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THE ASYMPTOTICS OF THE RAY-SINGER ANALYTIC TORSION OF THE SYMMETRIC POWERS OF A POSITIVE VECTOR BUNDLE

by J.-M. BISMUT and E. VASSEROT

Let X be a compact complex manifold, equipped with a smooth Hermitian metric. Let $(E,\| \ \|_E)$, $(\xi,\| \ \|_\xi)$ be holomorphic Hermitian vector bundles on X. Assume that $(E,\| \ \|_E)$ is positive, i.e. if L^E is the curvature of the holomorphic Hermitian connection on $(E,\| \ \|_E)$, for any $U \in TX\backslash\{0\}$, $e \in E\backslash\{0\}$, then $\langle L^E(U,\bar{U})e,\bar{e}\rangle > 0$. By [K] Theorem III 6.19, E is an ample vector bundle on X.

For $p \in \mathbb{N}$, let $S^p(E)$ be the p^{th} symmetric tensor power of E. Then by a result of Le Potier [LP], [K] Theorem III.6.25, for p large enough, and $q > 0$, $H^q(S^p(E)\otimes\xi) = 0$. Let τ_p be the Ray-Singer analytic torsion of the Dolbeault complex $\Omega^{(0,\cdot)}(S^p(E)\otimes\xi)$ [RS]. The purpose of this paper is to establish an asymptotic formula for $\operatorname{Log}(\tau_p)$ as $p \to +\infty$. This extends an earlier result by ourselves [BV] Theorem 8, in the case where E is a positive line bundle.

The general strategy is the same as in [BV]. Namely if $\bar{\square}_p^{X,q}$ denotes the Hodge Laplacian acting on $\Omega^{(0,q)}(S^p(E)\otimes\xi)$, we first establish in Theorem 1 an asymptotic formula for $\operatorname{Tr}\left[\exp\left(-\frac{t}{p}\bar{\square}_p^{X,q}\right)\right]$ as $p \to +\infty$. In Theorem 8 we prove that if λ_p^q is the lowest eigenvalue of $\bar{\square}_p^{X,q}$, if $q > 0$, as $p \to +\infty$, λ_p^q grows at least like p. The combination of these two results leads us in Theorem 11 to an asymptotic formula for $\operatorname{Log}(\tau_p)$ very much like in [BV].

To establish these intermediary results, we use a trick due to Getzler [Ge] in a different context. In [Ge], Getzler extended a result of Bismut [B2] on the asymptotics of certain heat equation operators, which

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is valid for line bundles, to vector bundles, associated with representations of the structure group with weight $p\lambda$ as $p \to +\infty$. Here if μ is the dual of the universal line bundle on $\mathbb{P}(E^*)$, we consider $S^p(E)$ as the direct image of $\mu^{\otimes p}$ by the map $\pi : \mathbb{P}(E^*) \to X$. We then use Getzler's trick to lift our initial problem to a corresponding problem on the line bundle $\mu^{\otimes p}$ on $\mathbb{P}(E^*)$, to which the techniques of [BV] can be applied.

Our paper is organized as follows. In § 1, we introduce our main assumptions and notation. In § 2, we calculate the asymptotics of $\operatorname{Tr}\left[\exp\left(-\frac{t}{p}\bar{\square}_p^{X,q}\right)\right]$ as $p \to \infty$. Finally in § 3, we establish our main result on the asymptotics of $\operatorname{Log}(\tau_p)$ as $p \to +\infty$.

As was pointed out by the referee, the results contained in this paper can be extended to other irreducible representations of E, which are associated with the weights pa (where a is a given weight) when p tends to $+\infty$. The corresponding vector bundles can be expressed as direct images of the p^{th} power of a certain line bundle over the corresponding flag manifold. Arguments of Demailly [De] Lemma 3.7, can then be used to prove the positivity of this Hermitian line bundle when $(E, \| \ \|_E)$ is positive, and the trick of Getzler [Ge] together with the techniques used in our paper still apply. This extension of our main result is left to the reader.

1. Assumptions and notation.

Let X be a compact complex manifold of complex dimension ℓ. Let TX be the complex holomorphic tangent space.

Let E be a holomorphic vector bundle on X, of complex dimension k. Let E^* be the dual of E. For $p \in N$, $S^p(E)$ denotes the p^{th} symmetric tensor power of E.

Let $\mathbb{P}(E^*)$ denote the projectivization of E^*, and let π be the projection $\mathbb{P}(E^*) \to X$. Let μ be the dual of the universal line bundle on $\mathbb{P}(E^*)$.

Let $\| \ \|_E$ be a smooth Hermitian metric on E. Let $\| \ \|_{S^p(E)}$, $\| \ \|_{E^*}$, $\| \ \|_\mu$ be the Hermitian metrics on $S^p(E)$, E^*, μ induced by the metric $\| \ \|_E$.

Let ∇^E, ∇^{E^*} be the holomorphic Hermitian connections on $(E, \| \ \|_E)$, $(E^*, \| \ \|_{E^*})$, and let L^E, L^{E^*} be the corresponding curvatures. Let ∇^μ be the holomorphic Hermitian connection on $(\mu, \| \ \|_\mu)$ and let r be its curvature.

We first calculate r. Let $T^V\mathbb{P}(E^*)$ be the relative tangent bundle to the fibres of $\pi : \mathbb{P}(E^*) \to X$. The connection ∇^{E^*} induces a horizontal subbundle $T^H\mathbb{P}(E^*)$ of $T\mathbb{P}(E^*)$.

Let r^V be the restriction of r to $T^V\mathbb{P}(E^*)$. r^V is explicitly known by a formula given in [GrH] p. 30, and defines the Fubini-Study metric along the fibres of $\mathbb{P}(E^*)$. r^V extends into a (1,1) form on $\mathbb{P}(E^*)$ such that if $U \in T^H\mathbb{P}(E^*)$, $i_U r^V = 0$.

Set

$$r^H = -\pi^* \frac{\langle L^{E^*} y, \bar{y} \rangle}{|y|^2}; \qquad y \in E^* \backslash \{0\}. \tag{1}$$

Then r^H is a (1,1) form on $\mathbb{P}(E^*)$. Also by [K] p. 90,

$$r = r^V + r^H \tag{2}$$

r^H is then the restriction of r to $T^H\mathbb{P}(E^*) \times T^H\mathbb{P}(E^*)$.

For $p \geqslant 1$, the connection ∇^E induces on $S^p(E)$ the holomorphic Hermitian connection $\nabla^{S^p(E)}$ on $(S^p(E), \| \ \|_{S_p(E)})$.

Let $(\xi, \| \ \|_\xi)$ be a holomorphic Hermitian vector bundle on X. Let ∇^ξ be the corresponding holomorphic Hermitian connection.

Let $\| \ \|_{TX}$ be a Hermitian metric on X.

We then equip $\Lambda(T^{*(0,1)}X) \otimes S^p(E) \otimes \xi$ with the tensor product of the metrics induced by $\| \ \|_{TX}$ on $\Lambda(T^{*(0,1)}X)$, of the metric $\| \ \|_{S^p(E)}$ and of the metric $\| \ \|_\xi$.

For $0 \leqslant q \leqslant \ell$, let $\Omega^{(0,q)}(S^p(E) \otimes \xi)$ be the set of C^∞ sections of $\Lambda^q(T^{*(0,1)}X) \otimes S^p(E) \otimes \xi$ over X. Set $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi) = \bigoplus_{q=0}^{\ell} \Omega^{(0,q)}(S^p(E) \otimes \xi)$.

Let dx be the volume form on X associated with the metric $\| \|_{TX}$. We equip $\Omega^{(0,\cdot)}(S^p(E) \otimes \xi)$ with the L_2 Hermitian product

$$\alpha, \alpha' \in \Omega^{(0,\cdot)}(S^p(E) \otimes \xi) \to \langle \alpha, \alpha' \rangle = \int_X \langle \alpha, \alpha' \rangle (x) \frac{dx}{(2\pi)^{\dim X}}. \tag{3}$$

Let $\bar{\partial}_p^X$ be the Dolbeault operator acting on $\Omega^{(0,\cdot)}(S^p(E)\otimes\xi)$, and let $\bar{\partial}_p^{X*}$ be the formal adjoint of $\bar{\partial}_p^X$ with respect to the Hermitian product (3). Set

$$\bar{\square}_q^X = (\bar{\partial}_p^X + \bar{\partial}_p^{X*})^2. \tag{4}$$

For $0 \leqslant q \leqslant \ell$, let $\bar{\square}_p^{X,q}$ be the restriction of $\bar{\square}_p^X$ to $\Omega^{(0,q)}(S^p(E)\otimes\xi)$.

2. The asymptotics of the trace of certain heat kernels as $p \to +\infty$.

If $U \in T_R X$, let U^H be its horizontal lift in $T_R^H \mathbb{P}(E)$, so that $U^H \in T_R^H \mathbb{P}(E)$, $\pi_* U^H = U$.

Let $w_1, \ldots, w_\ell$ be an orthonormal base of TX, let $w^1, \ldots, w^\ell$ be the corresponding dual base of T^*X. If $z \in \mathbb{P}(E^*)$, r_z acts as a derivation $r_{d,z}^H$ of $\Lambda_{\pi z}^{(0,1)}(T^{*(0,1)}X)$ by the formula

$$r_{d,z}^H = -\sum r_z(w_i^H, \bar{w}_j^H)\bar{w}^j \wedge i_{\bar{w}_i}. \tag{5}$$

We identify r_z^H with the self-adjoint matrix $\mathring{r}_z^H \in \mathrm{End}_{\pi x}(\overline{TX})$ such that if $U, V \in T_{\pi z}X$,

$$r_z^H(U^H, \bar{V}^H) = \langle U^H, \mathring{r}_z^H \bar{V}^H \rangle.$$

For $0 \leqslant q \leqslant \ell$, let $r_d^{H,q}$ be the restriction of r_d^H to $\Lambda^q(T^{*(0,1)}X)$. As $t \to 0$, we have the asymptotic expansion

$$\int_{\mathbb{P}(E^*)} \frac{\mathrm{Tr}\,[e^{tr_d^{H,q}}]}{\det(1 - e^{-t\mathring{r}^H})} \exp\left(\frac{-r}{2i\pi}\right) = \sum_{j=-\ell}^{m} a_j^q t^j + o(t^m). \tag{6}$$

Also for $p \in \mathbb{N}$, $0 \leqslant q \leqslant \ell$, $t > 0$, let $\mathrm{Tr}[\exp(-t\,\bar{\square}_p^{X,q})]$ be the trace of the operator $\exp(-t\,\bar{\square}_p^{X,q})$. For any $m \in \mathbb{N}$, as $t \to 0$

$$p^{-(\dim X + \dim E - 1)}\,\mathrm{Tr}\left[\exp\left(\frac{-t}{p}\bar{\square}_p^{X,q}\right)\right] = \sum_{j=-\ell}^{m} a_{p,j}^q t^j + o(t^m). \tag{7}$$

THEOREM 1. — *For any $t > 0$, $0 \leqslant q \leqslant \ell$, the following identity holds*

$$\lim_{p\to+\infty} p^{-(\dim X + \dim E - 1)}\,\mathrm{Tr}\left[\exp\left(\frac{-t}{p}\bar{\square}_p^{X,q}\right)\right] = rk(\xi)\int_{\mathbb{P}(E^*)} \frac{\mathrm{Tr}\,[e^{tr_d^{H,q}}]}{\det(1 - e^{-t\mathring{r}^H})} \exp\left(\frac{-r}{2i\pi}\right) \tag{8}$$

and the convergence in (8) *is uniform as* t *varies in compact sets of* R_+^*. *For any* $j \geqslant -\dim X$, $0 \leqslant q \leqslant \dim X$, *as* $p \to +\infty$

$$a_{p,j}^q = rk(\xi)\, a_j^q + O\left(\frac{1}{\sqrt{p}}\right). \tag{9}$$

In (7), *for any* $m \in \mathbb{N}$, $o(t^m)$ *is uniform with respect to* $p \in \mathbb{N}$.

Proof. – By [GrH] p. 165 and [K] Theorem III.4.10, for any $x \in X$, $p \in \mathbb{N}$

$$\begin{aligned} H^q(\mathbb{P}_x(E^*), \mu^{\otimes p}_{|\mathbb{P}_x(E^*)}) &= S^p(E)_x \quad \text{if} \quad q = 0 \\ H^q(\mathbb{P}_x(E^*), \mu^{\otimes p}_{|\mathbb{P}_x(E^*)}) &= 0 \quad \text{if} \quad q > 0. \end{aligned} \tag{10}$$

To prove (8), we will use (10) together with a procedure used by Getzler [Ge] in a similar situation, to transform the initial problem into a corresponding problem on $\mathbb{P}(E^*)$ associated with $\mu^{\otimes p}$, to which we can apply results of Bismut [B2] and Bismut-Vasserot [BV].

Let $U(E^*)$ be the bundle of orthonormal frames in E^*. We identify $U(E^*)$ with the set of linear isometries from $\mathbb{C}^k$ into E^*. Clearly

$$\mathbb{P}(E^*) = U(E^*) \times_{U(k)} \mathbb{P}(\mathbb{C}^k).$$

The connection ∇^{E^*} on $U(E^*)$ induces a connection on the fibration $\pi : \mathbb{P}(E^*) \to X$. The associated horizontal subbundle of $T\mathbb{P}(E^*)$ is exactly the vector bundle $T^H\mathbb{P}(E^*)$ considered in § 1.

We then have the identification of C^∞ vector bundles

$$\begin{aligned} T\mathbb{P}(E^*) &\cong T^H\mathbb{P}(E^*) \oplus T^V\mathbb{P}(E^*) \\ T^H\mathbb{P}(E^*) &\cong \pi^* TX. \end{aligned} \tag{11}$$

From (11), we deduce the identification of C^∞ vector bundles

$$\Lambda(T^{*(0,1)}\mathbb{P}(E^*)) \cong \pi^*(\Lambda(T^{*(0,1)}X)) \widehat{\otimes} \Lambda(T^{V*(0,1)}\mathbb{P}(E^*)). \tag{12}$$

We equip $T^V\mathbb{P}(E^*)$ with the Fubini-Study metric $\| \ \|_{T^V\mathbb{P}(E^*)}$. Let $\| \ \|_{T\mathbb{P}(E^*)}$ be the metric on $T\mathbb{P}(E^*) \cong T^H\mathbb{P}(E^*) \oplus T^V\mathbb{P}(E^*)$ which is the orthogonal sum of $\pi^*\| \ \|_{TX}$ and $\| \ \|_{T^V\mathbb{P}(E^*)}$. For $p \in \mathbb{N}$, set

$$\xi_p = \mu^{\otimes p} \otimes \pi^* \xi. \tag{13}$$

Let $\bar{\partial}_p^V$ be the $\bar{\partial}$ operator along the fibres of $\mathbb{P}(E^*)$ acting on smooth sections of $\pi^*(\Lambda(T^{*(0,1)}X))\widehat{\otimes}\Lambda(T^{V*(0,1)}\mathbb{P}(E^*))\otimes\xi_p$, and let $\bar{\partial}_p^{V*}$ be its formal adjoint with respect to the considered metrics.

Let $\nabla^{T^V\mathbb{P}(E^*)}$, ∇^{ξ_p} be the holomorphic Hermitian connections on $T^V\mathbb{P}(E^*)$, ξ_p respectively. These connections induce a natural connection on $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*))\otimes\xi_p$, which we note $\nabla^{\Lambda(T^{V*(0,1)}\mathbb{P}(E^*))\otimes\xi_p}$.

Definition 2. — *If α is a smooth section of $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*))\otimes\xi_p$ over $\mathbb{P}(E^*)$, if $U\in T_RX$, set*

$$\tilde{\nabla}_{p,U}\,\alpha = \nabla_{U^H}^{\Lambda(T^{V*(0,1)}\mathbb{P}(E^*))\otimes\xi_p}\alpha. \tag{14}$$

We extend $\tilde{\nabla}_p$ to a differential operator acting on smooth sections of $\pi^*(\Lambda(T_R^*X))\widehat{\otimes}\Lambda(T^{V*(0,1)}\mathbb{P}(E^*))\otimes\xi_p$, with the convention that if ω is a smooth section of $\Lambda(T_R^*X)$, and if α is a smooth section of $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*))\otimes\xi_p$, then

$$\tilde{\nabla}_p(\omega\alpha) = \pi^*(d\omega)\,\alpha + (-1)^{\deg\omega}\,\omega\wedge\tilde{\nabla}_p\alpha. \tag{15}$$

Let $\tilde{\nabla}'_p, \tilde{\nabla}''_p$ be the holomorphic and antiholomorphic parts of $\tilde{\nabla}_p$, so that

$$\tilde{\nabla}_p = \tilde{\nabla}'_p + \tilde{\nabla}''_p. \tag{16}$$

For $0\leqslant q'\leqslant k+\ell-1$, let $\Omega^{(0,q)}(\xi_p)$ be the set of smooth sections of $\Lambda(T^{*(0,1)}\mathbb{P}(E^*))\otimes\xi_p$ over $\mathbb{P}(E^*)$. Set $\Omega^{(0,\bullet)}(\xi_p) = \bigoplus_0^{k+\ell-1}\Omega^{(0,q)}(\xi_p)$. Let $\bar{\partial}_p^{\mathbb{P}(E^*)}$ be the classical Dolbeault operator acting on $\Omega^{(0,\bullet)}(\xi_p)$.

Using the identification (12), it is clear that $\tilde{\nabla}''_p, \bar{\partial}_p^V, \bar{\partial}_p^{V*}$ act on $\Omega^{(0,\bullet)}(\xi_p)$.

If A, B are operators acting on the $\mathbb{Z}$-graded vector space $\Omega^{(0,\bullet)}(\xi_p)$, $[A,B]$ denotes the supercommutator of A and B in the sense of [Q].

Proposition 3. — *The following identities of operators acting on $\Omega^{(0,\bullet)}(\xi_p)$ hold*

$$\begin{gathered}(\tilde{\nabla}''_p)^2 = 0\\ [\tilde{\nabla}''_p,\bar{\partial}_p^V] = [\tilde{\nabla}''_p,\bar{\partial}_p^{V*}] = 0\\ \bar{\partial}_p^{\mathbb{P}(E^*)} = \tilde{\nabla}''_p + \bar{\partial}_p^V.\end{gathered} \tag{17}$$

Proof. – Assume first that $\xi = \mathbb{C}$. Let F be the vector space of smooth sections of $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \mu^{\otimes p}$ over $\mathbb{P}(E^*)$. As in [B1], Section 1*f*), we view F as an infinite dimensional vector bundle over X. If $x \in X$, the fibre F_x is simply the vector space of smooth sections $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \mu^{\otimes p}$ on the fibre $\mathbb{P}(E^*)_x$.

Clearly F is a $U(k)$-equivariant vector bundle on X, in the sense it comes from a representation space for $U(k)$. $\tilde{\nabla}_p$ is then a connection on the vector bundle F, which is inherited from the original connection ∇^E on $U(E)$. Since $(\nabla^{E''})^2 = 0$, then $(\tilde{\nabla}_p'')^2 = 0$. It is now trivial to prove in full generality the equation $(\tilde{\nabla}_p'')^2 = 0$.

Since $U(k)$ acts on $\mathbb{P}(\mathbb{C}^{k*})$ by holomorphic isometries which lift unitarily to the dual of the universal bundle on $\mathbb{P}(\mathbb{C}^{k*})$, we get

$$[\tilde{\nabla}_p, \bar{\partial}_p^V] = 0 \,; \qquad [\tilde{\nabla}_p, \bar{\partial}_p^{V*}] = 0 . \tag{18}$$

In particular, the second equation in (17) holds.

If $B \in \mathrm{End}(E^*)$ is skew-adjoint, let By be the holomorphic Killing vector field on $\mathbb{P}(E^*)$ induced by the corresponding vector field on E^*. The vector field By lies in $T^V\mathbb{P}(E^*)$. Then $L^{E^*}y$ is a (1,1) form on X taking values in vector fields in $T^V\mathbb{P}(E^*)$. $L^{E^*}y$ lifts to a (1,1) form on $\mathbb{P}(E^*)$.

Assume first that $p = 0$, and $\xi = \mathbb{C}$. Let d^V be the de Rham operator along the fibres of $\mathbb{P}(E^*)$, and let d be the de Rham operator on $\mathbb{P}(E^*)$. Similarly $\tilde{\nabla}_0$ can be made to act on the de Rham complex of $\mathbb{P}(E^*)$. Then by [B1] eq. (1.30) and [BGS1] eq. (1.26), we find that

$$d = \tilde{\nabla}_0 + d^V + i_{L^{E^*}y} . \tag{19}$$

Since L^{E^*} is of type (1,1), we deduce from (19) that we have the identity of operators acting on $\Omega^{(0,\cdot)}(\xi_0)$

$$\bar{\partial}_0^{\mathbb{P}(E^*)} = \tilde{\nabla}_0'' + \bar{\partial}^V . \tag{20}$$

Extending (20) to $\Omega^{(0,\cdot)}(\xi_p)$ is easy and is left to the reader. □

Remark 4. – The fibration $\pi : \mathbb{P}(E^*) \to X$ is locally Kähler in the sense of Bismut-Gillet-Soulé [BGS1], [BGS2]. Part of the identities in (17) follows from [BGS1], Theorem 2.6.

Let $\tilde{\nabla}_p^{\prime\prime *}$, $\bar{\partial}_p^{V*}$, $\bar{\partial}_p^{\mathbb{P}(E^*)*}$, be the formal adjoints of $\tilde{\nabla}_p^{\prime\prime}$, $\bar{\partial}_p^{V}$, $\bar{\partial}_p^{\mathbb{P}(E^*)}$, with respect to the obvious Hermitian product on $\Omega^{(0,\cdot)}(\xi_p)$ associated to the various metrics.

Observe that $\bar{\partial}_p^{V*}$ restricts on each fibre of $\pi : \mathbb{P}(E^*) \to E^*$ to the fibrewise adjoint of $\bar{\partial}_p^{V}$.

THEOREM 5. — *The following identities of operators acting on* $\Omega^{(0,\cdot)}(\xi_p)$ *hold*

$$(21) \qquad \begin{gathered}(\bar{\partial}_p^{\mathbb{P}(E^*)}+\bar{\partial}_p^{\mathbb{P}(E^*)*})^2 = (\tilde{\nabla}_p^{\prime\prime}+\tilde{\nabla}_p^{\prime\prime *})^2 + (\bar{\partial}_p^{V}+\bar{\partial}_p^{V*})^2 \\ [(\tilde{\nabla}_p^{\prime\prime}+\tilde{\nabla}_p^{\prime\prime *})^2, (\bar{\partial}_p^{V}+\bar{\partial}_p^{V*})^2] = 0.\end{gathered}$$

Proof. — (21) follows from Proposition 3. □

For $0 \leqslant q \leqslant \dim X$, $0 \leqslant q' \leqslant \dim E - 1$, let $\Omega^{(0,q,q')}(\xi_p)$ be the set of smooth section of $\pi^*(\Lambda^q(T^{*(0,1)}X)) \widehat{\otimes} \Lambda^{q'}(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$. By (12), we know that $\Omega^{(0,q,q')}(\xi_p)$ is a vector subspace of $\Omega^{(0,q+q')}(\xi_p)$. More precisely, for any q, $0 \leqslant q \leqslant \dim X + \dim E - 1$

$$(22) \qquad \Omega^{(0,q)}(\xi_p) = \bigoplus_{q'+q''=q} \Omega^{(0,q',q'')}(\xi_p).$$

Set

$$(23) \qquad \bar{\square}_p^{\mathbb{P}(E^*)} = (\bar{\partial}_p^{\mathbb{P}(E^*)}+\bar{\partial}_p^{\mathbb{P}(E^*)*})^2.$$

By Theorem 5, we find that

$$(24) \quad \bar{\square}_p^{\mathbb{P}(E^*)} = \tilde{\nabla}_p^{\prime\prime}\tilde{\nabla}_p^{\prime\prime *} + \tilde{\nabla}_p^{\prime\prime *}\tilde{\nabla}_p^{\prime\prime} + \bar{\partial}_p^{V}\bar{\partial}_p^{V*} + \bar{\partial}_p^{V*}\bar{\partial}_p^{V}.$$

From (24), it is clear that the operator $\bar{\square}_p^{\mathbb{P}(E^*)}$ acts on each $\Omega^{(0,q,q')}(\xi_p)$. Let $\bar{\square}_p^{\mathbb{P}(E^*),q,q'}$ be the restriction of $\bar{\square}_p^{\mathbb{P}(E^*)}$ to $\Omega^{(0,q,q')}(\xi_p)$.

We now have the following result directly inspired by Getzler [Ge].

THEOREM 6. — *For any* $p \in N$, $0 \leqslant q \leqslant \dim X$, $t > 0$, *the following identity holds*

$$(25) \quad \operatorname{Tr}[\exp(-t\bar{\square}_p^{X,q})] = \sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr}[\exp(-t\bar{\square}_p^{\mathbb{P}(E^*),q,q'})].$$

Proof. — Let F_p be the vector space of smooth sections of $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)$. As in the proof of Proposition 3, we regard F_p as an infinite dimensional vector bundle over X. If $x \in X$, the fibre $F_{p,x}$ is the set of smooth sections of $\Lambda(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)_x$. F_p is a $U(k)$-equivariant Hermitian vector bundle on X, and $\tilde{\nabla}_p$ is the corresponding holomorphic Hermitian connection. Also $\Omega^{(0,\cdot)}(F_p)$ is canonically isomorphic to $\Omega^{(0,\cdot)}(\xi_p)$.

The operator $(\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2$ is $U(k)$-equivariant. Therefore the spectrum of $(\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2$ acting on a fibre $F_{p,x}$ does not depend on $x \in X$. In the sequel $\lambda \geqslant 0$ varies in the spectrum of $(\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2$.

The vector bundle F_p over X then splits into a direct orthonormal sum of finite dimensional vector spaces F_p^λ which are eigenspaces of $(\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2$ associated with the eigenvalues λ, i.e.

$$F_p = \bigoplus_{\lambda \geqslant 0} F_p^\lambda. \tag{26}$$

For $0 \leqslant q' \leqslant \dim E - 1$, let $F_p^{q'}$ be the set of smooth sections of $\Lambda^{q'}(T^{V*(0,1)}\mathbb{P}(E^*)) \otimes \xi_p$ over $\mathbb{P}(E^*)$. Clearly $F_p = \bigoplus_{q'=0}^{\dim E-1} F_p^{q'}$. Also the operator $(\bar{\partial}_p^V + \bar{\partial}_p^{V*})^2$ preserves each $F_p^{q'}$. To the splitting (26) of F_p corresponds the splitting

$$F_p^{q'} = \bigoplus_{\lambda \geqslant 0} F_p^{q',\lambda} \tag{27}$$

of each $F_p^{q'}$. Using (10) and Hodge theory, we know that

$$\begin{aligned} F_p^{q',\{0\}} &= S^p(E) \otimes \xi \quad \text{if} \quad q' = 0 \\ &= 0 \qquad\qquad\qquad\quad q' > 0. \end{aligned} \tag{28}$$

Moreover since $U(k)$ acts irreducibly on $S^p(E)$, the metric on $S^p(E)$ induced from the L_2 metric on the fibers of $\mathbb{P}(E^*)$ coincides (up to an irrelevant constant) with the metric $\| \ \|_{S^p(E)}$.

Let $\tilde{\Box}_p^{q,q',\lambda}$ be the restriction of the operator $(\tilde{\nabla}_p'' + \tilde{\nabla}_p''^*)^2$ to the set of smooth sections of $\Lambda^q(T^{*(0,1)}X) \widehat{\otimes} F_p^{q',\lambda}$ over X. From Theorem 5, we get

$$\begin{aligned} &\sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr}[\exp(-t\,\bar{\Box}_p^{\mathbb{P}(E^*),q,q'})] \\ &\qquad = \sum_{\lambda \geqslant 0} \exp(-t\lambda) \sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr}[\exp(-t\,\tilde{\Box}_p^{q,q',\lambda})]. \end{aligned} \tag{29}$$

From (28) and from the considerations which follow, we find that

$$(30) \quad \sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr}[\exp(-t\tilde{\square}_p^{q,q',0})] = \operatorname{Tr}[\exp(-t\square_p^{X,q})].$$

On the other hand, for $\lambda > 0$, we have a $U(k)$-equivariant exact sequence of vector bundles on X

$$(31) \qquad 0 \to F_p^{0,\lambda} \xrightarrow[\bar\partial_p^V]{} F_p^{1,\lambda} \to \cdots \xrightarrow[\bar\partial_p^V]{} F_p^{\dim E-1,\lambda} \to 0.$$

From (31), we easily deduce that for $\lambda > 0$

$$(32) \qquad \sum_{q'=0}^{\dim E-1} (-1)^{q'} \operatorname{Tr}[\exp(-t\tilde{\square}_p^{q,q',\lambda})] = 0.$$

Using (30), (32), we get (25). □

Remark 7. — As $t \to 0$, the left-hand side of (25) has a singularity $t^{-\dim X}$. A priori, the right-hand side has a singularity $t^{-(\dim X + \dim E - 1)}$. Therefore a cancellation process occurs in the right-hand side of (25) as $t \to 0$.

Proof of Theorem 1. — Let r_d be the analogue of r_d^H on $\mathbb{P}(E^*)$. Namely if $w'_1, \ldots, w'_{\ell+k-1}$ is an orthonormal base of $T\mathbb{P}(E^*)$, if $w'^1, \ldots, w'^{\ell+k-1}$ is the corresponding base of $T^*\mathbb{P}(E^*)$, set

$$(33) \qquad r_d = -\sum r(w'_i, \bar{w}'_j)\bar{w}'^j \wedge i_{\bar{w}_i}.$$

Then r_d acts as a derivation of

$$\Lambda(T^{*(0,1)}\mathbb{P}(E^*)) = \pi^*(\Lambda(T^{*(0,1)}X)) \widehat{\otimes} \Lambda(T^{V^*(0,1)}\mathbb{P}(E^*)).$$

We identify r with the self-adjoint matrix $\mathring{r} \in \operatorname{End}(\overline{T\mathbb{P}(E)})$ such that $U, V \in T\mathbb{P}(E)$

$$(34) \qquad r(U,V) = \langle U, \mathring{r}\bar{V}\rangle.$$

By (1), (2), it is clear that r_d preserves $\pi^*(\Lambda^q(T^{*(0,1)}X)) \otimes \Lambda^{q'}(T^{V^*(0,1)}\mathbb{P}(E^*))$. Let $r_d^{q,q'}$ be the corresponding restriction of r_d.

Let dz be the volume form on $\mathbb{P}(E^*)$ with respect to the metric $\| \ \|_{T\mathbb{P}(E^*)}$.

Clearly as $t \to 0$, we have the asymptotic expansion

$$(35)\quad (2\pi)^{-(\dim X+\dim E-1)} \int_{\mathbb{P}(E^*)} \det(\mathring{r}) \frac{\operatorname{Tr}[e^{\mathring{r}_d^{q,q'}}]}{\det(1-e^{-t\mathring{r}})}\, dz = \sum_{j=-\ell-k+1}^{m} b_j^{q,q'} t^j + o(t^m).$$

For any $p \in \mathbb{N}$, $0 \leqslant q \leqslant \dim X$, $0 \leqslant q' \leqslant \dim E - 1$, as $t \to 0$, we have the asymptotic expansion

$$(36)\quad p^{-(\dim X+\dim E-1)} \operatorname{Tr}\left[\exp\left(-\frac{t}{p}\,\overline{\square}_p^{\mathbb{P}(E^*),q,q'}\right)\right] = \sum_{-\ell-k+1}^{m} b_{p,j}^{q,q'} t^j + o(t^m).$$

By a straightforward adaptation of [B2] Theorem 1.5, and [BV] Theorem 2, we know that for any $t > 0$

$$(37)\quad \lim_{p\to+\infty} p^{-(\dim X+\dim E-1)} \operatorname{Tr}\left[\exp\left(-\frac{t}{p}\,\overline{\square}_p^{\mathbb{P}(E^*),q,q'}\right)\right]$$
$$(2\pi)^{-(\dim X+\dim E-1)}\, rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\det(\mathring{r}) \operatorname{Tr}[e^{\mathring{r}_d^{q,q'}}]}{\det(1-e^{-t\mathring{r}})}\, dz$$

and the convergence is uniform as t varies in compact subsets of R_+^*. Also as $p \to +\infty$

$$(38)\quad b_{p,j}^{q,q'} = rk(\xi) b_j^{q,q'} + O\left(\frac{1}{\sqrt{p}}\right).$$

Moreover in (36), $o(t^m)$ is uniform with respect to $p \in \mathbb{N}$.

By (2) $\mathring{r}$ map $\overline{T^V\mathbb{P}(E^*)}$ into itself. Let $\mathring{r}^V$ be the restriction of $\mathring{r}$ to $\overline{T^V\mathbb{P}(E^*)}$. We then find that

$$(39)\quad \sum_{0}^{\dim E-1} (-1)^{q'} \operatorname{Tr}[e^{t\mathring{r}_d^{q,q'}}] = \operatorname{Tr}[e^{t\mathring{r}_d^{H,q}}] \det(1-e^{-t\mathring{r}^V})$$
$$\det(1-e^{-t\mathring{r}}) = \det(1-e^{-t\mathring{r}^H}) \det(1-e^{-t\mathring{r}^V}).$$

By (25), (37), (39), we get

$$(40)\quad \lim_{p\to+\infty} p^{-(\dim X+\dim E-1)} \operatorname{Tr}\left[\exp\left(-\frac{t}{p}\,\overline{\square}_p^{X,q,q'}\right)\right] = rk(\xi) \int_{\mathbb{P}(E^*)} \frac{\operatorname{Tr}[e^{t\mathring{r}_d^{H,q}}]}{\det(1-e^{-t\mathring{r}^H})} \det\left(\frac{\mathring{r}}{2\pi}\right) dz.$$

Clearly

$$\det\left(\frac{\mathring{r}}{2\pi}\right)dz = \left[\exp\left(\frac{-r}{2i\pi}\right)\right]^{\max}. \tag{41}$$

Using (40), (41), we get (8). From the previous considerations, we also obtain the full proof of Theorem 1. □

3. The asymptotics of the Ray-Singer analytic torsion as $p \to \infty$.

From now on, we assume that the holomorphic Hermitian vector bundle $(E, \| \ \|_E)$ is positive, i.e. that if $U \in TX \backslash \{0\}$, $e \in E \backslash \{0\}$

$$\langle L^E(U,\bar{U})e,\bar{e}\rangle > 0. \tag{42}$$

From (2), we find that if $y \in E^* \backslash \{0\}$ represent $z \in \mathbb{P}(E^*)$, then

$$r_z = r^V + \pi^* \frac{\langle L^E \bar{y}, y\rangle}{|y|^2}. \tag{43}$$

Classically [GrH] p. 30, the restriction of the line $(\mu, \| \ \|_\mu)$ to the fibres $\mathbb{P}(E^*)$ is positive, i.e. if $U \in T^V\mathbb{P}(E^*)\backslash\{0\}$, $r^V(U,\bar{U}) > 0$. From (43), we deduce that the Hermitian line bundle $(\mu, \| \ \|_\mu)$ is positive on $\mathbb{P}(E^*)$. This is of course a well-known result [K] Theorem III 6.19.

THEOREM 8. — *There exists $C > 0$, $c > 0$, $c' > 0$ such that for any $p \in \mathbb{N}$, $1 \leqslant q \leqslant \ell$, $t \geqslant 1$, then*

$$p^{-(\dim X + \dim E - 1)} \operatorname{Tr}\left[\exp\left(-\frac{t}{p}\bar{\square}_p^{X,q,q}\right)\right] \leqslant C\exp\left(-\left(c - \frac{c'}{p}\right)t\right). \tag{44}$$

Proof. — By [BV] Theorems 1 and 2, there exist $C > 0$, $c > 0$, $c' > 0$ such that for $p \in \mathbb{N}$, $0 \leqslant q \leqslant \ell$, $0 \leqslant q' < k-1$, $q + q' \geqslant 1$, $t \geqslant 1$

$$\operatorname{Tr}\left[\exp\left(-\frac{t}{p}\bar{\square}_p^{\mathbb{P}(E^*),q,q'}\right)\right] \leqslant C\exp\left(-\left(c - \frac{c'}{p}\right)t\right). \tag{45}$$

Using (25) and (45), (44) follows. □

Remark 9. – Let λ_p^q be the lowest eigenvalue of $\bar{\square}_p^{X,q}$. From (44), we deduce that if $q \geqslant 1$

$$\lambda_p^q \geqslant cp - c'. \tag{46}$$

(46) is also an easy consequence of Theorem 5, of the considerations in the proof of Theorem 6 and of [BV], Theorem 1. [BV] Theorem 1 is itself a consequence of the Bochner-Kodaira-Nakano formula of Demailly [De] for the operator $\bar{\square}_p^{\mathbb{P}(E^*)}$. Strangely enough, (46) does not seem to be a straightforward consequence of a similar formula for $\bar{\square}_p^X$.

By Theorem 8 or by (46), there exists $p_0 \in \mathbb{N}$ such that if $p \geqslant p_0$, $1 \leqslant q \leqslant \ell$, the operator $\bar{\square}_p^{\mathbb{P}(E^*),q}$ is invertible.

DEFINITION 10. – *For $p \geqslant p_0$, $s \in \mathbb{C}$, $\operatorname{Re}(s) \geqslant \ell$, set*

$$\zeta_p(s) = \frac{-1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \left(\sum_1^{\ell} (-1)^q q \operatorname{Tr}\left[\exp\left(-t\, \bar{\square}_p^{X,q}\right)\right] \right) dt. \tag{47}$$

By a well-known result of Seeley [Se], $\zeta_p(s)$ extends into a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$. By definition $\exp(-\zeta_p'(0))$ is the Ray-Singer analytic torsion [RS] of the Hermitian vector bundle $S^p(E) \otimes \xi$.

We now state the main result of this paper.

THEOREM 11. – *As $p \to +\infty$*

$$\zeta_p'(0) = rk(\xi) \frac{1}{2} \int_{\mathbb{P}(E^*)} \operatorname{Log}\left[\det\left(\frac{p\mathring{r}^H}{2\pi} \right) \right] \exp\left(\frac{-pr}{2i\pi} \right) + o\left(p^{(\dim X + \dim E - 1)}\right). \tag{48}$$

In particular as $p \to +\infty$

$$\zeta_p'(0) = O(p^{\dim X + \dim E - 1} \operatorname{Log} p). \tag{49}$$

Proof. – In view of Theorems 1 and 8, which are the obvious extensions of [BV] Theorem 2, the proof of Theorem 11 proceeds formally as the proof of [BV] Theorems 4 and 8. Details are left to the reader. □

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