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# ANALYTIC EXTENSION FROM NON-PSEUDOCONVEX BOUNDARIES AND $A(D)$ -CONVEXITY

by Ch. LAURENT-THIÉBAUT and E. PORTEN

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## 1. Introduction.

Let  $D \subset \subset \mathbb{C}^n$  be a domain with smooth boundary. A compact  $K \subset \partial D$  is called removable if every continuous CR-function  $f$  on  $\partial D \setminus K$  has a holomorphic extension to all of  $D$ . In this paper we are interested in the link between removability and convexity properties of  $K$ . For any compact  $L \subset \overline{D}$ , we define its  $A(D)$ -convex hull as

$$A(D)\text{-hull}(L) = \{z \in \overline{D} : |f(z)| \leq \max_L |f|, \forall f \in A(D)\},$$

where  $A(D)$  denotes as usual the space of holomorphic functions which are continuous up to the boundary. If  $L = A(D)\text{-hull}(L)$ ,  $L$  is called  $A(D)$ -convex. Following [5], we call a compact  $K \subset \partial D$  CR-convex, if it satisfies  $A(D)\text{-hull}(K) \cap \partial D = K$ .

The main result of this paper is the following.

**THEOREM 1.** — *Let  $D$  be a bounded domain in  $\mathbb{C}^n, n \geq 2$ , with connected boundary of class  $\mathcal{C}^2$  and  $K \subset \partial D$  be a compact CR-convex set such that  $\partial D \setminus K$  is connected. Then each continuous CR-function  $u$  on  $\partial D \setminus K$  admits a holomorphic extension  $\tilde{u} \in \mathcal{O}(D \setminus A(D)\text{-hull}(K)) \cap \mathcal{C}((D \setminus A(D)\text{-hull}(K)) \cup (\partial D \setminus K))$ .*

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The following removability result is an immediate consequence.

**COROLLARY 2.** — *Let  $D$  be a bounded domain in  $\mathbb{C}^n$ ,  $n \geq 2$ , with connected boundary of class  $\mathcal{C}^2$  and  $K \subset \partial D$  be a compact set such that  $\partial D \setminus K$  is connected. If  $K$  is  $A(D)$ -convex, then  $K$  is removable.*

The theorem is best commented in its historical context.

For the case that  $\overline{D}$  is a Stein compact, analogous results were proved by E. L. Stout [14] and G. Lupacciolu [9], but with hypotheses formulated with respect to the  $\mathcal{O}(\overline{D})$ -hull, which can be larger than the  $A(D)$ -hull in the situation of Theorem 1. However in these papers more general results were stated, in [14] for weakly pseudoconvex domains, in [9] even for non-pseudoconvex domains. Later J. M. Ortega [12] discovered that the construction of the integral kernels used in [14] and [9] requires that  $\overline{D}$  possess a Stein neighborhood basis.

In [8], the first author gave the first complete proof of a result of this type for non-pseudoconvex domains, namely Theorem 4 quoted below. In contrast to [14], [9], the convexity condition on  $K$  is formulated with respect to the holomorphic functions defined on some uniform neighborhood of  $\overline{D}$ , and not with respect to  $\mathcal{O}(\overline{D})$ .

In [5], B. Jöricke proved Theorem 1 for weakly pseudoconvex domains, assuming sharper hypotheses on  $A(D)$ -convex hulls. Furthermore she was the first to attack the problem by a global version of the continuity principle, in contrast to the integral formula methods in the preceding articles. The reader may consult [5] for explanations of additional features in the pseudoconvex case. In particular, no assumption on connectedness of  $\partial D \setminus K$  is needed.

As a starting point to the non-pseudoconvex setting, let us look at two examples illustrating the assumptions in Theorem 1.

*Example 3.* — a) Consider  $D = B(0, 1) \setminus B(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ , where  $B(z, r) \subset \mathbb{C}^2$  denotes the ball of radius  $r$  centered at  $z$ . The compact  $K = \{(z_1, z_2) \in \partial D : x_1 \geq \frac{1}{2}\}$  is  $A(D)$ -convex. Then  $\partial D \setminus K$  has two components, whereas  $D \setminus A(D)\text{-hull}(K) = \{z \in D : x_1 < \frac{1}{2}\}$  is connected. Hence CR functions do not extend analytically from  $\partial D \setminus K$  to  $D \setminus A(D)\text{-hull}(K)$  in general. By rounding off the corner we get an example with smooth boundary.

b) If we drop CR-convexity, we run into monodromy problems: Consider  $D = Q \setminus (\{|z_1| \leq \epsilon, |x_2 - \frac{1}{2}| \leq \epsilon\} \cup \{|z_1| \leq \epsilon, |x_2 + \frac{1}{2}| \leq \epsilon, y_2 \leq$

$1 - \epsilon\}$ , where  $Q = (-1, 1) \times i(-1, 1) \times (-1, 1) \times i(-1, 1)$  and  $\epsilon > 0$  is a small constant. If  $K = \partial Q \cap \{y_2 = 0\}$ , then  $\partial D \setminus K$  is connected. Since  $A(D)\text{-hull}(K) = D \cap \{y_2 = 0\}$ ,  $\partial D \setminus A(D)\text{-hull}(K)$  has three connected components  $T_1 = \partial D \cap \{y_2 < 0\}$ ,  $T_2 = \partial\{|z_1| < \epsilon, |x_2 + \frac{1}{2}| < \epsilon, y_2 < 1 - \epsilon\} \cap \{y_2 > 0\}$ ,  $T_3 = (\partial D \cap \{y_2 > 0\}) \setminus T_2$ . We can define a univalent holomorphic function  $u$  near  $\partial D \setminus K$  which coincides near every  $z \in \partial D \setminus K$  with some branch of  $\log(z_2)$ . But we cannot extend  $u$  to  $D \setminus A(D)\text{-hull}(K)$  without losing the coincidence near  $T_2$  or  $T_3$ .  $\square$

Theorem 1 was first proved by the second author in his thesis [13]. His method combines the continuity principle with special constructions relying on geometrical properties of Stein manifolds. The essential point both in [5] and in [13] are monodromy problems of increasing difficulty, which are typical, if one tries to construct hulls by direct application of the continuity principle. Therefore techniques designed to handle this aspect should be of independent interest. This provided the motivation for further joint research of B. Jöricke and the second author [7], which led to a new version of the proof of Theorem 1, where all essential steps are executed by extension along suitable families of complex curves.

The argument presented in the present paper goes back to the observation of the first author how to adapt the integral formula constructions of [8] to the case at hand. In the present paper we do not make explicit use of integral formulas but reduce Theorem 1 by geometric arguments to the main result of [8], which we state for later reference.

**THEOREM 4.** — *Let  $\Omega$  be a Stein manifold of complex dimension  $n \geq 2$ ,  $K \subset \Omega$  an  $\mathcal{O}(\Omega)$ -convex compact subset, and  $D \subset \Omega$  a relatively compact domain such that  $\partial D \setminus K$  is a connected hypersurface of class  $\mathcal{C}^1$ . Then every continuous CR-function  $u$  on  $\partial D \setminus K$  admits a holomorphic extension  $\tilde{u} \in \mathcal{O}(D \setminus K) \cap \mathcal{C}((D \setminus K) \cup (\partial D \setminus K))$ .*

The main argument is explained in Section 3. In Section 2, we use a device of [5] in order to reduce the problem to the extension of holomorphic boundary values.

Finally a comment on dimensions is in order. Some authors ([14], [5], [7]) prefer to state results of the type of Theorem 1 only in complex dimension 2. The reason is that it is known that in dimension  $n \geq 3$  additional extension phenomena occur, which are principally overlooked by assumptions on the hull of the singularity. In fact the continuity principle tells that families of complex curves give rise to holomorphic extension.

On the other hand the proofs based on the continuity principle [5], [7], exhibit Theorem 1 as a result on extension along families of complex *hypersurfaces*. This difference between  $n = 2$  and  $n \geq 3$  also gets transparent in the characterization of removability for strongly pseudoconvex domains, where holomorphic convexity of the singularity is only adequate in dimension 2 (cf. [3]), whereas in higher dimension the relevant properties are of cohomological nature (cf. [10]).

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## 2. Extension to one-sided neighborhoods.

In this section we reduce the proof of Theorem 1 to an analogous statement with holomorphic boundary data.

Let  $H \subset \mathbb{C}^n$  be a hypersurface. A one-sided neighborhood of  $H$  is an open set  $V$  such that, for every  $z \in H$  and for every euclidean ball  $B$  of sufficiently small radius centered at  $z$ , at least one of the two components of  $B \setminus H$  is contained in  $V$ . A piecewise differentiable curve  $\gamma : [a, b] \rightarrow H$  is called CR-curve, if its (one-sided) derivatives are contained in  $T^c H$ . For  $z \in H$ , the CR-orbit  $\mathcal{O}_z$  of  $z$  in  $H$  is defined as the set of all points  $w \in H$  which can be joint with  $z$  by a CR-curve in  $H$ . For detailed information on CR-orbits the reader may consult [15], [6].

We shall use the following lemma from [5].

LEMMA 5. — *Let  $D, K$  be as in Theorem 1. Then there is a connected one-sided neighborhood  $V$  of  $\partial D \setminus K$  such that every continuous CR-function on  $\partial D \setminus K$  extends to  $V$ .*

For the reader's convenience we give a proof, which makes explicit use of the by now familiar concept of CR-orbits.

*Proof of Lemma 5.* — We observe that it is enough to show that  $\partial D \setminus K$  has only one CR-orbit. Indeed, in this case  $\partial D \setminus K$  contains a point

$z$  in which  $\partial D \setminus K$  is minimal in the sense of Tumanov, i.e., there is no local holomorphic hypersurface  $X \subset \partial D$  passing through  $z$  (otherwise the complex tangent bundle  $T^c \partial D \setminus K$  would be Frobenius integrable, and all orbits would be complex hypersurfaces immersed in  $M$ ). The theorem of Trépreau [16] yields one-sided analytic extension near  $z$ , and this property propagates along CR-curves to all points of  $\partial D \setminus K$  by propagation results of [17], [5].

Assume that  $\partial D \setminus K$  has several CR-orbits. It is well-known that CR-orbits of a real hypersurface are either subdomains or injectively immersed smooth holomorphic hypersurfaces. The union of all lower-dimensional CR-orbits is relatively closed and forms a lamination. Hence the connectedness of  $\partial D \setminus K$  implies the existence of a lower-dimensional CR-orbit  $\mathcal{O}$ . By general properties of laminations the relative closure  $L$  of  $\mathcal{O}$  in  $\partial D \setminus K$  is a union of lower-dimensional orbits, and each of these is dense in  $L$ .

Pick some  $z \in L$  and a function  $f \in A(D)$  with  $f(z) = 1$  and  $\max_K |f| < 1/2$ . Then the modulus of  $f|_L$  attains a maximum at some  $z' \in L$ . By the maximum principle,  $f$  is constant on  $\mathcal{O}_{z'}$  and, by density, also on  $L$ . So  $L$  has positive distance from  $K$  and must be compact. But a non-void compact union of holomorphic submanifolds of positive dimension is impossible. Indeed, for such an  $L$  one finds a closed euclidean ball  $B$  containing  $L$  such that  $L \cap \partial B \neq \emptyset$ . But this leads to a contradiction to the maximum principle on the varieties passing through points of  $L \cap \partial B$  (cf. [4], § V, Lemma 5). □

*Remark 6.* — As the referee pointed out, the original proof of [5], which was given for  $n = 2$ , can be extended to boundaries of class  $C^1$  by using techniques from [2]. For  $n > 2$ , some additional slicing arguments would be necessary. □

Let  $V$  be as in the lemma. After shrinking  $V$  conveniently, we can suppose that (i)  $D' = \text{int}(\overline{D \cup V})$  is a domain which contains  $K$  in its boundary, and that (ii) every  $f \in A(D)$  extends to a function in  $A(D')$ . For (ii), we have to observe that  $f$  extends holomorphically through a point  $z \in \partial D \setminus K$  if  $V$  contains near  $z$  the exterior side of  $\partial D$ . By (ii),  $A(D)$  can be identified with  $A(D')$ , and we have  $A(D)\text{-hull}(K) = A(D')\text{-hull}(K)$ .

Near every point  $z \in \partial D \setminus K$ ,  $V$  contains at least one side of  $\partial D$ . Slightly deforming  $\partial D \setminus K$  into  $V$ , we can construct a third domain  $D''$  such that  $\partial D''$  is of class  $C^2$  and  $\partial D'' \setminus K \subset D'$ . As  $A(D) = A(D')$ , we can

choose  $\partial D'' \setminus K$  so close to  $D \setminus K$  that

$$A(D')\text{-hull}(K) \cap \partial D = K$$

holds true. Observe that the analytic extension to  $V$  of a given CR-function on  $D \setminus K$  induces holomorphic data on  $\partial D'' \setminus K$ . Writing again  $D$  instead of  $D''$ , the proof of Theorem 1 is now reduced to the following intermediate statement.

**PROPOSITION 7.** — *Let  $D$  be a bounded domain in  $\mathbb{C}^n$  with boundary of class  $\mathcal{C}^2$  and  $K \subset \partial D$  be a compact set with  $\partial D \setminus K$  connected. Let  $D' \supset D$  be a bounded domain satisfying  $K \subset \partial D'$ ,  $K = A(D')\text{-hull}(K) \cap \partial D$ , and  $\overline{D} \setminus K \subset D'$ . Then each function  $u$  which is holomorphic in a neighborhood  $U$  of  $\partial D \setminus K$  has (after shrinking  $U$  if necessary) a holomorphic extension to  $D \setminus A(D')\text{-hull}(K)$ .*

### 3. Proof of Theorem 1.

As explained in Section 2, it is enough to prove Proposition 7.

First we observe that we cannot immediately apply Theorem 4 by taking  $\Omega$  as the envelope of holomorphy of  $D'$ . Of course  $D$  need not be relatively compact in  $\Omega$ . In what follows we will derive Proposition 7 by an exhaustion argument.

For every  $w \in D \setminus A(D')\text{-hull}(K)$ , there is a function  $f_w \in A(D')$  with  $f_w(w) = 1$  and  $\max_{z \in K} |f_w(z)| \leq 1/4$ . By a standard covering argument, we select a subsequence  $f_1, f_2, \dots$ , such that

$$D \setminus A(D')\text{-hull}(K) = \bigcup_{i=1}^{\infty} \{z \in D : |f_i(z)| > 3/4\}.$$

Note that a given compact subset of  $D \setminus A(D')\text{-hull}(K)$  is already contained in the finite union  $\bigcup_{i=1}^k \{z \in D : |f_i(z)| > 3/4\}$ , if  $k$  is sufficiently large.

Let  $\rho \in \mathcal{C}^2(\overline{D})$  be a defining function of  $D$ , i.e., a function with  $\nabla \rho \neq 0$  on  $\partial D$  and  $D = \{\rho < 0\}$ . We can choose a strictly decreasing sequence of positive numbers  $r_j \downarrow 0$  and domains  $D_j = \{\rho < -r_j\}$  so that for every  $j$  the intersection  $\partial D_j \cap \bigcup_{i=1}^j \{z \in D : |f_i(z)| > 1/2\}$  is contained in  $U$  (the neighborhood of  $\partial D \setminus K$  where  $u$  is defined). If  $\Omega$  denotes the envelope of holomorphy of  $D'$ , we can consider each  $D_j$  as a relatively compact subdomain of  $\Omega$ . Define

$$K_j = \mathcal{O}(\Omega)\text{-hull}\left(\bigcap_{i=1}^j \{z \in \overline{D_j} : |f_i(z)| \leq\right.$$

$3/4\}$ ).

As

$\Omega$  is holomorphically convex,  $K_j$  is compact in  $\Omega$ . By construction, we have  $K_j \cap \overline{D_j} = \bigcap_{i=1}^j \{z \in \overline{D_j} : |f_i(z)| \leq 3/4\}$  and  $\partial D_j \setminus K_j \subset \partial D_j \cap \bigcup_{i=1}^j \{z \in D : |f_i(z)| > 1/2\} \subset U$ . For later use we remark  $\overline{D_j} \setminus K_j \subset D_{j+1} \setminus K_{j+1}$ . Observe that we cannot immediately apply Theorem 4 because (i)  $D_j \setminus K_j$  may have several components, and (ii) the intersection of the closure of a component of  $D_j \setminus K_j$  with  $\partial D_j$  need not be connected.

In order to meet (i), we choose subdomains  $G_j \subset D_j$  in the following way: Let  $G_1$  be an arbitrary component of  $\{z \in D_1 : |f_1(z)| > 3/4\}$ . By induction we choose, for every  $j > 1$ ,  $G_j$  as the unique component of  $\bigcup_{i=1}^j \{z \in D_j : |f_i(z)| > 3/4\}$  with  $G_{j-1} \subset G_j$ . Because  $\partial D \setminus K$  is connected and  $D \setminus A(D')$ -hull( $K$ ) has, by the maximum principle, no components which are relatively compact in  $D$ ,  $D \setminus A(D')$ -hull( $K$ ) is connected. As mentioned above, every compact arc in  $D \setminus A(D')$ -hull( $K$ ) is contained in almost every finite union  $\bigcup_{i=1}^j \{z \in D : |f_i(z)| > 3/4\}$ . Since  $D \setminus A(D')$ -hull( $K$ ) is connected, we deduce  $D \setminus A(D')$ -hull( $K$ ) =  $\bigcup_{j=1}^\infty G_j$ .

As indicated in (ii), there is no reason for  $\partial G_j \setminus K_j$  to be connected. We shall use an elementary topological property of Stein manifolds to handle this difficulty.

LEMMA 8. — *Let  $\Omega$  be a Stein manifold of dimension  $n \geq 2$  and  $L$  a compact  $\mathcal{O}(\Omega)$ -convex subset. Let  $M$  be a connected, properly embedded, orientable real hypersurface of  $\Omega \setminus L$  of class  $\mathcal{C}^1$  such that  $M \cup L$  is compact. Then  $\Omega \setminus (M \cup L)$  contains exactly one relatively compact component.*

Of course Lemma 8 is a byproduct of E. M. Chirka’s relative version of the Harvey-Lawson theorem ([1], Theorem 19.6.2). For the reader’s convenience, we provide an elementary proof, based on an argument communicated by N. Shcherbina. The assumption that  $M$  be orientable is not necessary, since our proof also works with  $\mathbb{Z}/2\mathbb{Z}$ -valued intersection numbers.

*Proof.* — By the maximum principle,  $\Omega \setminus L$  has no relatively compact components. Hence every component of  $\Omega \setminus (M \cup L)$  has accumulation points on  $M$ . As  $M$  is connected,  $\Omega \setminus (M \cup L)$  has at most two components. Because  $\Omega$  has only one end and  $M \cup L$  is compact, there is a unique unbounded component. Consequently the second component, if it exists, has to be relatively compact.

Assume that there is no relatively compact component of  $\Omega \setminus (M \cup L)$ . Then we can easily construct a smoothly embedded oriented loop  $\gamma \subset \Omega \setminus L$  which intersects  $M$  transversely in only one point. Hence the intersection number of  $\gamma$  and  $M$  equals  $\pm 1$ , the sign depending on the orientations we choose for  $\gamma$  and  $M$ . We shall obtain a contradiction to the homotopy invariance of intersection numbers by deforming  $\gamma$  within  $\Omega \setminus L$  to a loop contained in  $\Omega \setminus (M \cup L)$ .

As  $L$  is  $\mathcal{O}(\Omega)$ -convex, an elementary construction gives a smooth plurisubharmonic non-negative exhaustion function  $\phi$  of  $\Omega$  such that  $L = \{\phi = 0\}$  and  $\phi$  is strictly plurisubharmonic on  $\Omega \setminus L$  (being exhaustive means that  $\{\phi < c\}$  is relatively compact for any  $c \in \mathbb{R}$ ). Fix  $c_1, c_2 > 0$  such that  $M \cup L \subset \{\phi < c_2\}$  and  $\gamma \subset \{c_1 < \phi < c_2\}$ . After a slight modification, we can assume that  $\phi$  is a Morse function on a neighborhood of  $\{c_1 \leq \phi \leq c_2\}$  (for information on Morse theory we refer to [11]). Then there are finitely many critical points  $q_1, \dots, q_k$  in  $\{c_1 \leq \phi \leq c_2\}$ . It is a well-known consequence of strict plurisubharmonicity that the Morse-indices at the points  $q_1, \dots, q_k$  cannot exceed  $n$ . This implies that the associated stable manifolds

$$S_i = \{p \in \Omega : \lim_{t \rightarrow +\infty} \Phi_{\nabla\phi,t}(p) = q_i\}$$

are at most of dimension  $n$ , in particular of codimension at least  $n \geq 2$ . Here  $\Phi_{X,t}$  denotes the time- $t$ -map of a vectorfield  $X$  and  $\nabla\phi$  the gradient with respect to some fixed riemannian metric on  $\Omega$ . After an arbitrarily small deformation of  $\gamma$ , we can assume  $\gamma \cap \bigcup_{i=1}^k S_i = \emptyset$ . This means that, for every  $p \in \gamma$ , we have  $\phi(\Phi_{\nabla\phi,t}(p)) > c_2$ , if  $t$  is sufficiently large. Now a compactness argument yields that there exists  $T > 0$  such that  $\phi(\Phi_{\nabla\phi,t}(\gamma))$  is disjoint from  $\{\phi \leq c_2\} \supset M \cup L$  for  $t > T$ , a contradiction to the homotopy invariance of intersection numbers.  $\square$

In the situation of Lemma 8, we shall call the relatively compact component of  $\Omega \setminus (M \cup L)$  the inner domain of  $M \cup L$ . For  $j$  fixed, let  $\mathcal{C}_j$  be the set of connected components of  $\partial G_j \setminus K_j$ . Since  $K_j$  is  $\mathcal{O}(\Omega)$ -convex, Lemma 8 associates to any  $T \in \mathcal{C}_j$  the inner domain  $B_T$  of  $T \cup K_j$ . Hence we may introduce a partial order on  $\mathcal{C}_j$  by writing  $T_1 \prec T_2$  if  $B_{T_1} \subset B_{T_2}$ . Reflexivity and transitivity are obvious. The following lemma contains antisymmetry and the existence of a unique maximal element.

LEMMA 9. — a) If  $T_1 \prec T_2, T_2 \prec T_1$ , for  $T_1, T_2 \in \mathcal{C}_j$ , then  $T_1 = T_2$ .

b) There is a unique maximal component  $M_j \in \mathcal{C}_j$ , which is the unique element of  $\mathcal{C}_j$  which belongs to the closure of the unbounded

connected component of  $\Omega \setminus (G_j \cup K_j)$ . Moreover its inner domain  $B_{M_j}$  contains  $G_j$  and all the other inner domains  $B_T, T \in \mathcal{C}_j$ .

*Proof.* — a) If we assume  $T_1 \prec T_2, T_2 \prec T_1$ , and  $T_1 \neq T_2$ , then the definition of  $\prec$  implies  $T_1 \subset B_{T_2}$  and  $T_2 \subset B_{T_1}$ . Hence  $B_{T_1} \cup B_{T_2}$  is a relatively compact domain whose boundary is contained in  $K_j$ , a contradiction to the  $\mathcal{O}(\Omega)$ -convexity of  $K_j$  and the maximum principle.

b) Lemma 8 implies that there is at most one  $M_j \in \mathcal{C}_j$  belonging to the closure of the unbounded connected component of  $\Omega \setminus (G_j \cup K_j)$ . If there were no such  $M_j$ , then  $G_j$  would be contained in a relatively compact component of  $\Omega \setminus K_j$ , which is in contradiction with the maximum principle. Hence  $M_j$  is uniquely defined, and we have  $B_{M_j} \supset G_j$ .

Let  $T$  be another component of  $\partial G_j \setminus K_j$ . Then  $(G_j \cup T) \subset B_{M_j}$  by connectedness of  $G_j$ , and  $B_T \subset B_{M_j}$  by definition of  $B_T$  and  $B_{M_j}$ .  $\square$

By Theorem 4, the restriction of  $u$  to a sufficiently small neighborhood of  $M_j$  extends to a function  $u_j \in \mathcal{O}(B_{M_j}) \subset \mathcal{O}(G_j)$  which coincides with  $u$  near  $M_j$ . It is not yet clear, whether  $u_j$  coincides with  $u$  near all components of  $\partial G_j \setminus K_j$ . So we must carefully check that we can produce the desired extension of  $u$  by gluing the  $u_j$ .

For this purpose we fix some compact subset  $L \subset D \setminus A(D')$ -hull( $K$ ). By construction,  $L \subset G_j$  for sufficiently large  $j$ . According to the following lemma, the sequence  $\{u_j\}$  gets stable near  $L$  thus suggesting a natural candidate for the final extension near  $L$ .

LEMMA 10. — *There is  $k_L \in \mathbb{N}$  and a neighborhood  $V$  of  $L$  such that the functions  $u_j$  coincide on  $V$ , for all  $j \geq k_L$ .*

*Proof.* — Choose  $c \in \mathbb{C}, |c| > 3/4$ , such that  $C = \{p \in \Omega : f_1(p) = c\}$  is a smooth complex curve which intersects  $\partial D \setminus K$  transversally. Since  $\Omega$  does not contain compact complex curves, there is some  $p_0 \in \partial D \setminus K$  which is an accumulation point of a non-relatively compact component of  $C \setminus \overline{D}$ .

Near  $p_0$ ,  $C$  intersects the hypersurfaces  $\partial D$  and  $\partial D_j$ , for  $j$  sufficiently large, in a family of almost parallel short segments  $\lambda_j$  which are all contained in  $U$ . Hence for large  $j$ , the segments  $\lambda_j$  are adherent to an unbounded component of  $C \setminus \overline{D_j}$  by transversality. Fix some  $j_0$  for a moment. Then  $\lambda_{j_0}$  is contained in all  $G_j$ , for  $j \geq j_1$ , if  $j_1$  is sufficiently large. For  $j \geq j_1$ , we deduce  $\lambda_j \subset \partial G_j \setminus K_j$ . As  $\lambda_j$  lies in the closure of an unbounded component of  $C \setminus \overline{D_j}$ , we even obtain  $\lambda_j \subset M_j$ .

If we take  $j_2 \geq j_1$  so large that  $L \subset G_j, j \geq j_2$ , then all the functions  $u_j, j \geq j_2$ , coincide near  $L$ . Indeed we can connect a given point  $p \in L$  with  $\lambda_{j_2}$  by an arc  $\gamma \subset G_{j_2} \cup \lambda_{j_2}$  and compare the functions  $u_j, j \geq j_2$ , along  $\gamma$ . Hence we can take  $k_L = j_2$ .  $\square$

Proposition 7 follows from Lemma 10 without difficulties: Let us take some exhaustion of  $D \setminus A(D')$ -hull( $K$ ) by compact sets  $L_1 \subset L_2 \subset \dots$  satisfying  $L_j \subset \text{int}(L_{j+1})$ . By Lemma 10 we get near every  $L_j$  a natural candidate  $\tilde{u}_j$  by taking the restriction of some  $u_k$  for  $k \geq k_{L_j}$ . Since the  $L_j$  are monotonously increasing sets, it is clear from Lemma 10 that the  $\tilde{u}_j$  glue to a well-defined function  $\tilde{u} \in \mathcal{O}(D) \setminus A(D')$ -hull( $K$ ). Finally the connectedness of  $\partial D \setminus K$  implies the coincidence of  $u$  and  $\tilde{u}$  near  $\partial D \setminus K$ . Proposition 7 and Theorem 1 are proved.  $\square$

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