ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 28, nº 2 (1978), p. 85-105
<http://www.numdam.org/item?id=AIF_1978__28_2_85_0>

Ann. Inst. Fourier, Grenoble
28, 2 (1978), 85-105.

ON THE GREEN TYPE KERNELS ON THE HALF SPACE IN $\mathbf{R}^n$

by Masayuki ITÔ

1. Let $\mathbf{R}^n$ be the $n(\geqslant 2)$-dimensional Euclidian space and D be the half space $\{x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}^n;\ x_1>0\}$. For a point $x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}^n$, we write

$$\bar{x}=(-x_1, x_2,\ldots, x_n) \qquad \text{and} \qquad |x|=\left(\sum_{j=1}^{n} x_j^2\right)^{1/2}.$$

When $n\geqslant 3$, we put $G_2(x,y)=|x-y|^{2-n}-|x-\bar{y}|^{2-n}$ in $D\times D$. Then G_2 is the Green kernel on D. Analogously we set, for a number α with $0<\alpha<n$,

$$G_\alpha(x,y)=|x-y|^{\alpha-n}-|x-\bar{y}|^{\alpha-n}$$

in $D\times D$, and we call it the Green type kernel of order α on D. The following question was proposed to me in a letter by H. L. Jackson : Does G_α also satisfy the domination principle provided that $0<\alpha<2$.

This paper is inspired by this question. Let $C_c(D)$ and $C(D)$ be the usual topological vector space of real-valued continuous functions in D with compact support and the usual topological vector space of real-valued continuous functions in D, respectively. We set

$$C_c^+(D)=\{f\in C_c(D);\ f\geqslant 0\}$$

and $C^+(D)=\{f\in C(D);\ f\geqslant 0\}$. For a given Hunt convolution kernel $\varkappa$ on $\mathbf{R}^n$, we define the linear operator

$$V_\varkappa : C_c(D) \ni f \to (\varkappa * f - \varkappa * \bar{f})_D \in C(D) \ (^1),$$

where $\bar{f}$ is the reflection of f about the boundary ∂D of D and where $(\varkappa * f - \varkappa * \bar{f})_D$ is the restriction of

$$\varkappa * f - \varkappa * \bar{f}$$

to D. If $V_\varkappa$ is positive (that is, $f \geqslant 0 \Longrightarrow V_\varkappa f \geqslant 0$), we say that $V_\varkappa$ is the Green type kernel associated with $\varkappa$.

The purpose of this paper is to show the following two theorems.

Theorem 1. — *Let $\varkappa$ be a Hunt convolution kernel on $\mathbf{R}^n$ and $(\varkappa_p)_{p \geqslant 0}$ be the resolvent associated with $\varkappa$. Suppose that $\varkappa$ is symmetric with respect to ∂D. Then the following two conditions are equivalent:*

(1) *$V_\varkappa$ is a Hunt kernel on D.*

(2) *For each $p > 0$, $\frac{\partial}{\partial x_1} \varkappa_p \leqslant 0$ in the sense of distributions in D.*

Theorem 2. — *Let $\varkappa$ be a Dirichlet convolution kernel on $\mathbf{R}^n$ and α be the singular measure (the Lévy measure) associated with $\varkappa$. Suppose that $\varkappa$ is also symmetric with respect to ∂D. Then the following two conditions are equivalent:*

(1) *$V_\varkappa$ is a Dirichlet kernel on D.*

(2) *$\frac{\partial}{\partial x_1} \alpha \leqslant 0$ in the sense of distributions in D.*

This theorem gives immediately that the question raised by H. L. Jackson is affirmatively solved.

2. Let $\varkappa$ be a convolution kernel on $\mathbf{R}^n$ (2). Similarly we define $V_\varkappa$. When $V_\varkappa$ is positive, we set

$$\mathscr{D}^+(V_\varkappa) = \{f \in C^+(D);\ V_\varkappa f \in C^+(D)\},$$

where

$$V_\varkappa f(x) = \sup \{V_\varkappa g(x);\ g \in C_c^+(D),\ g \leqslant f\}$$

(1) An $f \in C_c(D)$ may be considered as a finite continuous function in $\mathbf{R}^n$ with compact support $\subset D$.

(2) In potential theory, a convolution kernel means a positive measure.

in D. Put $\mathcal{D}(V_\varkappa) = \{f \in C(D); f^+, f^- \in \mathcal{D}^+(V_\varkappa)\}$ and, for an $f \in \mathcal{D}(V_\varkappa)$, $V_\varkappa f = V_\varkappa f^+ - V_\varkappa f^-$. Then $V_\varkappa$ is a linear operator from $\mathcal{D}(V_\varkappa)$ into $C(D)$.

Lemma 3. — *Let $\varkappa$ and $\varkappa'$ be two convolution kernels on $\mathbf{R}^n$. Suppose that $\varkappa$ and $\varkappa'$ are symmetric with respect to ∂D and that the convolution $\varkappa * \varkappa'$ is defined. If $V_\varkappa$ is positive, then, for any $f \in C_c(D)$, $V_{\varkappa'} f \in \mathcal{D}(V_\varkappa)$ and*

$$V_\varkappa(V_{\varkappa'} f) = (\varkappa * \varkappa' * f - \varkappa * \varkappa' * \bar{f})_D .$$

Proof. — We may assume that $f \geqslant 0$. Since $\varkappa * \varkappa'$ is defined and $|V_{\varkappa'} f| \leqslant \varkappa' * f + \varkappa' * \bar{f}$, we have $V_{\varkappa'} f \in \mathcal{D}(V_\varkappa)$. Our convolution kernels $\varkappa$ and $\varkappa'$ being symmetric with respect to ∂D, $\varkappa * \bar{f}(\bar{x}) = \varkappa * f(x)$ and

$$\varkappa' * \bar{f}(\bar{x}) = \varkappa' * f(x).$$

For the sake of simplicity, we write $h(x) = V_{\varkappa'} f(x)$ in D and $h(x) = 0$ on $\mathbf{R}^n - D$. Then, for a $g \in C_c^+(D)$, we have

$$\begin{aligned}
\int V_\varkappa(V_{\varkappa'} f)(x) g(x)\, dx
&= \int (\varkappa * h(x) - \varkappa * \bar{h}(x)) g(x)\, dx \\
&= \int h(x) \check{\varkappa} * g(x)\, dx - \int \bar{h}(x) \check{\varkappa} * g(x)\, dx \\
&= \int_D (\varkappa' * f(x) - \varkappa' * \bar{f}(x)) \check{\varkappa} * g(x)\, dx \\
&\qquad - \int_{\mathbf{R}^n - D} (\varkappa' * \bar{f}(x) - \varkappa' * f(x)) \check{\varkappa} * g(x)\, dx \\
&= \int \varkappa' * f(x) \check{\varkappa} * g(x)\, dx - \int \varkappa' * \bar{f}(x) \check{\varkappa} * g(x)\, dx \\
&= \int \varkappa * \varkappa' * (f - \bar{f})(x) g(x)\, dx ,
\end{aligned}$$

where $\check{\varkappa}$ is the adjoint convolution kernel of $\varkappa$; that is, $\check{\varkappa}(E) = \varkappa(\{-x;\, x \in E\})$ for any Borel set E. Since g is arbitrary, we obtain the required equality.

Remark 4. — In the above lemma, we have $V_\varkappa f \in \mathcal{D}(V_{\varkappa'})$ and $V_\varkappa(V_{\varkappa'} f) = V_{\varkappa'}(V_\varkappa f)$ provided that $V_{\varkappa'}$ is also positive.

Lemma 5. — *Let $\varkappa$ be a convolution kernel on $\mathbf{R}^n$. Suppose that $\varkappa$ is symmetric with respect to ∂D. Then $V_\varkappa$ is positive if and only if $\dfrac{\partial}{\partial x_1} \varkappa \leqslant 0$ in the sense of distributions in D.*

Proof. — First we shall show the « if » part. For a $t \in (0, \infty)$, put $H_t = \{x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n;\ x_1 = t\}$ and

$$D' = \left\{ x = (x_1, x_2, \cdots, x_n) \in D;\ \int_{H_{2x_1}} d\varkappa = 0 \right\}.$$

It suffices to prove that, for any $f \in C_c^+(D)$ and any $x \in D'$, $\varkappa * f(x) \geqslant \varkappa * f(\bar{x})$, because $\int_{D-D'} dx = 0$ and

$$\varkappa * f(\bar{x}) = \varkappa * \bar{f}(x).$$

We choose a sequence $(\varphi_k)_{k=1}^{\infty}$ of non-negative, spherically symmetric and infinitely differentiable functions such that $\int \varphi_k\, dx = 1$ and that the support of φ_k, $\operatorname{supp}(\varphi_k)$, is contained in $\{x \in \mathbf{R}^n;\ |x| < 1/k\}$. Then $\varkappa * \varphi_k$ is symmetric with respect to ∂D and $\dfrac{\partial}{\partial x_1} \varkappa * \varphi_k(x) \leqslant 0$ in

$$\{x \in \mathbf{R}^n;\ x_1 \geqslant 1/k\}.$$

Let $f \in C_c^+(D)$ and $x = (x_1, x_2, \cdots, x_n) \in D'$. Then

$$\int_{|y_1 - x_1| \geqslant 1/m} f(y) \varkappa * \varphi_k(x - y)\, dy \geqslant \int_{|y_1 - x_1| \geqslant 1/m} f(y) \varkappa * \varphi_k(\bar{x} - y)\, dy$$

provided with $0 < m \leqslant k$. By letting $k \to \infty$ and $m \to \infty$, we obtain that

$$\begin{aligned}
\varkappa * f(x) &= \int f(y)\, d\check{\varkappa} * \varepsilon_x(y) \\
&\geqslant \int_{\mathbf{R}^n - H_{x_1}} f(y)\, d\check{\varkappa} * \varepsilon_x(y) \\
&\geqslant \int_{\mathbf{R}^n - H_{x_1}} f(y)\, d\check{\varkappa} * \varepsilon_{\bar{x}}(y) \\
&\geqslant \varkappa * f(\bar{x}) - \left(\sup_{z \in \mathbf{R}^n} |f(z)| \right) \int_{H_{2x_1}} d\varkappa = \varkappa * f(\bar{x})
\end{aligned}$$

where ε_x denote the unit measure at x. Since f and x are arbitrary, the « if » part is true.

Next we shall show the « only if » part. Suppose that the « only if » part is false. Then there exist a number $t > 0$, a point $x = (x_1, x_2, \cdots, x_n) \in D$ with $x_1 > t$ and a non-negative, spherically symmetric and infinitely differentiable function φ in $\mathbf{R}^n$ with $\operatorname{supp}(\varphi) \subset \{x \in \mathbf{R}^n;\ |x| < t\}$ such that $\dfrac{\partial}{\partial x_1} \varkappa * \varphi(x) > 0$. Hence we can choose a number

$s > 0$ such that $s < x_1 - t$ and that, for every $y \in D$ with $|y| < s$, $\varkappa * \varphi(x - y) < \varkappa * \varphi(x - \bar{y})$. Since

$$\varkappa * \varphi(x - \bar{y}) = \varkappa * \varphi(\bar{x} - y),$$

we have, for an $f \neq 0 \in C_c^+(D)$ satisfying

$$\operatorname{supp}(f) \subset \{y \in \mathbf{R}^n ;\ |y| < s\},$$
$$\varkappa * f * \varphi(x) < \varkappa * f * \varphi(\bar{x}) = \varkappa * \bar{f} * \varphi(x).$$

But this contradicts the inequality $\varkappa * f \geqslant \varkappa * \bar{f}$ in D. Thus we see that the « only if » part is true.

In the same manner as above, we obtain the following

Lemma 6. — *Let* α *be a positive measure in* $\mathbf{R}^n - \{0\}$. *Suppose that* α *is symmetric with respect to* ∂D. *If* $\dfrac{\partial}{\partial x_1}\alpha \leqslant 0$ *in the sense of distributions in* D, *then, for any* $f \in C_c^+(D)$,

$$\int f(x - y)\, d\alpha(y) \geqslant \int \bar{f}(x - y)\, d\alpha(y)$$

in $D \cap \complement \operatorname{supp}(f)$.

3. We say that a convolution kernel $\varkappa$ on $\mathbf{R}^n$ is a Hunt convolution kernel if $\varkappa = \int_0^\infty \alpha_t\, dt$, where $(\alpha_t)_{t \geqslant 0}$ is a vaguely continuous semi-group of positive measures in $\mathbf{R}^n$; that is, $\alpha_0 = \varepsilon$ (the Dirac measure), $\alpha_t * \alpha_s = \alpha_{t+s}$ $(\forall t \geqslant 0,\ \forall s \geqslant 0)$ and the application $\mathbf{R}^+ = [0, \infty) \ni t \to \alpha_t$ is vaguely continuous. In this case, $(\alpha_t)_{t \geqslant 0}$ is uniquely determined (see, for example, [3]) and called the vaguely continuous semi-group associated with $\varkappa$. For a $p \in \mathbf{R}^+$, put

$$\varkappa_p = \int_0^\infty \exp(-pt)\alpha_t\, dt\ ;$$

then $(\varkappa_p)_{p \geqslant 0}$ is called the resolvent associated with $\varkappa$. This is characterized by a family $(\varkappa_p)_{p \geqslant 0}$ of convolution kernels on $\mathbf{R}^n$ satisfying

$$\varkappa_p - \varkappa_q = (q - p)\varkappa_p * \varkappa_q (\forall p \geqslant 0,\ \forall q > 0)$$

and $\lim_{p \to 0} \varkappa_p = \varkappa_0 = \varkappa$ (vaguely).

LEMMA 7 (see [3] or Theorem 5 in [6]). — *Let* $\varkappa$, $(\alpha_t)_{t\geq 0}$ *and* $(\varkappa_p)_{p\geq 0}$ *be the same as above. For a* $p > 0$ *and a* $t > 0$, *put*

$$\alpha_{p,t} = \exp(-pt) \sum_{k=0}^{\infty} \frac{p^k t^k}{k!} (p\varkappa_p)^k \qquad \text{and} \qquad \alpha_{p,0} = \varepsilon ;$$

then $(\varkappa_{p,t})_{t\geq 0}$ *is a vaguely continuous semi-group of positive measures and we have*

$$\varkappa + \frac{1}{p}\varepsilon = \int_0^\infty \alpha_{p,t}\, dt \quad \text{and} \quad \lim_{p\to\infty} \alpha_{p,t} = \alpha_t \quad (\text{vaguely}) \quad (t \geq 0).$$

LEMMA 8. — *Let* $\varkappa = \int_0^\infty \alpha_t\, dt$ *be a Hunt convolution kernel on* $\mathbf{R}^n$ *and* $(\varkappa_p)_{p\geq 0}$ *be the resolvent associated with* $\varkappa$. *If* $\varkappa$ *is symmetric with respect to* ∂D , *then, for any* p *and any* t , $\varkappa_p$ *and* α_t *are also symmetric with respect to* ∂D .

Proof. — For a $p \geq 0$, we denote by $\bar{\varkappa}_p$ the reflection of $\varkappa_p$ about ∂D . Evidently $(\bar{\varkappa}_p)_{p\geq 0}$ is the resolvent associated with $\bar{\varkappa}$. By using $\varkappa = \bar{\varkappa}$ and the unicity of the resolvent associated with $\varkappa$, we have, for each $p \geq 0$, $\varkappa_p = \bar{\varkappa}_p$. This means that $\varkappa_p$ is symmetric with respect to ∂D . This gives also that, for any $f \in C_c(D)$,

$$\int_0^\infty \exp(-pt) f\, d\alpha_t\, dt = \int_0^\infty \exp(-pt) \bar{f}\, d\alpha_t\, dt \quad (\forall p \geq 0).$$

The Laplace transformation being injective, we have, for each $t \geq 0$, $\int f\, d\alpha_t = \int \bar{f} d\alpha_t$. Hence, f being arbitrary, we see that α_t is symmetric with respect to ∂D .

Similarly we have the following

Remark 9. — If $\varkappa$ is symmetric with respect to the origin 0 (resp. spherically symmetric), then $\varkappa_p$ and α_t are also symmetric with respect to 0 (resp. spherically symmetric).

Let $\varkappa$ be a convolution kernel on $\mathbf{R}^n$. We say that $\varkappa$ is a Dirichlet convolution kernel if the (generalised) Fourier transformation $\hat{\varkappa}$ of $\varkappa$ is defined and equal to $\frac{1}{\psi}$, where ψ is a real-valued negative definite function in $\mathbf{R}^n$ such that $\frac{1}{\psi}$

is locally summable. By virtue of the Lévy-Khinchine theorem, we have, for any $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$,

$$\psi(x) = c + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \int (1 - \cos(2\pi x \cdot y))\, d\alpha(y),$$

where c is a non-negative constant, $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ is a positive semi-definite form, $x \cdot y$ is the inner product in $\mathbf{R}^n$ and where α is a positive measure in $\mathbf{R}^n - \{0\}$ symmetric with respect to 0 and satisfying $\int |x|^2/(1+|x|^2)\, d\alpha(x) < \infty$. It is well-known that the above decomposition of ψ is unique. The positive measure α in $\mathbf{R}^n - \{0\}$ is called the *singular measure* associated with $\varkappa$. Since, for each $t \geqslant 0$, $\exp(-t\psi)$ is of positive type in $\mathbf{R}^n$, there exists a positive measure α_t in $\mathbf{R}^n$ such that $\hat{\alpha}_t = \exp(-t\psi)$. Evidently $(\alpha_t)_{t \geqslant 0}$ is a vaguely continuous semi-group of positive measures and $\varkappa = \int_0^\infty \alpha_t\, dt$. Hence a Dirichlet convolution kernel is a Hunt convolution kernel and symmetric with respect to 0.

4. A positive linear operator $V : C_c(D) \to C(D)$ is called a continuous kernel on D (Evidently V is continuous). Similarly as in the section 2, we define $\mathscr{D}^+(V)$ and $\mathscr{D}(V)$. We say that V is a Hunt kernel on D if $V = \int_0^\infty \tilde{V}_t\, dt$ (that is, for any $f \in C_c(D)$, $Vf(x) = \int_0^\infty \tilde{V}_t f(x)\, dt$ in D), where $(\tilde{V}_t)_{t \geqslant 0}$ is a continuous semi-group of continuous kernels on D; that is, $\tilde{V}_0 = I$ (the identity), for any $t \geqslant 0$, $s \geqslant 0$ and any $f \in C_c(D)$, $\tilde{V}_t f \in \mathscr{D}(\tilde{V}_s)$, $\tilde{V}_s(\tilde{V}_t f) = \tilde{V}_t(\tilde{V}_s f) = \tilde{V}_{t+s} f$ and the application $\mathbf{R}^+ \ni t \to \tilde{V}_t f$ is continuous in $C(D)$. Similarly as in [3], we see that $(\tilde{V}_t)_{t \geqslant 0}$ is uniquely determined, and we call it the continuous semi-group associated with V. For a $p \geqslant 0$, put $V_p = \int_0^\infty \exp(-pt) V_t\, dt$; then we call $(V_p)_{p \geqslant 0}$ the resolvent associated with V. It is known that, for any $p \geqslant 0$, $q > 0$ and any $f \in C_c(D)$, $V_p f \in \mathscr{D}(V_q)$, $V_q f \in \mathscr{D}(V_p)$,

$$V_p f - V_q f = (q - p) V_q(V_p f) = (q - p) V_p(V_q f)$$

(the resolvent equation) and $\lim_{p \to 0} V_p f = V_0 f = Vf$ in $C(D)$.

Let V_1 and V_2 two continuous kernels on D. If, for any $f \in C_c(D)$, $V_2 f \in \mathcal{D}(V_1)$, the application $C_c(D) \ni f \to V_1(V_2 f)$ is positive linear, we denote it by $V_1 \cdot V_2$.

Remark 10 (see [2]). — A Hunt kernel V on D satisfies the domination principle; that is, for two $f, g \in C_c^+(D)$, $Vf \leqslant Vg$ on $\operatorname{supp}(f)$ implies the same inequality on D.

5. We shall show Theorem 1 mentioned in the section 1.

(1) $\Longrightarrow$ (2). By Lemmas 5 and 8, it suffices to prove that, for each $p > 0$, $V_{\varkappa_p}$ is positive. Let $(V_p)_{p \geqslant 0}$ be the resolvent associated with $V_\varkappa$. Then, for an $f \in C_c^+(D)$ and a $p > 0$, $V_\varkappa f = (pV_\varkappa + I)(V_p f)$. On the other hand, Lemmas 3 and 8 give the $V_{\varkappa_p} f \in \mathcal{D}(V_\varkappa)$ and

$$V_\varkappa f = (\varkappa * (f - \bar{f}))_D = ((p\varkappa + \varepsilon) * \varkappa_p * (f - \bar{f}))_D = (pV_\varkappa + I)(V_{\varkappa_p} f).$$

By using the resolvent equation, we have

$$V_p f - V_{\varkappa_p} f = (I - pV_p)((pV_\varkappa + I)(V_p f - V_{\varkappa_p} f)) = 0 .$$

The function f being arbitrary, we have $V_p = V_{\varkappa_p}$, and hence $V_{\varkappa_p}$ is positive.

(2) $\Longrightarrow$ (1). By Lemma 5, $V_{\varkappa_p}$ is positive $(\forall p > 0)$. Let $\alpha_{p,t}$ be the positive measure defined in Lemma 7 $(\forall p > 0, \forall t \geqslant 0)$ and $(\alpha_t)_{t \geqslant 0}$ be the vaguely continuous semi-group associated with $\varkappa$. By Lemmas 3 and 7,

$$V_{\alpha_{p,t}} = \exp(-pt) \sum_{k=0}^{\infty} \frac{p^k t^k}{k!} (pV_{\varkappa_p})^k,$$

where $(pV_{\varkappa_p})^0 = I$, $(pV_{\varkappa_p})^1 = pV_{\varkappa_p}$ and

$$(pV_{\varkappa_p})^{n+1} = (pV_{\varkappa_p})^n \cdot (pV_{\varkappa_p}).$$

Therefore $V_{\alpha_{p,t}}$ is positive. From Lemma 7, it follows that, for any $f \in C_c(D)$, $\lim_{p \to \infty} V_{\alpha_{p,t}} f = V_{\alpha_t} f$ in $C(D)$ $(\forall t \geqslant 0)$. Hence V_{α_t} is positive. By using Lemma 3, we see that $(V_{\alpha_t})_{t \geqslant 0}$ is a continuous semi-group of continuous kernels on D and that $V_\varkappa = \int_0^\infty V_{\alpha_t}\, dt$. Consequently $V_\varkappa$ is a Hunt kernel on D. This completes the proof.

Question 11. — Let $\varkappa$ be a Hunt convolution kernel on $\mathbf{R}^n$ satisfying $\varkappa = \bar{\varkappa}$. Is it true that $V_\varkappa$ is a Hunt kernel on D provided that $V_\varkappa$ is positive?

Remark 12. — Let $k(x)$ be a non-negative continuous function in the wide sense in $\mathbf{R}^n$ satisfying $k(x) = k(\bar{x})$. Suppose that $\varkappa = k(x)\,dx$ is a Hunt convolution kernel and that $V_\varkappa$ is also a Hunt kernel on D. Put

$$G(x,y) = k(x-y) - k(x-\bar{y}) \quad \text{in} \quad D \times D.$$

If the function kernel $k(x-y)$ satisfies the continuity principle [3], then G satisfies the domination principle; that is, for two positive measures μ and ν in D with compact support and with $\int G\mu\, d\mu < \infty$, then $G\mu \leqslant G\nu$ on supp (μ) implies the same inequality in D, where

$$G\mu(x) = \int G(x,y)\, d\mu(y).$$

It is known that $k(x-y)$ satisfies the continuity principle when $\varkappa$ is a Dirichlet convolution kernel (see [4]).

We show this remark. We see that G also satisfies the continuity principle. Therefore it suffices to prove that, for a positive measure μ in D with compact support and an $x \in D$, $G\mu \leqslant G\varepsilon_x$ in D provided that $G\mu \leqslant G\varepsilon_x$ on supp (μ) and that $G\mu$ is finite continuous (see [8]). Since $V_\varkappa$ is a Hunt kernel, there exists $f \in C_c^+(D)$ such that $V_\varkappa f = Gf \geqslant 1$ on supp (μ), where $Gf(y) = \int G(y,z) f(z)\, dz$. Here we remark that μ is considered as a positive measure in $\mathbf{R}^n$. For a given positive number δ, there exists a neighborhood U of 0 such that, for any finite continuous function $\varphi \geqslant 0$ in $\mathbf{R}^n$ with supp $(\varphi) \subset U$ with $\int \varphi\, dx = 1$, $\mu * \varphi$, $\varepsilon_x * \varphi \in C_c^+(D)$ and $G(\mu * \varphi) \leqslant G(\varepsilon_x * \varphi) + \delta Gf$ on supp $(\mu * \varphi)$. By letting $\varphi\, dx \to \varepsilon$ (vaguely) and $\delta \downarrow 0$, we have $G\mu \leqslant G\varepsilon_x$.

(3) This means that, for a positive measure μ in $\mathbf{R}^n$ with compact support, the function $\int k(x-y)\, d\mu(y)$ of x is finite continuous provided that its restriction to supp (μ) is finite continuous.

6. Theorem 1 gives the following

COROLLARY 13. — *Let* $\varkappa = \int_0^\infty \alpha_t \, dt$ *be a Hunt convolution kernel on* $\mathbf{R}^n$. *Then* $\varkappa$ *is symmetric with respect to* ∂D *and* $V_\varkappa$ *is a Hunt kernel on* D *if and only if, for each* $t \geqslant 0$, α_t *is symmetric with respect to* ∂D *and* $\frac{\partial}{\partial x_1} \alpha_t \leqslant 0$ *in the sense of distribution in* D.

COROLLARY 14. — *Let* $\varkappa = \int_0^\infty \alpha_t \, dt$ *be a Hunt convolution kernel on* $\mathbf{R}^n$ *and* μ *be a Hunt convolution kernel on* $\mathbf{R}^1$ *supported by* $\mathbf{R}^+$. *Suppose that* $\varkappa_\mu = \int_0^\infty \alpha_t \, d\mu(t)$ *is defined (in the sense of measures) and that* $\varkappa$ *is symmetric with respect to* ∂D. *If* $V_\varkappa$ *is a Hunt kernel on* D, *then* $V_{\varkappa_\mu}$ *is also a Hunt kernel on* D.

Proof. — We denote by $(\mu_p)_{p \geqslant 0}$ the resolvent associated with μ. Since $\mu_p \leqslant \mu$, $\varkappa_{\mu,p} = \int \alpha_t \, d\mu_p(t)$ is defined $(\forall p \geqslant 0)$. It is known that $\varkappa_\mu$ is a Hunt convolution kernel on $\mathbf{R}^n$ and that $(\varkappa_{\mu,p})_{p \geqslant 0}$ is the resolvent associated with $\varkappa_\mu$ (see Theorem 1 in [5]). By Theorem 1 and Corollary 13, α_t is symmetric with respect to ∂D and $\frac{\partial}{\partial x_1} \alpha_t \leqslant 0$ in the sense of distributions in D. Hence $\varkappa_\mu$ is also symmetric with respect to ∂D and $\frac{\partial}{\partial x_1} \varkappa_{\mu,p} \leqslant 0$ in the sense of distributions in D $(\forall p \geqslant 0)$. Consequently Theorem 1 gives this corollary.

In the same manner as above, we have the following

COROLLARY 15. — *Let* $(\alpha_t)_{t \geqslant 0}$ *be a vaguely continuous semi-group of positive measures in* $\mathbf{R}^n$ *and* μ *be a Hunt convolution kernel on* $\mathbf{R}^1$ *supported by* $\mathbf{R}^+$. *Suppose that* $\int_0^\infty \alpha_t \, d\mu(t)$ *is defined and that, for each* $t \geqslant 0$, α_t *is symmetric with respect to* ∂D *and* $\frac{\partial}{\partial x_1} \alpha_t \leqslant 0$ *in the sense of distributions in* D. *Then* $V_{\varkappa_\mu}$ *is a Hunt kernel on* D, *where*

$$\varkappa_\mu = \int_0^\infty \alpha_t \, d\mu(t).$$

We shall show that the question raised by H. L. Jackson is affirmatively solved.

Remark 16. — Let ν be a positive measure in $(0, 2)$ such that $\int_0^2 \frac{1}{\alpha}\, d\nu(\alpha) < \infty$ and c_0, c_1 be non-negative constants. Put

$$\varkappa = \begin{cases} c_0\varepsilon + \left(\int |x|^{\alpha-n}\, d\nu(\alpha)\right) dx & \text{if } n = 2 \\ c_0\varepsilon + \left(\int |x|^{\alpha-n} d\nu(\alpha) + c_1|x|^{2-n}\right) dx & \text{if } n \geqslant 3. \end{cases}$$

Then $V_\varkappa$ is a Hunt kernel.

In fact, we have, with a positive constant $c(\alpha)$,

$$|x|^{\alpha-n} = c(\alpha) \int_0^\infty \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{2t}\right) t^{\alpha/2-1}\, dt$$

$(0 < \alpha < 2$ if $n = 2$, $0 < \alpha \leqslant 2$ if $n \geqslant 3)$. Evidently the function $c(\alpha)$ of α is finite continuous. Put

$$\mu = \begin{cases} c_0\varepsilon + \left(\int c(\alpha) t^{\alpha/2-1}\, d\nu(\alpha)\right) dt & \text{if } n = 2 \\ c_0\varepsilon + \left(\int c(\alpha) t^{\alpha/2-1}\, d\nu(\alpha) + c_1 c(2)\right) dt & \text{if } n \geqslant 3 \end{cases}$$

in $\mathbf{R}^1$. Since $\int_0^2 \frac{1}{\alpha}\, d\nu(\alpha) < \infty$, $\varkappa_\mu$ is a convolution kernel on $\mathbf{R}^n$ and

$$\varkappa_\mu = \left(\int \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{2t}\right) d\mu(t)\right) dx .$$

Hence μ is a convolution kernel on $\mathbf{R}^1$ supported by $\mathbf{R}^+$. Then μ is a Hunt convolution kernel on $\mathbf{R}^1$ (cf. [5]), and Corollary 14 gives our remark.

Let G_α be the Green type kernel of order α in D. Put

$$G(x,y) = \begin{cases} \int G_\alpha(x,y)\, d\nu(\alpha) & \text{if } n = 2 \\ \int G_\alpha(x,y)\, d\nu(\alpha) + c_1 G_2(x,y) & \text{if } n \geqslant 3. \end{cases}$$

Then Remarks 12 and 16 give that G satisfies the domination principle.

7. Let $L_{\mathrm{loc}}(D)$ be the usual Fréchet space of real-valued locally summable functions in D. A Hilbert space $H(D)$

contained in $L_{loc}(D)$ is called a Dirichlet space on D if the following three conditions are satisfied:

(1) For each compact set K in D, there exists a constant $A(K) > 0$ such that, for any $u \in D$, $\int_K |u|\, dx \leqslant A(K)\|u\|$.

(2) $C_c(D) \cap H(D)$ is dense both in $C_c(D)$ and in $H(D)$.

(3) For any normalized contraction T on $\mathbf{R}^1$ (4) and any $u \in H(D)$, $T \cdot u \in H(D)$ and $\|T \cdot u\| \leqslant \|u\|$.

This is the definition by A. Beurling and J. Deny (see [1]). Here we denote by $\|\cdot\|$ and by $(\cdot,\cdot)$ the norm in $H(D)$ and the associated inner product, respectively. For an $f \in C_c(D)$, (1) gives that there exists uniquely $u_f \in H(D)$ such that, for any $u \in H(D)$, $(u_f,u) = \int uf\, dx$.

Let V be a linear operator from $C_c(D)$ into $L_{loc}(D)$. We say that V is a Dirichlet kernel on D if there exists a Dirichlet space $H(D; V)$ on D such that, for any

$$f \in C_c(D), \qquad Vf = u_f.$$

Evidently $H(D; V)$ is uniquely determined. We call $H(D; V)$ the Dirichlet space associated with V and V the kernel of $H(D; V)$. For a Dirichlet kernel V on D, we set

$$\mathscr{D}(V) = \left\{ f \in L_{loc}(D); \quad \sup \left\{ \frac{\left|\int uf\, dx\right|}{\|u\|}; \; u \neq 0 \in C_c(D) \cap H(D;\, V) \right\} < \infty \right\}$$

and $\mathscr{D}^+(V) = \{f \in \mathscr{D}(V);\ f \geqslant 0\}$, where $\|\cdot\|$ denote the norm in $H(D; V)$. By virtue of (2), for an $f \in \mathscr{D}(V)$, there exists uniquely $Vf \in H(D; V)$ such that, for any

$$u \in C_c(D) \cap H(D;\, V), \quad (Vf,u) = \int uf\, dx,$$

where $(\cdot,\cdot)$ denote the inner product in $H(D; V)$. Thus V may be considered as a linear operator from $\mathscr{D}(V)$ into $H(D; V)$. It is known that V is positive (that is,

$$f \in \mathscr{D}^+(V) \Longrightarrow Vf \geqslant 0 \text{ a.e.}) \quad \text{(see [1])}.$$

(4) This means that T is an application: $\mathbf{R}^1 \to \mathbf{R}^1$ such that $\mathbf{R}(0) = 0$ and $|Ta - Tb| \leqslant |a - b|$ $(\forall a, \forall b \in \mathbf{R}^1)$.

LEMMA 17. — *Let* $\varkappa$ *be a Hunt convolution kernel on* $\mathbf{R}^n$ *satisfying* $\varkappa = \bar{\varkappa}$. *If* $V_\varkappa$ *is a Dirichlet kernel on* D , *then* $V_\varkappa$ *is a Hunt kernel.*

Proof. — For the sake of simplicity, we write $H = H(D; V_\varkappa)$. Denote by $\|\cdot\|$ and by $(\cdot,\cdot)$ the norm in H and the inner product in H , respectively. Let $L^2(D)$ be the Hilbert space of real-valued square summable functions in D . For a $p \geqslant 0$, H_p denotes the Hilbert space associated to the norm $\|u\|_p = \left(p\int |u|^2\,dx + \|u\|^2\right)^{1/2}$ on $H \cap L^2(D)$. Evidently H_p is a Dirichlet space on D . Let $f \in C_c(D)$. For any $u \in C_c(D) \cap H$, we have

$$\begin{aligned}\int V_pf(x)u(x)\,dx &= \frac{1}{p}\left((V_pf,u)_p - (V_pf,u)\right)\\ &= \frac{1}{p}\left((V_\varkappa f,u) - (V_pf,u)\right)\\ &\leqslant \frac{1}{p}\left(\|V_\varkappa f\| + \|V_pf\|\right)\|u\|,\end{aligned}$$

where V_p is the kernel of H_p and where $(\cdot,\cdot)_p$ is the inner product in H_p. Hence $V_pf \in \mathcal{D}(V)$. Since, for any $u \in C_c(D) \cap H$,

$$\begin{aligned}p(V_\varkappa(V_pf),u) &= p\int u(x)V_pf(x)\,dx\\ &= (V_pf,u)_p - (V_pf,u) = (V_\varkappa f - V_pf,u),\end{aligned}$$

(2) gives $V_\varkappa f - V_pf = pV_\varkappa(V_pf)$ a.e. in D . Let $(\varkappa_p)_{p\geqslant 0}$ be the resolvent associated with $\varkappa$. By Lemmas 3 and 8, we have $V_\varkappa f - V_{\varkappa_p}f = pV_\varkappa(V_{\varkappa_p}f)$. In the same manner as in the proof of Theorem 1, we have $V_pf = V_{\varkappa_p}f$ a.e. in D , and hence $V_{\varkappa_p}$ is positive ($\forall p > 0$) . By Theorem 1 and Lemma 5, we see that $V_\varkappa$ is a Hunt kernel.

We shall prove Theorem 2 mentioned in the section 1.

(1) $\Longrightarrow$ (2). Let $(\varkappa_p)_{p\geqslant 0}$ be the resolvent associated with $\varkappa$. Then it is known that $p^2\varkappa_p \to \alpha$ vaguely in $\mathbf{R}^n - \{0\}$ as $p \to \infty$ (see [1]), and hence theorem 1 and Lemma 17 give that $\frac{\partial}{\partial x_1}\alpha \leqslant 0$ in the sense of distributions in D .

(2) $\Longrightarrow$ (1). Since $p^2\varkappa_p \to \alpha$ vaguely in $\mathbf{R}^n - \{0\}$ as $p \to \infty$, Lemma 8 gives that α is symmetric with respect to ∂D . Let A be the diagonal set of $D \times D$ and β be the

positive measure in $D \times D - A$ defined by

$$\iint f(x)g(y)\, d\beta(x,y) = \iint (f(x-y) - \bar{f}(x-y))g(x)\, d\alpha(y)\, dx$$

for any couple f, $g \in C_c(D)$ with $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$ (see Lemma 6). For any p, $\varkappa_p$ being symmetric with respect to the origin, we have $\alpha = \check{\alpha}$, and hence β is symmetric with respect to A. Let $C_c^\infty(D)$ be the topological vector space of real-valued and infinitely differentiable functions in D with compact support (we identify an element of $C_c^\infty(D)$ and an infinitely differentiable function in $\mathbf{R}^n$ with compact support in D).

Let $f \in C_c^\infty(D)$. Consider the approximation of the function $|f(x) - f(y)|^2$ of (x,y) by the functions of form $\sum_i \varphi_i(x)\psi_i(y)$ in $D \times D$, where $\varphi_i \in C_c^\infty(D)$ and $\psi_i \in C_c^\infty(D)$ with

$$\mathrm{supp}(\varphi_i) \cap \mathrm{supp}(\psi_i) = \emptyset .$$

Then we see that

$$\begin{aligned} 0 \leqslant \iint |f(x) - f(y)|^2\, d\beta(x,y) + \int |f(x)|^2 a(x)\, dx \\ = \iint |f(x-y) - f(x)|^2\, d\alpha(y)\, dx \\ - \iint (\bar{f}(x-y) - \bar{f}(x))(f(x-y) - f(x))\, d\alpha(y)\, dx < \infty \ (^5) \end{aligned}$$

where, for $x = (x_1, x_2, \cdots, x_n) \in D$, $a(x) = 2\int_{|y_1| \geqslant x_1} d\alpha(y)$. Let $\tilde{H}$ be the specialized Dirichlet space with the kernel $\varkappa$ (see [1]). We denote by $|||\cdot|||$ and by $((\cdot,\cdot))$ the norm in $\tilde{H}$ and the associated inner product. For a couple f, $g \in C_c^\infty(D)$, we put

$$\begin{aligned} (f,g) = \int fg\left(\frac{a}{2} + c\right) dx + \frac{1}{4\pi^2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \int \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}\, dx \\ + \frac{1}{2} \iint (f(x) - f(y))(g(x) - g(y))\, d\beta(x,y) \\ = ((f - \bar{f}, g)) = ((f, g - \bar{g})) = \frac{1}{2}((f - \bar{f}, g - \bar{g})), \end{aligned}$$

(5) The author would like to express his hearty thanks to Prof. F. Hirsch for the correction of this formula.

where $\hat{\varkappa} = \left(c + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_i x_j + \int (1 - \cos(2\pi x\cdot y))\, d\alpha(y)\right)^{-1}$.
Then $(\cdot,\cdot)$ is an inner product in $C_c^\infty(D)$. For a compact set K in D, we have

$$\sup_{\substack{u\in C_c^\infty(D)\\ u\neq 0}} \frac{\int_K |u|\, dx}{\|u\|} = \sup_{\substack{u\in C_c^\infty(D)\\ u\neq 0}} \frac{\sqrt{2}\int_K |u-\bar{u}|\, dx}{|||u-\bar{u}|||} < \infty ,$$

where $\|u\| = (u,u)^{1/2}$. Hence the completion H of $C_c^\infty(D)$ by $\|\cdot\|$ is contained in $L_{loc}(D)$. Evidently, for any $u \in C_c^\infty(D)$ and any normalized contraction T on $\mathbf{R}^1$, $T\cdot u \in H$ and $\|T\cdot u\| \leqslant \|u\|$. For a $u \in H$, we choose a sequence $(u_k)_{k=1}^\infty \subset C_c^\infty(D)$ such that

$$\lim_{k\to\infty} \|u_k - u\| = 0 .$$

Since $(T\cdot u_k)_{k=1}^\infty$ converges weakly to $T\cdot u$ in H as $k \to \infty$ (see [1]), we have $T\cdot u \in H$ and $\|T\cdot u\| \leqslant \|u\|$. Hence H is a Dirichlet space on D. We shall show that $V_\varkappa$ is the kernel of H. For an integer $m \geqslant 1$, let T_m denote the projection from $\mathbf{R}^1$ into $\left[-\frac{1}{m}, \frac{1}{m}\right]$. Let $f \in C_c(D)$; then $\varkappa * (f - \bar{f}) - T_m \cdot \varkappa * (f-\bar{f}) \in \tilde{H}$ and

$$V_\varkappa f - T_m \cdot V_\varkappa f \in C_c(D),$$

because $\varkappa * (f-\bar{f}) = 0$ on ∂D and $\lim_{|x|\to\infty} \varkappa * (f-\bar{f})(x) = 0$. Therefore there exists a neighborhood V_m of the origin such that, for any non-negative, spherically symmetric and infinitely differentiable function φ in $\mathbf{R}^n$ with $\operatorname{supp}(\varphi) \subset V_m$ and $\int \varphi\, dx = 1$, $f * \varphi \in C_c^\infty(D)$ and

$$(V_\varkappa f - T_m \cdot V_\varkappa f) * \varphi \in C_c^\infty(D).$$

Since

$$(\varkappa * (f-\bar{f}) - T_m \cdot \varkappa * (f-\bar{f})) * \varphi = (V_\varkappa f - T_m \cdot V_\varkappa f) * \varphi - \overline{(V_\varkappa f - T_m \cdot V_\varkappa f) * \varphi}$$

and, for a $u \in \tilde{H}$,

$$|||u*\varphi|||^2 = \iint ((u*\varepsilon_x, u*\varepsilon_y))\varphi(x)\varphi(y)\, dx\, dy \leqslant |||u|||^2,$$

we have

$$\|(V_\varkappa f - T_m \cdot V_\varkappa f) * \varphi\|^2$$
$$\leqslant \frac{1}{2}|||\varkappa * (f - \bar{f}) - T_m \cdot \varkappa * (f - \bar{f})|||^2 \leqslant 2|||\varkappa * (f - \bar{f})|||^2.$$

By letting $\varphi\, dx \to \varepsilon$ (vaguely) and $m \to \infty$, we see that $V_\varkappa f \in H$ and, for any $u \in C_c^\infty(D)$,

$$(V_\varkappa f, u) = ((\varkappa * (f - \bar{f}), u)) = \int u(f - \bar{f})\, dx = \int uf\, dx .$$

This implies immediately that, for any $u \in H$,

$$(V_\varkappa f, u) = \int uf\, dx .$$

Consequently $V_\varkappa$ is the kernel of the Dirichlet space H. This completes the proof.

Theorem 2 gives also that the question raised by H. L. Jackson is affirmatively solved. In fact, the singular measure associated with the convolution kernel $r^{\alpha-n}$ is equal to $c_\alpha|x|^{-\alpha-n}\, dx$ provided that $0 < \alpha < 2$, where c_α is a positive constant, where $|x|^{\alpha-n}\, dx$ is symbolically denoted by $r^{\alpha-n}$ $(0 < \alpha < n)$.

We denote now by Δ the laplacian on $\mathbf{R}^n$. We say that a convolution kernel $\varkappa$ on $\mathbf{R}^n$ is a Frostman-Kunugui kernel if $\varkappa$ is spherically symmetric, vanishes at infinity (6), and if $\Delta\varkappa \geqslant 0$ in the sense of distributions outside the origin 0. Theorem 2 and Theorem 1 in [7] give the following

COROLLARY 18. — *Suppose* $n \geqslant 3$. *Then the following two statements hold.*

(1) *For a Frostman-Kunugui kernel* $\varkappa \neq 0$ *on* $\mathbf{R}^n$ *satisfying* $\frac{\partial}{\partial x_1}\Delta\varkappa \leqslant 0$ *in the sense of distributions in* D, *there exists uniquely a spherically symmetric Dirichlet convolution kernel* $\varkappa'$ *on* $\mathbf{R}^n$ *such that* $V_{\varkappa'}$ *is a Dirichlet kernel on* D *and that, for any* $f \in C_c(D)$, $V_\varkappa(V_{\varkappa'}f)(x) = V_{\varkappa'}(V_\varkappa f)(x) = G_2 f(x)$ *in* D.

(2) *For a spherically symmetric Dirichlet kernel* $\varkappa$ *on* $\mathbf{R}^n$ *such that* $V_\varkappa$ *is a Dirichlet kernel on* D, *there exists uniquely*

(6) This means that, for any finite continuous function f in $\mathbf{R}^n$ with compact support, $\varkappa * f(x) \to 0$ as $|x| \to \infty$.

a Frostman-Kunugui kernel $\varkappa'$ *on* $\mathbf{R}^n$ *such that* $\frac{\partial}{\partial x_1}\Delta\varkappa \leqslant 0$ *in the sense of distributions in* D *and that, for any* $f \in C_c(D)$, $V_\varkappa(V_{\varkappa'}f)(x) = V_{\varkappa'}(V_\varkappa f)(x) = G_2 f(x)$ *in* D.

Proof. — First we shall show (1). By Theorem 1 in [7], there exists uniquely a spherically symmetric Dirichlet kernel $\varkappa'$ on $\mathbf{R}^n$ such that $\varkappa * \varkappa' = r^{2-n}$. We have, with a positive constant c, $(\Delta\varkappa) * \varkappa' = -c\varepsilon$ in the sense of distributions in $\mathbf{R}^n$. This implies that the singular measure associated with $\varkappa'$ is equal to $\frac{1}{c}\Delta\varkappa$ outside 0. Theorem 2 and our assumption give that $V_{\varkappa'}$ is a Dirichlet kernel on D. Since $\Delta\varkappa \geqslant 0$ in the sense of distributions in $\mathbf{R}^n - \{0\}$ and $\varkappa$ vanishes at infinity, $\frac{\partial}{\partial x_1}\varkappa \leqslant 0$ in the sense of distributions in D. By Lemma 5, $V_\varkappa$ is positive, and by Lemma 3 and Remark 4, we obtain the required equality. Let's show the uniqueness of $\varkappa'$. Let $\varkappa''$ be a Dirichlet convolution kernel on $\mathbf{R}^n$ which is possessed of the same properties as of $\varkappa'$. Since $\varkappa$ is injective (see Theorem 1 in [7]) (7) and

$$\varkappa * (V_{\varkappa'} f - \overline{V_{\varkappa'} f}) = \varkappa * (V_{\varkappa''} f - \overline{V_{\varkappa''} f})$$

in $\mathbf{R}^n$ (8), we have $V_{\varkappa'}f = V_{\varkappa''}f$ $(\forall f \in C_c(D))$. This implies that, for any $f \in C_c(D)$, $(\varkappa' - \varkappa'')f = (\varkappa' - \varkappa'') * \bar{f}$. In the same manner as in Lemma 5, we have $\frac{\partial}{\partial x_1}(\varkappa' - \varkappa'') = 0$ in the sense of distributions in D. Since $\varkappa' - \varkappa''$ is spherically symmetric and vanishes at the infinity, we have $\varkappa' = \varkappa''$. Thus we see that (1) holds.

Next we shall show (2). By Theorem 1 in [7], there exists uniquely a Frostman-Kunugui kernel $\varkappa'$ on $\mathbf{R}^n$ such that $\varkappa * \varkappa' = r^{2-n}$. Since the singular measure associated with $\varkappa$ is equal to $\frac{1}{c}\Delta\varkappa'$ outside 0, Theorem 2 gives that $\frac{\partial}{\partial x_1}\Delta\varkappa' \leqslant 0$ in the sense of distributions in D. Similarly as

(7) This means that, for an $f \in C(D)$, $f = 0$ provided that $\varkappa * |f|$ is defined and that $\varkappa * f = 0$.

(8) We may assume that $V_{\varkappa'}f$ is a continuous function in $\mathbf{R}^n$ with support $\subset \overline{D}$.

above, we see that $V_{\varkappa'}$ is positive and the required equality holds. Since $\varkappa$ is also injective (see, for example, [1]), we can similarly show the uniqueness of $\varkappa'$.

Remember the Riesz decomposition formula

$$r^{\alpha-n} * r^{(2-\alpha)-n} = a_\alpha r^{2-n} \quad (0 < \alpha < 2),$$

where a_α is a positive constant (see [9]). Then, by this corollary, we see that G_α satisfies the domination principle provided with $n \geqslant 3$ and $0 < \alpha < 2$.

Remark 19. — For a spherically symmetric convolution kernel $\varkappa$ on $\mathbf{R}^n$, $\frac{\partial}{\partial x_1}\varkappa \leqslant 0$ in the sense of distributions in D if and only if $\frac{\partial}{\partial r}\varkappa \leqslant 0$ in the sense of distributions in $\mathbf{R}^n - \{0\}$, where $r = |x|$. In this case, $\varkappa$ is absolutely continuous outside 0.

By using Theorem 1, Corollary 13 and this remark 19, we have the following

Remark 20. — Let $\varkappa = \int_0^\infty \alpha_t \, dt$ be a spherically symmetric Dirichlet kernel on $\mathbf{R}^n$. Then $V_\varkappa$ is a Dirichlet kernel on D if and only if, for any $t \geqslant 0$, α_t is of form

$$\alpha_t = c_t \varepsilon + k_t(|x|)\, dx,$$

where c_t is a non-negative constant and k_t is a non-negative decreasing (in the wide sense) function on $\mathbf{R}^+$.

8. First we shall show that the inverse of the question raised by H. L. Jackson is also affirmative.

Proposition 21. — *If the Green type kernel* G_α $(0 < \alpha < n)$ *on* D *satisfies the domination principle, then* $0 < \alpha \leqslant 2$.

Proof. — Since G_α satisfies the domination principle, G_α also satisfies the balayage principle (see, for example, [8]); that is, for a positive measure μ in D with compact support and a compact set F in D, there exists a positive measure μ'_F supported by F such that $G_\alpha\mu \geqslant G_\alpha\mu'_F$ in D and

$G_\alpha\mu = G_\alpha\mu'_F$ G_α-n.e. on F [9]. Let $\mu \neq 0$ and F be a closed ball contained in D such that $\operatorname{supp}(\mu) \cap F = \varnothing$. Suppose that $\alpha > 2$. Let t be positive integer satisfying $0 < \alpha - 2t \leqslant 2$ and $\beta = \alpha - 2t$. Then

$$G_\alpha(x,y) = \int G_{2t}(x,z)G_\beta(z,y)\,dz$$

(see Lemma 3). Since $G_{2t}(G_\beta\mu) = G_{2t}(G_\beta\mu'_F)$ a.e. on F, we have $G_\beta\mu = G_\beta\mu'_F$ a.e. on F, because

$$\Delta^t(G_{2t}(G_\beta\mu) - G_{2t}(G_\beta\mu'_F)) = (-c)^t(G_\beta\mu - G_\beta\mu'_F)$$

in the sense of distributions in D, where c is the positive constant satisfying $\Delta r^{2-n} = -c\varepsilon$. Since $G_\beta\mu$ is continuous on F and $G_\beta\mu'_F$ is lower semi-continuous, we have $G_\beta\mu \geqslant G_\beta\mu'_F$ on F, and so $\int G_\beta\mu'_F\,d\mu'_F < \infty$. The function kernel G_β satisfying the domination principle, we have $G_\beta\mu \geqslant G_\beta\mu'_F$ in D. By virtue of the injectivity of G_β, we have $G_\beta\mu \neq G_\beta\mu'_F$. But this contradicts the equality $G_{2t}(G_\beta\mu) = G_{2t}(G_\beta\mu'_F)$ G_α-n.e. on F. Thus we achieve the proof.

We raise a question.

Question 22. — Let $\varkappa$ be a convolution kernel on $\mathbf{R}^n$ satisfying $\varkappa = \bar{\varkappa}$. Suppose that $V_\varkappa$ is a Hunt kernel on D. Then is it true that $\varkappa$ is the sum of a Hunt convolution kernel and of a non-negative constant ?

The following proposition shows that the answer is « yes » in a special case.

PROPOSITION 23. — *Let $\varkappa$ be a convolution kernel on $\mathbf{R}^n$ satisfying $\varkappa = \bar{\varkappa}$. Suppose that $V_\varkappa$ is a Hunt kernel on D. If $\int d\varkappa < \infty$ and $\varkappa$ is absolutely continuous outside 0, then $\varkappa$ is a Hunt convolution kernel.*

Proof. — We may assume that $\int d\varkappa < 1$. For a $p \in (0,1]$, we put

$$\varkappa_p = \sum_{k=0}^{\infty} (-p)^k(\varkappa)^{k+1};$$

[9] We write $G_\alpha\mu = G_\alpha\mu'_E$ G_α-n.e. on F if, for any positive measure ν in D with $\operatorname{supp}(\nu) \subset F$ and $\int G_\alpha\nu\,d\nu < \infty$, $\int G_\alpha\mu\,d\nu = \int G_\alpha\mu'_F\,d\nu$.

then $\varkappa_p$ is a real measure in $\mathbf{R}^n$, absolutely continuous outside 0, $\varkappa_p = \bar{\varkappa}_p$ and $\int d|\varkappa_p| < \infty$, where $|\varkappa_p|$ denote the total variation of $\varkappa_p$. Since $(p\varkappa + \varepsilon) * \varkappa_p = \varkappa$, Lemma 3 gives that, for any $f \in C_c(D)$, $(pV_\varkappa + I)(V_{\varkappa_p} f) = V_\varkappa f$. Let $(V_p)_{p \geqslant 0}$ the resolvent associated with $V_\varkappa$. In the same manner as in Theorem 1, we have, for any $f \in C_c(D)$, $V_p f = V_{\varkappa_p} f$ in D. Hence $V_{\varkappa_p}$ is positive. In the same manner as in Lemma 5, we have $\frac{\partial}{\partial x_1} \varkappa_p \leqslant 0$ in the sense of distributions in D. We show that $\varkappa_p$ is a convolution kernel. It suffices to prove that, for any $f \in C_c^+(D)$, $\int_D f\, d\varkappa_p \geqslant 0$, because

$$\varkappa_p(\{0\}) = \frac{\varkappa(\{0\})}{1 + p\varkappa(\{0\})} \geqslant 0, \qquad \varkappa_p = \bar{\varkappa}_p$$

and $\varkappa_p$ is absolutely continuous outside 0. For each integer $k \geqslant 1$, we choose a non-negative, spherically symmetric and infinitely differentiable function φ_k in $\mathbf{R}^n$ such that $\int \varphi_k\, dx = 1$ and $\operatorname{supp}(\varphi_k) \subset \left\{x \in R^n;\ |x| < \frac{1}{k}\right\}$. Since $\frac{\partial}{\partial x_1} \varkappa_p * \varphi_k(x) \leqslant 0$ in the set

$$\left\{x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n;\ x_1 \geqslant \frac{1}{k}\right\}$$

and $\lim_{|x| \to \infty} \varkappa_p * \varphi_k(x) = 0$, we have $\varkappa_p * \varphi_k(x) \geqslant 0$ in the above set. Hence, for any $f \in C_c^+(D)$,

$$\int_D f\, d\varkappa_p = \lim_{k \to \infty} \int_{x_1 \geqslant \frac{1}{k}} f(x) \varkappa_p * \varphi_k(x)\, dx \geqslant 0 .$$

Consequently $\varkappa_p$ is a convolution kernel ($\forall p \in (0,1]$). Since $\varkappa - \varkappa_p = p\varkappa * \varkappa_p$, $\varkappa \geqslant \varkappa_p$. For a $p \in (1, 2]$, we put

$$\varkappa_p = \sum_{k=0}^{\infty} (1 - p)^k (\varkappa_1)^{k+1};$$

then $\varkappa_p$ is also a real measure in $\mathbf{R}^n$, absolutely continuous outside 0, $\varkappa_p = \bar{\varkappa}_p$, $\int d|\varkappa_p| < \infty$ and $\varkappa - \varkappa_p = p\varkappa * \varkappa_p$. In the same manner as above, $\varkappa_p$ is a convolution kernel. Inductively we obtain a family $(\varkappa_p)_{p \geqslant 0}$ of convolution kernels satisfying $\varkappa - \varkappa_p = p\varkappa * \varkappa_p$ and $\lim_{p \to 0} \varkappa_p = \varkappa$ (vaguely). By Lemma 3.2 in [6], we obtain that, for each $p \geqslant 0$ and $q > 0$, $\varkappa_p - \varkappa_q = (q - p)\varkappa_p * \varkappa_q$ and $\lim_{p \to 0} \varkappa_p = \varkappa$ (vaguely), where $\varkappa_0 = \varkappa$. Since $V_\varkappa$ is a Hunt kernel on D, $\varkappa \neq 0$, and hence, for any $x \neq 0 \in \mathbf{R}^n$, $\varkappa \neq \varkappa * \varepsilon_x$, because

$$\lim_{|x| \to \infty} \varkappa * f(x) = 0$$

for any finite continuous function f in $\mathbf{R}^n$ with compact support. Hence, by Corollary 1 of Theorem 5 in [6], $\varkappa$ is a Hunt convolution kernel. This completes the proof.

Remark 24. — In the above proposition, if $\varkappa$ is spherically symmetric, the same conclusion holds without the assumption that $\varkappa$ is absolutely continuous outside 0. See Remark 19.

BIBLIOGRAPHY

[1] A. Beurling and J. Deny, Dirichlet spaces, *Proc. Nat. Acad. U.S.A.*, 45 (1959), 208-215.

[2] J. Deny, Éléments de la théorie du potentiel par rapport à un noyau de Hunt, *Sém. Brelot-Choquet-Deny* (Théorie du potentiel), 5e année, 1960-1961, nº 8.

[3] J. Deny, Noyaux de convolution de Hunt et noyaux associés à une famille fondamentale, *Ann. Inst. Fourier* (Grenoble), 12 (1962), 643-667.

[4] M. Itô, Sur la régularité des noyaux de Dirichlet, *C.R. A. S.* Paris, 286 (1969), 867-868.

[5] M. Itô, Sur la famille sous-ordonnée au noyau de convolution de Hunt donné, *Nagoya Math. J.*, 51 (1973), 45-56.

[6] M. Itô, Sur le principe relatif de domination pour les noyaux de convolution, *Hiroshima Math. J.*, 5 (1975), 293-350.

[7] M. Itô, Sur les noyaux de Frostman-Kunugui et les noyaux de Dirichlet, *Ann. Inst. Fourier* (Grenoble), 27, 3 (1977), 45-95.

[8] M. Kishi, Maximum principle in the potential theory, *Nagoya Math. J.*, 23 (1963), 165-187.

[9] M. Riesz, Intégrales de Riemann-Liouville et potentiels, *Acta Sc. Math.*, Szeged, 9 (1938), 1-42.

Manuscrit reçu le 10 janvier 1977
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