ANNALES DE L'INSTITUT FOURIER

MAURICE DE GOSSON

Microlocal regularity at the boundary for pseudo-differential operators with the transmission property (I)

Annales de l'institut Fourier, tome 32, nº 3 (1982), p. 183-213

<http://www.numdam.org/item?id=AIF_1982__32_3_183_0>

Ann. Inst. Fourier, Grenoble
32, 3 (1982), 183-213

MICROLOCAL REGULARITY AT THE BOUNDARY FOR PSEUDO-DIFFERENTIAL OPERATORS WITH THE TRANSMISSION PROPERTY (1)

by Maurice DE GOSSON

1. Introduction and results.

When studying boundary value problems – even in the differential case – one is almost systematically confronted with the question of the behaviour, near the boundary of associated pseudo-differential operators. Since these operators are not local, one cannot expect, in general, simple results of regularity up to the boundary.

Fortunately, the pseudo-differential operators associated to these problems have the peculiarity of behaving almost as well as ordinary differential operators "on" the boundary; they have the so-called "transmission property". This class of operators was introduced in the 70's by L. Boutet de Monvel, in relation with the theory of Wiener-Hopf operators (see [2]). In this work was proved, among other properties, the following essential result:

If $P \in L^m(\mathbf{R}^n)$ has the transmission property and $\overline{\Omega}$ (resp Ω) denotes the half-space $\overline{\mathbf{R}^n_+}$ (resp $\mathbf{R}^n_+$) with boundary $\partial\Omega\ (= \mathbf{R}^{n-1})$, we have a continuous mapping:

$$C_0^\infty(\overline{\Omega}) \longrightarrow C^\infty(\overline{\Omega})$$

defined by:

$$u \longrightarrow P(u^0)_{|\Omega}$$

(here $C_0^\infty(\overline{\Omega})$ and $C^\infty(\overline{\Omega})$ are the usual spaces of functions C^∞ up to $\partial\Omega$, and u^0 is the 0-extension of u across $\partial\Omega$).

We shall in this work not only generalize this result to the case of the Sobolev spaces, but also give microlocal statements. To do this one has to define a notion of "boundary singular spectrum".

In our case, we shall mainly use the "∂WF set" as introduced, for instance in Melrose-Sjöstrand [12] or Chazarain [3] (the notation of whom is being used here). This set (which is defined as a subset of the cotangent bundle to the boundary), measures in some way the lack of "tangential regularity" of a distribution. A major problem which arises when dealing with this concept, is that $\partial WF(u)$ isn't generally intrinsically defined for an arbitrary distribution $u \in D'(\mathbf{R}^n_+)$, unless u is solution of a non-characteristic partial differential equation $Pu = f$, with f "normally regular" up to the boundary in every coordinate system (see the precited paper of Melrose-Sjöstrand). This isn't really an obstruction for our purpose, since we shall only consider here equations having this property. It should nevertheless also be noted that R. Melrose has recently (see [11] and the references therein) defined new classes of distributions which allow an invariant definition of $\partial WF(u)$ even in the non-characteristic case.

This work is structured as follows:

– In § 2, we recall the standard definitions and properties concerning partially regular distributions; nothing really new in this section, except may be (1.9) which states that a pseudo-differential operator transforms a normally regular distribution into a distribution having the same property.

– In § 3 is proved the result which is the key to our regularity theorems of the following sections:

(3.1) Let $P \in L^m(\mathbf{R}^n)$ be properly supported and have the transmission property. Let $A \in L^0(\partial\Omega)$ be a properly supported tangential operator, elliptic in a conic neighborhood of $\sigma \in T^*_0(\partial\Omega)$. Then, if $\bar{\gamma}$ is an arbitrary conic subset of γ such that $\bar{\gamma} \subset\subset \gamma$, we have:

$$\left.\begin{array}{l} u \in H^{s,t'}_{loc}(\mathbf{R}^n) \\ Au \in H^{s,t}_{loc}(\mathbf{R}^n) \\ u_{|\Omega} = 0 \end{array}\right\} \Longrightarrow BPu_{|\Omega} \in H^{s-m+\infty, t-\infty}_{loc}(\bar{\Omega})$$

for every properly supported $B \in L^0(\partial\Omega)$ such that $WF(B) \subset \bar{\gamma}$.

The proof of (3.1) is reduced by using the calculus of tangential operators to the case $A = B = I_d$, $t = t'$. The corresponding "non microlocal" result has already been proved in a previous work (de Gosson [4]), where it was used to derive partial hypoellipticity results. We have chosen to reproduce it here, although it is rather long and technical, not only for the reader's convenience, but also because it is highly instructive, since it contains the main difficulties which appear when one has to prove precise results of partial regularity.

– In § 4, we recall the definition of the C^∞-boundary singular spectra ∂WF and WF_b, and extend these definitions to the case of Sobolev spaces. The main result of this section is (in the C^∞-case):

(4.13) if P has the transmission property, and if

$$u \in C^\infty(\overline{\mathbf{R}_+}, D'(\partial\Omega)),$$

then we have:

$$WF_b[P(u^0)_{|\Omega}] \subset WF_b(u)$$

u^0 being the canonical 0-extension of u across $\partial\Omega$.

The proof of (4.13) is a consequence of the following corollary of theorem (3.1), which can be written:

$$\partial WF[P(g \otimes \delta)_{|\Omega}] \subset WF(g)$$

in the C^∞-case (here $g \in D'(\partial\Omega)$, and δ is the Dirac measure) (see theorem (4.8)).

– Finally, § 5 is devoted to a precise study of the singularities of noncharacteristic Cauchy problems of principal type, to which we add a condition of transversality for the bicharacteristics. We thus obtain, by using the techniques developed in § 4, a non trivial generalization of proposition (4.16) in Andersson-Melrose [1]. Our result can be stated in the following way: If $u \in D'(\overline{\Omega})$ solves the Cauchy problem:

$$(c) \begin{cases} Pu = f \\ \gamma_j u = g_j \end{cases}$$

for $f \in C^\infty(\overline{\mathbf{R}_+}, D'(\partial\Omega))$, $g_j \in D'(\partial\Omega)$ $(j = 0, 1, \dots, m-1)$ we have:

$$(5.1) \quad \left.\begin{matrix} \sigma \notin WF_{s+m-j-\frac{1}{2}}(g_j) \\ \sigma \notin \partial WF(f) \end{matrix}\right\} \Longrightarrow \sigma \notin \partial WF_{s+m-\frac{3}{2}}(u) \quad \text{near } \partial\Omega .$$

Some of the results in this paper have been announced in de Gosson [5]. See also de Gosson [6] for a generalization of the transmission property to operators with quasi-homogeneous symbols.

2. Partially regular distributions.

Let X be an open set in $\mathbf{R}^n$, and consider a distribution $u \in D'(X)$. If $x = (y, z) \in \mathbf{R}^p \times \mathbf{R}^{n-p}$, we denote the dual variable by $\xi = (\eta, \zeta)$, and say that u is partially regular (in y) if, for every $\varphi \in C_0^\infty(X)$ and every $N \in \mathbf{N}$, we can find constants $\mu_{\varphi,N} \in \mathbf{R}$ and $C_{\varphi,N} > 0$ such that the Fourier transform φu of $\varphi u \in E'(X)$ satisfies:

(2.1) $|\widehat{\varphi u}(\xi)| \leqslant C_{\varphi,N}(1 + |\eta|)^{-N}(1 + |\zeta|)^{\mu_{\varphi,N}}$ for every $\xi \in \mathbf{R}^n$.

Remark. – When, in the latter definition, the constant $\mu_{\varphi,N}$ does not depend on N, the distribution u is said to be strongly regular in the variable y. The example (with $X = \mathbf{R}^2$) of the distribution $u(x', x_n) = \delta(x' - x_n)$ shows that a distribution partially regular in all the variables need not to be a smooth function. Nevertheless, one has the following result (F. Trèves [13]):

(2.2) If $u \in D'(X)$ is regular in y and strongly regular in z, then $u \in C^\infty(X)$.

In practice, we will only consider distributions regular in the "normal variable", that is, in $x_n \in \mathbf{R}$ (we have taken here $X = \mathbf{R}^n$); it can be proved, by using the closed graph theorem, that the space of all normally regular distributions can be canonically identified with the space $C^\infty(\mathbf{R}, D'(\partial\Omega))$ of infinitely differentiable functions, valued in $D'(\partial\Omega)$, (see Treves [13] or Melrose [11]), for this reason we will not make any distinctions between these two spaces, and only use the above notation. Generally, the definition of partial regularity is not, unfortunately, coordinate invariant, and doesn't therefore make sense when we replace X by a manifold. There is nevertheless one very important case where this can be done, namely:

(2.3) Let $u \in D'(\mathbf{R}^n)$, and suppose that

$$WF(u) \cap \{(x', x_n, 0, \xi_n)/(x', x_n) \in \mathbf{R}^n, \xi_n \neq 0\} = \emptyset .$$

Then $u \in C^\infty(\mathbf{R}, D'(\partial\Omega))$ in every coordinate system.

Finally, an extendible distribution $u \in D'(\overline{\Omega})$ is said to be regular (or smooth) up to the boundary in the normal variable, if there is a distribution $\widetilde{u} \in C^\infty(\mathbf{R}, D'(\mathbf{R}^{n-1}))$ extending u. Using for instance a Seeley extension of u, one can prove again that there is a canonical corresponding between the space of distributions smooth up to the boundary in x_n, and the space $C^\infty(\overline{R_+}, D'(\partial\Omega))$ of distribution valued functions which are C^∞ up to the boundary, and thus identify these two spaces. It is easily seen, by the definitions, that one has the following Fourier characterization:

(2.4) $u \in C^\infty(\overline{R_+}, D'(\partial\Omega))$ if and only if for every $\varphi \in C_0^\infty(\partial\Omega)$, and every $N \in \mathbf{N}$, there are constants $C_{\varphi,N} > 0$ and $\mu_{\varphi,N} \in \mathbf{R}$ such that:

$$|D_{x_n}^N \widehat{\varphi u}(\xi', x_n)| \leqslant C_{\varphi,N}(1 + |\xi'|)^{\mu_{\varphi,N}}$$

for $\xi' \in \mathbf{R}^{n-1}$ and $x_n \in \overline{\mathbf{R}_+}$.

Here $\widehat{\varphi u}(\xi', x_n)$ denotes the partial Fourier transform of φu with respect to the tangential variable x'.

Useful tools in the study of partial regularity are provided by the double-indexed Sobolev spaces $H^{s,t}$: We say that a distribution $u \in S'(\mathbf{R}^n)$ belongs to $H^{s,t}(\mathbf{R}^n)$ (s and t being real numbers) if the Fourier transform $\hat{u}$ of u is a function satisfying:

$$(2.5) \qquad (\textstyle\int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s (1 + |\xi'|^2)^t \, d\xi)^{1/2} < \infty .$$

(Note that $H^{s,0}(\mathbf{R}^n)$ is the usual Sobolev space $H^s(\mathbf{R}^n)$).

$H^{s,t}(\mathbf{R}^n)$ is clearly a complex vector space, and the left side of (1.5) defines a norm, denoted by $\|u\|_{s,t}$ on this space. In the same way, $H^{s,t}(\overline{\Omega})$ is the space of all restrictions, to the half space Ω, of the elements in $H^{s,t}(\mathbf{R}^n)$, and it is equipped with the quotient topology, defined, for $u \in H^{s,t}(\overline{\Omega})$, by: $\|u\|_{s,t} = \inf \|\widetilde{u}\|_{s,t}$, the infimum being taken over all $\widetilde{u} \in H^{s,t}(\mathbf{R}^n)$ extending u. The corresponding local spaces $H^{s,t}_{\text{loc}}$, $H^{s,t}_{\text{comp}}$ are defined as usually. We now list the basic properties of these spaces. For details and proofs, the reader should consult Hörmander [7], and [8]:

$$(2.6) \qquad H^{s,t} \subset H^{s',t'} \iff \begin{cases} s \geqslant s' \\ s + t \geqslant s' + t' \end{cases}$$

$$(2.7) \qquad H^{s,t}_{\text{loc}}(\mathbf{R}^n) \subset C^k\left(\mathbf{R}, H^{s+t-k-\frac{1}{2}}_{\text{loc}}(\partial\Omega)\right)$$

and $\qquad H^{s,t}_{\text{loc}}(\overline{\Omega}) \subset C^k\left(\overline{\mathbf{R}_+}, H^{s+t-k-\frac{1}{2}}(\partial\Omega)\right)$

if $0 \leqslant k < s - \frac{1}{2}$.

Setting, as usually, $H^{s+\infty,t} = \cap H^{s+k,t}$ and $H^{s,t-\infty} = \cap H^{s,t-k}$ (the intersections being taken for all $k \in \mathbf{R}$), we have, in particular:

$$(2.7') \qquad H_{\mathrm{loc}}^{s+\infty,t-\infty}(\mathbf{R}^n) \subset C^\infty\left(\mathbf{R}, H_{\mathrm{loc}}^{s+t-\frac{1}{2}}(\partial\Omega)\right)$$

and

$$H_{\mathrm{loc}}^{s+\infty,t-\infty}(\overline{\Omega}) \subset C^\infty\left(\overline{\mathbf{R}_+}, H_{\mathrm{loc}}^{s+t-\frac{1}{2}}(\partial\Omega)\right).$$

Let us now describe briefly the action of pseudo-differential operators on the $H^{s,t}$ spaces. First, for operators in all the variables, we have the following $H^{s,t}$-continuity result, the proof of which is a trivial adaptation of the usual continuity result in ordinary Sobolev spaces :

(2.8) If $P \in L^m(\mathbf{R}^n)$ is a properly supported operator, then:

$$P : H_{\mathrm{loc}}^{s,t}(\mathbf{R}^n) \longrightarrow H_{\mathrm{loc}}^{s-m,t}(\mathbf{R}^n).$$

In particular, if $P \in L^{-\infty}(\mathbf{R}^n)$:

$$P : H_{\mathrm{loc}}^{s,t}(\mathbf{R}^n) \longrightarrow C^\infty(\mathbf{R}^n).$$

Using (2.8) one can easily prove the following important result:

(2.9) If $P \in L^m(\mathbf{R}^n)$ is properly supported, then:

$$P : C^\infty(\mathbf{R}, D'(\partial\Omega)) \longrightarrow C^\infty(\mathbf{R}, D'(\partial\Omega)).$$

Proof. – By localizing, it is of course sufficient to prove that P maps $C_0^\infty(\mathbf{R}, E'(\partial\Omega))$ in itself. But, if $u \in C_0^\infty(\mathbf{R}, E'(\partial\Omega))$, for every integer N, we can find, in view of (2.1), constants $C_N > 0$ and $\mu_N \in \mathbf{R}$ such that:

$$|\hat{u}(\xi)| \leqslant C_N(1 + |\xi'|)^{\mu_N}(1 + |\xi_n|)^{-N}, \quad \forall \xi \in \mathbf{R}^n$$

that is, for some new constant $C_{N,\epsilon}$:

$$(*) \quad |\hat{u}(\xi)| \leqslant C_{N,\epsilon}(1 + |\xi'|)^{\mu_N}(1 + |\xi|)^{-N} \quad \text{if} \quad |\xi'| \leqslant \epsilon\, |\xi_n|,$$

ϵ being any fixed positive number.

On the other hand, since u is compactly supported, the easy part of Paley-Wiener's theorem implies the existence of constant $\mu \in \mathbf{R}$ and $C_0 > 0$ such that:

$$|\hat{u}(\xi)| \leqslant C_0(1 + |\xi|)^{\mu}, \quad \forall \xi \in \mathbf{R}^n$$

that is, for every $N \in \mathbf{N}$:

$$|\hat{u}(\xi)| \leqslant C_0(1 + |\xi|)^{\mu+N}(1 + |\xi|)^{-N} .$$

Since $|\xi| \sim |\xi'|$ in the cone defined by $|\xi'| > \epsilon |\xi_n|$, the latter inequality can be written:

$$(**) \quad |\hat{u}(\xi)| \leqslant C'_\epsilon(1 + |\xi'|)^{\mu+N}(1 + |\xi|)^{-N} \quad \text{if} \quad |\xi'| > \epsilon |\xi_n| ,$$

for some new constant C'; finally by combining (*) and (**) we have:

$$(***) \qquad |\hat{u}(\xi)| \leqslant C_{N,\epsilon}(1 + |\xi'|)^{\nu_N}(1 + |\xi|)^{-N}, \quad \forall \xi \in \mathbf{R}^n$$

where $\nu_N = \sup(\mu_N, \mu + N)$ and $C_N = \sup(C_{N,\epsilon}\,; C'_\epsilon)$; from this it follows immediately the existence of a strictly decreasing sequence $(\omega_N)_N$ such that:

$$(2.9') \qquad u \in H_C^{N,\omega_N}(\mathbf{R}^n) \quad \text{for every } N,$$

that is, in view of (2.8):

$$Pu \in H_C^{N-m,\omega_N}(\mathbf{R}^n) \quad \text{for every } N$$

which proves our assertion, in view of (2.7).

The following proposition will be fundamental in the proof of the main theorem (3.1). It shows in some way how to obtain partial regular distributions by using pseudo-differential operators:

(2.10) Let $P \in L^m(\mathbf{R}^n) \cap L^{-\infty}(\Gamma)$, where Γ is a conic neighborhood of $\{(x', 0; 0, \xi_n)/x' \in \partial\Omega, \xi_n \in \mathbf{R}\}$, the conormal bundle to $\partial\Omega$, be properly supported. We can then find a neighborhood $\widetilde{\Omega}$ of $\partial\Omega$ in $\mathbf{R}^n$, such that:

$$u \in H_{loc}^{s,t}(\mathbf{R}^n) \Longrightarrow Pu \in H_{loc}^{s-m+\infty,\, t-\infty}(\widetilde{\Omega}).$$

Proof. – For each $x' \in \partial\Omega$, there is an open neighborhood $\Omega_{x'}$ of $(x', 0)$ in $\mathbf{R}^n$, and a positive constant $C_{x'}$ such that:

$$\Omega_{x'} \times \{\xi/|\xi'| < C_{x'}|\xi_n|\} \subset \Gamma .$$

Let us set $\widetilde{\Omega} = \cup\, \Omega_{x'}$, the union being taken for all $x' \in \partial\Omega$. If $\varphi \in C_0^\infty(\widetilde{\Omega})$, we can recover $\mathrm{supp}(\varphi)$ by a finite number of sets $\Omega_{x'}$; hence there exists a constant $C > 0$ such that:

$$p_1 = \varphi p \in S^{-\infty}(\{\xi/|\xi'| < C|\xi_n|\})$$

(p being the symbol of P). We are going to show that the operator $P_1 = p_1(x, D)$ is continuous from $H^{s,t}(\mathbf{R}^n)$ to $H^{s-m+k,t-k}(\mathbf{R}^n)$ for every $k \in \mathbf{N}$.

Define $\theta \in C^\infty(\mathbf{R}^n)$ $(0 \leqslant \theta \leqslant 1)$ by:

$$\theta(\xi) = \begin{cases} 0 & \text{if } |\xi| \leqslant \frac{1}{2} \\ 1 & \text{if } |\xi| \geqslant 1 \end{cases}$$

and $\chi \in C^\infty(\mathbf{R}^n \backslash 0)$ $(0 \leqslant \chi \leqslant 1)$, homogeneous of degree 0, by:

$$\chi(\xi) = \begin{cases} 0 & \text{if } |\xi'| \leqslant \frac{1}{2} C |\xi_n| \\ 1 & \text{if } |\xi'| \geqslant \frac{3}{4} C |\xi_n| \end{cases}$$

for some positive C. We then have:

$$p_1(x, \xi) = \theta(\xi)\,\chi(\xi)\,p(x, \xi) \pmod{S^{-\infty}}$$

hence, in view of (1.8) it is now sufficient to prove:

(2.11) $$u \in H^{s,t}(\mathbf{R}^n) \Longrightarrow Op(\theta\chi)\,u \in H^{s+k,t-k}(\mathbf{R}^n)$$

for every k.

Setting $v = Op(\theta\chi)\,u$, we have:

$$\|v\|^2_{s+k,t-k} = \int |\theta(\xi)\,\chi(\xi)|^2\, |\hat{u}(\xi)|^2\, (1+|\xi|^2)^{s+k}\, (1+|\xi'|^2)^{t-k}\, d\xi$$

$$= \int |\theta(\xi)\,\chi(\xi)|^2\, |\hat{u}(\xi)|^2 \left(\frac{1+|\xi|^2}{1+|\xi'|^2}\right)^k (1+|\xi|^2)^s\,(1+|\xi'|^2)^t\, d\xi$$

that is, finally:

$$\|v\|^2_{s+k,t-k} \leqslant C_k\, \|u\|_{s,t}$$

for some constant $C_k > 0$, since $|\theta\chi| \leqslant 1$ and

$$\chi(\xi)\left(\frac{1+|\xi|^2}{1+|\xi'|^2}\right)^k \sim \chi(\xi),$$

and this ends the proof of (1.11).

We shall also use throughout this work "tangential" pseudo-differential, that is, operators acting only in the x'-variable, the class of which is denoted by $L^{m'}(\partial\Omega)$ $(m' \in \mathbf{R})$. These operators can be extended as operators acting on $D'(\mathbf{R}^n)$ (resp. $D'(\overline{\Omega})$) by:

$$(2.12) \qquad Au(x) = \int e^{ix'\xi'} a(x', \xi')\, \hat{u}(\xi', x_n)\, d\xi'$$

for $u \in C_0^\infty(\mathbf{R}^n)$ (resp. $u \in C_0^\infty(\overline{\Omega})$); here $a \in S^{m'}(\partial\Omega)$, and $\xi' \longrightarrow \hat{u}(\xi', \cdot)$ denotes the partial Fourier transform with respect to $x' \in \partial\Omega$.

We have, for these operators the following continuity property, the proof of which is standard (it is a generalization of th. (2.5.2) in Hörmander [7]):

(2.13) If $A \in L^{m'}(\partial\Omega)$ is properly supported, we have a continuous mapping:

$$A : H^{s,t}_{\mathrm{loc}}(\mathbf{R}^n) \longrightarrow H^{s,t-m'}_{\mathrm{loc}}(\mathbf{R}^n)$$

for all $(s, t) \in \mathbf{R}^2$.

An easy consequence of (2.13) is:

(2.14) If $A \in L^{m'}(\partial\Omega)$ is properly supported, then:

$$A : H^{s,t}_{\mathrm{loc}}(\overline{\Omega}) \longrightarrow H^{s,t-m'}_{\mathrm{loc}}(\overline{\Omega}).$$

Proof. – Let $u \in H^{s,t}_c(\overline{\Omega})$; by the definition of this space, there is an extension $\widetilde{u} \in H^{s,t}_c(\mathbf{R}^n)$ of u; by (2.12) we have $A\widetilde{u} \in H^{s,t-m'}_c(\mathbf{R}^n)$, hence the result, since $A\widetilde{u}_{|\Omega} = Au$, in view of (2.12).

Finally, combining (2.13), (2.14), and (2.9′), we have the important:

(2.15) If $A \in L^{-\infty}(\partial\Omega)$, then:

i) $A : H^{s,t}_{\mathrm{loc}}(\overline{\Omega}) \longrightarrow H^{s,\infty}_{\mathrm{loc}}(\overline{\Omega})$

ii) $A : C^\infty(\overline{\mathbf{R}_+}, D'(\partial\Omega)) \longrightarrow C^\infty(\overline{\Omega})$

and i), ii) still hold when $\overline{\Omega}$ (resp. $\overline{\mathbf{R}_+}$) is replaced by $\mathbf{R}^n$ (resp. $\mathbf{R}$).

Also note the following recursive characterization of $H^{s,t}$, which is in some sense the simplest of all results in partial hypoellipticity:

(2.16) Let $H^{s,t}$ denote either $H^{s,t}_{loc}(\overline{\Omega})$ or $H^{s,t}_{loc}(\mathbf{R}^n)$. We then have:

$$u \in H^{s,t} \iff \begin{cases} D_{x_n} u \in H^{s-1,t} \\ u \in H^{s-1,t+1} . \end{cases}$$

Finally, we define the kind of operators which will be used in the following sections.

It has been known for a long time (see Boutet de Monvel [2]) that one cannot expect to obtain reasonable properties of regularity near the boundary for pseudo-differential operators, unless one makes the assumption that these operators have the transmission property: More precisely, let $P \in L^m(\mathbf{R}^n)$ (m positive or negative integer) be a classical pseudo-differential operator: the complete symbol p of P admits an asymptotic expansion:

$$p \sim \sum_{j \geq 0} p_j$$

where the p_j are symbols, positive homogeneous of degree $m - j$ in the dual variable. P is said to have the transmission property with respect to $\partial\Omega$, if the following conditions hold:

(2.17) $D^{\alpha'}_{\xi'} D^p_{x_n} p_j(x', 0; 0, -1) = (-1)^{m-j-|\alpha'|} D^{\alpha'}_{\xi'} D^p_{x_n} p_j(x', 0; 0, 1)$

for every x', j, α' and p.

These are not very restrictive. They are clearly fulfilled by differential operators. Moreover, if P is a noncharacteristic differential operator (or more generally a noncharacteristic pseudo-differential operator with the transmission property), one can prove that every microlocal parametrix Q of P also has the transmission property (see de Gosson [4], where an explicit construction of Q is given).

3. Microlocal regularity at the boundary.

As announced in the introduction, we are going to prove in this section the central result of this work:

(3.1) THEOREM. – *Let* $P \in L^m(\mathbf{R}^n)$ *be a properly supported pseudo-differential operator with the transmission property with respect to* *the hyperplane* $\partial\Omega$. *Let* $A \in L^0(\partial\Omega)$ *be a properly supported tangential operator, elliptic in a conic neighborhood of* $\sigma \in T_0^*(\partial\Omega)$. *Then, if* $\bar{\gamma}$ *is an arbitrary conic subset of* γ *in* $T_0^*\partial\Omega$, *with* $\bar{\gamma} \subset\subset \gamma$, *we have:*

$$\left.\begin{array}{l} u\ H_{loc}^{s,t'}(\mathbf{R}^n) \\ Au\ H_{loc}^{s,t}(\mathbf{R}^n) \\ u_{|\Omega} = 0 \end{array}\right\} \Longrightarrow \mathrm{BP}u_{|\Omega} \in H_{loc}^{s-m+\infty,t-\infty}(\bar{\Omega})$$

for every properly supported $B \in L^0(\partial\Omega)$ *satisfying* $WF(B) \subset \bar{\gamma}$, *and all* $s, t \in \mathbf{R}$.

(3.1) will be proved by reduction to the case $A = B = I_d$, $t = t'$. More precisely, we shall first prove the following result:

(3.1′) THEOREM. – *Let* P *be as in* (3.1). *Then for all real numbers* $s, t \in \mathbf{R}$, *we have:*

$$\left.\begin{array}{l} u \in H_{loc}^{s,t}(\mathbf{R}^n) \\ u_{|\Omega} = 0 \end{array}\right\} \Longrightarrow \mathrm{P}u_{|\Omega} \in H_{loc}^{s-m+\infty,t-\infty}(\bar{\Omega}).$$

The proof of (3.1′) is unfortunately long and technical; it will be made in several steps, and is mainly based on Taylor expansions of the symbol of P in order to use the transmission property. The main difficulty arises when one has to estimate the Taylor remainder corresponding to the expansion at $x_n = 0$; it will be necessary to introduce a kernel which is in some sense "smoothing" in the normal variable. Let $\sigma_p(x, \xi) \sim \sum_{j \geq 0} p_j(x, \xi)$ be an asymptotic development of the complete symbol of P, of course the p_j are positively homogeneous of degree $m - j$ in the dual variable, so that $p_j \in S_c^{m-j}$. In view of the hypothesis on P, each p_j has the transmission property, so it is of course, in view of the inclusions (2.7) sufficient, in order to simplify the notations, to restrict ourselves to the case where the complete symbol of P is $p = p_0$, positively homogeneous of degree m.

Define now two functions θ and ψ by:

i) $\theta \in C^\infty(\mathbf{R}^n)$, $0 \leq \theta \leq 1$

and:

$$\begin{cases} \theta(\xi) = 0 & \text{if} \quad |\xi| \leqslant \frac{1}{2} \\ \theta(\xi) = 1 & \text{if} \quad |\xi| \geqslant 1 \end{cases}$$

ii) $\psi \in C^\infty(\mathbf{R}^n \backslash 0)$, $0 \leqslant \psi \leqslant 1$, χ homogeneous of degree 0, and:

$$\begin{cases} \psi(\xi) = 1 & \text{if} \quad \xi \in \Gamma_1 = \{\xi / |\xi'| \leqslant C\, |\xi_n|\} \\ \psi(\xi) = 0 & \text{if} \quad \xi \in \Gamma_2 = \{\xi / |\xi'| \geqslant 2C\, |\xi_n|\} . \end{cases}$$

Writing:

$$p(x, \xi) = \theta(\xi)\, \psi(\xi)\, p(x, \xi) - \theta(\xi)\, (\psi(\xi) - 1)\, p(x, \xi) + (1 - \theta(\xi))\, p(x, \xi)$$

we see at once that we can neglect the last term, since we evidently have, $(1 - \theta)\, p$ being in $S^{-\infty}(\mathbf{R}^n)$:

$$\mathrm{O}p((1 - \theta)\, p)\, u \in C^\infty(\mathbf{R}^n) \subset H_{\mathrm{loc}}^{s-m+\infty, t-\infty}(\mathbf{R}^n), \quad \forall u \in D'(\mathbf{R}^n) .$$

On the other hand it is clear that $\theta(\xi)\, (\psi(\xi) - 1)$ vanishes in a full neighborhood of $\{(x', x_n, 0, \xi_n)\}$, so we may apply (2.10) (with $\widetilde{\Omega} = \mathbf{R}^n$), and:

$$\mathrm{O}p[\theta(1 - \psi)\, p]\, u \in H_{\mathrm{loc}}^{s-m+\infty, t-\infty}(\Omega) .$$

We are thus, finally reduced to the case where the symbol of P is $\widetilde{p}(x, \xi) = \theta(\xi)\, \psi(\xi)\, p(x, \xi)$, p being homogeneous of degree m.

Using now a Taylor expansion at $\xi' = 0$ for p, until an arbitrary order N', we have:

$$\widetilde{p}(x, \xi) = \theta(\xi)\, \psi(\xi) \sum_{|\alpha'| < N'} \frac{1}{\alpha'!}\, \partial_{\xi'}^{\alpha'} p(x; 0, \xi_n)\, \xi'^{\alpha'} + \sum_{|\alpha'| = N'} \frac{N'}{\alpha'!}\, r_{\alpha'}(x, \xi)\, \xi'^{\alpha'}$$

where:

$$r_{\alpha'}(x, \xi) = \int_0^1 (1 - t)^{N'-1}\, \partial_{\xi'}^{\alpha'} a(x; t\xi', \xi_n)\, dt .$$

(Remark that this equality makes sense even for $\xi_n = 0$, since $\psi(\xi)$ vanishes in a neighborhood of the ξ' axis).

We first study the contribution of a term $r_{\alpha'}(x, \xi)\, \xi'^{\alpha'}$ ($|\alpha'| = N'$) from the Taylor remainder. Trivial estimates show that we have estimates:

$$|\partial_\xi^\omega \partial_x^\beta r_\alpha(x,\xi)| \leqslant C_{\omega,\beta,k}(1+|\xi_n|)^{m-|\omega|-N'}$$

uniformly for $x \in K$ (compact set), as soon as $m - N' < 0$; clearly the same estimates hold when r_α is replaced by $\theta\psi r_\alpha$; but $\psi(\xi)$ vanishes outside the cone $\Gamma_2 : |\xi'| \geqslant 2C|\xi_n|$, so we have in fact, since $|\xi| \sim |\xi_n|$ if $|\xi'| \leqslant 2C|\xi_n|$:

$$|\partial_\xi^\omega \partial_x^\beta \theta(\xi)\,\psi(\xi)\,r_\alpha(x,\xi)| \leqslant C'_{\omega,\beta,k}(1+|\xi|)^{m-|\omega|-N'}.$$

Thus, $\theta\psi r_{\alpha'} \in S^{m-N}(\mathbf{R}^n)$ if $N > m$; and from this, we conclude, using (2.8) and (2.13), that:

$$Op(\theta(\xi)\,\psi(\xi)\,r_{\alpha'}(x,\xi)\,\xi'^{\alpha'})(u) = Op(\theta(\xi)\,\psi(\xi)\,r_{\alpha'}(x,\xi))(D_{x'}^{\alpha'}u)$$

belongs to $H_{\mathrm{loc}}^{s-(m-N'),t-N'}(\mathbf{R}^n)$, that is, to $H_{\mathrm{loc}}^{s-m+k,t-k}(\mathbf{R}^n)$ as soon as $N' \geqslant k$.

Let us now look at the terms $\partial_{\xi'}^{\alpha'} p(x;0,\xi_n)\,\xi'^{\alpha'}$ of the Taylor sum. To simplify the notations, we set:

$$A_{\alpha'} = B_{\alpha'} \circ D_{x'}^{\alpha'}$$

where $B_{\alpha'}$ is the operator with symbol

$$\theta(\xi)\,\psi(\xi)\,\partial_{\xi'}^{\alpha'} p(x;0,\xi_n) \in S^{m-|\alpha'|}.$$

Since $D_{x'}^{\alpha'}u \in H_{\mathrm{loc}}^{s,t-|\alpha'|}(\mathbf{R}^n)$ vanishes in Ω, we see, replacing u by $D_{x'}^{\alpha'}u$, $A_{\alpha'}$ by $B_{\alpha'}$, m by $m-|\alpha'|$ and k by $k+|\alpha'|$, that it is sufficient to study A_0 (corresponding to the case $\alpha' = 0$); for the sake of simplicity we drop the subscript in A_0 and denote this operator by A.

We have thus reduced the proof of theorem (3.1) to the case where P is the operator A, with symbol

$$\sigma_A(x,\xi) = \theta(\xi)\,\psi(\xi)\,a(x;0,\xi_n),$$

which is a typical transmission operator of degree m. A being defined as above, let us make a Taylor expansion of $p(x;0,\xi_n)$ at $x_n = 0$, until an arbitrarily order N. We get, for $\xi_n \neq 0$:

$$p(x;0,\xi_n) = \sum_{P<N} \frac{1}{P!}\,\partial_{x_n}^p p(x',0;0,\xi_n)\,x_n^p + \frac{1}{N!}\,r_N(x,\xi_n)\,x_n^N$$

with $\quad r_N(x,\xi_n) = \int_0^1 (1-t)^{N-1}\,\partial_{x_n}^p p(x',tx_n;0,\xi_n).$

Let us first study the contributions of the terms of the Taylor sum. It is at this point that we shall use the transmission property. We have, in fact, in view of the transmission formulas (2.17):

$$\partial^p_{x_n} p(x', 0; 0, \xi_n) = \partial^p_{x_n} p(x', 0; 0, 1)\, \xi_n^m \quad (\xi_n \neq 0).$$

Since multiplication by the C^∞ function $\partial_{x_n} p(x', 0; 0, 1)\, x_n^p$ maps $H^{s,t}_{\mathrm{loc}}(\mathbf{R}^n)$ into itself, we only have to prove:

(3.2) LEMMA. – *Let* $A_m = Op(\theta(\xi)\,\psi(\xi)\,\xi_n^m)$, *with* $m \in \mathbf{Z}$. *Then, if* $u \in H^{s,t}_{\mathrm{loc}}(\mathbf{R}^n)$ *vanishes in* Ω, *we have:*

$$A_m u_{|\Omega} \in H^{s-m+\infty,\, t-\infty}_{\mathrm{loc}}(\bar{\Omega}).$$

Proof of the lemma. –

1) $m \in \mathbf{N}$. We have:

$$A_m u = F^{-1}(\theta\,\psi\,\widehat{D^m_{x_n} u}) = F^{-1}(\theta(\psi - 1)\,\widehat{D^m_{x_n} u}) + F^{-1}((\theta - 1)\,\widehat{D^m_{x_n} u}) + D^m_{x_n} u$$

where F^{-1} denotes the inverse Fourier transform. Since $D^m_{x_n} u_{|\Omega} = 0$, $\theta - 1 \in C^\infty_0(\mathbf{R}^n)$, and the symbol $\theta(\psi - 1)$ satisfies the conditions of (2.10), we have: $A_m u_{|\Omega} \in H^{s-m+\infty,\, t-\infty}_{\mathrm{loc}}(\bar{\Omega})$.

2) $m < 0$. We shall prove the result by induction, both on m and k. We suppose that

$$(*) \qquad A_m u_{|\Omega} \in H^{s-m+k,\, t-k}_{\mathrm{loc}}(\bar{\Omega})$$

for every k; let us show that then:

$$A_{m-1} u_{|\Omega} \in H^{s-m+1+k,\, t-k}_{\mathrm{loc}}(\bar{\Omega}).$$

In view of (2.16), this is equivalent to prove that:

i) $\quad D_{x_n} A_{m-1} u_{|\Omega} \in H^{s-m+k,\, t-k}_{\mathrm{loc}}(\bar{\Omega}), \quad \forall k \in \mathbf{N}$

ii) $\quad A_{m-1} u_{|\Omega} \in H^{s-m+k,\, t-k+1}_{\mathrm{loc}}(\bar{\Omega}), \quad \forall k \in \mathbf{N}$.

But i) is immediate in view of the induction hypothesis, since $D_{x_n} A_{m-1} = A_m$. We now prove ii) by induction on k:

– ii) is true for $k = 0$, since we obviously have, in view of (2.8): $A_{m-1} u \in H^{s-(m-1),\, t}_{\mathrm{loc}}(\mathbf{R}^n)$.

— Suppose it is true for $k-1$, that is:

$$(**)\qquad A_{m-1}\, u_{|\Omega} \in H_{\mathrm{loc}}^{s-m+k-1,\,t-k+2}(\bar{\Omega})$$

and let us show that we then have:

$$A_{m-1}\, u_{|\Omega} \in H_{\mathrm{loc}}^{s-m+k,\,t-k+1}(\bar{\Omega})\,.$$

But, still in view of the conditions (2.16), this will be true if and only if:

a) $D_{x_n} A_{m-1}\, u_{|\Omega} \in H_{\mathrm{loc}}^{s-m+k-1,\,t-k+1}(\bar{\Omega})$

b) $A_{m-1}\, u_{|\Omega} \in H_{\mathrm{loc}}^{s-m+k-1,\,t-k+2}(\bar{\Omega})$.

But a) once again immediately results from (*), with k replaced by $k-1$, since $D_{x_n} A_{m-1} = A_m$; and b) is exactly (**), and the lemma is proved.

Let us now go to the last step in the proof of theorem (3.1). We have to evaluate the contribution of the Taylor remainder $r_N(x, \xi_n)\, x_n^N$. Since p is positively homogeneous of degree m in ξ_n, we can split (modulo products by C^∞ functions) this term in a superposition of terms of the type:

$$\left.\begin{aligned} &x_n^N\, \theta(\xi)\, \psi(\xi)\, y(\xi_n)\, \xi_n^m \\ &x_n^N\, \theta(\xi)\, \psi(\xi)\, y(-\xi_n)\, \xi_n^m \end{aligned}\right\}$$

where $y(\xi_n)$ $(= 1$ if $\xi_n \geqslant 0$, 0 if $\xi_n < 0)$ is Heavisides function. It is of course sufficient to study only the first term, since:

$$\begin{aligned} x_n\, \theta(\xi)\, \psi(\xi)\, y(-\xi_n)\, \xi_{n-}^m &= x_n^N\, \theta(\xi)\, \psi(\xi)\, \xi_n^m - x_n\, \theta(\xi)\, \psi(\xi)\, y(\xi_n)\, \xi_{n+}^m \\ &= x_n^N\, A_m - x_n\, \theta(\xi)\, \psi(\xi)\, y(\xi_n)\, \xi_{n+}^m \end{aligned}$$

and we know, by our previous calculations the properties of A_m. We need:

(3.3) LEMMA. — *Let* $R_{m,N}$ *be the operator with symbol:*

$$x_n^N\, \theta(\xi)\, \psi(\xi)\, y(\xi_n)\, \xi_{n+}^m\,,\ m \in \mathbf{Z}\,.$$

If $u \in H_{\mathrm{loc}}^{s,t}(\mathbf{R}^n)$ *vanishes in* Ω, *we have:*

$$R_{m,N}\, u_{|\Omega} \in H_{\mathrm{loc}}^{s-m+k,\,t-k}(\bar{\Omega})$$

if $k < N - |s| - 3$.

Theorem (3.1) will then be proved, since N can be taken arbitrarily big in (3.3). Let us prove the lemma.

a) $m = 0$. We then have: $R_{0,N}\, u(x) = x_n^N\, F^{-1}(\theta\, \psi\, y \hat{u})(x)$ where F^{-1} is the inverse Fourier transform; we thus obtain:

$$R_{0,N}\, u(x) = x_n^N\, F^{-1}(y\hat{u})(x) + x_n^N\, F^{-1}((\theta\,\psi - 1)\, y\hat{u})(x).$$

The second term is very easy to handle. In fact, since $\theta\,\psi - 1 = \theta(\psi - 1) + \theta - 1$, we have:

$$F^{-1}(\theta(\psi - 1)\, y\hat{u}) + F^{-1}((\theta - 1)\, y\hat{u}) \in H^{s+k,\, t-k} + C^\infty(\mathbf{R}^n)$$

for every $k \in \mathbf{N}$, since we can apply again (2.10) to the symbol $\theta(\psi - 1)\, y \in S^0$ and since $(\theta - 1) \in C_0^\infty(\mathbf{R}^n)$ defines a smoothing operator.

Let us now look at the term $x_n^N F^{-1}(y\hat{u})(x)$ which will be much more troublesome.

In view of a classical well-known formula, we have:

$$F^{-1}(y) = \frac{i}{2\pi}\left(vp\, \frac{1}{x}\right) + \frac{1}{2}\, \delta$$

thus:

$$F^{-1}(y\hat{u}) = \frac{i}{2\pi}\left(vp\, \frac{1}{x_n} \otimes \delta_{x'}\right) * u + \frac{1}{2}\, u\,.$$

Since $\frac{1}{2}\, u_{|\Omega} = 0$, we only have to consider the convolution term. In fact, we shall prove that the map:

$$u \longrightarrow x_n^N \left[\left(vp\, \frac{1}{x_n} \otimes \delta_{x'}\right) * u\right]_{|\Omega}$$

is continuous from $\{u \in H_c^{s,t}(\mathbf{R}^n)/u_{|\Omega} = 0\}$ in $H_{\mathrm{loc}}^{s+k,\, t-k}(\overline{\Omega})$, for every $k < N - |s| - 3$. Let $E = \{u \in C_0^\infty(\mathbf{R}^n)/u_{|\Omega} = 0\}$, and $\varphi_1, \varphi_2 \in C_0(\mathbf{R}^n)$. Setting

$$U = \varphi_1\, x_n^N \left[\left(vp\, \frac{1}{x_n} \otimes \delta_{x'}\right) * \varphi_2\, u\right]$$

we have, for $x_n > 0$:

$$U(x', x_n) = \int_{-\infty}^{0} K(x_n, y_n)\, u(x', y_n)\, dy_n$$

where K is the kernel defined by:

$$K(x_n, y_n) = \varphi_1(x_n)\,\varphi_2(x_n)\,\frac{x_n^N}{x_n - y_n}.$$

A trivial computation shows that: $K \in C_0^{N-1}(\overline{\mathbf{R}_+} \times \overline{\mathbf{R}_-})$ $(N \geqslant 1)$ thus we may extend K to a kernel $L \in C_0^{N-1}(\mathbf{R}^2)$.

Now, define V by: $V(x', x_n) = \int L(x_n, y_n)\, u(x', y_n)\, dy_n$.

Obviously $U_{|\Omega} = V_{|\Omega'}$ and by the definition of the normed space $H_{\rm loc}^{s+k,t-k}(\overline{\Omega})$, we have: $\| U_{|\Omega} \|_{s+k,t-k} \leqslant \| V \|_{s+k,t-k}$.

Application of Plancherel's theorems immediately yields:

$$V(x) = \int \hat{L}(x_n, -\eta_n)\, \hat{u}(x', \eta_n)\, d\eta_n$$

hence, the Fourier transform of V is:

$$\hat{V}(\xi) = \int \hat{L}(\xi_n, -\eta_n)\, \hat{u}(\xi', \eta_n)\, d\eta_n$$

and, we may estimate the $H^{s+k,t-k}$ norm of V by:

$$\| V \|_{s+k,t-k}^2 = \int (1 + |\xi|^2)^{s+k} (1 + |\xi'|^2)^{t-k}\, |\hat{V}(\xi)|^2\, d\xi$$

$$\leqslant d\xi \left[\int (1 + |\xi|^2)^{\frac{s+k}{2}} (1 + |\xi'|^2)^{\frac{t-k}{2}} |L(\xi_n, -\eta_n)|^2\, \hat{u}(\xi', \eta_n)|^2\, d\eta_n \right]^2.$$

Using Peetre's inequality, we obtain:

$$\| V \|_{s+k,t-k}^2 \leqslant C \int d\xi \left[\int (1 + |\xi|^2)^{k/2} (1 + |\xi'|^2)^{-k/2} (1 + |\xi_n - \eta_n|^2)^{|s|/2} \right.$$
$$\left. \cdot\, |L(\xi_n, -\eta_n)|\, (1 + |\xi'|^2 + \eta_n^2)^{s/2} (1 + |\xi'|^2)^{t/2}\, |\hat{u}(\xi', \eta_n)|^2\, d\eta_n \right]^2$$

and we obtain, using finally Cauchy-Schwarz' inequality:

$$\| V \|_{s+k,t-k}^2 \leqslant C \int F(\xi')\, G(\xi_n)\, d\xi'\, d\xi_n$$

where we have set:

$$F(\xi') = \int |\hat{u}(\xi', \eta_n)|^2 (1 + |\xi'|^2 + \eta_n^2)^s (1 + |\eta'|^2)^t\, d\eta_n$$

$$G(\xi_n) = \int |L(\xi_n, -\eta_n)|^2 (1 + |\xi_n|^2)^k (1 + |\xi_n - \eta_n|^2)^{|s|}\, d\eta_n.$$

Hence, we may write:

$$\| V \|_{s+k,t-k}^2 \leqslant C \int G(\xi_n)\, d\xi_n \cdot \| u \|_{s,t}^2.$$

But, since $L \in C_0^{N-1}(\mathbf{R}^2)$, the integral $\int G(\xi_n)\, d\xi_n$ is convergent as soon as $2k + 2|s| - 2(N-2) < -2$, that is: $k < N - |s| - 3$.

That's the way it is in the case $m = 0$.

b) The case $m > 0$ is an immediate consequence of the latter, since we obviously have: $R_{m,N} = R_{0,N} \circ D^m_{x_n}$. Thus, all we have to do now, is to examinate $R_{m,N}$ with $m < 0$. We want to prove, that:

$$(*) \qquad R_{m,N}\, u \in H_{\mathrm{loc}}^{s-m+k,\, t-k}(\bar{\Omega}) \quad \text{if} \quad k < N - |s| - 3 .$$

We first note that we may prove (*) with $R_{m,N}$ replaced by the operator: $\widetilde{R}_{m,N} = Op(\theta\, \psi\, \xi^m_{n+}) \circ x^N_n$.

In fact, for every $\varphi \in C_0^\infty(\mathbf{R}^n)$, we have, integrating by parts:

$$\begin{aligned} R_{m,N}\, \varphi(x) &= \int e^{ix\xi}\, x_n^N\, \theta(\xi)\, \psi(\xi)\, \xi^m_{n+}\, \hat{\varphi}(\xi)\, d\xi \\ &= (-1)^N \int e^{ix\xi}\, D^N_{\xi_n}\, [\theta(\xi)\, \psi(\xi)\, \xi^m_{n+}\, \varphi(\xi)]\, d\xi . \end{aligned}$$

By Leibniz' formula, we get:

$$D^N_{\xi_n}(\theta(\xi)\, \psi(\xi)\, \xi^m_{n+}\, \hat{\varphi}(\xi)) = \sum_{0 \leq p \leq N} C^p_N\, D^p_{\xi_n}(\theta\, \psi\, \xi^m_{n+})\, x_n^{N-p}\, \varphi(\xi) .$$

– Now, if $p > 0$, the sum contains terms with factors of the type:

$$D^p_{\xi_n}(\theta\, \psi\, \xi^m_{n+}) = C_p \cdot \theta\, \psi\, \xi^{m-p}_{n+} + r_{m-p}$$

where r_{m-p} is a symbol with degree $m - p$, vanishing in a conic neighborhood of the ξ_n axis; in view of (2.10) we have:

$$Op(r_{m-p})\, u \in H_{\mathrm{loc}}^{s-(m-p)+\infty,\, t-\infty}(\mathbf{R}^n) .$$

That is:

$$Op(r_{m-p})\, u \in H_{\mathrm{loc}}^{s-m+\infty,\, t-\infty}(\mathbf{R}^n)$$

in view of the inclusions (2.7).

– If $p = 0$, we have the term $\theta\, \psi\, \xi^m_{n+}\, x_n^N\, \varphi$, and:

$$Op(\theta\, \psi\, \xi^m_{n+})(x_n^N\, \varphi) = \widetilde{R}_{m,N}\, \varphi$$

which proves our assertion.

Let us then prove that:

$$\widetilde{R}_{m-p,N}\, u_{|\Omega} \in H_{loc}^{s-(m-p)+k,\, t-k}(\overline{\Omega}) \ (\subset H_{loc}^{s-m+k,t-k}(\overline{\Omega}))\,.$$

A trivial computation shows that we have:

$$D_{x_n} \circ \widetilde{R}_{m-p-1,N} = \widetilde{R}_{m-p,N}$$

and the proof is now finished, by using exactly the same introduction argument as in lemma (3.2), where we had, in the same way, the relation $D_{x_n} \circ A_{m-1} = A_m$.

Let us now show, as announced in the beginning of this section, how we derive Theorem (3.1) from (3.1′).

First we observe that if the condition $Au \in H_{loc}^{s,t}(\mathbf{R}^n)$ holds for one operator A, elliptic in γ, it will also hold with A replaced by any other properly supported operator $A_1 \in L^0(\partial\Omega)$ such that $WF(A_1) \subset \gamma_1 \subset\subset \gamma$. In fact, by the classical calculus of pseudo-differential operators, we know that we can find a microlocal inverse of A in γ_1, that is, a properly supported $\overline{A} \in L^0(\partial\Omega)$, such that:

$$\overline{A}A - I = R', \quad \text{and} \quad R' \in L^0(\partial\Omega) \cap L^{-\infty}(\overline{\gamma}_1)\,.$$

Thus, we have:

$$A_1 u = A_1 \overline{A}Au - A_1 R'u$$

and since $Au \in H_{loc}^{s,t}(\mathbf{R}^n)$, $A_1 \overline{A} \in L^0(\partial\Omega)$, we get, in view of (1.13): $A_1 \overline{A}Au \in H_{loc}^{s,t}(\mathbf{R}^n)$.

On the other hand if $WF(A_1) \subset \gamma_1$, we obviously have:

$$A_1 R' \in L^{-\infty}(\partial\Omega)$$

and by using (2.13) once again, we obtain, since $u \in H_{loc}^{s,t'}(\mathbf{R}^n)$ (for some $t' \in \mathbf{R}$):

$$A_1 R'u \in H_{loc}^{s,+\infty}(\mathbf{R}^n) \subset H_{loc}^{s,t}(\mathbf{R}^n)$$

and, finally: $A_1 u \in H_{loc}^{s,t}(\mathbf{R}^n)$. Now, choose $A = I$ in a small conic neighborhood $\overline{\gamma} \subset\subset \overline{\gamma}_1$ of σ; we have: $Pu = PA_1 u + P(I - A)$ and since $A_1 u \in H_{loc}^{s,t}(\Omega)$ vanishes in Ω, theorem (3.1) shows that: $PA_1 u_{|\Omega} \in H_{loc}^{s-m+\infty,t-\infty}(\overline{\Omega})$ and, in view of (2.13), for every properly supported $B \in L^0(\partial\Omega)$ we also have:

$$BPA_1 u_{|\Omega} \in H_{loc}^{s-m+\infty,t-\infty}(\overline{\Omega})\,.$$

All we have to do now is to study the term $P(I - A_1)$. But $I - A$ vanishes in $\overline{\gamma}$, hence, if B satisfies $WF(B) \subset \overline{\gamma}$, we know, since u belongs to the space $H^{s,t'}_{loc}(\mathbf{R}^n)$ that:

$$(*) \qquad BP(I - A_1)u \in H^{s,\infty}_{loc}(\mathbf{R}^n)$$

(see the remark following the proof of the proposition).

On the other hand $(1 - A_1)u \in H^{s,t'}_{loc}(\mathbf{R}^n)$ and vanishes in Ω, so we may use theorem (3.1) again, which yields:

$$P(I - A_1)\, u_{|\Omega} \in H^{s-m+\infty,t'-\infty}_{loc}(\overline{\Omega})$$

that is, also:

$$(**) \qquad BP(I - A_1)u_{|\Omega} \in H^{s-m+\infty,t'-\infty}_{loc}(\overline{\Omega}) .$$

Combining (*) and (**), and using the obvious fact that:

$$H^{s,\infty}_{loc} \cap H^{s-m+\infty,t'-\infty} \subset H^{s-m+\infty,t-\infty}$$

we finally have:

$$BPu_{|\Omega} \in H^{s-m+\infty,t-\infty}_{loc}(\overline{\Omega})$$

which proves theorem (3.1), since $\overline{\gamma}$ can be chosen in an arbitrary way, if one takes $\overline{\gamma}_1 \subset\subset \overline{\gamma}$ large enough.

Remark. – (*) is a consequence of the more general following result:

(3.4) If $P \in L^m(\mathbf{R}^n)$, $A \in L^0(\partial\Omega)$ are properly supported tangential operators, with $A = I$ in a conical neighborhood γ of $\sigma \in T^*\partial\Omega$, then, for every properly supported $B \in L^0(\partial\Omega)$, with $WF(B) \subset \overline{\gamma} \subset\subset \gamma$, one has for every $(s, t) \in \mathbf{R}^2$:

$$u \in H^{s,t}_{loc}(\mathbf{R}^n) \Longrightarrow BP(I - A) \in H^{s,\infty}_{loc}(\mathbf{R}^n) .$$

The proof of (3.4) is standard. Using a straightforward generalization of the usual asymptotic expansion formulas for the symbol of the compose of two pseudo-differential operators (see, for instance, J. Sjöstrand [14], Appendix, p. 50) it is easily seen that one can write, modulo a smoothing operator in $L^{-\infty}(\mathbf{R}^n)$:

$$BP(I - A) \equiv R_1$$

where R_1 is defined by:

$$R_1\,\varphi(x) = \int e^{ix\xi}\, r(x\,,\xi)\,\hat{\varphi}(\xi)\,,\ \varphi \in C_0^\infty(\mathbf{R}^n),$$

and $r \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ satisfies for every $N \in \mathbf{R}$, $\alpha, \beta \in \mathbf{N}^n$, estimates of the form:

$$(***) \qquad |\partial_\xi^\alpha \partial_x^\beta r(x, \xi)| \leq C_{\alpha,\beta,N}(1 + |\xi'|)^{-N}(1 + |\xi|)^{m-|\alpha|}$$

uniformly in x on each compact subset $K \subset \mathbf{R}^n$.

Using (***), one proves (3.4) using exactly the same classical arguments used to prove (2.8) and (2.13).

4. Wave fronts sets near the boundary.

Let $u \in C^\infty(\overline{\mathbf{R}_+}, D'(\partial\Omega))$ be a normally regular distribution, and a real number ω. We shall say that an element $\sigma (= (x_0', \xi_0'))$ of $T_0^*\partial\Omega$ does not belong to $\partial WF_\omega(u)$ (the "tangential singular spectrum" of u), if we can find a conic neighborhood γ of σ, and a tangential operator $A \in L^0(\partial\Omega)$, properly supported, and elliptic in γ, such that:

(4.1) $Au \in H^{s,t}_{loc}(\overline{\Omega})$ near $\partial\Omega$, $\forall (s, t) \in \mathbf{R}^2$ such that $s + t = \omega + \frac{1}{2}$.

The interest of this definition comes from:

(4.2) $\sigma \notin \partial WF_\omega(u) \Longrightarrow \sigma \notin WF_{\omega-k}(D^k_{x_n} u(\cdot, x_n))$

for fixed $x_n \geq 0$ and $0 \leq k < \omega$.

(Here $D^k_{x_n} u(\cdot, x_n)$ is the k-th sectional trace of $D^k_{x_n} u$ on the hyperplane $t = x_n$; $WF_{\omega-k}$ is the "$H^{\omega-k}$ wave front set" of Hörmander, [9]).

To prove (4.2), we only remark that (2.7) implies that $Au \in C^k(\overline{\mathbf{R}_+}, H^{\omega-k}_{loc}(\partial\Omega))$; in fact we have a much more precise result (see Hörmander [7]): there exists a constant $C > 0$, not depending on x_n, such that:

$$\| D^k_{x_n} Au(\cdot, x_n)\|_{\omega-k} \leq C \| Au \|_{s,t}$$

if $s + t = \omega + \frac{1}{2}$, $0 \leq k < \omega$, $s - k > \frac{1}{2}$, which proves the assertion.

The following property is an immediate consequence of the definition and of (2.8):

(4.3) $\sigma \notin \partial WF_{\omega}(u) \Longrightarrow \sigma \notin \partial WF_{\omega-k}(D^k_{x_n} u)$ for every integer k.

We now prove an important result which shows that one has a great liberty in choosing the operator A in definition (4.1):

(4.4) If $Au \in H^{s,t}_{loc}(\overline{\Omega})$ $\left(s + t = \omega + \frac{1}{2}\right)$, for A elliptic in γ, then the same relation holds when we replace A by any other properly supported $\overline{A} \in L^0(\partial\Omega)$, such that $WF(\overline{A}) \subset \overline{\gamma}$, $\overline{\gamma}$ being a conic subset of $T^*_0\partial\Omega$, with $\overline{\gamma} \subset\subset \gamma$.

Proof. – By the standard calculus of pseudo-differential operators, we know that for every $\overline{\gamma} \subset\subset \gamma$, there is an operator $B \in L^0(\partial\Omega)$, inverting microlocally A in $\overline{\gamma}$, that is:

$$BA - I = R \in L^0(\partial\Omega) \cap L^{-\infty}(\overline{\gamma}).$$

– Now, let $\overline{A} \in L^0(\partial\Omega)$ satisfy the conditions of (4.4); we then have, since B of course can be chosen properly supported:

$$\overline{A}u = \overline{A}B\,Au - \overline{A}Ru .$$

In view of (1.13), $\overline{A}B\,Au \in H^{s,t}_{loc}(\overline{\Omega})$ if $s + t = \omega + \frac{1}{2}$, since this is true for Au; on the other hand, since $WF(\overline{A}) \cap WF(R) = \emptyset$, we have $AR \in L^{-\infty}(\partial\Omega)$, and this ends the proof, in view of (2.15), u being normally regular.

Let us finally define:

(4.5) $$\partial WF_{\infty}(u) = \bigcap_{\omega \in \mathbf{R}} \partial WF_{\omega}(u) .$$

Of course, (4.5) is equivalent to:

(4.5′) $\sigma \notin \partial WF_{\infty}(u)$ if and only if there is an operator A satisfying the same ellipticity condition as before, and such that: $Au \in C^{\infty}(\overline{\Omega})$.

Thus, $\partial WF_{\infty}(u)$ is nothing else than the notion of singular spectrum used in Chazarain [3], Andersson-Melrose [1], Sjöstrand-Melrose [12]. Of course, (4.4) still remains valid in this case. In the sequel we will write $\partial WF_{\infty}(u) = \partial WF(u)$.

It is also possible to give an equivalent definition of $\partial WF(u)$, using the Fourier transform:

(4.5″) $(x'_0, \xi'_0) \notin \partial\mathrm{WF}(u)$ if and only if there is a neighborhood $\mathrm{U}(x'_0)$ of x'_0 in $\partial\Omega$, and a conic neighborhood $\Gamma(\xi'_0)$ of ξ'_0 in $\mathbf{R}^{n-1}\backslash 0$, such that for all integers k, N and $\alpha \in \mathrm{C}_0^\infty(\mathrm{U}(x'_0))$, we have estimates:

$$|\mathrm{D}_{x_n}^k \widehat{\alpha u}(\xi', x_n)| \leqslant \mathrm{C}_{\alpha,k,\mathrm{N}}(1 + |\xi'|)^{-\mathrm{N}}, \quad \forall \xi' \in \Gamma(\xi'_0)$$

(uniformly in $x_n \in \overline{\mathbf{R}}$).

Thus, $\partial\mathrm{WF}(u)$ appears, in view of (2.2), as the natural way to measure the lack of strong regularity in the tangential variables for u. It should be observed, indeed, that:

$$\partial\mathrm{WF}(u) = \emptyset \iff u \in \mathrm{C}^\infty(\overline{\Omega}) \tag{4.6}$$

and, more generally:

$$\partial\mathrm{WF}_\omega(u) = \emptyset \iff u \in \bigcap_{s+t=\omega+\frac{1}{2}} \mathrm{H}^{s,t}_{\mathrm{loc}}(\overline{\Omega}). \tag{4.7}$$

Let us now prove:

(4.8) THEOREM. – *If* $\mathrm{Q} \in \mathrm{L}^m(\mathbf{R}^n)$ *(properly supported) has the transmission property, then, for* $g \in \mathrm{H}^{-\infty}_{\mathrm{loc}}(\partial\Omega)$, *we have, for* $\omega \in \mathbf{R} \cup \{+\infty\}$: $\partial\mathrm{WF}_{\omega-1}\, \mathrm{Q}(g \otimes \delta)_{|\Omega} \subset \mathrm{WF}_{\omega+m}(g)$.

Proof. – We first note that the tangential singular spectrum of $\mathrm{Q}(g\otimes\delta)_{|\Omega}$ makes sense, since $\mathrm{Q}(g\otimes\delta)_{|\Omega} \in \mathrm{C}^\infty(\mathbf{R}_+, \mathrm{D}'(\partial\Omega))$ in view of (3.1′). Secondly, it is of course sufficient to prove (4.8) with $\omega \in \mathbf{R}$. Let $\sigma \notin \mathrm{WF}_{\omega+m}(g)$; hence we can find a properly supported $\mathrm{A} \in \mathrm{L}^0(\partial\Omega)$, elliptic at σ and such that $\mathrm{A}u \in \mathrm{H}^{\omega+m}_{\mathrm{loc}}(\partial\Omega)$. An easy calculation now shows, that:

$$\mathrm{A}(g\otimes\delta) \in \mathrm{H}^{s,t}_{\mathrm{loc}}(\mathbf{R}^n) \iff \begin{cases} s < -\dfrac{1}{2} \\[2mm] s + t \leqslant \omega + m - \dfrac{1}{2} \end{cases}$$

and we thus have, for fixed $s < -\dfrac{1}{2}$:

$$\mathrm{A}(g\otimes\delta) \in \mathrm{H}^{s,\omega+m-s-\frac{1}{2}}_{\mathrm{loc}}(\mathbf{R}^n).$$

On the other hand, $\mathrm{A}g \in \mathrm{H}^\alpha_{\mathrm{loc}}(\partial\Omega)$ for some α, that is also:

$$A(g \otimes \delta) \in H^{s,\alpha-s-\frac{1}{2}}(\mathbf{R}^n).$$

Since $A(g \otimes \delta)_{|\Omega} = 0$ we can apply theorem (3.1), which yields:

(4.9) $$BQ(g \otimes \delta)_{|\Omega} \in H_{loc}^{s-m+k,\,\omega+m-s-\frac{1}{2}-k}(\overline{\Omega})$$

for every $k \in \mathbf{R}$ and a properly supported $B \in L^0(\partial\Omega)$, elliptic in a sufficient small conic neighborhood of σ; (4.9) now proves (4.8) since we have $(s - m + k) + \left(\omega + m - s - \frac{1}{2} - k\right) = \omega - \frac{1}{2}\cdot$

The following proposition is the analogue of the classical H^s-pseudo local property, for the tangential singular spectrum:

(4.10) If $P\, L^m(\mathbf{R}^n)$ is properly supported and has the transmission property, we have, for every $u \in C^\infty(\overline{\mathbf{R}_+}, D'(\partial\Omega))$:

$$\partial WF_{\omega-m}\, P(u^0)_{|\Omega} \subset \partial WF_\omega(u)$$

(u^0 denoting the canonical extension of u by 0 outside $\overline{\Omega}$).

In view of the definitions, (4.10) is an evident consequence of the following stronger result:

(4.11) P being defined as above, let $u \in H_{loc}^{s,-\infty}(\overline{\Omega})$ with $s > -\frac{1}{2}\cdot$ Then, we have:

$$Au \in H_{loc}^{s,t}(\overline{\Omega}) \Longrightarrow BP(u^0)_{|\Omega} \in H_{loc}^{s-m,t}(\overline{\Omega})$$

the tangential operators being defined as in the proof of (4.6).

Proof. – We first note that the condition $s > -\frac{1}{2}$ insures us that u^0 is canonically defined (see for instance Lions-Magenes [10]).

1) Let us first prove (4.11) for $-\frac{1}{2} < s < \frac{1}{2}$. By the same argument as used in (4.4), we see that it is sufficient to prove our assertion with $A = I$ in a small conic neighborhood γ in $T_0^*(\partial\Omega)$, and vanishing outside $\overline{\gamma} \supset \gamma$. We then have:

$$P(u^0) = AP(u^0) + (I - A)\,P(u^0).$$

We know (see Lions-Magenes [10], for instance), that if $-\frac{1}{2} < s < \frac{1}{2}$ we have $u \in H_{loc}^{s,t}(\overline{\Omega}) \Longrightarrow u^0 \in H_{loc}^{s,t}(\mathbf{R}^n)$; thus

$$AP(u^0)_{|\Omega} \in H^{s-m,t}_{loc}(\bar{\Omega}).$$

Now, choose $B \in L^0(\partial\Omega)$, elliptic in a conic neighborhood $\gamma_0 \subset \gamma$, and with $WF(B) \subset \gamma$; we then have $B(I - A) \in L^{-\infty}(\partial\Omega)$, that is: $BP(u^0)_{|\Omega} \in H^{s-m,t}_{loc}(\bar{\Omega}) + H^{s-m,\infty}_{loc}(\bar{\Omega})$ which proves 1).

2) We now suppose that (5.9) is true for $-\frac{1}{2} + k < s < \frac{1}{2} + k$ ($k \in \mathbf{N}$), and prove it is also true with k replaced by $k+1$. In view of the fact that:

$$(1 + |\xi|^2)^{\frac{s+1}{2}} \sim (1 + |\xi|^2)^{\frac{s}{2}} + \sum_{j=1}^{n} \xi_j (1 + |\xi|^2)^{\frac{s}{2}}$$

we have the following norm equivalence in $H^{s+1,t}(\bar{\Omega})$

$$\|\varphi\|_{s+1,t} \sim \|\varphi\|_{s,t} + \sum_{j=1}^{n} \|D_{x_j}\varphi\|_{s,t}.$$

By the induction hypothesis we have: $\| BP(u^0)_{|\Omega} \|_{s-m,t} < \infty$, thus all we have to do is to prove that: $\| D_{x_j} BP(u^0)_{|\Omega} \|_{s-m,t} < \infty$.

a) If $j \neq n$, we have: $D_{x_j} B = BD_{x_j} + [D_{x_j}, B]$, $[\ ,\ ]$ being the commutator. But $[D_{x_j}, B] \in L^{-1}(\partial\Omega) \subset L^0(\partial\Omega)$, and we have $WF([D_{x_j}, B]) \subset WF(B)$, so it is sufficient to consider the term $BD_{x_j} P(u^0)_{|\Omega}$. But we also have:

$$D_{x_j} P(u^0) = P((D_{x_j} u)^0) + [D_{x_j}, P](u^0)$$

and $\| BD_{x_j} P(u^0)_{|\Omega} \|_{s-m,t} < \infty$ since $D_{x_j} u \in H^{s,t}(\bar{\Omega})$, and $[D_{x_j}, P] \in L^m(\mathbf{R}^n)$ has the transmission property.

b) If $j = n$:

$$\begin{aligned} D_{x_n} BP(u^0) &= BD_{x_n} P(u^0) \\ &= BP(D_{x_n}(u^0)) + B[D_{x_n}, P](u^0) \end{aligned}$$

and in view of the same argument as above, it is sufficient to evaluate $BP(D_{x_n}(u^0))$. But:

$$D_{x_n}(u^0) = (D_{x_n} u)^0 + u(x', 0) \otimes \delta$$

and $u(x', 0)$ is well defined since we are in the case $s > \frac{1}{2}$. Now, $D_{x_n} u \in H^{s,t}_{loc}(\bar{\Omega})$, and, in view of the induction hypothesis,

$$BP((D_{x_n} u)^0)_{|\Omega} \in H^{s-m,t}_{loc}(\bar{\Omega}).$$

On the other hand, the proof of (4.10) shows that

$$P(u(\cdot, 0) \otimes \delta)_{|\Omega} \in H^{s-m+k,\, t-k}_{\mathrm{loc}}(\overline{\Omega})$$

for every k, since $Au \in H^{s+1,\, t}_{\mathrm{loc}}(\overline{\Omega})$ implies $Au(\cdot, 0) \in H^{s+t+\frac{1}{2}}(\partial\Omega)$ in view of (1.7).

In order to measure the smoothness of $u \in C^{\infty}(\overline{\mathbf{R}_+}, D'(\partial\Omega))$ not only in terms of tangential regularity, but to take also in account the (micro-) local behaviour of u away from the boundary, it is customary (see Melrose-Sjöstrand [12], and the references therein) to use the notion of "boundary singular spectrum" $WF_b(u)$ (sometimes also denoted by $SS_b(u)$). The generalization of this concept to Sobolev spaces is straightforward: Let $\sim$ be the equivalence relation on $\partial T^*\Omega \backslash N^*\partial\Omega = \{(x', 0, \xi', \xi_n)/\xi' \neq 0\}$ defined by

$$(x'_0, 0\,; \xi'_0, \xi_{n_0}) \sim (x'_1, 0, \xi'_1, \xi_{n_1}) \iff (x'_0, \xi'_0) = (x'_1, \xi'_1)$$

(that is if $i^*(x_0, \xi_0) = i^*(x_1, \xi'_1)$, where i^* is the canonical projection of $T^*\overline{\Omega}_{|\Omega}$ on $T^*\partial\Omega$). We then define the homogeneous topological space: $B\overline{\Omega} = T^*_0\overline{\Omega} \backslash N^*\partial\Omega/\sim$ which will always be identified to $T^*_0\partial\Omega \cup T^*_0\Omega$. Now, if $u \in C^{\infty}(\overline{\mathbf{R}_+}, D'(\partial\Omega))$, we set by definition:

(4.12) If $s \in \mathbf{R} \cup \{+\infty\}$, then:

$$WF_{b,s}(u) = WF_s(u) \cup \partial WF_{s-\frac{1}{2}}(u).$$

In view of the remarks made above, we may identify $WF_{b,s}(u)$ to a conic subset (with the obvious meaning of this) of $B(\overline{\Omega})$, and by combining theorem (4.10) with the properties of the usual H^s-wave front set, we obtain, if $P \in L^m(\mathbf{R}^n)$ has the transmission property:

(4.13) $$WF_{b,s-m}[P(u^0)_{|\Omega}] \subset WF_{b,s}(u).$$

5. The Microlocal Cauchy Problem.

In this last section we apply the methods of § 3 and 4 to the study of the singularities of Cauchy problems of real principal type in the half space $\overline{\Omega}$ generalizing the classical local solvability results on $\mathbf{R}^n$ for these operators (see, for instance, Hörmander [9]).

From now on, we suppose that P is a partial differential operator of real principal type, the boundary $\partial\Omega$ being non-characteristic. Then

(5.1) THEOREM. – *Let* $u \in D'(\bar{\Omega})$ *be a solution of the Cauchy problem:*

$$(C) \qquad \begin{cases} Pu = f \\ \gamma_j u = g_j \end{cases}$$

for $f \in C^\infty(\overline{R_+}, D'(\partial\Omega))$ *and* $g_j \in D'(\partial\Omega)$, $j = 0, 1, \ldots, m-1$. *Let* $\sigma \in T_0^*(\partial\Omega)$, *and suppose that there is a conic neighborhood* γ *of* σ *such that every (null) bicharacteristic strip of* P *through* $i^{*-1}(\sigma)$ (i^* *being the canonical mapping* $\partial T^*\Omega \longrightarrow T^*\partial\Omega$) *is such that at least one point of its projection on the base space* $\mathbf{R}^n$ *is contained in* $\mathbf{R}^n \setminus \Omega$. *Then we have:*

$$\left.\begin{array}{l} \sigma \notin \partial WF_{s-\frac{1}{2}}(f) \\ \sigma \notin WF_{s+m-j-\frac{1}{2}}(g_j) \\ (j = 0, 1, \ldots, m-1) \end{array}\right\} \Longrightarrow \sigma \notin \partial WF_{s+m-\frac{3}{2}}(\varphi u)$$

for every $\varphi \in C_0^\infty([0, \epsilon])$, *for some* $\epsilon > 0$.

Proof. – The condition on f ensures us that $u \in C^\infty(\mathbf{R}_+, D'(\partial\Omega))$; let u^0 (resp. f^0) be the canonical 0-extension of u (resp. f) across $\partial\Omega$. Since P is a differential operator, we get, in view of the classical "jump formula":

$$P(u^0) = S + f^0$$

where S is a sum of terms $S_{\alpha',k,j}$ of the type $(D_{x'}^{\alpha'} \gamma_k u) \otimes \delta_{x_n}^{(j)}$ where the $a_\alpha \in C^\infty(\mathbf{R}^n)$, $|\alpha'| + k + j \leq m-1$, with coefficients in $C^\infty(\partial\Omega)$.

Now, $\partial\Omega$ being non characteristic for P, we can find (see § 4, [4]) a properly supported $Q \in L^{-m}(\mathbf{R}^n)$ with the transmission property, inverting microlocally P, that is:

$$QP = I + R$$

with $R \in L^0(\partial\Omega) \cap L^{-\infty}(\Gamma)$, Γ being a conic neighborhood of the conormal bundle $N_0^*(\partial\Omega)$. Thus, we get:

$$(5.2) \qquad u^0 = \Sigma\, Q(S_{\alpha',j,k}) + Q(f^0) - R(u^0).$$

In view of theorem (4.8) and of the hypothesis on the traces of u, we get, if $A \in L^0(\partial\Omega)$ is elliptic at σ, with sufficiently small conic support:

$$AQ(S_{\alpha',j,k})_{|\Omega} \in H_{loc}^{s+2m-k-|\alpha'|-j-1+\infty,-\infty}(\bar{\Omega})$$

that is, in particular:

$$AQ(S_{\alpha',j,k})_{|\Omega} \in H_{loc}^{s+m+\infty,-\infty}(\bar{\Omega})$$

for all α', j, k; this shows that:

$$\sigma \notin \partial WF_{s+m-\frac{1}{2}}[Q(S)_{|\Omega}]. \tag{5.3}$$

On the other hand, we also have

$$\sigma \notin \partial WF_{s+m-\frac{1}{2}} Q(f^0)_{|\Omega} \tag{5.4}$$

as an immediate consequence of theorem (4.10).

It now remains to evaluate the remainder term $R(u^0)$; it is at this stage that we shall make use of the assumptions on the bicharacteristics.

Noting that we have,

$$PR = R'P$$

where $R' \in L^0(\mathbf{R}^n)$ vanishes near $N_0^*(\partial\Omega)$, we get:

$$\begin{aligned} PRu^0 &= R'P(u^0) \\ &= R'(S) + R'(f^0). \end{aligned}$$

Now, since $\sigma \notin WF_{s+m-k-\frac{1}{2}}(\gamma_k u)$, we see by using the same argument as in the proof of (4.8), that, if $A_0 = I$ in a small conic neighborhood of σ: $A_0 S_{\alpha',j,k} \in H_{loc}^{\sigma,\tau}(\mathbf{R}^n)$ if $\sigma < -\frac{j+1}{2}$ and $\sigma + \tau = s + m - k - |\alpha'| - j - 1$; thus:

$$A_0 S \in H_{loc}^{\sigma,\tau}(\mathbf{R}^n) \quad \text{if} \quad \begin{cases} \sigma < -\dfrac{j+1}{2} \\ \sigma + \tau = s. \end{cases}$$

Writing now:

$$AR'S = AR'(A_0 S) + AR'(I - A_0) S$$

and taking the conic support of A small enough, the operator defined by $AR'(I - A_0)$ is regulari is regularizing near $\partial\Omega$; thus:

$$AR'(I - A_0)\, S \in C^\infty([-\epsilon, \epsilon] \times \partial\Omega) \quad \text{for some} \quad \epsilon > 0 .$$

On the other hand, we have $AR'A_0 = AR'$ since $AA_0 = A$ and R' is smoothing near $N_0^*\partial\Omega$, so we finally have:

$$(5.5) \qquad AR'(S) \in H^{\sigma,\tau}_{loc}(\mathbf{R}^n) \quad \text{if} \quad \sigma + \tau = s .$$

Making the same reasonment for Rf^0, we get similarly:

$$(5.6) \qquad AR'(f^0) \in H^{\sigma,\tau}_{loc}(\mathbf{R}^n) \quad \text{if} \quad \sigma + \tau = s .$$

Combining (5.5) and (5.6), we see, by taking $\tau = 0$, $\sigma = s$, that $(x_0, \xi_0) = (x'_0, x_{n_0}, \xi'_0, \xi_{n_0}) \notin WF_s(PR(u^0))$ if $|x_{n_0}| < \epsilon$, and $\xi_{n_0} \in \mathbf{R}$, with $(x'_0, \xi'_0) = \sigma$. Let us now choose a (null) bicharacteristic α passing through such a point, and let t_1, t_2 be real numbers such that $\alpha(t_1) \in C\overline{\Omega}$, $\alpha(t_2) \in \Omega$; hence the bicharacteristic segment $\alpha([t_1, t_2])$ cuts the boundary $\partial\Omega$. Since $R(u^0)_{|C\overline{\Omega}} \in C^\infty(C\overline{\Omega})$, Hörmander's well-known theorem of propagation of the regularity (Hörmander [9], th. (3.5.1)) shows that no point of $\alpha([t_1, t_2])$ belongs to $WF(Ru^0)$; in particular $(x_0, \xi_0) \notin WF(Ru^0)$; since this is true for all the (x_0, ξ_0) satisfying the condition above, we have:

$$WF_{s+m-1}(Ru^0) \cap \Gamma_\epsilon = \emptyset$$

where $\Gamma_\epsilon = \{(x, \xi)/(x', \xi') \in \gamma,\ \xi_n \in \mathbf{R},\ |x_n| < \epsilon\}$; since on the other hand $WF(Ru^0) \cap N_0^*(\partial\Omega) = \emptyset$, we get:

$$ARu^0 \in H^{s+m-1}_{loc}([-\epsilon, \epsilon] \times \partial\Omega)$$

and, in view of an easy extention of the proof of (2.10) :

$$ARu^0 \in H^{s+m-1+\infty,-\infty}_{loc}([-\epsilon, \epsilon] \times \partial\Omega)$$

that is, finally: $\sigma \notin \partial WF(\varphi R u^0_{|\Omega})$ for every $\varphi \in C_0^\infty([0, \epsilon] \times \partial\Omega)$, and this ends the proof of (5.1).

BIBLIOGRAPHY

[1] K.G. ANDERSSON, R.B. MELROSE, The propagation of singularities along gliding rays, *Invent. Math.*, 41 (1977).

[2] L. BOUTET DE MONVEL, Boundary problems for pseudo differential operators, *Acta. Math.*, 126 (1971).

[3] J. CHAZARAIN, Reflection of C^∞ singularities for a class of operators with multiple characteristics, Rims – Kyoto University, vol. 12, supp. 1977.

[4] M. DE GOSSON, Hypoellipticité partielle à la frontière pour les opérateurs pseudo-différentiels de transmission, *Annali di Mat. Pura ed Appl.* serie IV, t. cxxiii (1980).

[5] M. DE GOSSON, Parametrix de transmission pour des opérateurs de type parabolique etc, *C.R. Acad. Sc.*, Paris, t. 292.

[6] M. DE GOSSON, Résultats microlocaux en hypoellipticité partielle à la frontière pour les O.P.D. de transmission, *C.R. Acad Sc.*, Paris, t. 292.

[7] L. HÖRMANDER, *Linear partial differential operators.* Springer Verlag, 1964.

[8] L. HÖRMANDER, Pseudo-differential operators and non-elliptic boundary problems. *Ann. Math.*, 83 (1966).

[9] L. HÖRMANDER, On the existence and the regularity of solutions of linear pseudo-differential equations, *L'Ens. Math.*, t. XVII, fasc. 2 (1972).

[10] LIONS, MAGENES, *Problèmes aux limites non homogènes et applications,* vol. I, Dunod, 1968.

[11] R.B. MELROSE, *Transformations of boundary problems,* Preprint, *Acta Math.* (1980).

[12] R.B. MELROSE, J. SJÖSTRAND, Singularities of boundary problems, I, *Comm. on Pure and Appl. Math.*, XXXI (1978).

[13] F. TREVES, *Linear partial differential equations with constant coefficients,* Gordon and breach, 1963.

[14] J. SJÖSTRAND, Operators of principal type with interior boundary conditions, *Acta Math.*, 130 (1973).

Manuscrit reçu le 26 août 1981
révisé le 7 décembre 1981.

Maurice DE GOSSON,
Department of Mathematics
University of Bayreuth
D – 8580 Bayreuth (RFA).