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POINTWISE ESTIMATES FOR THE WEIGHTED BERGMAN PROJECTION KERNEL IN $\mathbb{C}^n$, USING A WEIGHTED L^2 ESTIMATE FOR THE $\bar{\partial}$ EQUATION

by Henrik DELIN

1. Introduction.

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain, and φ be a plurisubharmonic function in Ω. Consider the space $L^2_\varphi(\Omega) := L^2(\Omega, e^{-\varphi}d\lambda)$ of square integrable functions on Ω with the measure $e^{-\varphi}d\lambda$, where $d\lambda$ is the Lebesgue measure. The subspace $H^2_\varphi(\Omega)$ of holomorphic functions is then a closed subset of $L^2_\varphi(\Omega)$. We define the orthogonal projection and the corresponding integral kernel in the following way.

DEFINITION 1. — *The Bergman projection operator S_φ is the orthogonal projection*

$$S_\varphi : L^2_\varphi(\Omega) \to H^2_\varphi(\Omega).$$

The Bergman integral kernel, $S_\varphi(\cdot,\cdot)$, of this bounded operator is defined by the relation that, for $v \in L^2_\varphi(\Omega)$,

$$S_\varphi(v)(z) = \int S_\varphi(z,\zeta)v(\zeta)e^{-\varphi(\zeta)}d\lambda(\zeta).$$

As a reference for the basic properties of the (unweighted) Bergman kernel see e.g. the paper by Bergman [1].

Key words: Bergman kernel – Weighted L^2 estimates – $\bar{\partial}$ equation – $\bar{\partial}$-Neumann problem – Kähler metric.
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In Theorem 2, we will give the main result in this paper, which is a growth estimate of $S_\varphi(z,\zeta)$ in the whole of $\mathbb{C}^n$ when $|z-\zeta| \to \infty$. Previously, Hörmander [10] has given estimates for $S_\varphi(z,z)$ for points on the diagonal of $\Omega \times \Omega$, close to the boundary in strictly pseudoconvex domains. In 1991, Christ [4] proved pointwise estimates of various weighted kernels in $\mathbb{C}^1$. One of the kernels was the Bergman projection kernel $S_\varphi(\cdot,\cdot)$. The function φ was supposed to belong to a certain class of subharmonic functions in $\mathbb{C}^1$, satisfying some extra conditions on the measure $\Delta\varphi$. Theorem 2 of this paper is a partial generalisation of Christ's result on $S_\varphi(\cdot,\cdot)$ to several variables.

The unweighted kernel was studied by Kerzman [11], published 1972. He proved differentiability at the boundary in smooth, strictly pseudoconvex, bounded domains in $\mathbb{C}^2$. In his proof, he used the $\bar\partial$-Neumann solution operator to construct the Bergman kernel. We will essentially use the same idea, in the following way. The kernel may be written as a projection onto $H^2_\varphi(\Omega)$ of a given C_0^∞-function v supported around ζ. We may then write

$$v = S_\varphi v + u = S(\cdot,\zeta) + u,$$

where u is orthogonal to $S_\varphi v$, i.e., u is the L^2_φ-minimal, or canonical, solution to $\bar\partial u = \bar\partial v$. We use the solution to the $\bar\partial$-Neumann problem to obtain u, or at least a weighted estimate of u. This weighted estimate for the canonical solution to the $\bar\partial$ equation is stated in Theorem 1.

The technique by Kerzman was also used by McNeal [13], who extended Kerzman's results by giving estimates for the Bergman kernel and its derivatives in domains of finite type in $\mathbb{C}^2$. Nagel, Rosay, Stein and Wainger [15] also studied domains of finite type in $\mathbb{C}^2$ proving estimates similar to the ones of McNeal. We will not study this type of regularity of the kernel, though. The presence of the weight $e^{-\varphi}$ makes regularity results much more complicated.

As mentioned, the proof of Theorem 2 is based on Theorem 1, a weighted L^2 estimate for solutions to the $\bar\partial$ equation. There exists several variants of weighted L^2 estimates of Hörmander type for the $\bar\partial$ equation. In Donnelly-Fefferman [7], there can be found related ideas concerning the condition on the weight, though not used in the same way as in this paper. Ohsawa-Takegoshi [16] used ideas which are similar to ours, and the paper by Diederich and Ohsawa [6] include a result on the existence of solutions that satisfy a similar weighted L^2 estimate. Related papers, which also prove existence of solutions satisfying weighted estimates are McNeal [14] and Diederich-Herbort [5].

In [2], Berndtsson gave a proof of a weighted estimate for the L^2-minimal, canonical, solution, with a weight w. Unfortunately for us, it imposes a condition on $\partial\bar{\partial}w$, which does not fit the purposes of this paper. To prove Theorem 2, we would like w to depend on a certain distance function, and then a condition on the norm of dw is more suitable. To that end, we will prove the following result.

THEOREM 1. — *Assume that f is a closed $(0,1)$-form on a pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$ and that φ is strictly plurisubharmonic and C^2 there. Let w be a weight function on Ω satisfying $|\partial w|_{i\partial\bar{\partial}\varphi} \leq \varepsilon w$ for some $\varepsilon \in (0, \sqrt{2})$. Then the $L^2_\varphi(\Omega)$-minimal solution to $\bar{\partial}u = f$ satisfies*

$$\int_\Omega |u|^2 e^{-\varphi} w d\lambda \leq \frac{2}{(\varepsilon - \sqrt{2})^2} \int_\Omega |f|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} w d\lambda. \tag{1.1}$$

Here, $| \cdot |_{i\partial\bar{\partial}\varphi}$ denotes the norm in the Kähler metric with Kähler form $i\partial\bar{\partial}\varphi$. Generally, we will let ω denote a Kähler form, and then apply our results to the particular metric with form

$$\omega = i\partial\bar{\partial}\varphi.$$

Theorem 1 strengthens the mentioned estimate of Diederich and Ohsawa somewhat. The main differences are, that their estimate is not given for the L^2-minimal solution, and it also assumes the manifold is complete. The L^2-minimality of u is necessary for the way we use it to estimate the Bergman kernel.

Before stating the main theorem, we need some more notation. In order to replace the Kähler form ω by something simpler to handle, we let μ_z be a constant $(1,1)$-form that dominates ω close to z in the following way.

Given ω and a point $z \in \mathbb{C}^n$, let $\mu_z := \mu(\omega)(z)$ to be any constant hermitian $(1,1)$-form, such that

$$\mu_z \geq \omega(\zeta) \text{ for } \zeta \in B_{\mu_z}(z,1), \tag{1.2}$$

i.e., it majorises ω in the unit ball in the Kähler metric with metric form given by μ_z. If the inequality in (1.2) holds in (the larger set) $B_\omega(z,1)$, then certainly (1.2) holds, so there are no problems finding such a μ_z if ω is, say, continuous. A linear change of coordinates, that makes μ become the Euclidean metric, will make ω become bounded by the Euclidean metric. That is, $\omega \leq \beta$ on the Euclidean unit ball, where

$$\beta := \frac{i}{2} \sum dz_k \wedge d\bar{z}_k \tag{1.3}$$

is the Kähler form for the Euclidean metric.

Now, the eigenvalues of ω with respect to μ can be defined as follows. If ω and μ are seen as matrices through some basis, the eigenvalues of ω with respect to μ, denoted by $\lambda_i(\omega|\mu)$, are the solutions to

$$\det(\lambda\mu - \omega) = 0.$$

Since this equation is invariant under both left and right multiplication by nonsingular matrices, this definition is independent on the choice of coordinate representation for ω and μ. The minimal eigenvalue can also be given by

$$\lambda_{\min}(\omega|\mu) = \min_{v:\mu(v,\bar{v})=1} \omega(v,\bar{v}).$$

The relative determinant is

$$\det(\omega|\mu) = \prod \lambda_i(\omega|\mu).$$

When given with respect to the Euclidean metric, we omit this notation, writing $\det(\omega)$.

For further notational convenience, let

$$c_{\omega,\mu}(\zeta) := \inf_{\tilde{\zeta}\in B_{\mu_\zeta}(\zeta,1)} \lambda_{\min}(\omega(\tilde{\zeta})|\mu_\zeta)$$

be the minimal eigenvalue of ω with respect to μ_ζ that occurs in the given ball around ζ. Observe that by (1.2), $c_{\omega,\mu}(\zeta) \leq 1$, but it can also be a lot smaller. Finally, let $\rho_\omega(z,\zeta)$ denote the distance between z and ζ measured in the ω metric.

Now, we may state the following theorem.

THEOREM 2. — *Let φ be a strictly plurisubharmonic C^2 function on $\mathbb{C}^n$. Let μ_z satisfy (1.2), with $\omega = i\partial\bar{\partial}\varphi$, and assume that $0 < \varepsilon < \sqrt{2}$. Then the weighted Bergman kernel satisfies the estimate*

$$|S_\varphi(z,\zeta)|^2 \leq \frac{C}{(\sqrt{2}-\varepsilon)^2 c_{\omega,\mu}(\zeta)} \det(\mu_z)\det(\mu_\zeta)e^{\varphi(z)+\varphi(\zeta)-\varepsilon\rho_\omega(z,\zeta)}, \tag{1.4}$$

where the constant C only depends on the dimension n.

Remark. — Note that actually, $S_\varphi(\cdot,\cdot)$ is conjugate symmetric in the two variables. Hence, in the denominator, one could replace $c_{\omega,\mu}(\zeta)$ by $c_{\omega,\mu}(z)$ (or the larger one of these). I don't know if this factor is really necessary, or what would be the optimal factor in general. In some way, it measures how far ω is from being constant around a point.

As long as ω does not vary too much inside $B_\omega(z,1)$, staying uniformly equivalent to μ_z for every z, we could replace μ_z and μ_ζ in (1.4) by $\omega(z)$ and $\omega(\zeta)$, respectively. This would also imply that $\lambda_{\min}(\omega|\mu_z)$ is bounded from below by some constant. Under this (rather vague) condition on ω, we would obtain the more desirable estimate

$$|S_\varphi(z,\zeta)|^2 \leq \frac{C}{(\sqrt{2}-\varepsilon)^2}\det(\omega(z))\det(\omega(\zeta))e^{\varphi(z)+\varphi(\zeta)-\varepsilon\rho_\omega(z,\zeta)},$$

where now C also depends on the uniformity of ω.

In the case of one complex variable, and under the extra condition that $i\partial\bar{\partial}\varphi$ is a doubling measure, it is possible to make a slight improvement of Theorem 2. We will discuss this in Section 6.

Finally, I want to thank Prof. Bo Berndtsson for giving me the idea of this problem, and for valuable support and inspiration during the time I have been working on it.

2. Notation and preliminaries.

As seen in Section 1, the results in this paper will concern domains in $\mathbb{C}^n$. Since the estimate (1.4) involves the distance function with respect to the metric given by $i\partial\bar{\partial}\varphi$, it turns out to be natural to look at $\mathbb{C}^n$ with a Kähler metric, and obtain results in that context. For the proof, we shall use the usual L^2 techniques for the $\bar{\partial}$ equation, and the generalisation of these techniques to Kähler manifolds. I do not know how to obtain the desired estimate using these methods with respect only to the Euclidean metric, but there would certainly be some additional problems to overcome.

By $\langle\cdot,\cdot\rangle_{\omega,\varphi}$, we denote the scalar product on forms with values in the trivial line bundle with metric ω on the base manifold, and metric $e^{-\varphi}$ on the fibres, i.e., $\langle\cdot,\cdot\rangle_{\omega,\varphi} = \langle\cdot,\cdot\rangle_\omega e^{-\varphi}$. The subscript $\cdot_e$ will mean the Euclidean metric in $\mathbb{C}^n$.

Let $L^2_\varphi(\Omega, d\lambda)$ denote the L^2 space with scalar product

$$\int_\Omega \langle\cdot,\cdot\rangle_{e,\varphi} d\lambda = \int_\Omega \langle\cdot,\cdot\rangle_e e^{-\varphi} d\lambda,$$

and let $L^2_\varphi(\Omega,\omega,dV)$ denote the L^2 space with scalar product

$$\int_\Omega \langle\cdot,\cdot\rangle_{\omega,\varphi} dV = \int_\Omega \langle\cdot,\cdot\rangle_\omega e^{-\varphi} dV.$$

(We include ω in the notation $L^2_\varphi(\Omega, \omega, dV)$ to indicate the dependence on ω for forms of degree greater that 0.) We will use dV to denote the volume form induced by the metric given by ω, i.e.,

$$dV = \frac{\omega^n}{n!}.$$

We denote balls with radius R and centre x in the metric ω by $B_\omega(x, R)$.

Interior multiplication of differential forms will be denoted by $\lrcorner$, and is defined by

$$\langle \gamma \lrcorner \alpha, \beta\rangle = \langle \alpha, \bar\gamma \wedge \beta\rangle$$

for forms α, β and γ of matching degrees. This is a pointwise operation on forms and of course independent of the fibre metric $e^{-\varphi}$. Let $\Lambda^{(p,q)}$ be the space of alternating (p,q)-exterior products. Then we define the operator

$$\Lambda = \omega \lrcorner : \Lambda^{(p,q)} \longrightarrow \Lambda^{(p-1,q-1)}$$

which takes (p,q)-products to $(p-1, q-1)$-products.

For holomorphic line bundles, one cannot define the operators ∂ and d, since they are not preserved under holomorphic change of frames. Instead, the differential operator one should use is the connection on the cotangent bundle.

The connection on differential forms may then be written as a sum of a $(0,1)$ part and a $(1,0)$ part. Not going into detail here, we note that for the trivial line bundle with $e^{-\varphi}$ as the fibre metric, these parts of different bidegree turn out to be as follows. The $(0,1)$ part is just $\bar\partial$, which is well defined on holomorphic bundles. Its formal adjoint in $L^2_\varphi(\Omega, \omega, dV)$ will here be denoted by $\bar\partial^*_\varphi$, i.e., for smooth, compactly supported forms α and β,

$$\int \langle \bar\partial\alpha, \beta\rangle_{\omega,\varphi} dV = \int \langle \alpha, \bar\partial^*_\varphi \beta\rangle_{\omega,\varphi} dV.$$

The $(1,0)$ part turns out to depend on the fibre metric, or φ, and will be denoted by ∂_φ. Locally, or in our trivial case, we may write it as

$$\partial_\varphi = e^{\varphi}\partial e^{-\varphi} = \partial - \partial\varphi\wedge,$$

where ∂ is the $(1,0)$ part we get when $\varphi \equiv 0$. We denote the corresponding adjoint in $L^2_\varphi(\Omega, \omega, dV)$ by ∂^*, since it turns out not to depend on the fibre metric given by φ.

Let $\bar\partial^*$ denote the adjoint of $\bar\partial$ for $\varphi = 0$. One can show with a short integration by parts argument that

$$\bar\partial^*_\varphi = e^{\varphi}\bar\partial^* e^{-\varphi} = \bar\partial^* + \partial\varphi \lrcorner.$$

We define the complex "Box Laplacian" as

$$\Box'' := \bar{\partial}\bar{\partial}_\varphi^* + \bar{\partial}_\varphi^*\bar{\partial}.$$

The following commutator identity will also be useful. We may write

$$\partial^* = -i[\bar{\partial}, \Lambda]. \tag{2.1}$$

A proof of this formula can be found e.g. in Wells [20], in chapter V, section 3.

We will also have use for the Hodge $*$ operator. The complex linear $*$ operator that takes (p,q)-forms to $(n-q, n-p)$-forms is defined by the relation

$$\langle \alpha, \xi \rangle dV = \alpha \wedge \overline{*\xi}$$

for every α and ξ with the same degree.

Generally, in the sequel, ξ will be an $(n,1)$-form, and γ will be an $(n-1,0)$-form. Usually their relation will be that

$$\gamma = *\xi.$$

For the bidegrees we are interested in, we will make use of the following simple facts about the $*$ operator and Λ. For an $(n,0)$-form α, the $*$ operator is just multiplication by a scalar, since $* : \Lambda^{(n,0)} \to \Lambda^{(n,0)}$, and $\Lambda^{(n,0)}$ is a one dimensional space. For an orthonormal basis $\{\theta_i\}$ of $(1,0)$-forms in the metric ω, let $\theta := \theta_1 \wedge \cdots \theta_n$. Then

$$i^n(-1)^{\frac{n(n-1)}{2}} \theta \wedge \bar{\theta} = dV = \langle \theta, \theta \rangle_\omega dV = \theta \wedge \overline{*\theta}.$$

Conjugating, we see that

$$*\theta = i^n(-1)^{\frac{n(n+1)}{2}} \theta. \tag{2.2}$$

This must then hold for any $(n,0)$-form. Furthermore, if γ is an $(n-1,0)$-form, then

$$*\gamma = i^{n-1}(-1)^{\frac{n(n-1)}{2}} \omega \wedge \gamma. \tag{2.3}$$

This is seen in a similar way as for $(n,0)$-forms by looking at the elements of an orthonormal basis, this time for $(n-1,0)$-forms. If $\{\theta_j\}$ is as before, then

$$\widehat{\theta_j} := (-1)^{j-1}\theta_1 \wedge \dots \theta_{j-1} \wedge \theta_{j+1} \wedge \dots \theta_n,$$

is an orthonormal basis for $(n-1,0)$-forms. Now,

$$i^n(-1)^{\frac{n(n-1)}{2}} \theta \wedge \bar{\theta} = dV = \langle \widehat{\theta_j}, \widehat{\theta_j} \rangle_\omega dV = \widehat{\theta_j} \wedge \overline{*\widehat{\theta_j}}.$$

Since $\theta = \theta_j \wedge \widehat{\theta_j} = (-1)^{n-1}\widehat{\theta_j} \wedge \theta_j$, one can conclude that

$$\begin{aligned} *\widehat{\theta_j} &= -i^n(-1)^{\frac{n(n-1)}{2}}\bar{\theta}_j \wedge \theta \\ &= i^{n-1}(-1)^{\frac{n(n-1)}{2}}\sum_k i\theta_k \wedge \bar{\theta}_k \wedge \widehat{\theta_j} \\ &= i^{n-1}(-1)^{\frac{n(n-1)}{2}}\omega \wedge \widehat{\theta_j}. \end{aligned}$$

The last inequality follows from the fact that we may write $\omega = \sum i\theta_k \wedge \bar{\theta}_k$. This proves (2.3) for basis elements, and the rest follows from linearity.

The adjoint formulation of (2.3) is that if ξ is an $(n,1)$-form, then

$$*\xi = i^{n-1}(-1)^{\frac{n(n-1)}{2}}\Lambda\xi, \tag{2.4}$$

or

$$\Lambda\xi = -i^{n-1}(-1)^{\frac{n(n+1)}{2}} *\xi. \tag{2.5}$$

This follows from (2.3) by

$$\begin{aligned} \langle \alpha, *\xi\rangle_\omega dV &= \alpha \wedge \overline{**\xi} = \alpha \wedge \overline{i^{n-1}(-1)^{\frac{n(n-1)}{2}}\omega \wedge *\xi} \\ &= \langle \omega \wedge \alpha, i^{n-1}(-1)^{\frac{n(n-1)}{2}}\xi\rangle_\omega dV = \langle \alpha, i^{n-1}(-1)^{\frac{n(n-1)}{2}}\Lambda\xi\rangle_\omega dV. \end{aligned}$$

3. Some weighted L^2 identities.

In order to prove Theorem 1, we start by obtaining a weighted L^2 identity for $(n,1)$-forms satisfying the boundary conditions of the $\bar{\partial}$-Neumann problem. When the problem is formulated on a Kähler manifold, the right choice is using $(n,1)$-forms instead of $(0,1)$-forms. The identity obtained differs from what could be considered standard L^2 identities through the presence of the additional weight w.

By using some ideas from Siu [18], we first prove an identity for $(n-1,0)$-forms, given in Lemma 1. (Siu calls the method he uses the $\partial\bar{\partial}$-Bochner-Kodaira technique). Then, using the $*$ operator, which is an isometry between $(n,1)$-forms and $(n-1,0)$-forms, we obtain the desired identity given in Proposition 1. One should note that it is also possible to prove Propostion 1 more directly using the Bochner-Kodaira-Nakano identity. This argument can be found in Siu [19], in a proof of the Ohsawa-Takegoshi extension theorem.

One major difference between the identity for $(n-1,0)$-forms and the one for $(n,1)$-forms is that in terms of the $(n-1,0)$-form γ, this formulation is not dependent on any metric for the underlying manifold. In particular, it uses no Kähler assumption. Here, we have chosen to formulate the boundary term in a way that involves the metric, though it actually does not depend on it. That fact can be seen from the proof. The reason for the formulation given, is that it makes it easier to determine the sign of that term.

LEMMA 1. — *Let φ and w be sufficiently smooth real valued functions. Let γ be an $(n-1,0)$-form defined in a neighbourhood of a smooth bounded set $G=\{z:\nu(z)<0\}$. Here, ν is a defining function of G, with $d\nu\neq 0$ on ∂G. Assume further that γ satisfies the boundary condition $\partial\nu\wedge\gamma=0$ on ∂G. Then*

$$(3.1)\quad 2\,\mathrm{Re}\int_G i^n\bar\partial\partial_\varphi\gamma\wedge\bar\gamma e^{-\varphi}w$$
$$=\int_G i^n\partial\bar\partial\varphi\wedge\gamma\wedge\bar\gamma e^{-\varphi}w+(-1)^n\int_G i^n\bar\partial\gamma\wedge\overline{\bar\partial\gamma}e^{-\varphi}w$$
$$+(-1)^{n-1}\int_G i^n\partial_\varphi\gamma\wedge\overline{\partial_\varphi\gamma}e^{-\varphi}w+\int_G i^n\partial w\wedge\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi})$$
$$+\int_{\partial G} i^n *(\partial\bar\partial\nu\wedge\gamma\wedge\bar\gamma)e^{-\varphi}w\frac{dS}{|d\nu|}.$$

Proof. — This follows from the following direct computations of the differential forms and using Stokes theorem. We have, using the notation $\partial_\varphi=e^{\varphi}\partial e^{-\varphi}$ from Section 2, that

$$(3.2)\quad \partial\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi})$$
$$=\partial(\bar\partial\gamma\wedge\bar\gamma e^{-\varphi}+(-1)^{n-1}\gamma\wedge\overline{\partial_\varphi\gamma}e^{-\varphi})$$
$$=\big(\partial_\varphi\bar\partial\gamma\wedge\bar\gamma+(-1)^n\bar\partial\gamma\wedge\overline{\bar\partial\gamma}+(-1)^{n-1}\partial_\varphi\gamma\wedge\overline{\partial_\varphi\gamma}$$
$$+\gamma\wedge\overline{\bar\partial\partial_\varphi\gamma}\big)e^{-\varphi}.$$

Since

$$\partial_\varphi\bar\partial\gamma=-\bar\partial\partial_\varphi\gamma+\partial\bar\partial\varphi\wedge\gamma,$$

we obtain that the first term on the right hand side of (3.2) equals

$$(3.3)\qquad \partial_\varphi\bar\partial\gamma\wedge\bar\gamma=-\bar\partial\partial_\varphi\gamma\wedge\bar\gamma+\partial\bar\partial\varphi\wedge\gamma\wedge\bar\gamma.$$

Writing $\gamma\wedge\overline{\bar\partial\partial_\varphi\gamma}=(-1)^{n-1}\overline{\bar\partial\partial_\varphi\gamma\wedge\bar\gamma}$, we can see that after multiplication by i^n, the last term of (3.2) becomes the conjugate of the first term of (3.3). Thus (3.2) yields

$$(3.4)\quad i^n\partial\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi})=\big(-2\,\mathrm{Re}\,i^n\bar\partial\partial_\varphi\gamma\wedge\bar\gamma+i^n\partial\bar\partial\varphi\wedge\gamma\wedge\bar\gamma$$
$$+(-i)^n\bar\partial\gamma\wedge\overline{\bar\partial\gamma}+(-1)^{n-1}i^n\partial_\varphi\gamma\wedge\overline{\partial_\varphi\gamma}\big)e^{-\varphi}.$$

Now, we may multiply (3.4) with a weight function w and integrate. Using Stokes formula, the left hand side yields

$$(3.5)\quad \int_G wi^n\partial\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi}) = \int_{\partial G} i^n w\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi}) - \int_G i^n\partial w\wedge\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi}).$$

From the boundary condition

$$(3.6)\qquad \partial\nu\wedge\gamma = 0 \text{ for points on } \partial G,$$

it follows that for any $(0,n)$-form α, $\gamma\wedge\alpha$ vanishes as a differential form on the submanifold ∂G. If not, we would have $d\nu\wedge\gamma\wedge\alpha = \partial\nu\wedge\gamma\wedge\alpha$ nonzero at the boundary of G, and this is not the case by (3.6). Hence, we may write the boundary integral in (3.5) as

$$(3.7)\quad \begin{aligned}\int_{\partial G} i^n\bar\partial(\gamma\wedge\bar\gamma e^{-\varphi})w &= \int_{\partial G} i^n(\bar\partial\gamma\wedge\bar\gamma + (-1)^{n-1}\gamma\wedge\overline{\bar\partial_\varphi\gamma})we^{-\varphi}\\ &= \int_{\partial G} i^n\bar\partial\gamma\wedge\bar\gamma we^{-\varphi}.\end{aligned}$$

Now, to rewrite this boundary integral in a way which is more usable to us when determining the sign, we do the following. First, we note that (3.6) gives that $\partial\nu\wedge\gamma = \nu A$ for some $(n,0)$-form A which is bounded on $\overline{G}$. Then we see that

$$\bar\partial(\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi}) = \bar\partial(\nu A\wedge\bar\gamma e^{-\varphi}) = \bar\partial\nu\wedge A\wedge\bar\gamma e^{-\varphi} + \nu\bar\partial(A\wedge\bar\gamma e^{-\varphi}).$$

This vanishes for points on ∂G, the second term since ν vanishes at the boundary, and the first term since $\bar\partial\nu\wedge\bar\gamma = 0$ on ∂G by (3.6). Hence, on ∂G,

$$\begin{aligned}0 &= \bar\partial(\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi})\\ &= \bar\partial\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi} - \partial\nu\wedge\bar\partial\gamma\wedge\bar\gamma e^{-\varphi} + (-1)^n\partial\nu\wedge\gamma\wedge\overline{\bar\partial_\varphi\gamma}e^{-\varphi}\\ &= -\partial\bar\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi} - \partial\nu\wedge\bar\partial\gamma\wedge\bar\gamma e^{-\varphi},\end{aligned}$$

i.e.,

$$(3.8)\qquad \partial\bar\partial\nu\wedge\gamma\wedge\bar\gamma e^{-\varphi} = -\partial\nu\wedge\bar\partial\gamma\wedge\bar\gamma e^{-\varphi}\quad\text{on } \partial G.$$

Using the following formula; if β is a form of degree $2n-1$, then

$$\int_{\partial G}\beta = \int_{\partial G} *(d\nu\wedge\beta)\frac{dS}{|d\nu|},$$

we obtain that the boundary integral of (3.5) may be written as

$$\begin{aligned}\int_{\partial G} i^n\bar\partial\gamma\wedge\bar\gamma e^{-\varphi}w &= \int_{\partial G} i^n *(\partial\nu\wedge\bar\partial\gamma\wedge\bar\gamma)e^{-\varphi}w\frac{dS}{|d\nu|}\\ &= -\int_{\partial G} i^n *(\partial\bar\partial\nu\wedge\gamma\wedge\bar\gamma)e^{-\varphi}w\frac{dS}{|d\nu|}.\end{aligned}$$

This proves the lemma. □

Next, we translate (3.1) into an identity for $(n,1)$-forms using the $*$ operator defined by a Kähler metric.

PROPOSITION 1. — *Let ξ be an $(n,1)$-form on a smooth, bounded, open set $G = \{z : \nu(z) < 0\}$ in a Kähler manifold. Here, ν is a defining function of G, with $d\nu \neq 0$ on ∂G. Assume ξ satisfy $\partial\nu \lrcorner \xi = 0$ at the boundary, and that φ and w are sufficiently smooth real valued functions. Then*

$$\begin{aligned}(3.9)\qquad 2\,\mathrm{Re}\int \langle \bar\partial\bar\partial^*_\varphi \xi, \xi\rangle_\omega e^{-\varphi} w dV &= \int i\langle \partial\bar\partial\varphi \wedge \Lambda\xi, \xi\rangle_\omega e^{-\varphi} w dV + \int \left|\bar\partial^*_\varphi \xi\right|^2_\omega e^{-\varphi} w dV \\ &+ \int |\partial^*\xi|^2_\omega e^{-\varphi} w dV - \int \left|\bar\partial\xi\right|^2_\omega e^{-\varphi} w dV \\ &+ \int \Big(i\langle \partial w \wedge \Lambda\xi, \bar\partial^*_\varphi \xi\rangle_\omega + i\langle \partial w \wedge \bar\partial\Lambda\xi, \xi\rangle_\omega \Big) e^{-\varphi} dV \\ &+ \int_{\partial G} \langle i\partial\bar\partial\nu \wedge \Lambda\xi, \xi\rangle_\omega e^{-\varphi} w \frac{dS}{|d\nu|}.\end{aligned}$$

Proof. — This is basically just a reformulation of Lemma 1 in terms of ξ, where $\gamma = *\xi$, and keeping track of the signs. The major difference is the introduction of a Kähler metric, which will help us get control of the term $\bar\partial\gamma \wedge \overline{\bar\partial\gamma}$. Multiplying the entire identity (3.1) with the factor $(-1)^{n-1}(-1)^{\frac{n(n-1)}{2}} = -(-1)^{\frac{n(n+1)}{2}}$, the sign of each term will be the same as the corresponding one in (3.9).

To start with the boundary condition, assume that $\partial\nu \lrcorner \xi = 0$ on ∂G. Then we see that for any $(n,0)$-form α,

$$0 = \langle \alpha, \partial\nu \lrcorner \xi\rangle_\omega = \langle \bar\partial\nu \wedge \alpha, \xi\rangle_\omega = \bar\partial\nu \wedge \alpha \wedge \overline{*\xi} = \overline{\partial\nu \wedge \bar\alpha \wedge \gamma},$$

so the boundary condition of Lemma 1 is satisfied.

Now, the integrand in the left hand side in (3.9) yields, by (2.2),

$$\begin{aligned}\langle \bar\partial\bar\partial^*_\varphi \xi, \xi\rangle_\omega dV &= \langle \bar\partial(-*\partial_\varphi *\xi), \xi\rangle_\omega dV = -\bar\partial(*\partial_\varphi *\xi) \wedge \overline{*\xi} \\ &= -(-1)^{\frac{n(n+1)}{2}} i^n \bar\partial\partial_\varphi \gamma \wedge \bar\gamma.\end{aligned}$$

For the right hand side, we have the following for each term. The first integrand is by (2.5),

$$\begin{aligned}i\langle \partial\bar\partial\varphi \wedge \Lambda\xi, \xi\rangle_\omega dV &= i\partial\bar\partial\varphi \wedge -i^{n-1}(-1)^{\frac{n(n+1)}{2}} *\xi \wedge \overline{*\xi} \\ &= -(-1)^{\frac{n(n+1)}{2}} i^n \partial\bar\partial\varphi \wedge \gamma \wedge \bar\gamma.\end{aligned}$$

For the second term we can write, again using (2.2), and the fact that $** = (-1)^n$,

$$\langle\bar{\partial}^*_\varphi\xi, \bar{\partial}^*_\varphi\xi\rangle_\omega dV = *\partial_\varphi*\xi \wedge \overline{**\partial_\varphi*\xi} = i^n(-1)^{\frac{n(n+1)}{2}}(-1)^n\partial_\varphi\gamma \wedge \overline{\partial_\varphi\gamma}.$$

The third and fourth term of (3.9) cannot be dealt with in quite the same way, since they depend on the metric being Kähler. By writing the terms in normal coordinates, we will obtain that together they correspond to the $\bar{\partial}\gamma \wedge \overline{\bar{\partial}\gamma}$ term of (3.1). Since the terms involved consist of only first order derivatives of the forms, the following approach is valid when the metric is Kähler. Let z be normal coordinates at a point z_0 in G (which implies that $|dz_k|_\omega = 1$ and constitutes an orthonormal basis at the point. (This is in contrast to the ordinary coordinates z in $\mathbb{C}^n$, where, with our conventions, $|dz_k|_e = \sqrt{2}$). Then, we write

$$\xi = \sum \xi_k dz \wedge d\bar{z}_k.$$

One immediately obtains that, at z_0,

$$*\xi = \gamma = i^n(-1)^{\frac{n(n+1)}{2}}\sum \xi_k\widehat{dz_k},$$

and differentiating, which is possible in normal coordinates we have, still at z_0,

$$\begin{aligned}
\bar{\partial}\gamma \wedge \overline{\bar{\partial}\gamma} &= \sum \bar{\partial}_i\xi_k d\bar{z}_i \wedge \widehat{dz_k} \wedge \overline{\sum \bar{\partial}_i\xi_k d\bar{z}_i \wedge \widehat{dz_k}} \\
&= (-1)^{2n-1}\sum \bar{\partial}_i\xi_k\overline{\bar{\partial}_k\xi_i}dz \wedge d\bar{z} \\
&= \left(\frac{1}{2}\sum |\bar{\partial}_i\xi_k - \bar{\partial}_k\xi_i|^2 - \sum |\bar{\partial}_i\xi_k|^2\right) dz \wedge d\bar{z} \\
&= (|\bar{\partial}\xi|^2_\omega - |\bar{\partial}\gamma|^2_\omega)dz \wedge d\bar{z}.
\end{aligned}$$

Since $|\bar{\partial}\gamma| = |*\bar{\partial}\gamma| = |\partial^*\xi|$, we can conclude that

$$-(-1)^{\frac{n(n-1)}{2}}i^n\bar{\partial}\gamma \wedge \overline{\bar{\partial}\gamma} = (|\bar{\partial}\gamma|^2_\omega - |\bar{\partial}\xi|^2_\omega)dV = (|\partial^*\xi|^2_\omega - |\bar{\partial}\xi|^2_\omega)dV.$$

This gives the identity for the third and fourth term, at the arbitrary point z_0.

Further, the two terms arising from differentiation of the weight function w are, again by (2.5),

$$\begin{aligned}
i\langle\partial w \wedge \Lambda\xi, \bar{\partial}^*_\varphi\xi\rangle_\omega dV &= i^n(-1)^{\frac{n(n+1)}{2}}\langle\partial w \wedge *\xi, *\partial_\varphi*\xi\rangle_\omega dV \\
&= (-1)^{\frac{n(n+1)}{2}}i^n\partial w \wedge \gamma \wedge \overline{**\partial_\varphi\gamma} \\
&= (-1)^n(-1)^{\frac{n(n+1)}{2}}i^n\partial w \wedge \gamma \wedge \overline{\partial_\varphi\gamma}
\end{aligned}$$

and

$$i\langle \partial w \wedge \bar{\partial}\Lambda\xi, \xi\rangle_\omega dV$$
$$= -i^n(-1)^{\frac{n(n+1)}{2}}\langle \partial w \wedge \bar{\partial} * \xi, \xi\rangle_\omega dV = -i^n(-1)^{\frac{n(n+1)}{2}}\partial w \wedge \bar{\partial}\gamma \wedge \bar{\gamma}.$$

These two are exactly what one gets from the last term in (3.1).

And finally we turn to the the boundary integral. It is similar to the first term, giving

$$i\langle \partial\bar{\partial}\nu \wedge \Lambda\xi, \xi\rangle_\omega = *(i\langle \partial\bar{\partial}\nu \wedge \Lambda\xi, \xi\rangle_\omega dV) = -(-1)^{\frac{n(n+1)}{2}} i^n * (\partial\bar{\partial}\nu \wedge \gamma \wedge \bar{\gamma}).$$

This finishes the proof. □

4. Proof of the weighted L^2 estimate.

Now we are in a position to prove a weighted L^2 estimate for the minimal solution to the $\bar{\partial}$ problem with a suitable weight.

LEMMA 2. — *Assume that f is a closed $(n,1)$-form on a pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$. Let w be a weight function on Ω satisfying $|\partial w|_\omega \leq \varepsilon w$, for some $0 < \varepsilon < \sqrt{2}$. Let further φ be a strictly plurisubharmonic C^2 function on Ω and u be the $L^2_\varphi(\Omega, \omega, dV)$-minimal solution to $\bar{\partial}u = f$. Then*

$$\int |u|^2_\omega e^{-\varphi} w dV \leq \frac{2}{(\sqrt{2}-\varepsilon)^2}\int |f|^2_\omega e^{-\varphi} w dV, \tag{4.1}$$

where the metric is given by $\omega = i\partial\bar{\partial}\varphi$.

Theorem 1 is now an easy consequence of Lemma 2.

Proof of Theorem 1. — Apply Lemma 2 to $\tilde{f} = f dz$, where f is a closed $(0,1)$-form, and get the $L^2_\varphi(\Omega, \omega, dV)$-minimal solution $\tilde{u} = u dz$ to $\bar{\partial}\tilde{u} = \tilde{f}$. Then $\bar{\partial}u = f$, and we observe that by (2.2),

$$|\tilde{u}|^2_\omega dV = i^n(-1)^{\frac{n(n-1)}{2}}\tilde{u} \wedge \overline{\tilde{u}} = i^n(-1)^{\frac{n(n-1)}{2}} |u|^2 dz \wedge d\bar{z} = |u|^2 2^n d\lambda$$

and similarly that

$$|\tilde{f}|^2_\omega dV = |f|^2_\omega \langle dz, dz\rangle_\omega dV = |f|^2_\omega 2^n d\lambda.$$

Hence, the function u is the $L^2_\varphi(\Omega, d\lambda)$-minimal solution to $\bar{\partial}u = f$ for $(0,1)$-forms, and the desired inequality for u and f follows from

Lemma 2. This concludes the proof of Theorem 1 as soon as we have proved Lemma 2. □

Proof of Lemma 2. — First, assume that Ω is a smooth, bounded, and strictly pseudoconvex domain and that φ is smooth. Then we may apply Proposition 1 in the following way. For a given $(n,1)$-form f, let ξ be the $\bar\partial$-Neumann solution to

$$\Box''\xi = (\bar\partial\bar\partial^*_\varphi + \bar\partial^*_\varphi\bar\partial)\xi = f \quad \text{on} \quad \Omega. \tag{4.2}$$

When $\bar\partial f = 0$, it follows that

$$\bar\partial\xi = 0$$

and hence the second term in (4.2) vanishes. This yields

$$\Box''\xi = \bar\partial\bar\partial^*_\varphi\xi = f.$$

Now, we define u to be

$$u = \bar\partial^*_\varphi\xi.$$

Then u will be the $L^2_\varphi(\Omega,\omega,dV)$-minimal solution to $\bar\partial u = f$, since u is orthogonal to all holomorphic $(n,0)$-forms. Further, since $\bar\partial\xi = 0$, $\bar\partial\Lambda\xi = [\bar\partial,\Lambda]\xi = i\partial^*\xi$, by (2.1). Apply Proposition 1 to this ξ. By the pseudoconvexity of Ω, the boundary integral will be positive, and we obtain

$$\begin{aligned}2\operatorname{Re}\int\langle f,\xi\rangle_\omega e^{-\varphi}w dV \geq &\int i\langle\partial\bar\partial\varphi\wedge\Lambda\xi,\xi\rangle_\omega e^{-\varphi}w dV\\ &+\int|u|^2_\omega e^{-\varphi}w dV + \int|\partial^*\xi|^2_\omega e^{-\varphi}w dV\\ &+\int i\langle\partial w\wedge\Lambda\xi,u\rangle_\omega e^{-\varphi}dV\\ &-\int\langle\partial w\wedge\partial^*\xi,\xi\rangle_\omega e^{-\varphi}dV.\end{aligned} \tag{4.3}$$

To estimate the last two terms, we use the bound on ∂w. Taking absolute value, and observing that $|\partial w\wedge\Lambda\xi|^2_\omega \leq |\partial w|^2_\omega |\Lambda\xi|^2_\omega \leq \varepsilon^2 w^2 |\xi|^2_\omega$, the fourth term on the right hand side of (4.3) can be estimated by

$$\begin{aligned}&\left|\int i\langle\partial w\wedge\Lambda\xi,u\rangle_\omega e^{-\varphi}dV\right|\\ &\leq \frac{\delta}{2}\int|\partial w\wedge\Lambda\xi|^2_\omega \frac{e^{-\varphi}}{w}dV + \frac{1}{2\delta}\int|u|^2_\omega e^{-\varphi}w dV\\ &\leq \frac{\varepsilon^2\delta}{2}\int|\xi|^2_\omega e^{-\varphi}w dV + \frac{1}{2\delta}\int|u|^2_\omega e^{-\varphi}w dV,\end{aligned} \tag{4.4}$$

for $\delta > 0$ to be determined later.

For the last term of (4.3), the second with ∂w, we note that by Cauchy's inequality, valid for $(1,0)$- and $(n-1,1)$-forms like ∂w and $\partial^*\xi$,

$$|\partial w \wedge \partial^*\xi|_\omega^2 \leq |\partial w|_\omega^2\, |\partial^*\xi|_\omega^2 \leq \varepsilon^2 w^2\, |\partial^*\xi|_\omega^2\,,$$

and hence

$$\begin{aligned}(4.5)\quad &\left|\int \langle \partial w \wedge \partial^*\xi, \xi\rangle_\omega e^{-\varphi} dV\right| \\ &\leq \frac{1}{\varepsilon^2}\int |\partial w \wedge \partial^*\xi|_\omega^2 \frac{e^{-\varphi}}{w} dV + \frac{\varepsilon^2}{4}\int |\xi|_\omega^2 e^{-\varphi} w dV \\ &\leq \int |\partial^*\xi|_\omega^2 e^{-\varphi} w dV + \frac{\varepsilon^2}{4}\int |\xi|_\omega^2 e^{-\varphi} w dV.\end{aligned}$$

What needs to be estimated from (4.4) and (4.5) is $\int |\xi|_\omega^2 e^{-\varphi} w dV$, and it can be majorized by the $\partial\partial\varphi$ term in (4.3) by choosing an appropriate metric. If we choose as a metric for Ω the one with metric form $\omega = i\partial\bar\partial\varphi$, then

$$i\langle \partial\bar\partial\varphi \wedge \Lambda\xi, \xi\rangle_\omega = \langle \omega \wedge \Lambda\xi, \xi\rangle_\omega = |\Lambda\xi|_\omega^2 = |\xi|_\omega^2\,.$$

Hence, using (4.4) and (4.5), we get from (4.3) the following inequality. (The constants are chosen so that the terms with $|\partial^*\xi|_\omega^2$ cancel.) If $\omega = i\partial\bar\partial\varphi$, then

$$\begin{aligned}(4.6)\qquad &2\,\mathrm{Re}\int \langle f, \xi\rangle_\omega e^{-\varphi} w dV \\ &\geq \left(1 - \frac{\varepsilon^2\delta}{2} - \frac{\varepsilon^2}{4}\right)\int |\xi|_\omega^2 e^{-\varphi} w dV + \left(1 - \frac{1}{2\delta}\right)\int |u|_\omega^2 e^{-\varphi} w dV.\end{aligned}$$

On the other hand, the left hand side of (4.6) can be estimated from above by

$$(4.7)\quad \left|2\,\mathrm{Re}\int \langle f, \xi\rangle_\omega e^{-\varphi} w dV\right| \leq \frac{1}{\delta'}\int |f|_\omega^2 e^{-\varphi} w dV + \delta'\int |\xi|_\omega^2 e^{-\varphi} w dV.$$

Put (4.6) and (4.7) together, and choose $\delta > 1/2$ and $\delta' > 0$ such that

$$1 - \frac{\varepsilon^2\delta}{2} - \frac{\varepsilon^2}{4} = \delta'.$$

This is possible if $\varepsilon^2 < 4/(2\delta+1) < 2$. For a given ε with $0 < \varepsilon < \sqrt{2}$, it can be shown that $\delta^2 + 1/4 = 1/\varepsilon^2$ yields the best constant C in the lemma for this proof. We omit the elementary calculations here. With this choice of δ, we obtain the estimate

$$\begin{aligned}\int |u|_\omega^2 e^{-\varphi} w dV &\leq \frac{1}{(1 - \frac{\varepsilon^2\delta}{2} - \frac{\varepsilon^2}{4})(1 - \frac{1}{2\delta})}\int |f|_\omega^2 e^{-\varphi} w dV \\ &= \frac{2}{2 - \varepsilon\sqrt{4 - \varepsilon^2}}\int |f|_\omega^2 e^{-\varphi} w dV \\ &\leq \frac{2}{(\sqrt{2} - \varepsilon)^2}\int |f|_\omega^2 e^{-\varphi} w dV.\end{aligned}$$

This proves the desired result for smooth, bounded, strictly pseudoconvex domains.

For an arbitrary pseudoconvex set Ω, take an increasing sequence of smooth, strictly pseudoconvex domains Ω_k with $\cup\Omega_k = \Omega$. Let u_k be the L^2-minimal solution to $\bar{\partial}u_k = f$ on Ω_k. Then u_k satisfies the desired bound on Ω_k. Hence, for $k \geq k_0$,

$$\begin{aligned}\int_{\Omega_{k_0}} |u_k|_\omega^2 e^{-\varphi} w dV &\leq \int_{\Omega_k} |u_k|_\omega^2 e^{-\varphi} w dV \\ &\leq \frac{2}{(\sqrt{2}-\varepsilon)^2} \int_{\Omega_k} |f|_\omega^2 e^{-\varphi} w dV \\ &\leq \frac{2}{(\sqrt{2}-\varepsilon)^2} \int_{\Omega} |f|_\omega^2 e^{-\varphi} w dV.\end{aligned}$$

That is, (u_k) is a bounded sequence in $L^2(\Omega_{k_0}, \omega, e^{-\varphi} w dV)$. The special case with $w = 1$ as a weight, or standard L^2 estimates, yields (u_k) is a bounded sequence in $L^2(\Omega_{k_0}, \omega, e^{-\varphi} dV)$ as well. Further, the following sequence, denoted (A_k), satisfies

$$A_k := \int_{\Omega_k} |u_k|_\omega^2 e^{-\varphi} dV \leq \int_{\Omega} |f|_\omega^2 e^{-\varphi} dV,$$

i.e., (A_k) is bounded. Hence, for a given k_0, there is a subsequence of u_k (also denoted by u_k) with a weak limit u in $L^2(\Omega_{k_0}, \omega, e^{-\varphi} dV)$, and such that A_k converges. Since w is locally bounded, u is also a weak limit of u_k in $L^2(\Omega_{k_0}, \omega, e^{-\varphi} w dV)$. As a weak limit, u also solves $\bar{\partial}u = f$ on Ω_{k_0}. By taking further subsequences of (u_k) for increasing k_0, and then a diagonal sequence (still denoted by u_k) we may assume that $u_k \to u$ weakly in $L^2(\Omega_{k_0}, \omega, e^{-\varphi} dV)$ and in $L^2(\Omega_{k_0}, \omega, e^{-\varphi} w dV)$ for every k_0.

Since weak convergence decreases norms, we have that

$$\begin{aligned}\int_{\Omega_{k_0}} |u|_\omega^2 e^{-\varphi} w dV &\leq \lim_{k\to\infty} \int_{\Omega_{k_0}} |u_k|_\omega^2 e^{-\varphi} w dV \\ &\leq \frac{2}{(\sqrt{2}-\varepsilon)^2} \int_{\Omega} |f|_\omega^2 e^{-\varphi} w dV,\end{aligned} \tag{4.8}$$

and by monotone convergence, this also holds with Ω_{k_0} replaced by Ω. This is then the desired estimate for u, and all that remains is to prove that u actually is the $L^2_\varphi(\Omega, \omega, dV)$-minimal solution. This can be seen in the following way. As in (4.8),

$$\int_{\Omega_{k_0}} |u|_\omega^2 e^{-\varphi} dV \leq \lim_{k\to\infty} \int_{\Omega_{k_0}} |u_k|_\omega^2 e^{-\varphi} dV \leq \lim_{k\to\infty} A_k,$$

and the same inequality holds with Ω_{k_0} replaced by Ω. If u_0 is the minimal solution, then, since u_k is minimal on Ω_k,

$$A_k = \int_{\Omega_k} |u_k|^2_\omega \, e^{-\varphi} dV \leq \int_{\Omega_k} |u_0|^2_\omega \, e^{-\varphi} dV \leq \int_{\Omega} |u_0|^2_\omega \, e^{-\varphi} dV$$
$$\leq \int_{\Omega} |u|^2_\omega \, e^{-\varphi} dV \leq \lim_{k\to\infty} A_k.$$

Letting $k \to \infty$ in the left hand side yields

$$\int_\Omega |u_0|^2 \, e^{-\varphi} dV = \int_\Omega |u|^2 \, e^{-\varphi} dV,$$

and hence u has minimal norm. By uniqueness, $u = u_0$.

In case φ is not smooth, but only in C^2, we may use a similar limiting argument on $\Omega_k \subset\subset \Omega$ before letting $k \to \infty$: Convoluting with an approximate identity, we get a sequence of smooth $\varphi_\varepsilon \downarrow \varphi$, and $i\partial\bar\partial\varphi_\varepsilon \to i\partial\bar\partial\varphi$ uniformly on Ω_k by continuity. Then we pass to the limit using arguments similar to the ones above. Leaving out the details, this will give the theorem for φ in C^2 on $\Omega_k \subset\subset \Omega$, and thereafter we may let $\Omega_k \uparrow \Omega$. This concludes the proof of Lemma 2. □

5. Estimating the kernel.

To prove the pointwise estimate of the Bergman kernel $S_\varphi(z, \zeta)$ in Theorem 2, we will first make an L^2 estimate of the kernel by using the weighted L^2 estimate of Theorem 1. The weight will be used to obtain the factor with exponential decay relative to the distance between z and ζ in the $i\partial\bar\partial\varphi$ metric. First we need a result on the existence of a bounded potential when the metric is bounded. It can be found for example in Lelong and Gruman [12]. Since the proof is short, we sketch it for completeness.

LEMMA 3. — *Assume ω is a positive, bounded, continuous, d-closed $(1,1)$-form on a neighbourhood of a smooth, strictly pseudoconvex, star shaped domain. Then there exists a plurisubharmonic function ψ on the domain such that $i\partial\bar\partial\psi = \omega$, and $\|\psi\|_{L^\infty} \leq C\|\omega\|_{L^\infty}$, where the constant C depends only on the dimension and the domain.*

Proof. — Since ω is closed, the Poincaré lemma says there exists w such that $dw = \omega$. Decomposing w we may write $d(w_{0,1} + w_{1,0}) = \omega$, where

$$w_{0,1} = \sum_{j,k} \int_0^1 t\omega_{kj}(tz) z_j d\bar z_k$$

and

$$w_{1,0} = \overline{w_{0,1}} = \sum_{j,k} \int_0^1 t\omega_{kj}(tz)\bar{z}_k dz_j.$$

Hence $w_{0,1}$ and $w_{1,0}$ are bounded by $\|\omega\|_{L^\infty}$. Bidegree reasons give that $\bar{\partial} w_{0,1} = 0 = \partial w_{1,0}$. Since $w_{0,1} = \overline{w_{1,0}}$ it will be enough to solve $i\bar{\partial} v = w_{0,1}$ and let $\psi = 2\,\mathrm{Re}\, v$. Then $i\partial\bar{\partial}\psi = i\partial\bar{\partial}(v + \bar{v}) = (\partial w_{0,1} + \bar{\partial} w_{1,0}) = d(w_{0,1} + w_{1,0}) = \omega$. The proof will be finished if we can find a v which is bounded by $w_{0,1}$. The desired result can be found in the book by Henkin and Leiterer [9], Theorem 2.6.1, p. 82. Their theorem says even more. For a smooth, strongly pseudoconvex set D, there exists a constant C, such that if w is a continuous $(0, q)$-form on $\overline{D}$, with $\bar{\partial} w = 0$ in D, then there is a solution v to $\bar{\partial} v = w$ which a bounded Hölder norm,

$$\|v\|_{C^{1/2}(D)} \leq C\|w\|_{L^\infty(D)}.$$

Especially, $\|v\|_{L^\infty(D)} \leq C\|w\|_{L^\infty(D)}$. □

Now, remember that in (1.2), we let μ_ζ be an hermitian form that dominates the Kähler form ω. Then, to use Lemma 3 above, we will map μ_ζ with a complex linear mapping onto the Euclidean metric form β defined in (1.3), thus mapping ω to something with a known bound. For this, we will need the following observations on the behaviour of the kernel and of the metric under such mappings.

There is an elementary formula for how the Bergman kernel behaves under holomorphic mappings, which is valid also for the weighted kernel:

$$S_\varphi(z, \zeta) = J_{\mathbb{C}}\eta(z) S_{\varphi\circ\eta^{-1}}(\eta(z), \eta(\zeta)) \overline{J_{\mathbb{C}}\eta(\zeta)}, \tag{5.1}$$

where $J_{\mathbb{C}}\eta = \partial(\eta_1, \ldots, \eta_n)/\partial(z_1, \ldots, z_n)$ is the complex Jacobian determinant. The proof is principally the same as in the unweighted case, by variable substitution and using the uniqueness of the reproducing kernel. Calculations for the weighted case can be found in the paper by Dragomir [8], in Lemma 1. This paper also includes some further references on Bergman kernels with a more general weight than we consider in this paper.

Now, assume for simplicity that $\zeta = 0$, and let η be a complex linear map that takes $\mu = \mu_\zeta$ to the Euclidean metric, and the unit ball $B_\mu(0, 1)$ onto the Euclidean unit ball. Let L be the linear operator representing μ in the Euclidean metric, i.e., such that

$$\langle v, v\rangle_\mu = \langle v, Lv\rangle_e.$$

We see that we can take η as the square root of the positive operator L, then η is the linear operator for which $(\eta^{-1})^*\mu = \beta$. Then $J_{\mathbb{C}}\eta$ is constant,

and

$$S_\varphi(z,\zeta) = |J_{\mathbb{C}}\eta|^2 S_{\varphi\circ\eta^{-1}}(\eta(z),\eta(\zeta)) = \det(\mu) S_{\varphi\circ\eta^{-1}}(\eta(z),\eta(\zeta)). \tag{5.2}$$

For the metric and the distance function under the same mapping η, we observe that since $i\partial\bar{\partial}\varphi \le \mu$ on $B_\mu(0,1)$, $i\partial\bar{\partial}(\varphi\circ\eta^{-1}) = (\eta^{-1})^* i\partial\bar{\partial}\varphi \le \beta$ on $\eta(B_\mu(0,1)) = B_e(0,1)$. Also, if $\rho_\varphi(\cdot,\cdot)$ is the distance in the metric with potential φ, then

$$\rho_\varphi(z,\zeta) = \rho_{\varphi\circ\eta^{-1}}(\eta(z),\eta(\zeta)), \tag{5.3}$$

i.e., distances are preserved.

As mentioned in Section 1, the eigenvalues of a form with respect to another form is independent of the choice of coordinates. For example, we have

$$\lambda_{\min}(\omega|\mu_\zeta) = \lambda_{\min}((\eta^{-1})^*\omega|(\eta^{-1})^*\mu_\zeta) = \lambda_{\min}(i\partial\bar{\partial}\varphi\circ\eta^{-1}|\beta), \tag{5.4}$$

and likewise that

$$\det(\omega|\mu_\zeta) = \det(i\partial\bar{\partial}\varphi\circ\eta^{-1}|\beta). \tag{5.5}$$

5.1. An L^2 estimate.

From Theorem 1 we obtain the following estimate on the Bergman kernel.

PROPOSITION 2. — *Let $S_\varphi(z,\zeta)$ be the Bergman projection kernel in $L^2_\varphi(\mathbb{C}^n)$. Let further $\rho(z,\zeta)$ be the distance function of the metric with metric form $\omega = i\partial\bar{\partial}\varphi$ and let μ_{ζ_0} satisfy (1.2). Assume $0 < \varepsilon < \sqrt{2}$. Then, for any ζ_0, we have*

$$\begin{aligned}\int |S_\varphi(z,\zeta_0)|^2 e^{\varepsilon\rho(z,\zeta_0)} e^{-\varphi(z)} d\lambda(z) \\ \le \frac{C\det(\mu_{\zeta_0})e^{\varphi(\zeta_0)}}{(\sqrt{2}-\varepsilon)^2 \inf_{\zeta\in B_{\mu_{\zeta_0}}(\zeta_0,1)} \lambda_{\min}(\omega(\zeta)|\mu_{\zeta_0})}.\end{aligned} \tag{5.6}$$

The constant C depends only on the dimension n.

Proof. — First, assume that $\omega \le \beta$ in $B_e(\zeta_0,1)$. Then we may take μ_{ζ_0} as the Euclidean metric. Let $\chi : \mathbb{C}^n \to \mathbb{R}$ be a nonnegative radial function compactly supported in $B_e(\zeta_0, 1)$ with $\int \chi d\lambda = 1$. Then, for any harmonic function g,

$$g(\zeta_0) = \int_{\zeta \in B_e(\zeta_0,1)} \chi(\zeta) g(\zeta) d\lambda(\zeta). \tag{5.7}$$

We can use this to estimate $S_\varphi(z, \zeta)$. Namely, let $H(\zeta)$ be a holomorphic function in $B_e(\zeta_0, 1)$ with $H(\zeta_0) = 0$, and define

$$v(\zeta) = \chi(\zeta) e^{\varphi(\zeta) + \bar{H}(\zeta)}.$$

Since $S_\varphi(z, \zeta)$ is antiholomorphic in ζ, we have by (5.7) that for any z,

$$S_\varphi v(z) = \int_{B_e(\zeta_0,1)} S_\varphi(z,\zeta)\chi(\zeta) e^{\bar{H}(\zeta)} d\lambda(\zeta) = S_\varphi(z,\zeta_0) e^{\bar{H}(\zeta_0)} = S_\varphi(z, \zeta_0).$$

Now, let $f = \bar{\partial} v$, and let u be the $L^2_\varphi(\mathbb{C}^n, d\lambda)$-minimal solution to $\bar{\partial} u = f$. Then v can be orthogonally decomposed as

$$v = u + S_\varphi v,$$

or

$$S_\varphi(z, \zeta_0) = v(z) - u(z). \tag{5.8}$$

The main part to estimate here is u. v is an explicitly known function with compact support, and unless z and ζ_0 are close together, v vanishes. Generally, since χ is bounded,

$$|v(z)|^2 e^{-\varphi(z)} \leq C e^{\varphi(z) + 2 \operatorname{Re} H(z)}, \tag{5.9}$$

which will be dominated by the part coming from u.

To estimate the $L^2_\varphi(\mathbb{C}^n, d\lambda)$-minimal solution u to $\bar{\partial} u = f$, we have the weighted estimate of Theorem 1. For the right hand side of (1.1), we see that

$$f = \bar{\partial} v = (\bar{\partial}\chi + \chi \bar{\partial}(\varphi + \bar{H})) e^{\varphi + \bar{H}}.$$

The norm $|\cdot|_\omega$ on 1-forms is given by the inverse of ω, and can be estimated from above by the Euclidean norm and $\lambda^{-1}_{\min}(\omega|e)$. This yields

$$|f|^2_\omega e^{-\varphi} \leq \frac{2}{\lambda_{\min}(\omega|e)} \left(\left|\bar{\partial}\chi\right|^2_e e^{\varphi + 2 \operatorname{Re} H} + \chi^2 \left|\bar{\partial}(\varphi + \bar{H})\right|^2_e e^{\varphi + 2 \operatorname{Re} H} \right). \tag{5.10}$$

In the Euclidean metric, $\bar{\partial}\chi$ is bounded by a constant. Furthermore, since H is holomorphic and φ plurisubharmonic, one can see that

$$\left|\bar{\partial}(\varphi + \bar{H})\right|^2_e e^{\varphi + 2 \operatorname{Re} H} = \left|\bar{\partial}(\varphi + 2 \operatorname{Re} H)\right|^2_e e^{\varphi + 2 \operatorname{Re} H} \leq \Delta_e e^{\varphi + 2 \operatorname{Re} H}.$$

Since χ^2 has compact support, Green's formula allows us to apply the Δ_e on χ^2 instead of on the exponential factor. This yields

$$\int |f|^2_\omega \, e^{-\varphi} d\lambda \le \frac{2}{\inf_{\zeta \in B_e(\zeta_0,1)} \lambda_{\min}(\omega(\zeta)|e)} \int_{B_e(\zeta_0,1)} (C + \Delta_e \chi^2) e^{\varphi + 2\,\mathrm{Re}\, H} d\lambda.$$

Now, $\Delta_e \chi^2$ is bounded, and can be absorbed in C. Furthermore, on $B_e(\zeta_0, 1)$, $\rho_\omega(z, \zeta_0)$ is bounded by 1, since $\omega \le \beta$ there, and hence $w = e^{\varepsilon\rho(z,\zeta_0)}$ is bounded from above and below by positive constants. Thus it may be inserted into the integral, and we obtain, by Theorem 1,

$$\begin{aligned}
(5.11) \quad & \int |u(z)|^2 \, e^{-\varphi(z)} e^{\varepsilon\rho(\zeta_0,z)} d\lambda(z) \\
& \le \frac{2}{(\sqrt{2} - \varepsilon)^2} \int |f(\zeta)|^2_\omega \, e^{-\varphi(\zeta)} e^{\varepsilon\rho(\zeta_0,\zeta))} d\lambda(\zeta) \\
& \le \frac{C}{(\sqrt{2} - \varepsilon)^2 \inf_{\zeta \in B_e(\zeta_0,1)} \lambda_{\min}(\omega(\zeta))} \int_{B_e(\zeta_0,1)} e^{\varphi + 2\,\mathrm{Re}\, H} d\lambda.
\end{aligned}$$

By (5.9) the same estimate holds with $u(\cdot)$ replaced by $S_\varphi(\cdot, \zeta_0)$. The only thing that remains to complete the proof for the case $\omega \le \beta$ is to estimate

$$\int_{B_e(\zeta_0,1)} e^{\varphi + 2\,\mathrm{Re}\, H} d\lambda.$$

By Lemma 3 there exists a plurisubharmonic function ψ with $i\partial\bar{\partial}\psi = \omega = i\partial\bar{\partial}\varphi$, which is bounded by a constant on $B_e(\zeta_0, 1)$. Then, since $\psi - \varphi$ is pluriharmonic, there exists a holomorphic function $\tilde{H}$ such that $2\,\mathrm{Re}\, \tilde{H} = \psi - \varphi$. Now, let $H = \tilde{H} - \tilde{H}(\zeta_0)$, i.e., $2\,\mathrm{Re}\, H = \psi - \varphi - \psi(\zeta_0) + \varphi(\zeta_0)$. Then we may write

$$\int_{B_e(\zeta_0,1)} e^{\varphi + 2\,\mathrm{Re}\, H} d\lambda = \int_{B_e(\zeta_0,1)} e^{\psi - \psi(\zeta_0) + \varphi(\zeta_0)} \le C e^{\varphi(\zeta_0)}$$

since ψ is bounded. This finishes the proof for the case when $\omega \le \beta$ around ζ_0.

For the general case, let η be as in the discussion on page 984, with ζ_0 as the origin. We may then use, in the following order, (5.1), variable substitution and (5.3), the bounded case just proved, and then finally (5.4).

This yields

$$\begin{aligned}
\int |S_\varphi(z,\zeta_0)|^2 e^{\varepsilon\rho_\varphi(z,\zeta_0)} e^{-\varphi(z)} d\lambda(z) \\
&= \int \left|S_{\varphi\circ\eta^{-1}}(\eta(z),\eta(\zeta_0))\right|^2 e^{\varepsilon\rho_\varphi(z,\varphi_0)} e^{-\varphi(z)} \left|J_{\mathbb{C}}\eta\right|^4 d\lambda(z) \\
&= \int \left|S_{\varphi\circ\eta^{-1}}(x,\xi_0)\right|^2 e^{\rho_{\varphi\circ\eta^{-1}}(x,\xi_0)} e^{-\varphi\circ\eta^{-1}(x)} d\lambda(x) \left|J_{\mathbb{C}}\eta\right|^2 \\
&\le \frac{Ce^{-\varphi\circ\eta^{-1}(\xi_0)}\det(\mu_{\zeta_0})}{(\sqrt{2}-\varepsilon)^2 \inf_{\xi\in B_e(\xi_0,1)} \lambda_{\min}(\partial\bar\partial\varphi\circ\eta^{-1}(\xi)|e)} \\
&\le \frac{Ce^{-\varphi(\zeta_0)}\det(\mu_{\zeta_0})}{(\sqrt{2}-\varepsilon)^2 \inf_{\zeta\in B_{\mu_{\zeta_0}}(\zeta_0,1)} \lambda_{\min}(\omega|\mu_{\zeta_0})}.
\end{aligned}$$

This completes the proof of Proposition 2. □

5.2. Pointwise estimates.

To obtain pointwise estimates from Proposition 2, we will use a simple lemma that can be found e.g. in Berndtsson [2]. The proof from that paper will be sketched here for completeness.

LEMMA 4. — *Let φ be plurisubharmonic on $B = B_e(0,1)$. Define further*

$$M_\varphi = \{v \le 0 : \partial\bar\partial v = \partial\bar\partial\varphi \text{ on } B\}$$

and put $a_\varphi = \sup_{M_\varphi} v(0)$. Assume that $u \in L^2_{loc}(B)$ satisfies

$$\int_B |u|^2 e^{-\varphi} \le 1$$

and

$$\sup_B \left|\bar\partial u\right|^2 e^{-\varphi} \le 1.$$

Then

$$|u(0)|^2 e^{-\varphi(0)+a_\varphi} \le C,$$

where C is a universal constant.

Note. — In our case, we will have that $\partial\bar\partial\varphi$ is uniformly bounded, and Lemma 3 then says that a_φ is bounded by a constant. The conclusion is that

$$|u(0)|^2 \le Ce^{\varphi(0)}.$$

Proof of Lemma 4. — When $\varphi = 0$, take a cut-off function χ on B which equals 1 when $z < 1/2$. Let

$$K = c_n \partial \frac{1}{|z|^{2n-2}}$$

be the Bochner-Martinelli kernel. Then

$$u(0) = \int_B \bar{\partial}(\chi u) \cdot K = \int_B \chi \bar{\partial} u \cdot K + \int u \bar{\partial} \chi \cdot K.$$

The first term can be estimated by $\|\bar{\partial} u\|_{L^\infty}$, since $K \in L^1(B)$. The second term can be estimated by $\|u\|_{L^2}$ since $\bar{\partial}\chi = 0$ for $|z| < 1/2$. This gives us the lemma for $\varphi = 0$.

For an arbitrary plurisubharmonic φ, we let $v \in M_\varphi$. Then $v - \varphi$ is pluriharmonic, and hence there is a holomorphic H so that $v = \varphi + 2\operatorname{Re} H$. Put $u_H = ue^H$, then $\bar{\partial} u_H = e^H \bar{\partial} u$. Hence,

$$\int_B |u_H|^2 e^{-v} = \int_B |u|^2 e^{-\varphi} \leq 1,$$

and

$$|\bar{\partial} u_H|^2 e^{-v} = |\bar{\partial} u|^2 e^{-\varphi} \leq 1.$$

Since $e^{-v} > 1$, the same holds with that factor removed. The case $\varphi = 0$ now applies to u_H, and we find that

$$|u(0)|^2 e^{-\varphi(0)+v(0)} = |u_H(0)|^2 \leq C.$$

Taking supremum over all v now gives the lemma. $\square$

We are now in a position to prove Theorem 2.

Proof of Theorem 2. — To obtain a pointwise estimate from the left hand side of (5.6), we will use Lemma 4, working along the same lines as in the proof of Proposition 2. For a given point z_0, let $\mu = \mu_{z_0}$ be an hermitian form on the tangent space, satisfying (1.2). Let η be the complex linear mapping for this μ as defined on page 984. Furthermore, $\rho_\omega(\cdot, z_0) \leq 1$ on $B_\omega(z_0, 1) \supseteq B_{\mu_{z_0}}(z_0, 1)$, so by the triangle inequality,

$$e^{\varepsilon\rho(z,\zeta_0)} e^{-\varepsilon\rho(z_0,\zeta_0)} \geq e^{-\varepsilon\rho(z,z_0)} \geq e^{-\sqrt{2}} \text{ for } z \in B_\omega(z_0, 1),$$

and we may insert this into the right hand side along with a change of constant. This yields

$$\begin{aligned}\int_{B_e(\eta(z_0),1)} &\left|S_\varphi(\eta^{-1}(z),\zeta_0)\right|^2 e^{-\varphi(\eta^{-1}(z))}d\lambda(z)\\ &= \det(\mu_{z_0})\int_{B_{\mu_{z_0}}(z_0,1)} \left|S_\varphi(z,\zeta_0)\right|^2 e^{-\varphi(z)}d\lambda(z)\\ &\leq C\det(\mu_{z_0})\int \left|S_\varphi(z,\zeta_0)\right|^2 e^{\varepsilon\rho(z,\zeta_0)}e^{-\varphi(z)}d\lambda(z)e^{-\varepsilon\rho(z_0,\zeta_0)}\\ &\leq \frac{C\det(\mu_{z_0})\det(\mu_{\zeta_0})e^{\varphi(\zeta_0)}e^{-\varepsilon\rho(z_0,\zeta_0)}}{(\sqrt{2}-\varepsilon)^2\inf_{\zeta\in B_{\mu_{\zeta_0}}(\zeta_0,1)}\lambda_{\min}(\omega(\zeta)|\mu_{\zeta_0})}.\end{aligned}$$

Further, $\bar\partial S(\cdot,\zeta_0)=0$, so Lemma 4 applies for $S(\eta^{-1}(\cdot),\zeta_0)$ with the plurisubharmonic function $\varphi\circ\eta^{-1}$, yielding

$$\begin{aligned}&\left|S_\varphi(z_0,\zeta_0)\right|^2 e^{-\varphi\circ\eta^{-1}(\eta(z_0))+a_{\varphi\circ\eta^{-1}}\eta(z_0)}\\ &\qquad\leq \frac{C\det(\mu_{z_0})\det(\mu_{\zeta_0})e^{\varphi(\zeta_0)}e^{-\varepsilon\rho(z_0,\zeta_0)}}{(\sqrt{2}-\varepsilon)^2\inf_{\zeta\in B_{\mu_{\zeta_0}}(\zeta_0,1)}\lambda_{\min}(\omega(\zeta)|\mu_{\zeta_0})}.\end{aligned}$$

But $a_{\varphi\circ\eta^{-1}}$ was bounded, independently of φ, by Lemma 3, so Theorem 2 follows. □

6. A slightly improved version in $\mathbb{C}^1$.

In the case of one complex variable, we will see that, under an extra condition on φ, it is possible to strengthen the main result of this paper slightly. The condition we impose is that the $(1,1)$-form $\omega = i\partial\bar\partial\varphi$, which in the case of $\mathbb{C}^1$ is a measure, is a doubling measure.

DEFINITION 2. — *A measure ω is a doubling measure on $\mathbb{C}$ if there is a constant C_d such that*

$$\omega(B_e(z,2r)) \leq C_d\omega(B_e(z,r))$$

for every $z\in\mathbb{C}$ and $r>0$.

Let $\mu := \mu_{\zeta_0}$ be a constant $(1,1)$-form satisfying (1.2). In $\mathbb{C}^1$, μ is just a constant factor times the Euclidean volume form, and we identify μ with this constant factor. Then we let

$$\kappa := \kappa(\zeta_0) := \frac{1}{\pi}\int_{B_\mu(\zeta_0,1)} i\partial\bar\partial\varphi \tag{6.1}$$

denote the (normalised) $i\partial\bar{\partial}\varphi$-volume of the μ-unit ball around ζ_0. Note that the normalised μ-volume of this ball is 1 since μ is a multiple of β, the Euclidean form as defined in (1.3), and has no curvature. Since $i\partial\bar{\partial}\varphi \leq \mu$, it follows that $\kappa \leq 1$.

In some cases it may be possible to show that the μ-unit ball is equivalent to the $i\partial\bar{\partial}\varphi$-unit ball, and then we see that κ would be approximately the (normalised) volume of the unit ball. Generally, the volume of the unit ball is known to depend on the curvature of the metric. Results of this type can be found e.g. in Schoen and Yau [17], and Chavel [3], which are good references in this area.

Remember now, that we use Theorem 1 only for forms f supported in a ball with radius 1. Using this, together with the fact that in $\mathbb{C}$, it is possible to solve the $\partial\bar{\partial}$-equation in general (which then is just the Laplace equation), we will prove the following proposition, which is to serve as a replacement for Theorem 1.

PROPOSITION 3. — *Let φ be a strictly subharmonic C^2 function on $\mathbb{C}$, and assume that $\omega = i\partial\bar{\partial}\varphi$ is a doubling measure and that $\omega(\zeta) \leq \beta$ for $\zeta \in B_e(\zeta_0, 1)$, i.e., the Euclidean metric form $\mu = \beta$ satisfies (1.2). Let κ be defined as in (6.1). Let further u be the $L^2(\mathbb{C}, e^{-\varphi} d\lambda)$-minimal solution to $\bar{\partial} u = f$, and assume that* $\operatorname{supp}(f) \subseteq B_e(\zeta_0, 1)$. *Then,*

$$\int |u|^2 e^{-\varphi} e^{\varepsilon\rho_\omega(\zeta_0,\cdot)} d\lambda \leq \frac{C}{(\sqrt{2}-\varepsilon)\kappa} \int |f|_e^2 e^{-\varphi} d\lambda,$$

where the constant C depends only on the doubling constant of ω.

This will give us the following version of Theorem 2.

PROPOSITION 4. — *Let φ be a strictly subharmonic C^2 function on $\mathbb{C}$, and assume that $\omega = i\partial\bar{\partial}\varphi$ is a doubling measure. Let $0 < \varepsilon < \sqrt{2}$, and μ_z satisfy (1.2). Then the weighted Bergman kernel in $\mathbb{C}$ satisfies the estimate*

$$|S_\varphi(z,\zeta)|^2 \leq \frac{C}{(\sqrt{2}-\varepsilon)\kappa} \mu_z \mu_\zeta e^{\varphi(z)+\varphi(\zeta)-\varepsilon\rho_\omega(z,\zeta)},$$

where the constant C only depends on the doubling constant C_d of ω.

The proof of Proposition 4 is the same as for Theorem 2, with the following comments: Using Proposition 3, we will prove a variant of Proposition 2, which is given in Proposition 5. Proposition 5 can then be used to prove Proposition 4. □

Proof of Proposition 3. — Similarly to Theorem 1, we will obtain this result from the identity in Proposition 1. In $\mathbb{C}$, it can be seen that the identity is actually the same in every metric, a different metric corresponds to a substitution of the differential form ξ. From now on, we will use the convention to denote $\Delta = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$ for the Laplace operator. If we write down Proposition 1 directly in the Euclidean metric, identifying $\xi = \alpha dz \wedge d\bar{z}$ with α, we obtain the following identity:

$$
\begin{aligned}
(6.2)\qquad 2\operatorname{Re}\int f\bar{\alpha}e^{-\varphi}wd\lambda = \int \Delta\varphi\left|\alpha\right|^2 e^{-\varphi}wd\lambda + \int \left|u\right|^2 e^{-\varphi}wd\lambda \\
+\int\left|\frac{\partial\alpha}{\partial\bar{z}}\right|^2 e^{-\varphi}wd\lambda + \int\frac{dw}{dz}\alpha\bar{u}e^{-\varphi}d\lambda \\
+\int\frac{dw}{dz}\frac{d\alpha}{d\bar{z}}\bar{\alpha}e^{-\varphi}d\lambda.
\end{aligned}
$$

Now, assume, as in the proof of Lemma 2, that $\left|\partial w\right|_\omega \leq \varepsilon w$, that is,

$$\left|\frac{dw}{dz}\right|^2 \leq \varepsilon^2 w^2\Delta\varphi.$$

In analogy with the proof of Lemma 2, (essentially Theorem 1), we estimate the terms involving dw/dz as

$$\left|\frac{dw}{dz}\alpha\bar{u}\right| \leq \frac{\delta}{2}\left|\frac{dw}{dz}\right|^2\frac{\left|\alpha\right|^2}{w} + \frac{1}{2\delta}\left|u\right|^2 w \leq \frac{\varepsilon^2\delta}{2}w\Delta\varphi\left|\alpha\right|^2 + \frac{1}{2\delta}\left|u\right|^2 w,$$

and

$$\left|\frac{dw}{dz}\frac{d\alpha}{d\bar{z}}\bar{\alpha}\right| \leq \frac{1}{4}\left|\frac{dw}{dz}\right|^2\frac{\left|\alpha\right|^2}{w} + \left|\frac{d\alpha}{d\bar{z}}\right|^2 w \leq \frac{\varepsilon^2}{4}\Delta\varphi\left|\alpha\right|^2 w + \left|\frac{d\alpha}{d\bar{z}}\right|^2 w.$$

Thus, we obtain from (6.2), that

$$
\begin{aligned}
(6.3)\quad \left(1-\frac{\varepsilon^2\delta}{2}-\frac{\varepsilon^2}{4}\right)\int\Delta\varphi\left|\alpha\right|^2 e^{-\varphi}wd\lambda + \left(1-\frac{1}{2\delta}\right)\int\left|u\right|^2 e^{-\varphi}wd\lambda \\
\leq A\int_{B_e(\zeta_0,1)}\left|f\right|^2 e^{-\varphi}wd\lambda + \frac{1}{A}\int_{B_e(\zeta_0,1)}\left|\alpha\right|^2 e^{-\varphi}wd\lambda
\end{aligned}
$$

for any positive A, since f is supported on $B_e(\zeta_0,1)$. To obtain the desired estimate, we have to take care of the last integral on the right hand side. Previously, we did this with $\Delta\varphi\left|\alpha\right|^2$ on the left hand side, but this time we will use a different approach; we will use formula (6.2) again, with a different w, and add the result to (6.3). After a partial integration in (6.2) in the terms with dw/dz and discarding some terms, we obtain that, for

$v \geq 0$,

$$\begin{aligned}
(6.4)\quad \int \Delta\varphi\, |\alpha|^2 e^{-\varphi} v\, d\lambda &- \int \Delta v\, |\alpha|^2\, e^{-\varphi} d\lambda \\
&\leq 2\,\mathrm{Re} \int f \bar{\alpha} e^{-\varphi} v\, d\lambda \\
&\leq A \int_{B_e(\zeta_0,1)} |f|^2\, e^{-\varphi} v\, d\lambda + \frac{1}{A} \int_{B_e(\zeta_0,1)} |\alpha|^2\, e^{-\varphi} v\, d\lambda,
\end{aligned}$$

for any positive A. If we choose δ so that $1 - \varepsilon^2\delta/2 - \varepsilon^2/4 = 0$, i.e.,

$$\delta = \frac{2}{\varepsilon^2} - \frac{1}{2},$$

then the factor in front of the integral with u in (6.3) is

$$1 - \frac{1}{2\delta} = 1 - \frac{\varepsilon^2}{4 - \varepsilon^2} \geq \frac{1}{\sqrt{2}}(\sqrt{2} - \varepsilon).$$

By adding (6.3) and (6.4), we obtain that

$$\begin{aligned}
(6.5)\quad \int (v\Delta\varphi - \Delta v)\, |\alpha|^2\, e^{-\varphi} d\lambda &+ \frac{1}{\sqrt{2}}(\sqrt{2} - \varepsilon) \int |u|^2\, e^{-\varphi} w\, d\lambda \\
&\leq A \int_{B_e(\zeta_0,1)} (w + v)\, |f|^2\, e^{-\varphi} d\lambda + \frac{1}{A} \int_{B_e(\zeta_0,1)} (w + v)\, |\alpha|^2\, e^{-\varphi} d\lambda.
\end{aligned}$$

Since f is supposed to be supported in $B_e(\zeta_0, 1) \subset B_\omega(\zeta_0, 1)$, and $w = e^{\rho(\zeta_0,\cdot)}$, it follows that $1 \leq w \leq e$ on $B_e(\zeta_0, 1)$. From now on, assume that we can find a weight v such that $1 \leq v \leq K$ for some constant K, which is to be determined. Then, from (6.5) the following holds:

$$\begin{aligned}
(6.6)\quad \int \left(v\Delta\varphi - \Delta v - \left(\frac{e + K}{A}\right) \chi_{B_e(\zeta_0,1)} \right) |\alpha|^2\, e^{-\varphi} d\lambda& \\
+ \frac{1}{\sqrt{2}}(\sqrt{2} - \varepsilon) \int |u|^2\, e^{-\varphi} d\lambda &\leq A(e + K) \int |f|^2\, e^{-\varphi} d\lambda.
\end{aligned}$$

So, if v satisfies

$$v\Delta\varphi - \Delta v - \left(\frac{e + K}{A}\right) \chi_{B_e(\zeta_0,1)} \geq 0,$$

we would have a weighted estimate for u. This is achieved by solving

$$(6.7)\qquad \Delta v = \left(\Delta\varphi - \frac{e + K}{A} \right) \chi_{B_e(\zeta_0,1)}.$$

A necessary and sufficient condition for this to have a bounded solution is that the total mass of Δv is zero, and this is true if

$$A = 2\frac{e + K}{\kappa},$$

since $\Delta\varphi d\lambda = i\partial\bar{\partial}\varphi/2$. Now, with this A, a bounded solution to (6.7) is the logarithmic potential

$$\begin{aligned}\tilde{v} &= \frac{2}{\pi}\int_{B_e(\zeta_0,1)} \log|z-\zeta|\,(\Delta\varphi(\zeta)-\frac{\kappa}{2})d\lambda(\zeta)\\ &= \frac{1}{\pi}\int_{B_e(\zeta_0,1)} \log\left|1-\frac{\zeta}{z}\right|(\omega(\zeta)-\kappa d\lambda(\zeta)).\end{aligned}$$

When $z \to \infty$, clearly $\tilde{v} \to 0$, and since $\tilde{v}$ is harmonic outside $B_e(\zeta_0, 1)$, the maximum of $\tilde{v}$ must be attained in $\overline{B_e(\zeta_0, 1)}$.

To estimate $\tilde{v}$ in $\overline{B_e(\zeta_0,1)}$, we use the doubling property of ω. A doubling measure is known to have a polynomial behaviour; for $r \leq R$,

$$\omega(B_e(\zeta,r)) \leq C_1\left(\frac{r}{R}\right)^{\alpha}\omega(B_e(\zeta,R))$$

for some constants C_1 and $\alpha > 0$ depending on the doubling constant C_d. See Lemma 2.1 in Christ [4] for a proof of this result.

We may also note that the same is true when we replace ζ by a nearby point z in the left hand side. If $|z-\zeta| \leq 1$, then $B_e(z,1) \subseteq B_e(\zeta,2)$, and it follows that for $r \leq 1$,

$$\omega(B_e(z,r)) \leq C_1 r^{\alpha}\omega(B_e(z,1)) \leq C_1 r^{\alpha}\omega(B_e(\zeta,2)) \leq C_1 C_d r^{\alpha}\pi\kappa(\zeta). \tag{6.8}$$

When integrating $\log|z-\zeta|\,\omega$ in the definition of $\tilde{v}$, any negative contribution comes from integrating over the ball $B_e(z,1)$. Hence, to estimate

$$\int \log|z-\zeta|\,\omega(\zeta),$$

from above and below, the following inequalities suffice. When $|z-\zeta_0| \leq 1$,

$$\begin{aligned}\kappa\log 2 &\geq \frac{1}{\pi}\int_{B_e(\zeta_0,1)} \log|z-\zeta|\,\omega(\zeta)\\ &\geq \frac{1}{\pi}\int_{B_e(z,1)} \log|z-\zeta|\,\omega(\zeta)\\ &= \frac{1}{\pi}\sum_{j=0}^{\infty}\int_{2^{-j-1}\leq|z-\zeta|<2^{-j}} \log|z-\zeta|\,\omega(\zeta)\\ &\geq \frac{1}{\pi}\sum_{j=0}^{\infty}\log 2^{-j-1}\omega(B_e(z,2^{-j}))\\ &\geq -C_1C_d\sum_{j=0}^{\infty}(j+1)2^{-\alpha j}\omega(B_e(z,1)) \geq -\kappa C_2,\end{aligned}$$

where the last line follows from the inequality (6.8), and the assumption that $|z-\zeta_0| \le 1$. The constant C_2 depends only on the doubling constant C_d.

Also, a simple calculation shows that for $|z-\zeta_0| \le 1$,

$$0 \ge \frac{1}{\pi}\int_{B_e(\zeta_0,1)} \log|z-\zeta|\,\kappa d\lambda(\zeta) \ge -\frac{\kappa}{2},$$

so we have shown that

$$\kappa(1/2+\log 2) \ge \tilde{v} \ge -C_2\kappa.$$

Now, we let the weight function v be

$$v = \tilde{v} + C_2\kappa + 1.$$

Then we may bound v by

$$1 \le v \le K,$$

where $K = 3/2 + \log 2 + C_2$. This yields

$$A(e+K) = 2\frac{(e+K)^2}{\kappa} = \frac{C}{\kappa},$$

where C depends only on the doubling constant.

Thus we have proved everything that is needed to obtain Proposition 3 from (6.6). □

We may now apply Proposition 3 to prove the following version of Proposition 2.

PROPOSITION 5. — *Let $S_\varphi(z,\zeta)$ be the Bergman projection kernel in $L^2_\varphi(\mathbb{C})$. Let further $\rho(z,\zeta)$ be the distance function of the metric with metric form $\omega = i\partial\bar{\partial}\varphi$, and assume that it is a doubling measure. Let further μ_{ζ_0} satisfy (1.2), and assume that $0 < \varepsilon < \sqrt{2}$. Then, for any ζ_0, we have*

$$\int |S_\varphi(z,\zeta)|^2\, e^{\varepsilon\rho(z,\zeta_0)} e^{-\varphi(z)} d\lambda(z) \le \frac{C\mu_{\zeta_0} e^{\varphi(\zeta_0)}}{(\sqrt{2}-\varepsilon)\kappa(\zeta_0)},$$

where the constant C only depends on the doubling constant C_d.

Proof. — The proof is just like the proof of Proposition 2, with the following changes. Replace (5.10) with the estimate

$$|f|_e^2\, e^{-\varphi} \le 2\,|\bar{\partial}\chi|_e^2\, e^{\varphi+2\,\mathrm{Re}\,H} + 2\chi^2\,|\bar{\partial}(\varphi+\bar{H})|_e^2\, e^{\varphi+2\,\mathrm{Re}\,H}.$$

This will give us

$$\int |f|_e^2 e^{-\varphi} d\lambda \leq 2 \int_{B_e(\zeta_0,1)} (C + \Delta_e \chi^2) e^{\varphi + 2\operatorname{Re} H} d\lambda.$$

Then, using Proposition 3 instead of Theorem 1, we obtain, instead of (5.11), that

$$\begin{aligned} (6.9) \int |u(z)|^2 e^{-\varphi(z)} e^{\varepsilon\rho(\zeta_0,z)} d\lambda(z) &\leq \frac{C}{(\sqrt{2}-\varepsilon)\kappa(\zeta_0)} \int |f(\zeta)|_e^2 e^{-\varphi(\zeta)} d\lambda(\zeta) \\ &\leq \frac{C}{(\sqrt{2}-\varepsilon)\kappa(\zeta_0)} \int_{B_e(\zeta_0,1)} e^{\varphi+2\operatorname{Re} H} d\lambda. \end{aligned}$$

The proof is then completed as in the proof of Proposition 2, by estimating

$$\int_{B_e(\zeta_0,1)} e^{\varphi+2\operatorname{Re} H} d\lambda,$$

and then rescaling to the case when μ is not equal to β, and observing that κ is preserved under scalings. □

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