

$$\alpha_{\mathbb{G}} : \ell^\infty(\widehat{\mathbb{G}}) \longrightarrow \ell^\infty(\widehat{\mathbb{G}})\overline{\otimes}L^\infty(\mathbb{G}) : a \longmapsto \mathbb{V}(a \otimes 1)\mathbb{V}^*.$$

DEFINITION 5.3 ([13], Section 2.5). — *Let μ be a probability measure on $\text{Irred}(\mathbb{G})$. Set*

$$H^\infty(\widehat{\mathbb{G}}, \mu) = \left\{ a \in \ell^\infty(\widehat{\mathbb{G}}) \mid P_\mu(a) = a \right\}.$$

Equipped with the product defined by

$$(5.1) \quad a \cdot b := w^* \text{-} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_\mu^k(ab),$$

and the involution, norm and σ -weak topology inherited from $\ell^\infty(\widehat{\mathbb{G}})$, the space $H^\infty(\widehat{\mathbb{G}}, \mu)$ becomes a von Neumann algebra that we call the Poisson boundary of $\widehat{\mathbb{G}}$ with respect to μ .

Terminology 5.4. — *A probability measure μ on $\text{Irred}(\mathbb{G})$ is called generating if there exists, for every $x \in \text{Irred}(\mathbb{G})$, an $n \geq 1$ such that $\mu^{*n}(x) \neq 0$.*

Remark 5.5. — *The restriction of the co-unit $\hat{\varepsilon}$ yields a state on $H^\infty(\widehat{\mathbb{G}}, \mu)$, called the harmonic state. This state is faithful when μ is generating. In what follows, we always assume that μ is generating.*

DEFINITION 5.6. — *Let μ be a generating measure on $\text{Irred}(\mathbb{G})$. The Poisson boundary $H^\infty(\widehat{\mathbb{G}}, \mu)$ comes equipped with two natural actions, one of \mathbb{G} and one of $\widehat{\mathbb{G}}$:*

$$\begin{aligned} \alpha_{\mathbb{G}} : H^\infty(\widehat{\mathbb{G}}, \mu) &\longrightarrow H^\infty(\widehat{\mathbb{G}}, \mu)\overline{\otimes}L^\infty(\mathbb{G}) : \alpha_{\mathbb{G}}(a) = \mathbb{V}(a \otimes 1)\mathbb{V}^*, \\ \alpha_{\widehat{\mathbb{G}}} : H^\infty(\widehat{\mathbb{G}}, \mu) &\longrightarrow \ell^\infty(\widehat{\mathbb{G}})\overline{\otimes}H^\infty(\widehat{\mathbb{G}}, \mu) : \alpha_{\widehat{\mathbb{G}}}(a) = \hat{\Delta}(a). \end{aligned}$$

Note that $\alpha_{\mathbb{G}}$ is the restriction of the adjoint action of \mathbb{G} on $\ell^\infty(\widehat{\mathbb{G}})$, while $\alpha_{\widehat{\mathbb{G}}}$ is nothing else than the restriction of the comultiplication. The maps $\alpha_{\mathbb{G}}$ and $\alpha_{\widehat{\mathbb{G}}}$ are well defined because of the following equivariance formulae:

$$(5.2) \quad (\text{id} \otimes P_\mu)(\hat{\Delta}(a)) = \hat{\Delta}(P_\mu(a)) \quad \text{and} \quad (P_\mu \otimes \text{id})(\alpha_{\mathbb{G}}(a)) = \alpha_{\mathbb{G}}(P_\mu(a)).$$

Remark 5.7. — With the product defined by formula (5.1), the mappings $\alpha_{\mathbb{G}}$ and $\alpha_{\widehat{\mathbb{G}}}$ are multiplicative. This follows from the equivariance formulae (5.2). Hence $\alpha_{\mathbb{G}}$ and $\alpha_{\widehat{\mathbb{G}}}$ are actions on $H^\infty(\widehat{\mathbb{G}}, \mu)$. Because

$$(\widehat{\varepsilon} \otimes \text{id})\alpha_{\mathbb{G}}(a) = (\widehat{\varepsilon} \otimes \text{id})(\mathbb{V}(a \otimes 1)\mathbb{V}^*) = \widehat{\varepsilon}(a)1,$$

we see that $\widehat{\varepsilon}$ is an invariant state for the action

$$\alpha_{\mathbb{G}}: H^\infty(\widehat{\mathbb{G}}, \mu) \longrightarrow H^\infty(\widehat{\mathbb{G}}, \mu) \otimes L^\infty(\mathbb{G}).$$

Remark 5.8. — When $\widehat{\mathbb{G}}$ is a discrete group, the action $\alpha_{\mathbb{G}}$ is the trivial action on $\ell^\infty(\widehat{\mathbb{G}})$. In general, the fixed point algebra of $\alpha_{\mathbb{G}}$ is precisely the algebra of *central harmonic elements* $Z(\ell^\infty(\widehat{\mathbb{G}})) \cap H^\infty(\widehat{\mathbb{G}}, \mu)$. Since the Markov operator P_μ preserves the center $Z(\ell^\infty(\widehat{\mathbb{G}}))$, the commutative von Neumann algebra $Z(\ell^\infty(\widehat{\mathbb{G}})) \cap H^\infty(\widehat{\mathbb{G}}, \mu)$ with state $\widehat{\varepsilon}$, is exactly the Poisson boundary for the random walk on $\text{Irred}(\mathbb{G})$ with transition probabilities $p(x, y)$ and n -step transition probabilities $p_n(x, y)$ given by

$$(5.3) \quad p_x p(x, y) = p_x P_\mu(p_y), \quad p_x p_n(x, y) = p_x P_\mu^n(p_y).$$

Note that $p_n(e, y) = \mu^{*n}(y) = \psi_\mu^{*n}(p_y)$.

So, the action $\alpha_{\mathbb{G}}$ is *ergodic* if and only if there are no non-trivial central harmonic elements.

6. The Martin boundary of a discrete quantum group

The Martin boundary and the Martin compactification of a discrete quantum group have been defined by Neshveyev and Tuset in [19]. Fix a discrete quantum group $\widehat{\mathbb{G}}$ and a probability measure μ on $\text{Irred}(\mathbb{G})$. We have an associated Markov operator P_μ and a classical random walk on $\text{Irred}(\mathbb{G})$ with n -step transition probabilities given by (5.3).

DEFINITION 6.1. — *The probability measure μ on $\text{Irred}(\mathbb{G})$ is said to be transient if $\sum_{n=0}^\infty p_n(x, y) < \infty$ for all $x, y \in \text{Irred}(\mathbb{G})$.*

We suppose throughout that μ is a generating measure and that μ is transient.

Denote by $c_c(\widehat{\mathbb{G}}) \subset c_0(\widehat{\mathbb{G}})$ the algebraic direct sum of the algebras $\mathcal{L}(H_x)$. We define, for $a \in c_c(\widehat{\mathbb{G}})$,

$$G_\mu(a) = \sum_{n=0}^\infty P_\mu^n(a).$$

Observe that usually $G_\mu(a)$ is unbounded, but it makes sense in the multiplier algebra of $c_c(\widehat{\mathbb{G}})$, i.e. $G_\mu(a)p_x \in \mathcal{L}(H_x)$ makes sense for every $x \in \text{Irred}(\mathbb{G})$.

$\text{Irred}(\mathbb{G})$ because μ is transient. Moreover, $G_\mu(p_\varepsilon)$ is strictly positive and central. This allows to define the Martin kernel as follows.

Whenever μ is a measure on $\text{Irred}(\mathbb{G})$, we use the notation $\bar{\mu}$ to denote the measure given by $\bar{\mu}(x) = \mu(\bar{x})$.

DEFINITION 6.2 (Defs. 3.1 and 3.2 in [19]). — *Define*

$$K_\mu : c_c(\widehat{\mathbb{G}}) \longrightarrow \ell^\infty(\widehat{\mathbb{G}}) : K_\mu(a) = G_\mu(a)G_\mu(p_\varepsilon)^{-1}.$$

Define the Martin compactification \tilde{A}_μ as the C^* -subalgebra of $\ell^\infty(\widehat{\mathbb{G}})$ generated by $K_{\bar{\mu}}(c_c(\widehat{\mathbb{G}}))$ and $c_0(\widehat{\mathbb{G}})$. Define the Martin boundary A_μ as the quotient $\tilde{A}_\mu/c_0(\widehat{\mathbb{G}})$.

By Theorem 3.5 in [19], the adjoint action $\alpha_{\mathbb{G}}$ and the comultiplication $\hat{\Delta}$ define, by restriction

$$(6.1) \quad \alpha_{\mathbb{G}} : \tilde{A}_\mu \longrightarrow \tilde{A}_\mu \otimes C(\mathbb{G}) \quad \text{and} \quad \alpha_{\widehat{\mathbb{G}}} : \tilde{A}_\mu \longrightarrow M(c_0(\widehat{\mathbb{G}}) \otimes \tilde{A}_\mu).$$

By passing to the quotient, we get the following actions on the Martin boundary.

$$(6.2) \quad \gamma_{\mathbb{G}} : A_\mu \longrightarrow A_\mu \otimes C(\mathbb{G}) \quad \text{and} \quad \gamma_{\widehat{\mathbb{G}}} : A_\mu \longrightarrow M(c_0(\widehat{\mathbb{G}}) \otimes A_\mu).$$

Remark 6.3. — The actions $\alpha_{\mathbb{G}}$ and $\gamma_{\mathbb{G}}$ are reduced.

7. The correspondence between the actions of monoidally equivalent quantum groups

In this section, we prove that there is a *bijjective correspondence* between actions of monoidally equivalent compact quantum groups. Moreover, this correspondence preserves the spectral properties of the actions.

7.1. Construction of the bijjective correspondence

Notation 7.1. — Consider two compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 and a $*$ -algebra \mathcal{B} on which there exist two commuting actions $\delta_1 : \mathcal{B} \rightarrow \mathcal{C}(\mathbb{G}_1) \otimes_{\text{alg}} \mathcal{B}$ and $\delta_2 : \mathcal{B} \rightarrow \mathcal{B} \otimes_{\text{alg}} \mathcal{C}(\mathbb{G}_2)$. Given an action $\alpha : \mathcal{D} \rightarrow \mathcal{D} \otimes_{\text{alg}} \mathcal{C}(\mathbb{G}_1)$, we define

$$\mathcal{D} \boxtimes_{\text{alg}}^\alpha \mathcal{B} := \{a \in \mathcal{D} \otimes_{\text{alg}} \mathcal{B} \mid (\alpha \otimes \text{id})(a) = (\text{id} \otimes \delta_1)(a)\}.$$

When we consider everything on the von Neumann algebraic level, we denote in the same way

$$D \boxtimes^\alpha B := \{a \in D \overline{\otimes} B \mid (\alpha \otimes \text{id})(a) = (\text{id} \otimes \delta_1)(a)\}.$$

Also, if D is a C^* -algebra with reduced action α on it, then we denote $D \boxtimes_{red}^\alpha B$ as the norm closure of $\mathcal{D} \boxtimes_{alg}^\alpha \mathcal{B}$ in $D \boxtimes^\alpha B$.

LEMMA 7.2. — *The restriction of $\text{id} \otimes \delta_2$ to $\mathcal{D} \boxtimes_{alg}^\alpha \mathcal{B}$ gives an action of \mathbb{G}_2 on $\mathcal{D} \boxtimes_{alg}^\alpha \mathcal{B}$. We denote this action by $\text{id} \boxtimes \delta_2$.*

Proof. — From the following easy calculation, one can see that $\mathcal{D} \boxtimes_{alg}^{\alpha_1} \mathcal{B}$ is invariant under the action $\text{id} \otimes \delta_2$.

$$\begin{aligned} (\alpha \otimes \text{id} \otimes \text{id})(\text{id} \otimes \delta_2)(a) &= (\text{id} \otimes \text{id} \otimes \delta_2)(\alpha \otimes \text{id})(a) \\ &= (\text{id} \otimes \text{id} \otimes \delta_2)(\text{id} \otimes \delta_1)(a) \\ &= (\text{id} \otimes \delta_1 \otimes \text{id})(\text{id} \otimes \delta_2)(a) \end{aligned}$$

The last step is valid because δ_1 and δ_2 commute. Hence $\text{id} \boxtimes \delta_2$ is a well defined action of \mathbb{G}_2 on $\mathcal{D} \boxtimes_{alg}^\alpha \mathcal{B}$. □

Consider two monoidally equivalent compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 and a C^* -algebra D_1 . Suppose we have an action $\alpha_1: D_1 \rightarrow D_1 \otimes C(\mathbb{G}_1)$. As we stated in remark 2.11 and remark 3.9, the underlying Hopf*-algebra action carries all the relevant information. This means that we can work with this underlying Hopf *-algebra action $\alpha_1: \mathcal{D}_1 \rightarrow \mathcal{D}_1 \otimes_{alg} \mathcal{C}(\mathbb{G}_1)$. Consider a monoidal equivalence $\varphi: \mathbb{G}_2 \rightarrow \mathbb{G}_1$. Note that we have exchanged the roles of \mathbb{G}_1 and \mathbb{G}_2 . This will turn out to be more convenient in what follows. From theorem 4.2, we get a link algebra \mathcal{B} , unitaries $X^x \in \mathcal{L}(H_x, H_{\varphi(x)}) \otimes_{alg} \mathcal{B}$ and two commuting ergodic actions

$$\delta_1: \mathcal{B} \longrightarrow \mathcal{C}(\mathbb{G}_1) \otimes_{alg} \mathcal{B} \quad \text{and} \quad \delta_2: \mathcal{B} \longrightarrow \mathcal{B} \otimes_{alg} \mathcal{C}(\mathbb{G}_2)$$

given by

$$(7.1) \quad (\text{id} \otimes \delta_1)(X^x) = U_{12}^{\varphi(x)} X_{13}^x \quad \text{and} \quad (\text{id} \otimes \delta_2)(X^x) = X_{12}^x U_{13}^x.$$

The following theorem enables us to construct an action of \mathbb{G}_2 with the same spectral structure as α_1 .

THEOREM 7.3. — *The action $\alpha_2 := \text{id} \boxtimes \delta_2$ of \mathbb{G}_2 on $\mathcal{D}_2 := \mathcal{D}_1 \boxtimes_{alg}^{\alpha_1} \mathcal{B}$ has the following properties:*

- $a \mapsto a \otimes 1$ is a *-isomorphism between the fixed point algebras of α_1 and α_2 .
- The map $T_x: K_{\varphi(x)} \rightarrow K_x: v \mapsto v_{12} X_{13}^x$ is a bimodular isomorphism between the spectral subspaces of α_1 and α_2 . Moreover, T is a unitary element of $\mathcal{L}(K_{\varphi(x)}, K_x)$ for the inner products $\langle \cdot, \cdot \rangle_l$ and $\langle \cdot, \cdot \rangle_r$ defined by formulae (3.1) and (3.2).

- The set $(T_x)_{x \in \text{Irred}(\mathbb{G}_2)}$ respects the monoidal structure in the sense that for $x, y, z \in \text{Irred}(\mathbb{G}_2)$ and $V \in \text{Mor}(x \otimes y, z)$

$$T_x(X)_{13}T_y(Y)_{23}(V \otimes 1) = T_z(X_{13}Y_{23}(\varphi(V) \otimes 1))$$

for all $X \in K_{\varphi(x)}$, $Y \in K_{\varphi(y)}$.

- Suppose that α_1 is an ergodic action. Then the action α_2 as defined above is also an ergodic action. Moreover for all $x \in \text{Irred}(\mathbb{G}_2)$, $\text{mult}_q(x) = \text{mult}_q(\varphi(x))$.

Proof. — Suppose that $\alpha_2(a) = a \otimes 1$ for $a \in \mathcal{D}_2$. This means that $(\text{id} \otimes \delta_2)(a) = a \otimes 1$. By ergodicity of δ_2 there exists a $b \in \mathcal{D}_1$ such that $a = b \otimes 1$. But because $(\alpha_1 \otimes \text{id})(a) = (\text{id} \otimes \delta_1)(a)$, it follows that $b \in \mathcal{D}_1^{\alpha_1}$. So the map $\mathcal{D}_1^{\alpha_1} \rightarrow \mathcal{D}_2^{\alpha_2}: b \mapsto b \otimes 1$ is a *-isomorphism.

We now prove that the spectral subspaces of α_1 and α_2 are isomorphic as $\mathcal{D}_1^{\alpha_1}$ -bimodules. Denote by $K_{\varphi(x)}$ and K_x the spectral subspaces of respectively α_1 and α_2 for the representation $\varphi(x)$, respectively x . From remark 3.3, we know that the spectral subspaces have a natural bimodule structure over the fixed point algebra. We claim that the map

$$T: K_{\varphi(x)} \longrightarrow K_x: v \longmapsto v_{12}X_{13}^x$$

is the bimodule isomorphism we are looking for. If $v \in K_{\varphi(x)}$, then

$$(\text{id} \otimes \alpha_1 \otimes \text{id})T(v) = v_{12}U_{13}^{\varphi(x)}X_{14}^x = (\text{id} \otimes \text{id} \otimes \delta_1)T(v)$$

by definition 3.2 of the spectral subspace $K_{\varphi(x)}$ and the properties of X^x , so $T(v) \in \overline{H_x} \otimes \mathcal{D}_2$. Moreover, it is obvious that

$$(\text{id} \otimes \alpha_2)T(v) = (\text{id} \otimes \text{id} \otimes \delta_2)T(v) = v_{12}X_{13}^xU_{14}^x,$$

which means $T(v) \in K_x$. The $\mathcal{D}_1^{\alpha_1}$ -bilinearity of T is clear. Consider now the spectral subspaces K_x and $K_{\varphi(x)}$ as equipped with the left inner product as in (3.1). We show that T is a unitary element of $\mathcal{L}(K_{\varphi(x)}, K_x)$ for this inner product and obtain in this way that T actually gives an isomorphism between $K_{\varphi(x)}$ and K_x . Consider the map $S: K_x \rightarrow \overline{H_{\varphi(x)}} \otimes \mathcal{D}_1 \otimes \mathcal{B}: w \mapsto w(X_{13}^x)^*$. If $w \in K_x$, then

$$\begin{aligned} (\text{id} \otimes \text{id} \otimes \delta_2)S(w) &= (\text{id} \otimes \alpha_2)(w)(\text{id} \otimes \text{id} \otimes \delta_2)(X_{13}^x)^* \\ &= (w \otimes 1)U_{14}^x(U_{14}^x)^*(X_{13}^x)^* = S(w) \otimes 1. \end{aligned}$$

So, by ergodicity of δ_2 , we may conclude that $S(w) \in \overline{H_{\varphi(x)}} \otimes \mathcal{D}_1 \otimes \mathbb{C}$.

Because w has its second leg in \mathcal{D}_2 , we get that

$$\begin{aligned} (\text{id} \otimes \alpha_1 \otimes \text{id})S(w) &= (\text{id} \otimes \text{id} \otimes \delta_1)(w)(X_{14}^x)^* \\ &= (\text{id} \otimes \text{id} \otimes \delta_1)(w(X_{13}^x)^*)U_{13}^{\varphi(x)} \\ &= (\text{id} \otimes \text{id} \otimes \delta_1)(S(w))U_{13}^{\varphi(x)}. \end{aligned}$$

But we just proved that the third leg of $S(w)$ is scalar, so the last expression is nothing else than $(S(w) \otimes 1)U_{13}^{\varphi(x)}$. Thus, by the definition of $K_{\varphi(x)}$, we get that $S: K_x \rightarrow K_{\varphi(x)} \otimes \mathbb{C}$.

For every $v \in K_{\varphi(x)}$ and $w \in K_x$, we have that

$$\langle T(v), w \rangle_l = T(v)w^* = v_{12}X_{13}^xw^* = v_{12}S(w)^* = \langle v, S(w) \rangle_l.$$

So, S is actually the adjoint T^* of T in the sense of Hilbert C^* -modules. Moreover, it is trivial that $T^*T = 1 = TT^*$. Hence $T \in \mathcal{L}(K_{\varphi(x)}, K_x)$ is unitary for $\langle \cdot, \cdot \rangle_l$.

Next, we show that T is also a unitary element of $\mathcal{L}(K_{\varphi(x)}, K_x)$ for the right Hilbert C^* -module structure given by (3.2). From proposition 3.5 of [16], it suffices to show that T is isometric and surjective. The surjectivity follows from above. We use the orthogonality relations (2.3) and (4.3) for U^x and X^x to prove that T is indeed an isometry.

First notice that the conditional expectation $E_2: \mathcal{D}_2 \rightarrow \mathcal{D}_2^{\alpha_2}$ is nothing else than the map $a \mapsto (\text{id} \otimes \omega)(a) \otimes 1$, where ω is the invariant state for δ_1 and δ_2 . Indeed, for $a \in \mathcal{D}_2$,

$$E_2(a) = (\text{id} \otimes h_2)\alpha_2(a) = (\text{id} \otimes \text{id} \otimes h_2)(\text{id} \otimes \delta_2)(a) = (\text{id} \otimes \omega)(a) \otimes 1.$$

Consider now $v \in K_{\varphi(x)}$. On the one hand, we have that

$$\begin{aligned} 1 \otimes \langle T(v), T(v) \rangle_r &= (\text{id} \otimes E_2)((X_{13}^x)^*v_{12}^*v_{12}X_{13}^x) \\ &= (\text{id} \otimes \text{id} \otimes \omega)((X_{13}^x)^*v_{12}^*v_{12}X_{13}^x) \otimes 1 \\ &= \frac{1}{\dim_q(x)}(1 \otimes (\text{Tr} \otimes \text{id})((Q_{\varphi(x)}^{-1} \otimes 1)v^*v) \otimes 1) \end{aligned}$$

because of the orthogonality relations for X^x .

On the other hand

$$\begin{aligned} 1 \otimes \langle v, v \rangle_r &= (\text{id} \otimes E_1)(v^*v) = (\text{id} \otimes \text{id} \otimes h_1)(\text{id} \otimes \alpha_1)(v^*v) \\ &= (\text{id} \otimes \text{id} \otimes h_1)((U_{13}^x)^*v_{12}^*v_{12}U_{13}^x) \\ &= \frac{1}{\dim_q(x)}(1 \otimes (\text{Tr} \otimes \text{id})((Q_{\varphi(x)}^{-1} \otimes 1)v^*v)), \end{aligned}$$

where in the last step we used the orthogonality relations for U^x . Considering the map $\mathcal{D}_1^{\alpha_1} \rightarrow \mathcal{D}_2^{\alpha_2} : a \mapsto a \otimes 1$, the calculations above show that T is indeed isometric and hence unitary.

We now show that $(T_x)_{x \in \text{Ired}(\mathbb{G}_2)}$ preserves the monoidal structure. Take $v \in K_{\varphi(x)}$, $w \in K_{\varphi(y)}$ and $V \in \text{Mor}(x \otimes y, z)$. We calculate that

$$\begin{aligned} T_x(v)_{13} T_y(w)_{23} (V \otimes 1) &= v_{13} X_{14}^x w_{23} X_{24}^y (V \otimes 1) \\ &= v_{13} w_{23} (\varphi(V) \otimes 1) X_{13}^z = T_z(v_{13} w_{23} (\varphi(V) \otimes 1)), \end{aligned}$$

which proves the statement.

Finally, we prove the fourth part of the theorem. Recall the operators from formula (3.3).

$$R_{\varphi(x)} : v \mapsto (\varphi(t)^* \otimes 1)(1 \otimes v^*) \quad \text{and} \quad L_{\varphi(x)} = R_{\varphi(x)}^* R_{\varphi(x)}$$

with $v \in K_{\varphi(x)}$ and

$$R_x : w \mapsto (t^* \otimes 1)(1 \otimes w^*) \quad \text{and} \quad L_x = R_x^* R_x$$

with $w \in K_x$. Then

$$\begin{aligned} \langle v, L_{\varphi(x)} w \rangle 1 &= \langle R_{\varphi(x)} v, R_{\varphi(x)} w \rangle 1 \\ &= \langle (\varphi(t)^* \otimes 1)(1 \otimes v^*), (\varphi(t)^* \otimes 1)(1 \otimes w^*) \rangle \\ &= (\varphi(t)^* \otimes 1)(1 \otimes w^* v)(\varphi(t) \otimes 1) \end{aligned}$$

where $v, w \in K_{\varphi(x)}$. Remember the isomorphism $T_x : K_{\varphi(x)} \rightarrow K_x : v \mapsto v_{12} X_{13}^x$. Then:

$$\begin{aligned} \langle v_{12} X_{13}^x, L_x w_{12} X_{13}^x \rangle 1 &= \langle R_x v_{12} X_{13}^x, R_x w_{12} X_{13}^x \rangle 1 \\ &= (t^* \otimes 1)(1 \otimes (w_{12} X_{13}^x)^*(v_{12} X_{13}^x))(t \otimes 1) \\ &= (t^* \otimes 1 \otimes 1)((X_{24}^x)^* w_{23}^* v_{23} X_{24}^x)(t \otimes 1 \otimes 1) \\ &= (\varphi(t)^* \otimes 1 \otimes 1)(X_{14}^{\bar{x}} w_{23}^* v_{23} (X_{14}^{\bar{x}})^*)(\varphi(t) \otimes 1 \otimes 1) \\ &= (\varphi(t)^* \otimes 1 \otimes 1)(w_{23}^* v_{23})(\varphi(t) \otimes 1 \otimes 1) \\ &= (\varphi(t)^* \otimes 1)(1 \otimes w^* v)(\varphi(t) \otimes 1) \otimes 1. \end{aligned}$$

In this calculation, we have used that $X_{13}^{\bar{x}} X_{23}^x (t \otimes 1) = \varphi(t) \otimes 1$. This follows from the fact that $t \in \text{Mor}(\bar{x} \otimes x, \varepsilon)$. Again considering the map $\mathcal{D}_1^{\alpha_1} \rightarrow \mathcal{D}_2^{\alpha_2} : a \mapsto a \otimes 1$, we get that T_x intertwines L_x and $L_{\varphi(x)}$. It follows trivially from the definition 3.6 of quantum multiplicity that both quantum multiplicities are the same. This completes the proof of the theorem. \square

Remark 7.4. — It seems that the statement of the theorem cannot immediately be formulated on the C^* -algebraic level. If we define $D_2 = \{a \in D_1 \otimes B \mid (\alpha_1 \otimes \text{id})(a) = \text{id} \otimes \delta_1(a)\}$, as we did before for the algebraic and

the von Neumann algebraic case, it is not clear that $\alpha_2 = \text{id} \otimes \delta_2$ goes into $D_2 \otimes C(\mathbb{G}_2)$.

However, for von Neumann algebras, there is no problem. Suppose that $\alpha_1 : D_1 \rightarrow D_1 \overline{\otimes} L^\infty(\mathbb{G})$ is a von Neumann algebraic action and take the notations as before, where we now take the von Neumann algebraic link-algebra $B = (\mathcal{B}, \omega)''$. Here it does hold that $(\text{id} \otimes \delta_2)(D_1 \boxtimes^{\alpha_1} B) \subset D_1 \boxtimes^{\alpha_1} B \overline{\otimes} L^\infty(\mathbb{G}_2)$, as we only need to check that $(\text{id} \otimes \text{id} \otimes \mu)(\text{id} \otimes \delta_2)(a) \in D_1 \boxtimes^{\alpha_1} B$ for all $a \in D_1 \boxtimes^{\alpha_1} B$ and $\mu \in (L^\infty(\mathbb{G}_2))_*$. For C^* -algebras, this argument is not valid.

CLAIM. — *The algebra $\mathcal{D}_2 := D_1 \boxtimes_{\text{alg}}^{\alpha_1} B$ as defined in theorem 7.3 is precisely the spectral subalgebra of $(D_1 \boxtimes^{\alpha_1} B, \text{id} \boxtimes \delta_2)$.*

Proof. — Denote by $\widetilde{\mathcal{D}}_2$ the spectral subalgebra of $(D_1 \boxtimes^{\alpha_1} B, \text{id} \boxtimes \delta_2)$. It is clear that $D_1 \boxtimes_{\text{alg}}^{\alpha_1} B \subset \widetilde{\mathcal{D}}_2$.

On the other hand,

$$\widetilde{\mathcal{D}}_2 = \langle (\text{id} \otimes h)((\text{id} \boxtimes \delta_2)(a)(1 \otimes b)) \mid a \in D_1 \boxtimes^{\alpha_1} B, b \in \mathcal{C}(\mathbb{G}_2)_x, x \in \text{Irred}(\mathbb{G}) \rangle.$$

Because the elements of $\widetilde{\mathcal{D}}_2$ of course sit in $D_1 \boxtimes^{\alpha_1} B$, it is sufficient to prove that $\widetilde{\mathcal{D}}_2 \subset D_1 \otimes_{\text{alg}} B$.

If $a \in D_1 \boxtimes^{\alpha_1} B$, $b \in \mathcal{C}(\mathbb{G}_2)_x$ and $x \in \text{Irred}(\mathbb{G}_2)$, then $(\text{id} \otimes h)((\text{id} \boxtimes \delta_2)(a)(1 \otimes b))$ belongs to the strongly closed linear span of

$$\begin{aligned} & \{(\text{id} \otimes \text{id} \otimes h)((a \otimes \delta_2(d))(1 \otimes 1 \otimes b)) \mid a \in D_1, d \in B, b \in \mathcal{C}(\mathbb{G}_2)_x\} \\ & \subset D_1 \otimes_{\text{alg}} B_x. \end{aligned}$$

Since B_x is finite dimensional, $D_1 \otimes_{\text{alg}} B_x$ is already strongly closed in $D_1 \overline{\otimes} B$. Hence $c := (\text{id} \otimes h)((\text{id} \boxtimes \delta_2)(a)(1 \otimes b)) \in D_1 \otimes_{\text{alg}} B_x$ for all $a \in D_1 \boxtimes^{\alpha_1} B$ and $b \in \mathcal{C}(\mathbb{G}_2)_x$. On the other hand, $(\alpha_1 \otimes \text{id})(c) = (\text{id} \otimes \delta_1)(c)$, which implies that $c \in (D_1)_x \otimes_{\text{alg}} B_x \subset D_1 \otimes_{\text{alg}} B$. This ends the proof. \square

We can start from the comultiplication on \mathbb{G}_1 and apply the above construction. It is not surprising that we end up with the link algebra and the action δ_2 .

PROPOSITION 7.5. — *Consider two monoidally equivalent compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 . Then there exists a $*$ -isomorphism between $\mathcal{C}(\mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_1} B$ and the link algebra B . Moreover, this $*$ -isomorphism intertwines the action $\text{id} \boxtimes \delta_2$ with the action δ_2 .*

Proof. — We claim that $\delta_1 : B \rightarrow \mathcal{C}(\mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_1} B$ is the desired $*$ -isomorphism. From the definition of δ_1 , it follows that $\delta_1 : B \rightarrow \mathcal{C}(\mathbb{G}_1) \otimes_{\text{alg}} B$ is an injective $*$ -homomorphism. The image of δ_1 is contained in $\mathcal{C}(\mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_1} B$

because δ_1 is an action. Moreover, if $a \in \mathcal{C}(\mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_1} \mathcal{B}$ and ϵ_1 is the co-unit on $\mathcal{C}(\mathbb{G}_1)$, then

$$\delta_1((\epsilon_1 \otimes \text{id})(a)) = (\epsilon_1 \otimes \text{id} \otimes \text{id})(\Delta_1 \otimes \text{id})(a) = a$$

which means that $\delta_1(\mathcal{B}) = \mathcal{C}(\mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_1} \mathcal{B}$. So δ_1 is also surjective.

Because δ_1 and δ_2 commute, it is clear that this $*$ -isomorphism intertwines the actions δ_2 and $\text{id} \boxtimes \delta_2$. □

Now we consider the inverse monoidal equivalence $\varphi^{-1}: \mathbb{G}_1 \rightarrow \mathbb{G}_2$. According to theorem 4.2, we obtain the link algebra $\tilde{\mathcal{B}}$ generated by the coefficients of unitary elements $Y^x \in \mathcal{L}(H_{\varphi(x)}, H_x) \otimes_{\text{alg}} \tilde{\mathcal{B}}$ and two commuting ergodic actions

$$\gamma_2: \tilde{\mathcal{B}} \longrightarrow \mathcal{C}(\mathbb{G}_2) \otimes_{\text{alg}} \tilde{\mathcal{B}} \quad \text{and} \quad \gamma_1: \tilde{\mathcal{B}} \longrightarrow \tilde{\mathcal{B}} \otimes_{\text{alg}} \mathcal{C}(\mathbb{G}_1)$$

with

$$(7.2) \quad (\text{id} \otimes \gamma_2)(Y^x) = U_{12}^x Y_{13}^x \quad \text{and} \quad (\text{id} \otimes \gamma_1)(Y^x) = Y_{12}^x U_{13}^{\varphi(x)}.$$

Denote by $\tilde{\omega}$ the invariant state on $\tilde{\mathcal{B}}$. Then we get the following proposition.

PROPOSITION 7.6. — *Consider two monoidally equivalent compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 . Then we obtain a $*$ -isomorphism $\pi: \mathcal{C}(\mathbb{G}_2) \rightarrow \tilde{\mathcal{B}} \boxtimes_{\text{alg}}^{\gamma_1} \mathcal{B}$. This $*$ -isomorphism intertwines the comultiplication Δ_2 with the action $\text{id} \boxtimes \delta_2$.*

Proof. — We define the linear map $\pi: \mathcal{C}(\mathbb{G}_2) \rightarrow \tilde{\mathcal{B}} \otimes \mathcal{B}$ where

$$(\text{id} \otimes \pi)(U^x) = Y_{12}^x X_{13}^x.$$

Because

$$(\text{id} \otimes \gamma_1 \otimes \text{id})(Y_{12}^x X_{13}^x) = Y_{12}^x U_{13}^{\varphi(x)} X_{14}^x = (\text{id} \otimes \text{id} \otimes \delta_1)(Y_{12}^x X_{13}^x),$$

the image of π lies in $\tilde{\mathcal{B}} \boxtimes_{\text{alg}}^{\gamma_1} \mathcal{B}$.

Consider $x, y, z \in \text{Irred}(\mathbb{G}_2)$ and take $T \in \text{Mor}(x \otimes y, z)$. The multiplicativity of π follows from the following calculation:

$$\begin{aligned} (\text{id} \otimes \pi)(U_{13}^x U_{23}^y (T \otimes 1)) &= (\text{id} \otimes \pi)((T \otimes 1)U^z) = (T \otimes 1 \otimes 1)(Y_{12}^z X_{13}^z) \\ &= Y_{13}^x Y_{23}^y (\varphi(T) \otimes 1 \otimes 1) X_{13}^z \\ &= Y_{13}^x Y_{23}^y X_{14}^x X_{24}^y (T \otimes 1 \otimes 1) \\ &= (\text{id} \otimes \pi)(U^x)_{134} (\text{id} \otimes \pi)(U^y)_{234} (T \otimes 1 \otimes 1). \end{aligned}$$

Take now $t_x \in \text{Mor}(x \otimes \bar{x}, \varepsilon)$. Because

$$U_{13}^x U_{23}^{\bar{x}} (t_x \otimes 1) = t_x \otimes 1,$$

it follows that

$$\begin{aligned}
 (\text{id} \otimes \text{id} \otimes \pi)(U_{13}^{x*}(t_x \otimes 1)) &= (\text{id} \otimes \text{id} \otimes \pi)(U_{23}^{\bar{x}}(t_x \otimes 1)) \\
 &= Y_{23}^{\bar{x}} X_{24}^{\bar{x}}(t_x \otimes 1 \otimes 1) \\
 &= Y_{23}^{\bar{x}} X_{14}^{x*}(\varphi(t_x) \otimes 1 \otimes 1) \\
 &= X_{14}^{x*} Y_{13}^{x*}(t_x \otimes 1 \otimes 1).
 \end{aligned}$$

This proves that π also passes through the involution, so it is a $*$ -homomorphism. We now show that this map is the desired $*$ -isomorphism.

First we prove the injectivity. It is easy to show that $(\tilde{\omega} \otimes \omega)\pi = h$, with h the Haar measure of $\mathcal{C}(\mathbb{G}_2)$. Suppose now that for an $a \in \mathcal{C}(\mathbb{G}_2)$, $\pi(a) = 0$. Then also $\pi(a^*a) = 0$, which means that also $h(a^*a) = 0$. But h is faithful on $\mathcal{C}(\mathbb{G}_2)$, so $a = 0$.

To prove the surjectivity of π , we have to take a closer look at the elements of $\tilde{\mathcal{B}} \boxtimes_{\text{alg}}^{\gamma_1} \mathcal{B}$. From definition 3.4, we get that

$$\mathcal{B} = \bigoplus_{x \in \text{Irred}(\mathbb{G}_1)} \mathcal{B}_x \quad \text{and} \quad \tilde{\mathcal{B}} = \bigoplus_{x \in \text{Irred}(\mathbb{G}_1)} \tilde{\mathcal{B}}_x$$

where $\delta_1(\mathcal{B}_x) \subseteq \mathcal{C}(\mathbb{G}_1)_x \otimes_{\text{alg}} \mathcal{B}_x$ and $\gamma_1(\tilde{\mathcal{B}}_x) \subseteq \tilde{\mathcal{B}}_x \otimes_{\text{alg}} \mathcal{C}(\mathbb{G}_1)_x$. Suppose $b \in \tilde{\mathcal{B}} \boxtimes_{\text{alg}}^{\gamma_1} \mathcal{B}$. We claim that $b \in \bigoplus_{x \in \text{Irred}(\mathbb{G}_1)} \tilde{\mathcal{B}}_x \otimes_{\text{alg}} \mathcal{B}_x$. First notice that

$$\tilde{\mathcal{B}} \otimes_{\text{alg}} \mathcal{B} = \bigoplus_{x, y \in \text{Irred}(\mathbb{G}_1)} \tilde{\mathcal{B}}_x \otimes_{\text{alg}} \mathcal{B}_y.$$

So $b = \sum b_{xy}$ with $b_{xy} \in \tilde{\mathcal{B}}_x \otimes_{\text{alg}} \mathcal{B}_y$. Since $\delta_1(\mathcal{B}_y) \subseteq \mathcal{C}(\mathbb{G}_1)_x \otimes_{\text{alg}} \mathcal{B}_y$ and $\gamma_1(\tilde{\mathcal{B}}_x) \subseteq \tilde{\mathcal{B}}_x \otimes_{\text{alg}} \mathcal{C}(\mathbb{G}_1)_x$, it follows that $b_{xy} = 0$ if $x \neq y$.

So we only need to prove that $\pi(\mathcal{C}(\mathbb{G}_2)_x) = \tilde{\mathcal{B}}_x \boxtimes_{\text{alg}}^{\gamma_1} \mathcal{B}_x$,

Therefore, remember the formulas $(\text{id} \otimes \gamma_1)(Y^x) = Y_{12}^x U_{13}^{\varphi(x)}$ and $(\text{id} \otimes \delta_1)(X^x) = U_{12}^{\varphi(x)} X_{13}^x$ where $Y^x \in \mathcal{L}(H_{\varphi(x)}, H_x) \otimes_{\text{alg}} \tilde{\mathcal{B}}_x$ and $X^x \in \mathcal{L}(H_x, H_{\varphi(x)}) \otimes_{\text{alg}} \mathcal{B}_x$.

We know that a basis of \mathcal{B}_x (resp. $\tilde{\mathcal{B}}_x$) is given respectively by elements of the form $(\omega_{\widetilde{e_{k_x}, e_{l_x}}} \otimes \text{id})(X^x)$ and $(\omega_{\widetilde{e_{l_x}, e_{k_x}}} \otimes \text{id})(Y^x)$ with $\widetilde{e_{k_x}}$, $k_x \in \{1, \dots, \dim(\varphi(x))\}$ an orthonormal basis in $H_{\varphi(x)}$ and e_{l_x} , $l_x \in \{1, \dots, \dim(x)\}$ an orthonormal basis in H_x . Denote $(\omega_{\widetilde{e_{k_x}, e_{l_x}}} \otimes \text{id})(X^x) := b_{k_x, l_x}$ and $(\omega_{\widetilde{e_{l_x}, e_{k_x}}} \otimes \text{id})(Y^x) := \tilde{b}_{l_x, k_x}$. We also have a basis for $\mathcal{C}(\mathbb{G}_2)_x$ given by $(\omega_{e_{k_x}, e_{l_x}} \otimes \text{id})(U^x)$, again with e_{k_x} , $k_x \in \{1, \dots, \dim(x)\}$ an orthonormal basis in H_x . Denote by $(\omega_{e_{k_x}, e_{l_x}} \otimes \text{id})(U^x) := u_{k_x, l_x}$. In the following, we drop the subscript x . With these notations, we get that

$$\gamma_1(\tilde{b}_{kl}) = \sum_{p=1}^{\dim(\varphi(x))} \tilde{b}_{kp} \otimes u_{pl} \quad \text{and} \quad \delta_1(b_{ij}) = \sum_{q=1}^{\dim(\varphi(x))} u_{iq} \otimes x_{qj}$$

and

$$\pi(u_{kj}) = \sum_{l=1}^{\dim(\varphi(x))} \tilde{b}_{kl} \otimes b_{lj}.$$

An arbitrary element of $\mathcal{B}_x \otimes_{\text{alg}} \tilde{\mathcal{B}}_x$ has the form

$$a = \sum_{kli j} \lambda_{li}^{kj} \tilde{b}_{kl} \otimes b_{ij}.$$

We get that

$$(\gamma_1 \otimes \text{id})(a) = \sum_{klijp} \lambda_{li}^{kj} \tilde{b}_{kp} \otimes u_{pl} \otimes b_{ij}$$

equals

$$(\text{id} \otimes \delta_1)(a) = \sum_{klijq} \lambda_{li}^{kj} \tilde{b}_{kl} \otimes u_{iq} \otimes b_{qj}.$$

From this equality, we immediately see that $\lambda_{li}^{kj} = 0$ if $l \neq i$. We have also that $\lambda_{li}^{kj} = \lambda_{pp}^{kj}$ for all k, l, j, p . Indeed, the above equality provides us with the following equalities:

$$\sum_{lq} \lambda_{li}^{kj} \tilde{b}_{kl} \otimes u_{lq} \otimes b_{qj} = \sum_{lp} \lambda_{li}^{kj} \tilde{b}_{kp} \otimes u_{pl} \otimes b_{lj}$$

for every k, j . This can only happen when $\lambda_{li}^{kj} = \lambda^{kj}$ for every $l \in \{1, \dots, \dim(H_{\varphi(x)})\}$. So $a \in \tilde{\mathcal{B}}_x \boxtimes_{\text{alg}}^1 \mathcal{B}_x$ has the form

$$a = \sum_{kj} \lambda^{kj} \left(\sum_{l=1}^{\dim(\varphi(x))} \tilde{b}_{kl} \otimes b_{lj} \right)$$

which is a linear combination of the $\pi(u_{kj})$. This proves the surjectivity of π .

Moreover

$$\begin{aligned} (\text{id} \otimes \pi \otimes \text{id})(\text{id} \otimes \Delta_2)(U^x) &= (\text{id} \otimes \pi \otimes \text{id})(U_{12}^x U_{13}^x) \\ &= Y_{12}^x X_{13}^x U_{14}^x = (\text{id} \otimes (\text{id} \otimes \delta_2)\pi)(U^x), \end{aligned}$$

so the action $\text{id} \boxtimes \delta_2$ indeed corresponds to the comultiplication on \mathbb{G}_2 . \square

A combination of the two previous propositions now enables us to prove the reversibility of our construction.

PROPOSITION 7.7. — Consider two monoidally equivalent compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 and suppose also that $\alpha_1: \mathcal{D}_1 \rightarrow \mathcal{D}_1 \otimes \mathcal{C}(\mathbb{G}_1)$ is an action. Then \mathcal{D}_1 and $(\mathcal{D}_1 \boxtimes_{\text{alg}}^{\alpha_1} \mathcal{B}) \boxtimes_{\text{alg}}^{\alpha_2} \tilde{\mathcal{B}}$ are *-isomorphic. Moreover, this *-isomorphism intertwines the actions

$$\alpha_1 \quad \text{and} \quad \alpha'_1 = (\text{id} \otimes \text{id} \otimes \gamma_1) |_{(\mathcal{D}_1 \boxtimes_{\text{alg}}^{\alpha_1} \mathcal{B}) \boxtimes_{\text{alg}}^{\alpha_2} \tilde{\mathcal{B}}}.$$

Proof. — In exactly the same way as in proposition 7.6, we can prove that $\mathcal{C}(\mathbb{G}_1)$ is *-isomorphic to $\mathcal{B} \boxtimes_{\text{alg}}^{\delta_2} \tilde{\mathcal{B}}$. In this case, the *-isomorphism is given by $\pi: \mathcal{C}(\mathbb{G}_1) \rightarrow \mathcal{B} \boxtimes_{\text{alg}}^{\delta_2} \tilde{\mathcal{B}}$ where $(\text{id} \otimes \pi)(U^{\varphi(x)}) = X_{12}^x Y_{13}^x$. Also in the same way, we can prove that π intertwines the actions $\delta_1 \otimes \text{id} |_{\mathcal{B} \boxtimes_{\text{alg}}^{\delta_2} \tilde{\mathcal{B}}}$ and Δ_1 . From this, we get that $(\mathcal{D}_1 \boxtimes_{\text{alg}}^{\alpha_1} \mathcal{B}) \boxtimes_{\text{alg}}^{\alpha_2} \tilde{\mathcal{B}}$ is isomorphic to

$$\mathcal{D}'_1 := \{a \in \mathcal{D}_1 \otimes_{\text{alg}} \mathcal{C}(\mathbb{G}_1) \mid (\alpha_1 \otimes \text{id})(a) = (\text{id} \otimes \Delta_1)(a)\}.$$

From the calculation

$$\begin{aligned} (\text{id} \otimes \pi \otimes \text{id})(\text{id} \otimes \Delta_1)(U^{\varphi(x)}) &= (\text{id} \otimes \pi \otimes \text{id})(U_{12}^{\varphi(x)} U_{13}^{\varphi(x)}) = X_{12}^x Y_{13}^x U_{14}^{\varphi(x)} \\ &= (\text{id} \otimes \text{id} \otimes \gamma_1)(\text{id} \otimes \pi)(U^{\varphi(x)}), \end{aligned}$$

it follows that $(\text{id} \otimes \text{id} \otimes \gamma_1) |_{(\mathcal{D}_1 \boxtimes_{\text{alg}}^{\alpha_1} \mathcal{B}) \boxtimes_{\text{alg}}^{\alpha_2} \tilde{\mathcal{B}}}$ is equivalent with $\text{id} \otimes \Delta_1 |_{\mathcal{D}'_1}$ under the *-isomorphism $\text{id} \otimes \pi$.

In the same way as in proposition 7.5, we see that $\alpha_1: \mathcal{D}_1 \rightarrow \mathcal{D}'_1$ is a *-isomorphism. It is obvious that this *-isomorphism intertwines the actions $\text{id} \otimes \Delta_1 |_{\mathcal{D}'_1}$ and α_1 . Thus we get that

$$(7.3) \quad (\text{id} \otimes \pi) \circ \alpha_1: \mathcal{D}_1 \longrightarrow (\mathcal{D}_1 \boxtimes_{\text{alg}}^{\alpha_1} \mathcal{B}) \boxtimes_{\text{alg}}^{\alpha_2} \tilde{\mathcal{B}}$$

is a *-isomorphism and that it intertwines

$$\alpha_1 \quad \text{and} \quad \alpha'_1 = (\text{id} \otimes \text{id} \otimes \gamma_1) |_{(\mathcal{D}_1 \boxtimes_{\text{alg}}^{\alpha_1} \mathcal{B}) \boxtimes_{\text{alg}}^{\alpha_2} \tilde{\mathcal{B}}}.$$

This concludes the proof. □

Remark 7.8. — Theorem 7.3 and proposition 7.7 show that the assignment $\mathcal{D}_1 \rightarrow \mathcal{D}_1 \boxtimes_{\text{alg}}^{\alpha_1} \mathcal{B}$ and $\alpha_1 \rightarrow \text{id} \boxtimes \delta_2$ yields a bijective correspondence between actions of \mathbb{G}_1 and actions of \mathbb{G}_2 (up to conjugacy).

7.2. Special cases

The case where D_1 is a homogeneous space

Suppose we are given two monoidally equivalent compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 with monoidal equivalence $\varphi: \mathbb{G}_2 \rightarrow \mathbb{G}_1$. Denote by δ_1

and δ_2 again the ergodic actions of full quantum multiplicity on the corresponding link-algebra \mathcal{B} as in formula (7.1). Consider a quantum subgroup \mathbb{H}_1 of \mathbb{G}_1 and the corresponding action

$$\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1} : \mathcal{C}(\mathbb{H}_1 \setminus \mathbb{G}_1) \longrightarrow \mathcal{C}(\mathbb{H}_1 \setminus \mathbb{G}_1) \otimes \mathcal{C}(\mathbb{G}_1)$$

on the homogeneous space.

To a homogeneous space of a compact quantum group, naturally there corresponds a homogeneous space of the link algebra.

DEFINITION 7.9. — We define the homogeneous space $\mathcal{B}^{\mathbb{H}_1}$ by

$$\mathcal{B}^{\mathbb{H}_1} := \{a \in \mathcal{B} \mid (r_{\mathbb{H}_1} \otimes \text{id})\delta_1(a) = 1 \otimes a\}.$$

Note that δ_2 is an action on $\mathcal{B}^{\mathbb{H}_1}$ because δ_1 and δ_2 commute.

PROPOSITION 7.10. — There is a $*$ -isomorphism between $\mathcal{C}(\mathbb{H}_1 \setminus \mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1}} \mathcal{B}$ and $\mathcal{B}^{\mathbb{H}_1}$. Moreover, this $*$ -isomorphism intertwines the action $\text{id} \boxtimes \delta_2$ with the restriction of δ_2 to $\mathcal{B}^{\mathbb{H}_1}$.

Proof. — We prove, as in proposition 7.5, that

$$\delta_1 : \mathcal{B}^{\mathbb{H}_1} \longrightarrow \mathcal{C}(\mathbb{H}_1 \setminus \mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1}} \mathcal{B}$$

is a $*$ -isomorphism. Because

$$\begin{aligned} (r_{\mathbb{H}_1} \otimes \text{id})(\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1} \otimes \text{id})\delta_1(a) &= (r_{\mathbb{H}_1} \otimes \text{id})(\text{id} \otimes \delta_1)\delta_1(a) \\ &= (\text{id} \otimes \delta_1)(1 \otimes a) = 1 \otimes \delta_1(a), \end{aligned}$$

we get that $\delta_1(\mathcal{B}^{\mathbb{H}_1}) \subseteq \mathcal{C}(\mathbb{H}_1 \setminus \mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1}} \mathcal{B}$. The injectivity of δ_1 is clear. The surjectivity follows from the fact that for $a \in \mathcal{C}(\mathbb{H}_1 \setminus \mathbb{G}_1) \boxtimes_{\text{alg}}^{\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1}} \mathcal{B}$,

$$\begin{aligned} (7.4) \quad \delta_1(\epsilon \otimes \text{id})(a) &= (\epsilon \otimes \text{id} \otimes \text{id})(\text{id} \otimes \delta_1)(a) \\ &= (\epsilon \otimes \text{id} \otimes \text{id})(\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1} \otimes \text{id})(a) = a \end{aligned}$$

and

$$(r_{\mathbb{H}_1} \otimes \text{id})\delta_1(\epsilon \otimes \text{id})(a) = (r_{\mathbb{H}_1} \otimes \text{id})(a) = (\epsilon \otimes \text{id})(a),$$

where in the last step we used lemma 3.17. Because δ_1 and δ_2 commute, δ_1 intertwines $\text{id} \boxtimes \delta_2$ with the restriction of δ_2 to $\mathcal{B}^{\mathbb{H}_1}$. This ends the proof. \square

The case where α_1 is the adjoint action $\alpha_{\mathbb{G}_1}$

We now look at the special case where we are dealing with the adjoint action, introduced in section 5:

$$\alpha_{\mathbb{G}_1} : \ell^\infty(\widehat{\mathbb{G}}_1) \longrightarrow \ell^\infty(\widehat{\mathbb{G}}_1) \overline{\otimes} L^\infty(\mathbb{G}_1) : \alpha_{\mathbb{G}_1}(a) = \mathbb{V}_1(a \otimes 1) \mathbb{V}_1^*$$

We get the following proposition:

PROPOSITION 7.11. — *The mapping*

$$(7.5) \quad \rho : \ell^\infty(\widehat{\mathbb{G}}_2) \longrightarrow \ell^\infty(\widehat{\mathbb{G}}_1) \boxtimes^{\alpha_{\mathbb{G}_1}} B : \rho(b) = \mathbb{X}(b \otimes 1) \mathbb{X}^*$$

is a *-isomorphism. Moreover, this *-isomorphism intertwines the action $\text{id} \boxtimes \delta_2$ with the adjoint action $\alpha_{\mathbb{G}_2}$ on $\ell^\infty(\widehat{\mathbb{G}}_2)$. We also have that

$$\rho : \mathcal{L}(H_x) \longrightarrow \mathcal{L}(H_{\varphi(x)}) \boxtimes_{\text{alg}}^{\alpha_{\mathbb{G}_1}} \mathcal{B}$$

is a *-isomorphism.

Proof. — The following calculation

$$\begin{aligned} (\text{id} \otimes \delta_1)(\mathbb{X}(b \otimes 1) \mathbb{X}^*) &= (\mathbb{V}_1)_{12} \mathbb{X}_{13}(b \otimes 1 \otimes 1) \mathbb{X}_{13}^* (\mathbb{V}_1)_{12}^* \\ &= (\alpha_{\mathbb{G}_1} \otimes \text{id})(\mathbb{X}(b \otimes 1) \mathbb{X}^*) \end{aligned}$$

shows that $\rho : \ell^\infty(\widehat{\mathbb{G}}_2) \rightarrow \ell^\infty(\widehat{\mathbb{G}}_1) \boxtimes^{\alpha_{\mathbb{G}_1}} B$ is an injective *-homomorphism. For every $a \in \ell^\infty(\widehat{\mathbb{G}}_1) \boxtimes^{\alpha_{\mathbb{G}_1}} B$, we have that

$$\begin{aligned} (\text{id} \otimes \delta_1)(\mathbb{X}^* a \mathbb{X}) &= \mathbb{X}_{13}^* (\mathbb{V}_1)_{12}^* (\text{id} \otimes \delta_1)(a) (\mathbb{V}_1)_{12} \mathbb{X}_{13} \\ &= \mathbb{X}_{13}^* (\mathbb{V}_1)_{12}^* (\alpha_{\mathbb{G}_1} \otimes \text{id})(a) (\mathbb{V}_1)_{12} \mathbb{X}_{13} = (\mathbb{X}^* a \mathbb{X})_{13}. \end{aligned}$$

Because δ_1 is ergodic, it follows that $\mathbb{X}^* a \mathbb{X} = y \otimes 1$ with $y \in \ell^\infty(\widehat{\mathbb{G}}_2)$. From this; we get that for every $a \in \ell^\infty(\widehat{\mathbb{G}}_1) \boxtimes^{\alpha_{\mathbb{G}_1}} B$ there exists an $y \in \ell^\infty(\widehat{\mathbb{G}}_2)$ such that $a = \mathbb{X}(y \otimes 1) \mathbb{X}^*$. This proves the surjectivity of ρ . Note that ρ^{-1} is given by $\rho^{-1}(a) = (\text{id} \otimes \omega)(\mathbb{X}^* a \mathbb{X})$ for every $a \in \ell^\infty(\widehat{\mathbb{G}}_1) \boxtimes^{\alpha_{\mathbb{G}_1}} B$.

We prove now that $(\text{id} \boxtimes \delta_2) \circ \rho = (\rho \otimes \text{id}) \circ \alpha_{\mathbb{G}_2}$. This follows from the following calculation

$$\begin{aligned} (\text{id} \boxtimes \delta_2)(\rho(b)) &= (\text{id} \otimes \delta_2)(\mathbb{X}(b \otimes 1) \mathbb{X}^*) \\ &= \mathbb{X}_{12} (\mathbb{V}_2)_{13}(b \otimes 1 \otimes 1) (\mathbb{V}_2)_{13}^* \mathbb{X}_{12}^* = (\rho \otimes \text{id})(\alpha_{\mathbb{G}_2}(b)). \end{aligned}$$

It is also immediately clear that ρ sends $\mathcal{L}(H_x)$ to $\mathcal{L}(H_{\varphi(x)}) \boxtimes_{\text{alg}}^{\alpha_{\mathbb{G}_1}} \mathcal{B}$ and that ρ^{-1} sends $\mathcal{L}(H_{\varphi(x)}) \boxtimes_{\text{alg}}^{\alpha_{\mathbb{G}_1}} \mathcal{B}$ to $\mathcal{L}(H_x)$. This completes the proof. \square

Remark 7.12. — This result suggest strongly that Poisson and Martin boundaries of \mathbb{G}_2 are related to the bijective construction obtained in this section. In the next two sections, we show that this is indeed the case.

We conclude this section with an important remark about minimal actions of compact quantum groups and their applications to subfactors [3], [5]. An action is minimal if $M \cap (M^\alpha)' = \mathbb{C}1$ and if a faithfulness condition hold. By this, we mean that the irreducible representations contained in α must generate $\text{Irred}(\mathbb{G})$. Suppose we start with a minimal action α on a von Neumann algebra M . It follows from theorem 7.3 that the faithfulness condition holds also for $\text{id} \boxtimes \delta_2$. Again, from theorem 7.3, it follows that $(M \boxtimes^\alpha B)^{\text{id} \boxtimes \delta_2} = M^\alpha \otimes 1$. Using the fact that α is minimal and that δ_1 is ergodic, we get that $\text{id} \boxtimes \delta_2$ on $M \boxtimes^\alpha B$ is again a minimal action. This means that the construction of above preserves minimality of actions. By theorem 7.2 of [27], one might think that it is easier to construct minimal actions of compact quantum groups that are of Kac type. This is not the case. Indeed, the argument just given and theorem 4.5 show that constructing minimal actions of $A_o(n)$ is the same problem as for $SU_q(2)$.

8. Poisson boundaries of monoidally equivalent quantum groups

In this section we prove that the Poisson boundaries of two monoidally equivalent quantum groups correspond with each other through the construction of Theorem 7.3. Recall that, because of remark 7.4, we may do all computations immediately on the von Neumann algebraic level.

Consider two monoidally equivalent compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 where the monoidal equivalence is given by $\varphi: \mathbb{G}_2 \rightarrow \mathbb{G}_1$ with corresponding link algebra B and commuting actions δ_1 and δ_2 .

Notation 8.1. — From now on, we write respectively

$$\mathbb{V}_1 := \bigoplus_{x \in \text{Irred}(\mathbb{G}_1)} U^{\varphi(x)} \quad \text{and} \quad \mathbb{V}_2 := \bigoplus_{x \in \text{Irred}(\mathbb{G}_2)} U^x,$$

where $\{U^{\varphi(x)} \mid x \in \text{Irred}(\mathbb{G}_1)\}$ and $\{U^x \mid x \in \text{Irred}(\mathbb{G}_2)\}$ denote the set of irreducible representations of respectively \mathbb{G}_1 and \mathbb{G}_2 . We also denote by $B = (\mathcal{B}, \omega)''$ the von Neumann algebraic link algebra of the monoidal equivalence and by $\mathbb{X} := \bigoplus_{x \in \text{Irred}(\mathbb{G}_2)} X^x$. We denote the states φ_μ^1 and ψ_μ^1 , respectively φ_μ^2 and ψ_μ^2 on $\ell^\infty(\widehat{\mathbb{G}}_1)$, respectively $\ell^\infty(\widehat{\mathbb{G}}_2)$.

Let μ be a generating probability measure on $\text{Irred}(\mathbb{G}_1)$. Consider the Poisson boundary $H^\infty(\widehat{\mathbb{G}}_1, \mu)$ of \mathbb{G}_1 with adjoint action

$$\alpha_{\mathbb{G}_1}: H^\infty(\widehat{\mathbb{G}}_1, \mu) \longrightarrow H^\infty(\widehat{\mathbb{G}}_1, \mu) \overline{\otimes} L^\infty(\mathbb{G}_1): \alpha_{\mathbb{G}_1}(a) = \mathbb{V}_1(a \otimes 1)(\mathbb{V}_1)^*$$

We get the following theorem:

THEOREM 8.2. — Consider two monoidally equivalent compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 and let μ be a generating probability measure on $\text{Irred}(\mathbb{G}_1)$. Then the following

$$\rho: H^\infty(\widehat{\mathbb{G}}_2, \mu) \longrightarrow H^\infty(\widehat{\mathbb{G}}_1, \mu) \boxtimes^{\alpha_{\mathbb{G}_1}} B: \rho(b) = \mathbb{X}(b \otimes 1)\mathbb{X}^*$$

is a *-isomorphism. Moreover, this *-isomorphism intertwines the action $\text{id} \boxtimes \delta_2$ with the adjoint action $\alpha_{\mathbb{G}_2}$.

Proof. — In the next proposition, we prove that $(P_{\mu,1} \otimes \text{id}) \circ \rho = \rho \circ P_{\mu,2}$. Now, because $\rho: \ell^\infty(\widehat{\mathbb{G}}_2) \rightarrow \ell^\infty(\widehat{\mathbb{G}}_1) \boxtimes^{\alpha_{\mathbb{G}_1}} B$ is a *-isomorphism and because of the definition of the product on $H^\infty(\widehat{\mathbb{G}}_1, \mu)$ and $H^\infty(\widehat{\mathbb{G}}_2, \mu)$, we get that $\rho: H^\infty(\widehat{\mathbb{G}}_2, \mu) \rightarrow H^\infty(\widehat{\mathbb{G}}_1, \mu) \boxtimes^{\alpha_{\mathbb{G}_1}} B$ is a *-isomorphism. \square

PROPOSITION 8.3. — Let μ be a probability measure on $\text{Irred}(\mathbb{G}_1)$. We have that

$$(8.1) \quad (P_{\mu,1} \otimes \text{id})(\rho(b)) = \rho(P_{\mu,2}(b))$$

for every $b \in \ell^\infty(\widehat{\mathbb{G}}_2)$.

Proof. — Let $b \in \ell^\infty(\widehat{\mathbb{G}}_2)$. We claim that for $x, y \in \text{Irred}(\mathbb{G}_1)$,

$$(p_x \otimes p_y \otimes 1)(\widehat{\Delta}_1 \otimes \text{id})(\rho(b)) = X_{13}^x X_{23}^y (\widehat{\Delta}_2(b) \otimes 1) (X_{23}^y)^* (X_{13}^x)^*.$$

Take now $z \in \text{Irred}(\mathbb{G}_1)$ and $\varphi(T) \in \text{Mor}(x \otimes y, z)$. Then

$$\begin{aligned} (8.2) \quad & (p_x \otimes p_y \otimes 1)(\widehat{\Delta}_1 \otimes \text{id})(\rho(b))(\varphi(T) \otimes 1) \\ &= (\varphi(T) \otimes 1)(\rho(b)) \\ &= (\varphi(T) \otimes 1)(X^z(b \otimes 1)(X^z)^*) \\ &= X_{13}^x X_{23}^y (T \otimes 1)(b \otimes 1)(X^z)^* \\ &= X_{13}^x X_{23}^y (\widehat{\Delta}_2 \otimes \text{id})(b)(T \otimes 1)(X^z)^* \\ &= X_{13}^x X_{23}^y (\widehat{\Delta}_2(b) \otimes 1)(X_{23}^y)^* (X_{13}^x)^* (\varphi(T) \otimes 1) \end{aligned}$$

where (8.2) is valid because

$$(T \otimes 1)(X^z)^* = (X_{23}^y)^* (X_{13}^x)^* (\varphi(T) \otimes 1).$$

Then, we get that

$$(p_x \otimes 1)(P_{y,1} \otimes \text{id})(\rho(b)) = X^x (\text{id} \otimes \psi_y^1 \otimes \text{id})(X_{23}^y (\widehat{\Delta}_2(b) \otimes 1) (X_{23}^y)^*) (X^x)^*.$$

We prove that

$$(8.3) \quad (\psi_y^1 \otimes \text{id})((X^y)(d \otimes 1)(X^y)^*) = \psi_y^2(d)1$$

for every $d \in \ell^\infty(\widehat{\mathbb{G}}_2)$.

If this last equality (8.3) is valid, then we get

$$(8.4) \quad (p_x \otimes 1)(P_{y,1} \otimes \text{id})(\rho(b)) = X^x((p_x \otimes 1)(P_{y,2}(b) \otimes 1))(X^x)^* \\ = (p_x \otimes 1)\rho(P_{y,2}(b)).$$

This means that $(P_{y,1} \otimes \text{id})(\rho(b)) = \rho(P_{\mu,2}(b))$ for every $y \in \text{Irred}(\mathbb{G})$ and (8.1) is true.

The only thing left to prove is formula (8.3). Using the definition of ψ_y^1 (2.6), we get

$$(8.5) \quad (\psi_y^1 \otimes \text{id})((X^y)(d \otimes 1)(X^y)^*) \\ = (\eta \otimes h_1 \otimes \text{id})(U_{12}^{\varphi(y)} X_{13}^y (d \otimes 1 \otimes 1)(X_{13}^y)^*(U_{12}^{\varphi(y)})^*)$$

with η any normal state on $\ell^\infty(\widehat{\mathbb{G}}_1)$. Then

$$(8.5) = (\eta \otimes h_1 \otimes \text{id})((\text{id} \otimes \delta_1)(X^y(d \otimes 1)(X^y)^*)) \\ = (\eta \otimes \omega)(X^y(d \otimes 1)(X^y)^*).1 = \psi_y^2(d)1$$

where in the last step, we used the orthogonality relations (4.3). This proves equality (8.3) and the proof is complete. □

9. Applications to Tomatsu’s work on Poisson boundaries

The fundamental result obtained in the last section combined with recent work of Tomatsu on Poisson boundaries makes it possible to identify the Poisson boundary of a large class of quantum groups. This will be the content of this section. First, it provides us with an identification of the Poisson boundary of all duals of compact quantum groups that are monoidally equivalent with a q -deformation of a compact Lie group. Moreover, it will enable us to give concrete identifications of Poisson boundaries of some classes of non-amenable discrete quantum groups. Observe that monoidal equivalence does not preserve coamenability. We study in detail the Poisson boundary of the duals of the quantum automorphism groups $A_{\text{aut}}(D, \omega)$. When $\dim(D) \geq 5$, these are actually non-amenable.

9.1. Tomatsu’s work on Poisson boundaries

In [24], Tomatsu has proven that the Poisson boundary of the dual of a coamenable compact quantum group with commutative fusion rules can

be identified with the homogeneous space coming from its canonical Kac subgroup (see definition 3.18).

The main result is the following.

THEOREM 9.1 (Theorem 4.8 in [24]). — *Let \mathbb{G} be a coamenable compact quantum group with commutative fusion rules and \mathbb{H} its canonical Kac subgroup. Let μ be a generating measure on $\text{Irred}(\mathbb{G})$. The Izumi operator*

$$\Phi: L^\infty(\mathbb{H} \setminus \mathbb{G}) \longrightarrow H^\infty(\widehat{\mathbb{G}}, \mu): a \longmapsto (\text{id} \otimes h)(\mathbb{V}^*(1 \otimes a)\mathbb{V})$$

*is a *-isomorphism and intertwines the adjoint action $\alpha_{\mathbb{G}}$ with the action $\Delta_{\mathbb{H} \setminus \mathbb{G}}$ as defined in remark 3.16.*

Remark 9.2. — The mapping Φ is the Izumi operator, introduced in [13].

This gives us immediately the Poisson boundary of a whole class of quantum groups. Moreover, for q -deformations of classical Lie-groups, Tomatsu proves that the canonical Kac subgroup is just the maximal torus.

9.2. Identification of a large class of Poisson boundaries

Using the previous section, we obtain the Poisson boundary of every compact quantum group with commutative fusion rules which is monoidally equivalent to a coamenable one. Moreover, we obtain a concrete description of the Poisson boundary as a homogeneous space of the link algebra.

So, consider a compact quantum group (\mathbb{G}_1, Δ_1) which is coamenable and has commutative fusion rules. Let (\mathbb{G}_2, Δ_2) be monoidally equivalent with (\mathbb{G}_1, Δ_1) with monoidal equivalence given by $\varphi: \mathbb{G}_2 \rightarrow \mathbb{G}_1$. Again, we have the link algebra B and the two commuting actions δ_1 and δ_2 as before. Denote by \mathbb{H}_1 the canonical Kac group of \mathbb{G}_1 and by $r_{\mathbb{H}_1}: \mathcal{C}(\mathbb{G}_1) \rightarrow \mathcal{C}(\mathbb{H}_1)$ the corresponding restriction map.

Tomatsu’s result combined with theorem 8.2 and proposition 7.10 gives us the following theorem:

THEOREM 9.3. — *Consider a coamenable compact quantum group \mathbb{G}_1 with commutative fusion rules. Let \mathbb{G}_2 be a compact quantum group that is monoidally equivalent to \mathbb{G}_1 . Denote by B the von Neumann algebraic link algebra associated to the monoidal equivalence. Let \mathbb{H}_1 be the canonical Kac subgroup of \mathbb{G}_1 . Consider a generating measure μ on $\text{Irred}(\mathbb{G}_1)$. Then $(H^\infty(\widehat{\mathbb{G}}_2, \mu), \alpha_{\mathbb{G}_2})$ is isomorphic to $(B^{\mathbb{H}_1}, \delta_2)$. The isomorphism is given by the following generalized Izumi operator*

$$(9.1) \quad \Theta: B^{\mathbb{H}_1} \rightarrow H^\infty(\widehat{\mathbb{G}}_2, \mu): a \longmapsto (\text{id} \otimes \omega)(\mathbb{X}^*(1 \otimes a)\mathbb{X}).$$

This $*$ -isomorphism intertwines the adjoint action $\alpha_{\mathbb{G}_2}$ and the action δ_2 .

Proof. — By Tomatsu, $\Phi: L^\infty(\mathbb{H}_1 \setminus \mathbb{G}_1) \rightarrow H^\infty(\widehat{\mathbb{G}}_1, \mu)$ is a $*$ -isomorphism. Because Φ intertwines the actions $\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1}$ and $\alpha_{\mathbb{G}_1}$,

$$\Phi \otimes \text{id}: L^\infty(\mathbb{H}_1 \setminus \mathbb{G}_1) \boxtimes^{\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1}} B \longrightarrow H^\infty(\widehat{\mathbb{G}}_1, \mu) \boxtimes^{\alpha_{\mathbb{G}_1}} B$$

is also a $*$ -isomorphism. Combining this with Theorem 8.2, it follows that $L^\infty(\mathbb{H}_1 \setminus \mathbb{G}_1) \boxtimes^{\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1}} B$ and $H^\infty(\widehat{\mathbb{G}}_2, \mu)$ are $*$ -isomorphic through $\rho^{-1} \circ (\Phi \otimes \text{id})$. It follows from proposition 7.10 that $\delta_1: B^{\mathbb{H}_1} \rightarrow L^\infty(\mathbb{H}_1 \setminus \mathbb{G}_1) \boxtimes^{\Delta_{\mathbb{H}_1 \setminus \mathbb{G}_1}} B$ is a $*$ -isomorphism. Hence

$$\rho^{-1} \circ (\Phi \otimes \text{id}) \circ \delta_1: B^{\mathbb{H}_1} \longrightarrow H^\infty(\widehat{\mathbb{G}}_2, \mu)$$

is a $*$ -isomorphism.

Now we just need to prove that $\rho^{-1} \circ (\Phi \otimes \text{id}) \circ \delta_1 = \Theta$, which follows from the next obvious calculation.

$$\begin{aligned} \rho^{-1}(\Phi \otimes \text{id})\delta_1(a) &= (\text{id} \otimes \omega)(\mathbb{X}^*(\text{id} \otimes h_1 \otimes \text{id})((\mathbb{V}_1)_{12}^*(\delta_1(a))_{23}(\mathbb{V}_1)_{12})\mathbb{X}) \\ &= (\text{id} \otimes \omega)(\text{id} \otimes h_1 \otimes \text{id})(\text{id} \otimes \delta_1)(\mathbb{X}^*(1 \otimes a)\mathbb{X}) \\ &= (\text{id} \otimes \omega)((\text{id} \otimes \omega)(\mathbb{X}^*(1 \otimes a)\mathbb{X}) \otimes 1) = \Theta(a) \end{aligned}$$

for all $a \in B^{\mathbb{H}_1}$.

Moreover, Θ intertwines $\alpha_{\mathbb{G}_2}$ and δ_2 because for all $a \in B^{\mathbb{H}_1}$,

$$\begin{aligned} (\Theta \otimes \text{id})\delta_2(a) &= (\text{id} \otimes \omega \otimes \text{id})(\mathbb{X}_{12}^*\delta_2(a)_{23}\mathbb{X}_{12}) \\ &= (\text{id} \otimes \omega \otimes \text{id})((\mathbb{V}_2)_{13}(\text{id} \otimes \delta_2)(\mathbb{X}^*(1 \otimes a)\mathbb{X})(\mathbb{V}_2)_{13}^*) \\ &= \mathbb{V}_2((\text{id} \otimes \omega)(\mathbb{X}^*(1 \otimes a)\mathbb{X}) \otimes 1)(\mathbb{V}_2)^* = \alpha_{\mathbb{G}_2}(\Theta(a)). \end{aligned}$$

This completes the proof. □

Now we have enough material to identify the Poisson boundary of some classes of discrete quantum groups which are not amenable. A first important class of quantum groups that satisfy this are the duals of $A_o(F)$. If $\dim(F) \geq 3$, then $A_o(F)$ is not coamenable. The Poisson boundary of their dual was already obtained in a different way (but also using monoidal equivalence) by Vaes and the second author in [28]. In fact, they started by constructing a generalized Izumi operator as in formula (9.1) for the specific case of $A_o(F)$. They proved that this Izumi operator is multiplicative on $L^\infty(A_o(F, F_q))^{\mathbb{T}}$ by using the monoidal equivalence of $A_o(F)$ and $SU_q(2)$. Hence, they reduced the identification problem to a purely $SU_q(2)$ -problem. However, as every $A_o(F)$ is monoidally equivalent to some $SU_q(2)$, we can identify the Poisson boundary of $\widehat{A_o(F)}$ also using theorem 9.3, which is much more general.

Another, new class of examples will come from quantum automorphism groups $A_{\text{aut}}(D, \omega)$, which we explore in the next section.

9.3. Examples: Quantum automorphism groups.

In this section we identify the Poisson boundary for $A_{\text{aut}}(\widehat{D}, \omega)$ with D a C^* -algebra of finite dimension ≥ 4 and ω a δ -state on D . To do this, we make use of the previous section.

From theorem 4.7, it follows that every quantum automorphism group of this type is monoidally equivalent with one of the form $A_{\text{aut}}(M_2(\mathbb{C}), \text{Tr}(\cdot F))$, where $\text{Tr}(F^{-1}) = \delta^2$. Because of the quantum Kesten result (see [3]), it follows that $A_{\text{aut}}(M_2(\mathbb{C}), \text{Tr}(\cdot F))$ is coamenable. Moreover, it has the fusion rules of $SO(3)$ and those are commutative. Hence, we can apply theorem 9.1 of Tomatsu. We now prove that the canonical Kac subgroup is just the one-dimensional torus \mathbb{T} .

The canonical Kac subgroup of $A_{\text{aut}}(M_2(\mathbb{C}), \text{Tr}(\cdot P))$

Denote by \mathbb{G} the compact quantum group $A_{\text{aut}}(M_2(\mathbb{C}), \text{Tr}(\cdot P))$ and by \mathbb{H} its canonical Kac subgroup. Observe that we can take P a diagonal matrix. We consider only non-trivial P here. If P equals the identity matrix, we just get the compact group $SO(3)$ which is already Kac. Hence $\widehat{SO(3)}$ has trivial Poisson boundary. Denote by $\pi: C(\mathbb{G}) \rightarrow C(\mathbb{H})$ the canonical projection map. We denote by U the fundamental irreducible representation with label 1 and by Q the matrix corresponding to U as defined in 2.2, normalized such that $\text{Tr}(Q) = \text{Tr}(Q^{-1})$. The eigenvalues of Q are of the form $1, q, q^{-1}$.

Now $V := (\text{id} \otimes \pi)(U)$ is a representation of \mathbb{H} and because \mathbb{H} is Kac, $\bar{V} = (\text{id} \otimes \pi)(\bar{U})$ must be unitary. The matrix $F = \sqrt{Q^T}$, unitarizes \bar{U} , what in this case means $(F \otimes 1)\bar{U}(F^{-1} \otimes 1) = U$. We claim that V breaks up in 3 one-dimensional representations. As every representation of \mathbb{H} appears in a repeated tensor power of V , it follows that all irreducible representations of \mathbb{H} have dimension one.

Proof of claim. — As F^*F has 3 different eigenvalues, it suffices to prove that F^*F and V commute. It holds that $V = F\bar{V}F^{-1}$, so

$$VF = F\bar{V} \quad \text{and} \quad F^*V^* = \bar{V}^*F^*.$$

As \bar{V} is unitary, it follows that $VFF^*V^* = FF^*$, which means that $F^*F \in \text{End}(V)$.

Since all irreducible representations of \mathbb{H} have dimension 1, we conclude that \mathbb{H} is the dual of a discrete group Γ . Denote by u_g the irreducible representation of \mathbb{H} corresponding to $g \in \Gamma$. Since $V \cong \bar{V}$, there are two cases possible. The first is that there exist $g, h \in \Gamma$ such that

$$V \cong u_g \oplus u_h \oplus u_{g^{-1}}.$$

Observe that Γ is generated by g and h . We claim that Γ is abelian. Since U is a subrepresentation of $U^{\otimes 2}$, V is a subrepresentation of $V^{\otimes 2}$. But

$$V^{\otimes 2} = u_{g^2} \oplus u_{h^2} \oplus u_{g^{-2}} \oplus 2u_e \oplus u_{hg} \oplus u_{gh} \oplus u_{hg^{-1}} \oplus u_{g^{-1}h},$$

implying that $h \in \{g^2, h^2, g^{-2}, e, hg, gh, hg^{-1}, g^{-1}h\}$. Any of these possibilities for h imply that the group generated by g and h is abelian.

The second case is that $V = u_a \oplus u_b \oplus u_c$ with $a^2 = b^2 = c^2 = e$. It is easy to show that in this case, Γ is the Klein group, which is abelian too.

As Γ is commutative, \mathbb{H} is just a commutative compact group. Hence the maximal quantum subgroup of Kac type of \mathbb{G} is the maximal compact subgroup of \mathbb{G} .

Suppose $\chi: C(\mathbb{G}) \rightarrow \mathbb{C}$ is a character and $\alpha: M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C}) \otimes C(\mathbb{G})$ the canonical action of \mathbb{G} on $M_2(\mathbb{C})$ coming from U . Now $(\text{id} \otimes \chi)\alpha$ is an automorphism of $M_2(\mathbb{C})$, and hence implemented by a unitary matrix A . Moreover, as $\text{Tr}(\cdot P)$ is invariant under $(\text{id} \otimes \chi)\alpha$,

$$\text{Tr}(PAxA^*) = \text{Tr}(Px) \quad \text{for all } x \in M_2(\mathbb{C}).$$

Hence A is a diagonal matrix. But then $\text{Ad}(A) = \text{Ad}(\text{diag}(z, \bar{z}))$ for some $z \in \mathbb{T}$.

On the other hand, \mathbb{T} acts on $M_2(\mathbb{C})$ by $\text{Ad}(\text{diag}(z, \bar{z}))$. This action δ is $\text{Tr}(\cdot P)$ -invariant, so because of the universality of \mathbb{G} , there exists a morphism of quantum groups $\pi: C(\mathbb{G}) \rightarrow C(\mathbb{T})$ such that $(\text{id} \otimes \pi)\alpha = \delta$.

We may conclude that the maximal Kac subgroup of \mathbb{G} is the one-dimensional torus \mathbb{T} . □

The Poisson boundary of $A_{\text{aut}}(D, \omega)$

Theorem 4.7 gives a nice description of the link algebra of two monoidally equivalent quantum automorphism groups $A_{\text{aut}}(D_1, \omega_1)$ and $A_{\text{aut}}(D_2, \omega_2)$. Together with theorem 9.3 we obtain the following result:

THEOREM 9.4. — *Consider $A_{\text{aut}}(D, \omega)$ with D a C^* -algebra of finite dimension strictly bigger than 4 and φ a δ -state on D . Take $F \in M_2(\mathbb{C})$ such that $\text{Tr}(F^{-1}) = \delta^2$. Then the Poisson boundary of $A_{\text{aut}}(\widehat{D}, \omega)$ is given by $L^\infty(A_{\text{aut}}((D, \omega), (M_2(\mathbb{C}), \text{Tr}(\cdot F))))^\mathbb{T}$.*

The 4-dimensional case was considered above, except for the case $A_{\text{aut}}(\mathbb{C}^4)$. But this compact quantum group is coamenable by the quantum Kesten result and moreover Kac, so $\widehat{A_{\text{aut}}(\mathbb{C}^4)}$ has trivial Poisson boundary. This completes the identification of Poisson boundaries of the duals of quantum automorphism groups.

10. The Martin boundary of monoidally equivalent quantum groups

We prove that the Martin boundaries of the duals of two monoidally equivalent compact quantum groups are related to each other through the construction of theorem 7.3.

So, again, we start from a monoidal equivalence $\varphi: \mathbb{G}_2 \rightarrow \mathbb{G}_1$ with link algebra B and commuting actions δ_1 and δ_2 .

THEOREM 10.1. — *Consider two monoidally equivalent compact quantum groups \mathbb{G}_1 and \mathbb{G}_2 and let μ be a generating probability measure on $\text{Irred}(\mathbb{G}_1)$. Suppose $A_{\mu,1}$ is the Martin boundary of the discrete quantum group $\widehat{\mathbb{G}}_1$. Then*

$$\rho: \widetilde{A}_{\mu,2} \longrightarrow \widetilde{A}_{\mu,1} \boxtimes_{\text{red}}^{\alpha_{\mathbb{G}_1}} B: b \longmapsto \mathbb{X}(b \otimes 1)\mathbb{X}^*$$

are also $*$ -isomorphisms. This $*$ -isomorphism intertwines the action $\text{id} \boxtimes \delta_2$ with the action $\alpha_{\mathbb{G}_2}$. Moreover, we have that $\rho: c_0(\widehat{\mathbb{G}}_2) \rightarrow c_0(\widehat{\mathbb{G}}_1) \boxtimes_{\text{red}}^{\alpha_{\mathbb{G}_1}} B$ is a $*$ -isomorphism, by which

$$\rho: A_{\mu,2} \longrightarrow A_{\mu,1} \boxtimes_{\text{red}}^{\pi_{\mathbb{G}_1}} B: b \longmapsto \mathbb{X}(b \otimes 1)\mathbb{X}^*$$

is also a $*$ -isomorphism which intertwines the action $\text{id} \boxtimes \delta_2$ with the action $\pi_{\mathbb{G}_2}$.

Proof. — Recall that

$$\rho: \mathcal{L}(H_x) \longrightarrow \mathcal{L}(H_{\varphi(x)}) \boxtimes_{\text{alg}}^{\alpha_{\mathbb{G}_1}} \mathcal{B}$$

is a $*$ -isomorphism. So it follows that

$$\rho: c_c(\widehat{\mathbb{G}}_2) \longrightarrow c_c(\widehat{\mathbb{G}}_1) \boxtimes_{\text{alg}}^{\alpha_{\mathbb{G}_1}} \mathcal{B} \quad \text{and} \quad \rho: c_0(\widehat{\mathbb{G}}_2) \longrightarrow c_0(\widehat{\mathbb{G}}_1) \boxtimes_{\text{red}}^{\alpha_{\mathbb{G}_1}} B$$

is also a $*$ -isomorphism. Proposition 8.3 gives us that

$$(P_{\bar{\mu},1} \otimes \text{id})(\rho(b)) = \rho(P_{\bar{\mu},2}(b)) \quad \text{and thus} \quad (G_{\bar{\mu},1} \otimes \text{id})(\rho(b)) = \rho(G_{\bar{\mu},2}(b))$$

for every $b \in c_c(\mathbb{G}_2)$. Because $G_{\bar{\mu}}(p_\varepsilon) \in M(c_c(\text{Irred}(\mathbb{G})))$ for every $\mu \in \text{Irred}(\mathbb{G})$, we get

$$\begin{aligned} \rho(K_{\bar{\mu},2}(b)) &= \mathbb{X}(G_{\bar{\mu},2}(p_\varepsilon)^{-1}G_{\bar{\mu},2}(b) \otimes 1)\mathbb{X}^* \\ &= (G_{\bar{\mu},1}(p_\varepsilon)^{-1} \otimes 1)\rho(G_{\bar{\mu},2}(b)) = (K_{\bar{\mu},1} \otimes \text{id})(\rho(b)). \end{aligned}$$

This fact, combined with the fact that $\rho: \ell^\infty(\widehat{\mathbb{G}}_2) \rightarrow \ell^\infty(\widehat{\mathbb{G}}_1) \boxtimes^{\alpha_{\mathbb{G}_1}} B$ is a *-isomorphism, gives that

$$\rho: \widetilde{A}_{\mu,2} \longrightarrow \widetilde{A}_{\mu,1} \boxtimes_{red}^{\alpha_{\mathbb{G}_1}} B$$

is an injective *-homomorphism.

The only thing left to prove is that ρ is surjective. We consider the inverse monoidal equivalence $\varphi^{-1}: \mathbb{G}_1 \rightarrow \mathbb{G}_2$. So, in the same way as above, we get that

$$\widetilde{\rho}: \widetilde{A}_{\mu,1} \longrightarrow \widetilde{A}_{\mu,2} \boxtimes_{red}^{\alpha_{\mathbb{G}_2}} \widetilde{B}: \widetilde{\rho}(b) = \mathbb{Y}(b \otimes 1)\mathbb{Y}^*,$$

is an injective *-homomorphism. But we have that

$$\begin{aligned} \Lambda: (\widetilde{\rho} \otimes \text{id}) \circ \rho: \widetilde{A}_{\mu,2} &\longrightarrow (\widetilde{A}_{\mu,2} \boxtimes_{red}^{\alpha_{\mathbb{G}_2}} \widetilde{B}) \boxtimes_{red}^{\alpha_{\mathbb{G}_1}} B: \Lambda(b) \\ &= \mathbb{Y}_{12}\mathbb{X}_{13}(b \otimes 1 \otimes 1)\mathbb{X}_{13}^*\mathbb{Y}_{12}^* \end{aligned}$$

is just the natural *-isomorphism from proposition 7.7 for the monoidal equivalence $\varphi^{-1}: \mathbb{G}_1 \rightarrow \mathbb{G}_2$. Indeed, $\Lambda = (\text{id} \otimes \pi)\alpha_{\mathbb{G}_2}$ with

$$\pi: L^\infty(\mathbb{G}_2) \longrightarrow \widetilde{B} \boxtimes^{\gamma_1} B \quad \text{given by} \quad (\text{id} \otimes \pi)(\mathbb{V}_2) = \mathbb{Y}_{12}\mathbb{X}_{13}.$$

So, $\rho: \widetilde{A}_{\mu,2} \rightarrow \widetilde{A}_{\mu,1} \boxtimes_{red}^{\alpha_{\mathbb{G}_1}} B$ is also surjective and thus a *-isomorphism. In proposition 7.11, we showed that ρ intertwines the action $\text{id} \boxtimes \delta_2$ with the action $\alpha_{\mathbb{G}_2}$. We have also shown that $\rho: c_0(\widehat{\mathbb{G}}_2) \rightarrow c_0(\widehat{\mathbb{G}}_1) \boxtimes_{red}^{\alpha_{\mathbb{G}_1}} B$ is a *-isomorphism, such that

$$\rho: A_{\mu,2} \longrightarrow A_{\mu,1} \boxtimes_{red}^{\alpha_{\mathbb{G}_1}} B,$$

is a *-isomorphism which intertwines the actions $\text{id} \boxtimes \delta_2$ and $\pi_{\mathbb{G}_2}$. This completes the proof of the theorem.

Examples: The universal orthogonal quantum groups $A_o(F)$

Given an $A_o(F)$, there exists a $q \in]-1, 1[\setminus \{0\}$ such that $A_o(F) \underset{\text{mon}}{\sim} \underline{SU}_q(2)$. In [19], Neshveyev and Tuset identified the Martin boundary of $\underline{SU}_q(2)$ ($0 < |q| < 1$), under the restriction that the measure μ has finite first moment, with the Podlès sphere $C(\mathbb{T} \setminus SU_q(2))$. By theorem 10.1, we get that $A_{\mu,2} \cong C(\mathbb{T} \setminus SU_q(2)) \boxtimes_{red}^{\pi_{\mathbb{G}_1}} \mathcal{B}$. But $C(\mathbb{T} \setminus SU_q(2))$ is a quotient

space of $L^\infty(\mathbb{G}_1)$, so in exactly the same way as in theorem 9.3, we show that

$$C(\mathbb{T} \setminus SU_q(2)) \boxtimes_{red}^{\pi_{\mathbb{G}_1}} \mathcal{B} \cong B_r^\mathbb{T}$$

where $B_r^\mathbb{T}$ is the norm-closure of $\mathcal{B}^\mathbb{T} \subseteq B^\mathbb{T}$ and \mathcal{B} the link algebra. Theorem 4.5 says that $\mathcal{B} \cong \mathcal{C}(A_o(F, F_q))$. Combining these facts, we obtain the following theorem:

THEOREM 10.2. — *Let $\mathbb{G} = A_o(F)$. Then there exists $q \in]-1, 1[\setminus \{0\}$ such that $\mathbb{G} \underset{\text{mon}}{\sim} SU_q(2)$. Take μ a generating measure on $\text{Irred}(\mathbb{G})$ that is transient and has finite first moment:*

$$\sum_{x \in \mathbb{N}} x\mu(x) < \infty.$$

*Then the Martin boundary A_μ of \mathbb{G} is *-isomorphic with $(C(A_o(F, F_q)))_r^\mathbb{T}$. Moreover, this *-isomorphism intertwines the action δ_2 with the action $\pi_{\mathbb{G}_2}$.*

□

Remark 10.3. — The above identification was already obtained by Vaes and the second author in [28] by using another method supporting on techniques from [29] which allow, in the case of $A_o(F)$, to deduce the Martin boundary from the Poisson boundary. The result by which we obtain the identification here is more direct and much more general.

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Manuscrit reçu le 5 décembre 2007,
accepté le 17 juillet 2008.

An DE RIJDT
Sint-Michielswarande 60
6T4, 1040 Brussel (Belgium)
an.derijdt@gmail.com
Nikolas VANDER VENNET
Celestijnenlaan 200 B
3001 Heverlee (Belgium)
Nikolas.Vandervennet@math.univ-bpclermont.fr