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## ON HOLOMORPHIC FIELDS OF COMPLEX LINE ELEMENTS WITH ISOLATED SINGULARITIES

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### 1. Introduction.

Let  $V_d$  be a compact, connected, complex manifold of complex dimension  $d \geq 2$ . If we attach to each singular point (zero point) of a (continuous) field of tangent vectors with isolated singularities on  $V_d$  in a standard way an index of singularity, the index sum is always the same, namely equal to the Euler-Poincaré characteristic of  $V_d$ . An analogous result is no longer true for fields of complex *line elements* with isolated singularities. This can be seen already in the projective plane: a field obtained by joining all points with a fixed point has index sum 1, a field obtained from a « general » collineation has index sum 3. The question which number occur as index sums was answered by E. Kundert ([12]), who proved a.o. a general theorem on  $U(d)$ -bundles with fibre the  $(d-1)$ -dimensional complex projective space. As noticed by F. Hirzebruch, the results can be stated in simple terms, which read for the case of the tangent bundle  $\theta$  of  $V_d$  as follows: be given a field  $\mathfrak{F}$  of complex line elements with isolated singularities. Then there exists a (continuous) line bundle  $\xi$  on  $V_d$  and a continuous section  $s$  of the tensor product  $\theta \otimes \xi$ , which has zeros exactly at the singularities of  $\mathfrak{F}$ , of the same index as those singularities and which projects on  $\mathfrak{F}$  outside of little spherical neighbourhoods of the singularities. And conversely, given any continuous complex line bundle  $\xi$  on  $V_d$ ,  $\theta \otimes \xi$

has always sections with isolated zeros, giving a field of complex line elements with isolated singularities. It follows, that the integers occurring as index sums of fields of complex line elements are precisely the Chern numbers  $C_d(\theta \otimes \xi)$ , where  $\xi$  runs through all continuous complex line bundles on  $V_d$ .

It will be clear, that one can also define holomorphic fields of complex line elements with isolated singularities on  $V_d$  (the two examples given earlier are such fields), and it seems natural to ask questions of the following type:

(1) which index sums  $C_d(\theta \otimes \xi)$  can be represented by a holomorphic field with isolated singularities?

(2) can the holomorphic fields with isolated singularities, like in the topological case be classified in some sense by the holomorphic sections in vector bundles  $\theta \otimes \xi$ , where  $\xi$  now is a *holomorphic* line bundle on  $V$ ?

In this paper we consider questions of this type and partially answer them.

In fact, the second question can be answered completely by Theorem 4.2 of this paper:

*Let  $V_d$ ,  $d \geq 2$ , be a complex manifold,  $X$  a discrete subset of  $V$ ,  $\alpha$  a holomorphic  $d$ -vector bundle on  $V$ ,  $\beta$  the associated bundle with fibre  $P_{d-1}$ ,  $s$  a holomorphic section of  $\beta|_{V-X}$ . Then there is a holomorphic line bundle  $\xi$  on  $V$  and a holomorphic section  $s'$  of  $\alpha \otimes \xi$  such that on  $V-X$   $s = \pi(s')$ , where  $\pi$  is the canonical bundle map.*

It follows that holomorphic fields (with isolated singularities) on a compact complex manifold can be classified in a natural, unique way by (possible empty) Zariski-open subsets of the projective spaces belonging to the complex vector spaces  $H^0(V, \theta \otimes \xi)$ , where  $\xi$  runs through all holomorphic line bundles on  $V$ .

However, the answer is not completely satisfactory in as far as we have no *necessary and sufficient* criterion whether a bundle  $\theta \otimes \xi$ , having holomorphic sections, has also holomorphic sections *with isolated singularities*.

Nevertheless, our methods provide complete answers in many familiar cases. We give a few examples.

**THEOREM 3.1.** — *The product  $V = V^{(1)} \times V^{(2)} \times \dots \times V^{(k)}$  of the rational homogeneous manifolds  $V^{(1)}, \dots, V^{(k)}$  with second*

*Betti number 1 admits a holomorphic field of complex line elements without singularities if and only if at least one  $V^{(i)}$  is a  $P_1$ ; and in that case  $V$  admits only the trivial fields, i.e. the fields attaching to  $(x_1, \dots, x_k) \in V$  the line elements tangent to  $(x_1, \dots, x_{i-1}, P_1, x_{i+1}, \dots, x_k)$ .*

In particular, the odd-dimensional projective spaces and quadrics of dimension  $\geq 3$  have no holomorphic field of line elements without singularities, though they have continuous fields of such type. As another example, the 2-dimensional quadric  $P_1 \times P_1$  has only the two trivial fields.

**THEOREM 3.3.** — *The only holomorphic field of complex line elements without singularities on the Hirzebruch surfaces  $\Sigma_n$ ,  $n \geq 1$ , is the field along the fibres.*

On  $\Sigma_n$  there are many homotopy classes of continuous fields.

(COROLLARY TO THEOREM 4.4 AND 4.5) *A number  $i$  occurs as index sum of a continuous field of complex line elements with isolated singularities on  $\Sigma_n$ ,  $n \geq 0$ , if and only if  $i$  is even;  $i$  occurs as index sum of a holomorphic field if and only if  $i$  is even and non negative, with exception however of the case  $n = 0$ ,  $i = 2$ .*

This paper is written in the language of complex manifolds. Nevertheless, the main examples are algebraic and can, like the whole problem, be treated for characteristic  $p$  by essentially the same method.

We have used freely the classical, so called GAGA-results of W. L. Chow and J. P. Serre, for example about the equivalence of holomorphic and regular algebraic maps, in particular sections, and vector bundle classes for (projective) algebraic varieties (see [16]).

The author is indebted to R. Remmert, who suggested the proof of Lemma 4.1, and to Ch. Ehresmann and G. Reeb, who asked some questions which are answered by some of the theorems above.

At the time this lecture was delivered, the author learned, that Theorem 4.2 and the remarks after Theorem 4.3 were proved independantly by Mrs. F. Benzecri-Le Roy in her Paris thesis ([21]).

## 2. Preliminaries.

### Complex manifolds, vector bundles.

Let  $V_d$  be a compact, connected, complex manifold of complex dimension  $d$ . The complex structure defines in a natural way an orientation of  $V$ . With respect to this orientation we apply Poincaré duality  $H^i(V, \mathbb{Z}) \cong H_{2d-i}(V, \mathbb{Z})$ , carrying over the multiplicative structure on  $H^*(V, \mathbb{Z})$  onto  $H_*(V, \mathbb{Z})$ . According to A. Borel and A. Haefliger ([4]), it is possible to attach to each analytic cycle on  $V$  an element of  $H_*(V, \mathbb{Z})$ , such that the analytic intersection (if defined) corresponds to the homology product as introduced above. If  $W$  is another compact complex manifold, and  $f: V \rightarrow W$  a continuous map, we denote by  $f^*$  the induced homomorphism for homology.

In this homological framework, the Chern classes  $C_0(\xi), \dots, C_e(\xi)$  of a complex  $e$ -vector bundle  $\xi$  on  $V$  are homology classes,  $C_i(\xi) \in H_{2d-2i}(V, \mathbb{Z})$ .

But for the change from cohomology to homology, we shall use the notations and the basic results of [10]. In particular:

**PROPOSITION 2.1.** — *If  $\xi$  is a complex  $e$ -vector bundle and  $\eta$  a 1-vector bundle (line bundle) on  $V$ , then the total Chern class of the tensor product  $\xi \otimes \eta$  is given by*

$$C(\xi \otimes \eta) = (1 + C_1(\eta))^e + (1 + C_1(\eta))^{e-1} C_1(\xi) + \dots + C_e(\xi).$$

Let  $G_{n,e}$  be the Grassmann manifold of the  $(e)$ -dimensional linear subspaces  $\mathbf{C}_e$  of the  $(n+1)$ -dimensional complex vector space  $\mathbf{C}_n$ . Let the subset  $E \subset G \times \mathbf{C}_n$  be defined by

$$E = \{(g, x) \in G \times \mathbf{C}_n, x \in g\}$$

$E$  is the bundle space of a holomorphic  $(e)$ -vector bundle on  $G$ , which is called the universal subbundle  $\lambda$  on  $G$ . If  $\tau_n$  is the trivial  $(n+1)$ -vector bundle, then we have an exact sequence  $0 \rightarrow \lambda \rightarrow \tau_n$ . The quotient bundle  $\omega$  is called the universal quotient bundle on  $G$ . It is known that the Chern classes of  $\omega$  can be represented by certain Schubert cycles on  $G$ , all with a positive sign.

We say that a holomorphic  $e$ -vector bundle  $\xi$  on a complex manifold  $V$  is generated by its (global) sections if the restriction homomorphism  $H^0(V, \xi) \rightarrow H^0(p, \xi|_p)$  is surjective for all points  $p \in V$ .

**PROPOSITION 2.2.** — *Let  $V$  be a compact, complex manifold,  $\xi$  a holomorphic  $e$ -vector bundle on  $V$ , which is generated by its sections. Then the Chern classes of  $\xi$  can be represented by non negative analytic cycles.*

**PROOF.** — From the fact that  $\xi$  is generated by its sections, it follows that there exists a holomorphic map  $f: V \rightarrow G$ , where  $G$  is the Grassmann manifold of  $\mathbb{C}_{n-e}'$ s in a  $\mathbb{C}_n$ , such that  $f^*(\omega) = \xi([1])$ , p. 417). The statement now follows from Lemma III. 2 in [19], modified for the Borel-Haefliger homology.

**Fields of line elements.**

In this section we give only some definitions; the theory of the topological case can be found in [12] and [18].

For a (not necessarily compact) complex manifold  $V_d$  we denote by  $\theta_v$  the contravariant tangent bundle of  $V$ , and by  $\Omega_v$  the associated bundle with fibre  $\mathbb{P}_{d-1}$ , the complex projective space of complex dimension  $d-1$ . The points of the fibre over  $x \in V$  of this last bundle are called the complex line elements tangent to  $V$  at  $x$ .

Let  $A \subset V$  be a closed subset of  $V$  (possibly empty). A continuous (holomorphic) field of complex line elements with  $A$  as singular set or set of singular points will be a continuous (holomorphic) section of the restriction  $\Omega_v|_{V - A}$ . In this paper  $A$  is always a discrete subset of  $V$ , hence the fields of complex line elements considered here are always fields with isolated singularities. It is assumed that the field can not be extended to any point  $a \in A$ . Then each point  $a \in A$  has an index of singularity, different from zero, which can be described as follows; take a neighbourhood  $U$  of  $a$  on  $V$ , such that  $\Omega_v|_U$  is trivial. For any sphere with centre  $a$  (in any Riemannian metric on  $V$ ) the field defines a map of that sphere into  $\mathbb{P}_{d-1}$ , which gives (with a standard convention about signs) an element of  $\pi_{2d-1}(\mathbb{P}_{d-1})$ , hence an integer.

This integer is independent of the way the metric and the sphere are chosen.

In the case of a holomorphic field, the index of any singularity is always positive. This « well known » fact can be deduced from Lemma 4.1 of this paper and the analysis of a holomorphic map  $f$  of a neighbourhood of  $(0, \dots, 0) \in \mathbf{C}_n(y, \dots, y_n)$  into itself with  $f(y_1, \dots, y_n) = (0, \dots, 0)$ , if and only if  $(y_1, \dots, y_n) = (0, \dots, 0)$ .

### The surfaces $\Sigma_n$ .

We consider algebraic surfaces (sometimes called Hirzebruch surfaces) which are the bundle space of an algebraic  $P_1$  — bundle over  $P_1$ . It is known that there is an infinite number of biregularly inequivalent surfaces  $\Sigma_0, \Sigma_1, \dots, \Sigma_n, \dots$  all of them of course rational.  $\Sigma_n$  can be described as the bundle space of the associated  $P_1$  — bundle of the 2-vector bundle on  $P_1$ , which is the direct sum of the trivial line bundle and the line bundle of degree  $n$ . For  $n \geq 1$   $\Sigma_n$  can be characterised among the  $\Sigma_i$  by the fact that this surface admits a (unique) cross section  $K_n$  with  $K_n^2 = -n$ .  $\Sigma_0$  is just  $P_1 \times P_1$ .  $\Sigma_1$  is obtained from  $P_2$  by blowing up a point to  $K_1$ . The homology class  $X$  of the fibres and the homology class  $Y$  of  $K_n$  form a base for  $H_2(\Sigma_n, \mathbf{Z})$ . Let  $\xi$  and  $\eta$  be the  $\mathbf{C}^*$  — bundles on  $\Sigma_n$  with  $C_1(\xi) = x$  and  $C_1(\eta) = y$  respectively. Then each line bundle on  $\Sigma_n$  is one of the bundles  $\xi^a \otimes \eta^b$ ,  $a, b \in \mathbf{Z}$ . The bundle along the fibres is the bundle  $\xi^n \otimes \eta^2$  and there is an exact sequence

$$0 \rightarrow \xi^n \otimes \eta^2 \rightarrow \theta_{\Sigma_n} \rightarrow \xi^2 \rightarrow 0$$

from which it follows that  $C_1(\Sigma_n) = (n+2)x + 2y$  and  $C_2(\Sigma_n) = 4$ . We now show how to calculate  $\dim H^i(\Sigma_n, \xi^a \otimes \eta^b)$  for all  $n, i \geq 0$  and  $a, b \in \mathbf{Z}$  arbitrary. We give a general procedure, in particular cases there are simpler methods to get the result more quickly.

First of all, by general theorems ([11])

$$H^i(\Sigma_n, \xi^a \otimes \eta^b) = 0 \text{ for } i \geq 3.$$

Furthermore by Serre-duality (loc. cit.)

$$H^2(\Sigma_n, \xi^a \otimes \eta^b) \cong H^0(\Sigma_n, \xi^{-n-a-2} \otimes \eta^{-b-2}).$$

Once  $\dim H^0(\Sigma_n, \xi^a \otimes \eta^b)$  and  $\dim H^2(\Sigma_n, \xi^a \otimes \eta^b)$  are found, the Riemann Roch theorem gives  $\dim H^1(\Sigma_n, \xi^a \otimes \eta^b)$ . Therefore, it is sufficient to calculate  $\dim H^0(\Sigma_n, \xi^a \otimes \eta^b)$  for all  $a, b \in \mathbb{Z}$ . Now if  $b < 0$  there is no algebraic curve on  $\Sigma_n$  in the homology class  $ax + by$ , for  $(ax + by).x = b$  would be negative, which is impossible, because there is always fibre intersecting a given curve in a finite number of points. If we have a curve  $C$  in the homology class  $ax + by$  with  $a < nb$  (hence  $b \geq 1$ ), then its intersection with  $K_n$  is negative, and  $K_n$  is a component of  $C$ . Thus we have in this case :

$$\dim H^0(\Sigma_n, \xi^a \otimes \eta^b) = \dim H^0(\Sigma_n, \xi^a \otimes \eta^{b-1})$$

So we are left with the case  $a \geq nb \geq 0$ .

Let  $F$  be any fibre. From the cohomology sequences induced by the exact sequences  $0 \rightarrow \xi^{a-1} \rightarrow \xi^a \rightarrow \xi^a|_F \rightarrow 0$  and from  $H^1(\Sigma_n, \tau) = 0$  (the surfaces  $\Sigma_n$  are simply connected) we derive  $H^1(\Sigma_n, \xi^a) = 0$  for  $a \geq 0$ . (Here  $\tau$  denotes the trivial line-bundle on  $\Sigma_n$ ).

From the cohomology sequence induced by the exact sequence

$$0 \rightarrow \xi^a \otimes \eta^{d-1} \rightarrow \xi^a \otimes \eta^d \rightarrow \xi^a \otimes \eta^d|_{K_n} \rightarrow 0$$

for  $d = 1, \dots, b$  we find  $H^1(\Sigma_n, \xi^a \otimes \eta^b) = 0$  and

$$\begin{aligned} \dim H^0(\Sigma_n, \xi^a \otimes \eta^b) &= \dim H^0(\Sigma_n, \xi^a \otimes \eta^{b-1}) \\ &\quad + \dim H^0(K_n, \xi^a \otimes \eta^b|_{K_n}) \\ &= \dim H^0(\Sigma_n, \xi^a \otimes \eta^{b-1}) + (a - nb + 1). \end{aligned}$$

Starting from  $\dim H^0(\Sigma_n, \xi^a) = a + 1$  we get the desired result :

$$(2.3) \quad \dim H^0(\Sigma_n, \xi^a \otimes \eta^b) = \begin{cases} 0 & \text{if } a < 0 \text{ or } b < 0. \\ (b + 1) \left( a - \frac{1}{2} nb + 1 \right) & \text{if } 0 \leq nb \leq a \\ (c + 1) \left( a - \frac{1}{2} nc + 1 \right) & \text{if } 0 \leq a < nb, \end{cases}$$

where  $c$  is the largest number with  $a \geq nc$ .



**Holomorphic vector bundles on  $P_1$ .**

Let  $\xi$  be the line bundle on  $P_1$  with  $C_1(\xi) = 1$ . Then each line bundle on  $P$  is isomorphic to a bundle  $\xi^n = \xi \otimes \dots \otimes \xi$  ( $n$  times)  $n$  is called the degree of the bundle:  $\deg(\xi^n) = n$ .

**PROPOSITION 2.4.** — *Let  $\eta = \eta_1 \oplus \dots \oplus \eta_k$ , where  $\eta_i$  is a holomorphic 1-vector bundle on  $P_1$  of degree  $n_i$ , such that  $n_1 \geq n_2 \geq \dots \geq n_k$ . Then there exists a holomorphic 1-dimensional subbundle of  $\eta$  of degree  $d$  if and only if  $d = n_1$ , or  $d \leq n_2$ . If  $n_1 > n_2$ , then  $\eta_1$  is the only holomorphic 1-dimensional subbundle of  $\eta$  of degree  $n_1$ .*

**PROOF.** — Let  $\alpha$  be any holomorphic 1-dimensional subbundle of  $\eta$  of degree  $d$ . By taking a suitable meromorphic section of  $\alpha$  and projecting it on  $\eta_i$  we find that either  $d \leq n_i$  or that  $\alpha$  is contained in  $\eta_1 \oplus \dots \oplus \eta_{i-1} \oplus \eta_{i+1} \oplus \dots \oplus \eta_k$ . Since there is at least one  $i$  such that  $\alpha$  is not contained in  $\eta_1 \oplus \dots \oplus \eta_{i-1} \oplus \eta_{i+1} \oplus \dots \oplus \eta_k$ , we have  $d \leq n_i \leq n_1$ . Now if  $d > n_2$ , the projection of  $\alpha$  onto  $\eta_2 \oplus \dots \oplus \eta_k$  has to be zero, hence  $\alpha = \eta_1$ . These remarks already prove the only-part of our theorem. To prove the if-part, we can restrict ourselves to the case  $k = 2$ . We may assume that  $\eta_1$  is the trivial bundle and  $\eta_2 = \xi^{-n}$ ,  $n \geq 0$ .

The « closure at infinity »  $V$  of the bundle space  $W$  of  $\eta$  is a holomorphic  $P_1$  — bundle over  $P_1$ , namely the  $P_1$  — bundle associated with  $\eta$ , i.e.  $\Sigma_n$ . Let  $\alpha$  be any holomorphic 1-dimensional subbundle of  $\eta$ . Its infinity is a cross section  $C$  on  $V$ . We claim:  $C^2 = -2d - n$ . To see this, let  $\pi: V \rightarrow P_1$  be the bundle projection and  $\pi_1 = \pi|_C$ .  $\pi_1^*(\eta)$  has a canonical 1-dimensional subbundle  $\gamma$  and  $\gamma|_C = \pi_1^*(\alpha)$ . Now we have on an exact sequence ([5], § 7)

$$0 \rightarrow \tau \rightarrow \pi_1^*(\eta) \otimes \gamma^* \rightarrow \beta \rightarrow 0$$

where  $\tau$  denotes the trivial line bundle,  $\beta$  the bundle along the fibres and  $\gamma^*$  the dual bundle of  $\gamma$ .  $\beta|_C$  is the normal bundle of  $C$  in  $V$ , hence  $C_1(\beta|_C) = C^2$ . By restriction of the exact sequence to  $C$  we get therefore

$$C^2 = C_1(\pi_1^*(\eta) \otimes \gamma^*|_C) = \pi_1^*(C_1(\eta \otimes \alpha^*)) = -2d - n$$

by Proposition 2.1. In particular, if we take for  $\alpha$ ,  $\eta_1$ , we get a

section  $C$  with  $C^2 = -n$ , which has to be  $K_n$  for  $n \geq 1$ . The homology class of  $C$  is  $y - dx$ . We have to prove that this homology class can be represented by a holomorphic section if and only if  $d \leq -n$ , i.e. that there is an irreducible curve on  $\Sigma_n$  representing  $y + mx$  if and only if  $m \geq n$ . If a curve representing  $y + mx$  is reducible, it follows from the fact that  $\dim H^0(\Sigma_n, \xi^a \otimes \eta^b)$  for  $b < 0$  that it splits into another section and a number of fibres (with multiplicities). To prove our theorem we have to show that for  $m \geq n$

$$\dim H^0(\Sigma_n, \xi^m \otimes \eta) > \dim H^0(\Sigma_n, \xi^p \otimes \eta) + (m - p)$$

with  $n \leq p < m$ . This however follows from the general formula given above. For  $n = 0$  the argument is similar, but simpler.

### Rational homogeneous manifolds.

As was proved in [6] by A. Borel and R. Remmert, each compact homogeneous Kähler manifold is the product of a torus and a rational homogeneous manifold, i.e. a projective homogeneous manifold which is birationally equivalent to  $P_n$ .

Among the rational homogeneous manifolds are the «irreducible» ones, that are those with second Betti number 1. Their classification is known (see for example [5]), and among them are the projective spaces, more generally the Grassmann varieties, and the complex quadrics of dimension  $\geq 3$ . All other rational homogeneous manifolds are fibre bundles with a strictly lower dimensional homogeneous manifold as base and an irreducible one as fibre.

### A lemma on holomorphic vector fields on $P_n$ .

Let the non singular algebraic variety  $V_d$  be regularly embedded in  $P_n$ . By a general hyperplane section of  $V$  we mean a non singular hyperplane section  $V \cap P_{n-1}$ , along which  $V$  and  $P_{n-1}$  intersect simply. It is well known that for all  $P_{n-1} \in P_n^*$  (the projective space of  $P_n$ ) outside of a proper algebraic subset of  $P_n^*$ ,  $V \cap P_{n-1}$  is general.

LEMMA 2.5. — *Let the algebraic manifold  $V_d$ ,  $d \geq 2$ , be regularly embedded in  $P_n$ , such that  $V$  is not contained in any hyperplane of  $P_n$ . Then each holomorphic vector field on  $P_n$ ,*

tangent to  $V$  in each point of  $V$  and identically zero on one general hyperplane section  $H \cap V$  is identically zero on  $P_n$ .

PROOF. — Let  $\mathfrak{F}$  be a vector field as described in the lemma and  $g$  an element of the one parameter group of collineations, generated by  $\mathfrak{F}$ .  $V_d \cap H$  is not contained in any hyperplane  $P_{n-2} \subset H$ , otherwise the  $P_{n-1}$  through this  $P_{n-2}$  and a point of  $V$  outside  $H$  would contain  $V$ . Therefore, a general  $P_{n-2} \subset H$  will intersect  $V \cap H$  in a variety, not contained in any  $P_{n-3} \subset H$ . Since  $V \cap H$  is invariant under  $g$ , the  $P_{n-2}$  is invariant under  $g$ . So all  $P_{n-2} \subset H$  which intersect  $V \cap H$  generally are invariant under  $g$ . But the  $P_{n-2} \subset H$ , invariant under  $g$  form in the dual projective space of  $H$  also a Zariski-closed subset. It follows that all  $P_{n-2} \subset H$  are invariant under  $g$ , hence that  $g$  induces the identity on  $H$ , or that  $\mathfrak{F}$  is identically zero on  $H$ . Now, by elementary projective geometry, if  $\mathfrak{F}$  is not identically zero on  $P_n$ ,  $\mathfrak{F}$  has outside  $H$  either one or no zero. In the first case the 1-parameter group generated by  $\mathfrak{F}$  is just the group transforming each line through the fixpoint into itself and  $V$  would have to be a cone, which is impossible. In the second case, for each  $g$  there is a point  $q \in H$ , such that all the lines through  $q$  are transformed into itself. Now it is impossible that  $V$  has a finite number of points in common with such a line, because such a pointset is never invariant with respect to  $g$ . So  $V$  would again be a cone, which is impossible. It follows, that  $\mathfrak{F}$  is identically zero on  $P_n$ .

REMARK. — The theorem remains true for  $d = 1$  with one exception: a non singular conic in the plane.

#### An exact sequence of vector bundles on $P_n$ .

Let  $\xi$  be the line bundle on  $P_n$ , such that  $C_1(\xi)$  is the natural generator of  $H_{2n-2}(P_n, \mathbb{Z})$ , and let  $\tau_{n+1}$  be the trivial  $(n+1)$  — bundle on  $P_n$ . Then there is an exact sequence of holomorphic vector bundles

$$(2.6) \quad 0 \rightarrow \xi^* \rightarrow \tau_{n+1} \rightarrow \theta_{P_n} \otimes \xi^* \rightarrow 0.$$

For the proof see ([10], § 13) or [20], where more general results are given.

**3. Holomorphic fields of line elements without singularities.**

We start with

**THEOREM 3.1.** — *The product  $V = V^{(1)} \times \dots \times V^{(n)}$  of the irreducible rational homogeneous manifolds  $V^{(1)}, \dots, V^{(n)}$  admits a holomorphic field of complex line elements without singularities if and only if at least one  $V^{(i)}$  is a  $P_1$ ; and in that case  $V$  admits only the trivial fields, i.e. the fields attaching to the point  $(x_1, \dots, x_n)$  the line element tangent to  $(x_1, \dots, x_{i-1}, P_1, x_{i+1}, \dots, x_n)$ .*

**PROOF.** — The if-part of the theorem is trivial. To prove the only-part, we start with the case  $n = 1$ , stating that the only irreducible rational homogeneous manifold  $V_d$  admitting a holomorphic field of line elements is  $P_1$ . The assumption means that on  $V$  we have an exact sequence  $0 \rightarrow \xi \rightarrow \theta V$ , where  $\xi$  is a holomorphic line bundle on  $V$ . It is well known ([5], § 16), that  $H^1(V, Z) = 0$ ,  $H^2(V, Z) = Z$ , hence  $\xi = \eta^k$ , with  $C_1(\eta)$  equal to the natural generator of  $H_{2d-2}(V, Z)$ . We consider two cases: (a)  $k \leq 0$  and (b)  $k > 0$ . In case (a) we proceed as follows. We have an exact sequence  $0 \rightarrow \tau \rightarrow \theta_v \otimes \xi^{-1}$ , where  $\tau$  is the trivial line bundle on  $V$ . Hence  $\theta_v \otimes \eta^{-k}$  has a holomorphic section without zeros, or  $C_d(\theta_v \otimes \eta^{-k}) = 0$ . According to Proposition 2.1 this gives that the part of homological dimension 0 of

$$(1) \quad (1 + C_1(\eta^{-k}))^d + (1 + C_1(\eta^{-k}))^{d-1}C_1(\theta_v) + \dots + C_d(\theta_v) = 0.$$

Setting  $C_1(\eta) = h$  and writing out we get a polynomial in  $h, C_1(\theta_v), \dots, C_d(\theta_v)$  with non-negative coefficients only. But  $h^{d-i}C_i(\theta_v) \geq 0$  for  $0 \leq i \leq d$  because  $h$  is a hyperplane class and  $C_1(\theta), \dots, C_d(\theta)$  can be represented by non-negative algebraic cycles on  $V$  according to Proposition 2.2. On the left handside of (1) we get a sum of non-negative terms, of which at least one, namely  $C_d(\theta_v)$  is positive. For  $C_d(\theta_v)$  is the Euler Poincaré characteristic of  $V$  which is positive (the odd Betti numbers of  $V$  are all zero, see ([5], § 14 and § 16). But this would give a contradiction. So we have excluded case (a). In case (b) we consider separately the cases  $V_d = P_d$  and  $V^p$  otherwise. In the last case  $\xi = \eta^k, k \geq 1$ , defines a regular

embedding  $f: V_d \rightarrow P_N$  for suitable  $N$ , such that the hyperplane sections of  $W_d = f(V_d)$  are the zero sets of the sections of  $\xi$ . It is known<sup>(1)</sup> that each holomorphism of  $W$  is induced by a collineation of  $P_N$ , leaving  $W$  invariant. This means that each holomorphic vector field on  $W$  is the restriction to  $W$  of a vector field on  $P_N$ . Now if  $\xi$  would be a subbundle of  $\theta_V$ , there would be holomorphic vector fields on  $W$ , not identically zero, but zero on a general hyperplane section of  $W$ . The extension to  $P_N$  would fulfill the conditions of Lemma 2.5, without being identically zero, which is a contradiction. We are left with the case  $V_d = P_d$ . In this case we find from (1) as only possibility  $n$  odd and  $k = 2$ . Hence there would be holomorphic vector fields on  $P_d$ , zero on a non singular quadric, without being identically zero. It would follow that there exist collineations of  $P_d$  different from the identity and leaving a non singular quadric pointwise fixed, which is impossible for  $d \geq 2$ . After we have finished the proof for the case  $n = 1$ , we consider the general case. Let  $\pi_i: V \rightarrow V^{(i)}$  be the canonical projection of  $V$  onto  $V^{(i)}$ . In the exact sequence  $0 \rightarrow \xi \rightarrow \theta_V$ ,  $\xi = \eta_1^{k_1} \otimes \dots \otimes \eta_n^{k_n}$  where  $\eta_i = \pi_i^*(\mu_i)$ ,  $\mu_i$  being the line-bundle on  $V^{(i)}$  defined in the first part of this proof as  $\eta$ . First, we exclude the possibility that any  $k_i > 0$ . To that purpose, consider a « fibre »  $W^{(i)} = (x_1, \dots, x_{i-1}, V^{(i)}, x_{i+1}, \dots, x_n)$ .  $W^{(i)}$  is regularly embedded in  $V$  and biregularly equivalent to  $V^{(i)}$ . Clearly,  $\eta_j|W^{(i)}$  is the trivial bundle for  $j \neq i$  and (up to an obvious identification)  $\mu_i$  for  $j = i$ . Now  $\theta_V|W^{(i)}$  is the direct sum of the tangent bundle to  $W^{(i)}$  and a trivial bundle  $\tau^{(i)}$  (of dimension  $\dim V - \dim V^{(i)}$ ). Now, according to the result for  $n = 1$ , if  $W^{(i)}$  is not a  $P_1$ ,  $\xi|W^{(i)}$  is not a subbundle of  $\theta W^{(i)}$ . In any case, if  $k_i > 0$ ,  $\xi|W^{(i)}$  would have holomorphic sections, not identically zero, but zero on a divisor. The projection of such a section onto  $\tau^{(i)}$  would have the same properties, which is obviously impossible. Therefore,  $k_i \leq 0$  for  $i = 1, 2, \dots, n$ , unless our field is one of the trivial fields. We can finish the proof of our theorem in exactly the same way as we excluded the case  $k < 0$  for  $n = 1$ , provided that we justify the following remark: given an irreducible, positive algebraic cycle  $C$  on  $V$ , then for

<sup>(1)</sup> See A. Blanchard, Sur les variétés analytiques complexes. Ann. Ec. Norm. Sup. (3), 73, p. 174 (1957).

almost all positive divisors  $D$  in  $[\eta_i]$ , C.D is defined, *i.e.* consists of components of the right dimension only. This can be seen as follows. Set

$$\pi_i(C) = C' \quad \text{and} \quad C'' = \{p \in C' \mid \dim(\pi_i^{-1}(p) \cap C) > \dim C - \dim C'\}$$

$C''$  is a proper algebraic subset of  $C'$ . Almost all positive divisors  $E$  in  $[\mu_i]$ , intersect  $C'$  simply along  $D'$ ,  $D' \notin C''$  because  $[\mu_i]$  is a hyperplane section divisor class.  $D = \pi_i^{-1}(D')$  clearly fulfills our conditions, and since  $H^0(V^{(i)}, \mu_i) \cong H^0(V, \eta_i)$  in a canonical way, we have proved the remark.

REMARKS. — 1) The theorem was proved for  $n = 1$  and  $V_d = P_d$  in [20] by a different method. There are still other methods to prove the theorem in this special case, and we mention a few of them.

After it is found, as in the proof above, that  $k = 2$ , it is clear that the restriction of  $\xi$  to any line in  $P_d$  has also degree 2. The restriction of the tangent bundle of  $P_d$  to a line  $P_1 \subset P_d$  is the direct sum of one bundle of degree 2, namely  $\theta_{P_1}$ , and  $d - 1$  bundles of degree 1. From Proposition 2.4 it follows that  $\xi|_{P_1}$  has to be the tangent bundle to  $P_1$ . For  $d \geq 2$  this is absurd, because it would have to hold for any line  $P_1 \subset P_d$ .

Also it is possible to use the exact sequence (2.5), from which we get  $0 \rightarrow \eta^{-2} \rightarrow \tau_{d+1} \otimes \eta^{-1} \rightarrow \theta_{P_d} \otimes \eta^{-2} \rightarrow 0$  where  $\tau_{d+1}$  is the trivial  $(d + 1)$ -vector bundle on  $P_d$ . Now

$$H^0(P_d, \tau_{d+1} \otimes \eta^{-1}) = 0 \quad \text{and} \quad H^1(P_d, \eta^{-2}) = 0 \quad \text{for } d > 1,$$

hence  $H^0(P_d, \theta_{P_d} \otimes \eta^{-2}) = 0$  for  $d > 1$ , and  $\theta_{P_d} \otimes \eta^{-2}$  can have no trivial subbundle.

(2) It will be clear that a manifold  $V$  may have continuous fields of complex line elements not homotopic to any holomorphic field. According to section 1 the fibre preserving homotopy classes of such fields are in 1 — 1 — correspondence with the continuous line bundles  $\xi$  on  $V_d$  with  $C_d(\theta \otimes \xi) = 0$ . On the projective space  $P_d$  there is no such class for  $d$  even and exactly one for  $d$  odd. On the complex quadric  $Q_d$ ,  $d \geq 3$ , there is no such field for  $d$  even and one homotopy class for  $d$  odd. In the case  $V = P_1 \times P_1$ , let  $h_1$  and  $h_2$  be the set of generators of  $H_2(V, Z)$  represented by the « horizontal » and « vertical »

fibres,  $P_1 \times x_1$  and  $x_2 \times P_1$ . Then  $C_1(V) = 2(h_1 + h_2)$  and  $C_2(V) = 4$ . An element  $\alpha_1 h_1 + \alpha_2 h_2$  of  $H_2(V, \mathbf{Z})$  satisfies  $(\alpha_1 h_1 + \alpha_2 h_2)^2 - 2(h_1 + h_2)(\alpha_1 h_1 + \alpha_2 h_2) + 4 = 0$  if and only if  $\alpha_1 \alpha_2 - \alpha_1 - \alpha_2 + 2 = 0$ . This equation has two solutions:  $\alpha_1 = 2, \alpha_2 = 0$  and  $\alpha_1 = 0, \alpha_2 = 2$ . Therefore, in this case, there are only continuous fields homotopic to the trivial fields. From this the statement of the theorem follows immediately for this case, because the restriction of a 1-dimensional subbundle  $\xi$  of  $\theta$  with  $C_1(\xi) = 2h_1$  to a fibre  $P \times x_2$  has degree 2 and therefore has to coincide with the tangent bundle to  $P_1 \times x_2$  by Proposition 2.4.

It is possible that on a product  $V^{(1)} \times V^{(2)} \times \dots \times V^{(n)}$ , as considered in the theorem, the only continuous fields (up to a homotopy of course) are fields on a  $P_d$  or  $Q_d$ , fitted to the product. If this is true, and once we know the theorem for  $P_d$  and  $Q_d$ , it is clear that such a field can be holomorphic only if  $d = 1$  ( $P_1 = Q_1$ ) and the same trivial kind of argument as in the case of  $P_1 \times P_1$  will prove the theorem for the general case.

(3) Theorem 3.1 can be generalized in several directions: reducible rational homogeneous manifolds, other simple types of rational manifolds, homogeneous manifolds in general, in particular homogeneous Kähler manifolds, which are always the product of a rational homogeneous manifold and a torus, according to a theorem of A. Borel and R. Remmert ([6]). In the sequel of this section we consider three examples: flag manifolds, the surfaces  $\Sigma_n$ ,  $n = 1, 2, \dots$  and the product of  $P_1$  and an elliptic curve.

### Flag manifolds.

The set of all flags

$$C_{d_1} \subset C_{d_2} \subset \dots \subset C_{d_k} \subset C_d, \quad 1 \leq d_1 < d_2 < \dots < d_k \leq d - 1$$

is in a natural way a rational homogeneous manifold, called the flag manifold of type  $(d_1, d_2, \dots, d_k; d)$ . We shall denote it by  $F(d_1, \dots, d_k; d)$ , and call  $k$  the length of  $F$ . There are canonical projections of  $F(d_1, \dots, d_k; d)$  onto the flag manifolds  $F(d_{j_1}, \dots, d_{j_l})$ , where  $d_{j_1}, \dots, d_{j_l}$  is a subset of  $d_1, \dots, d_k$ . In particular, there are canonical projections  $\pi_i: F \rightarrow G_{d_i, \alpha}$ .

If  $x_i$  is a natural generator of  $H_{2\dim G-2}(G_{d_i, d_i}, Z)$  it is known that  $\pi_1^*(x_1), \dots, \pi_k^*(x_k)$  form a set of generators for

$$H_{2\dim F-2}(F, Z)$$

Finally, we shall denote by  $\rho_i$  the canonical projection of  $F(d_1, \dots, d_k; d)$  onto  $F(d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_k; d)$ .

**THEOREM 3.2.** — *The only holomorphic fields of complex line elements without singularities on a flag manifold  $F$  are fields tangent to the fibres of canonical projections of  $F$  with fibre  $P_1$  onto another flag manifold, in as far as such fiberings exist.*

**PROOF.** — Let  $\xi$  be a holomorphic line-subbundle of  $\theta_F$ . Then  $C_1(\xi) = \sum_{i=1}^k a_i x_i$ . In the same way as in the proof of Theorem 3.1 we find:  $a_i > 0$  for all  $i, i = 1, \dots, k$ . On the other hand, it can also be proved in the same way as in the case of Theorem 3.1 that, if in any point  $p \in F, \xi$  is tangent to none of the fibres

$$\rho_i^{-1}(p_i), p_i \in F(d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_k; d)$$

passing through  $p$ , then  $a_i \leq 0$  for all  $i$ . Hence  $\xi$  is tangent to the fibres of a  $\rho_i$  and application of Theorem 3.1 completes the proof.

For the surfaces  $\Sigma_n$  we use the notations of section 2.

**THEOREM 3.3.** — *The only holomorphic field of complex line elements without singularities on  $\Sigma_n, n \geq 1$ , is the field along the fibres.*

**PROOF.** — Let  $\gamma$  be a holomorphic 1-subbundle of  $\theta_{\Sigma_n}$ , and let  $\gamma'$  be the quotient bundle  $\theta/\gamma$ . Then  $C(\theta_{\Sigma_n}) = C_1(\gamma)C_1(\gamma')$ . If we set  $C_1(\gamma) = \alpha x + \beta y$ , we find by elimination of  $C_1(\gamma')$

$$(2) \quad \alpha = \frac{n\beta^2 - n\beta - 2\beta - 4}{2(\beta - 1)}$$

(the case  $\beta = 1$  is easily excluded). Now if  $\gamma$  is not the bundle along the fibres, application of Proposition 2.4 to the restriction of  $\gamma$  to a fibre and to  $K_n$  gives

$$(3) \quad \beta \leq 0, \text{ and either } \alpha = n\beta + 2 \text{ or } \alpha \leq n(\beta - 1).$$



An elementary calculation shows that (2) and (3) are not compatible for  $\beta \leq -1$ . Therefore, we are left with the case:  $n$  arbitrary,  $\alpha = 2$ ,  $\beta = 0$ , *i.e.* to finish our proof we have to show that  $\theta \otimes \xi^{-2}$  has no holomorphic sections with isolated zeros.

This will be proved when we show that each section of  $\theta$ , which is zero on two fibres is zero on  $K_n$ .

We can cover  $\Sigma_n$  by four maps, each biregularly equivalent to the affine plane  $\mathbf{C}_2$ , with coordinates  $(z_1, \xi_1)$ ,  $(z'_1, \eta_1)$ ,  $(z_2, \xi_2)$  and  $(z'_2, \eta_2)$  respectively, such that

(i) the fibres of  $\Sigma_n$  are given by  $z_1 = \text{constant}$  and  $z_2 = \text{constant}$ ;

(ii)  $K_n$  is given by  $\xi_1 = 0$  and  $\xi_2 = 0$ ;

(iii) the coordinate transformations are (*a.o.*) given by

$$(I) \quad \begin{cases} z'_1 = z_1 \\ \eta_1 = \xi_1^{-1} \end{cases} \quad (II) \quad \begin{cases} z_2 = z_1^{-1} \\ \xi_2 = z_1^n \xi_1 \end{cases} \quad (III) \quad \begin{cases} z'_2 = z_2 \\ \eta_2 = \xi_2^{-1} \end{cases}$$

everywhere where these expressions are defined. Then the restriction of a holomorphic vector  $\mathfrak{F}$  on  $\Sigma_n$  to  $\mathbf{C}_2(z_1, \xi_1)$  is given by  $\omega_1(z_1, \xi_1) \frac{\partial}{\partial z_1} + \omega_2(z_1, \xi_1) \frac{\partial}{\partial \xi_1}$  where  $\omega_1$  and  $\omega_2$  are polynomials in  $z_1$  and  $\xi_1$ . Now from (I) we get  $\frac{\partial}{\partial \xi_1} = -\eta_1^2 \frac{\partial}{\partial \eta_1}$ , so on  $\mathbf{C}_2(z'_1, \eta_1)$   $\mathfrak{F}$  is given by

$$\omega_1(z'_1, \eta_1^{-1}) \frac{\partial}{\partial z'_1} + \eta_1^2 \omega_2(z'_1, \eta_1^{-1}) \frac{\partial}{\partial \eta_1}.$$

It follows, that  $\omega_1$  is a polynomial of degree 0 in  $\xi_1$  and  $\omega_2$  a polynomial of degree  $\leq 2$  in  $\xi_1$ , *i.e.*

$$\omega_2(z_1, \xi_1) = A(z_1)\xi_1^2 + B(z_1)\xi_1 + C(z_1)$$

where  $A(z_1)$ ,  $B(z_1)$  and  $C(z_1)$  are polynomials in  $z_1$ . From (II) we derive

$$\begin{aligned} \frac{\partial}{\partial z_1} &= -z_2^2 \frac{\partial}{\partial z_2} + nz_2 \xi_2 \frac{\partial}{\partial \xi_2} \\ \frac{\partial}{\partial \xi_1} &= z_2^{-n} \frac{\partial}{\partial \xi_2} \end{aligned}$$

Therefore, on  $C_2(z_2, \xi_2)$   $\mathfrak{F}$  is given by

$$\begin{aligned} \omega_1(z_2^{-1}) \left( -z_2^2 \frac{\partial}{\partial z_2} + nz_2 \xi_2 \frac{\partial}{\partial \xi_2} \right) + \omega_2(z_2^{-1}, z_2^n \xi_2) z_2^{-n} \frac{\partial}{\partial \xi_2} \\ = -z_2^2 \omega_1(z_2^{-1}) \frac{\partial}{\partial z_2} + (nz_2 \xi_2 \omega_1(z_2^{-1}) + z_2^{-n} \omega_2(z_2^{-1}, z_2^n \xi_2)) \frac{\partial}{\partial \xi_2} \end{aligned}$$

It follows that degree  $\omega_1 \leq 2$ , i.e. that  $\omega_1(z_1) = az_1^2 + bz_1 + c$ , where  $a, b$  and  $c$  are constants, and that

$$\begin{aligned} nz_2 \xi_2 (az_2^2 + bz_2^{-1} + c) + z_2^{-n} (A(z_2^{-1}) z_2^{2n} \xi_2^2 + B(z_2^{-1}) z_2^n \xi_2 + c(z_2^{-1})) \\ = naz_2^{-1} \xi_2 + nb \xi_2 + nc z_2 \xi_2 + A(z_2^{-1}) z_2^{2n} \xi_2^2 \\ + B(z_2^{-1}) \xi_2 + z_2^{-n} C(z_2^{-1}) = A(z_2^{-1}) z_2^{2n} \xi_2^2 \\ + (naz_2^{-1} + nb + nc z_2 + B(z_2^{-1})) \xi_2 \\ + z_2^{-n} C(z_2^{-1}) \end{aligned}$$

is a polynomial in  $z_2$  and  $\xi_2$ . This means, that degree  $A(z_1) \leq n$ ,  $C = 0$  and  $B(z_1) = -naz_1 + \text{constant}$ . We conclude that the restriction of any field  $\mathfrak{F}$  to  $C_2(z_1, \xi_1)$  is given by

$$\omega_1 \frac{\partial}{\partial z_1} + \omega_2 \frac{\partial}{\partial \xi_1}, \text{ with}$$

$$\begin{aligned} \omega_1 &= az_1^2 + bz_1 + c \\ \omega_2 &= (d_n z^n + d_{n-1} z^{n-1} + \dots + d_0) \xi_1^2 + (-naz_1 + e) \xi_1 \end{aligned}$$

where  $a, b, c, d_0, \dots, d_n$  and  $e$  are constants (It is easy to check that these constants can be chosen arbitrary, from which it follows that  $\dim H^0(\Sigma_n, \theta) = n + 5$  for  $n \geq 1$ , but we do not need this fact). Now if  $\mathfrak{F}$  is identically zero on  $z_1 = 0$  and  $z_2 = 0$ , then  $a = b = c = 0$ , i.e.  $\mathfrak{F}$  is also identically zero on  $K_n$ .

REMARK. — On  $\Sigma_n$  there are several homotopy classes of continuous subbundles of  $\theta$ . For example, for  $n = 1$  we find, in addition to the case  $C_1(\xi) = x + 2y$  (of the field along the fibres) also the cases  $\alpha = 0, \beta = -1$ ;  $\alpha = 2, \beta = 0$ ;  $\alpha = 2, \beta = 3$ ;  $\alpha = -1, \beta = -2$ . According to Theorem 3.3 these homotopy classes cannot be represented by holomorphic subbundles of  $\theta$ .

**The product  $V$  of  $P_1$  and an elliptic curve  $E$ .**

Let  $x \in H_2(V, Z)$  be the homology class of the fibre  $p \times E$ , and  $y \in H_2(V, Z)$  the homology class of a fibre  $P_1 \times e$ ,

$e \in E$ . Then  $x$  and  $y$  form a base of  $H_2(V, \mathbf{Z})$ . Obviously,  $C_1(V) = 2x$  and  $C_2(V) = 0$ . From this it follows immediately that there are the following continuous subbundles  $\xi$  of  $\theta$ :

- (a)  $C_1(\xi) = ax, \quad a \in \mathbf{Z}$  arbitrary;  
 (b)  $C_1(\xi) = x + by, B \in \mathbf{Z}$  arbitrary.

In the second case, the restriction of  $\xi$  to a fibre  $P_1 \times e$  has degree 1. It follows from Proposition 2.4 that this is impossible if  $\xi$  is holomorphic. Therefore, all the homotopy classes (b) of continuous fields of complex line elements cannot be represented by holomorphic fields. On the other hand, the classes (a) can be represented by holomorphic fields if and only if  $a = 2$  or  $a \leq 0$ . In fact, since  $\theta_V|_P \times E$  and  $\xi|_P \times E$  are trivial vector bundles, each holomorphic 1-subbundle of  $\theta$  is the pull back of a holomorphic 1-subbundle  $\chi$  of  $\eta^2 \oplus \tau$ , where  $C_1(\eta)$  is the natural generator of  $H_0(P_1, \mathbf{Z})$  and  $\tau$  the trivial line bundle on  $P_1$ . (Observe, that  $\theta_V$  is the pull back on  $V$  of  $\eta^2 \oplus \tau$ ). But the holomorphic subbundles of  $\eta^2 \oplus \tau$  are known by 2.3 and Proposition 2.4. There is one of degree 2, namely  $\eta^2$  itself, and an infinity of rational families of them of each degree  $\leq 0$ . Since  $a = \text{degree}(\chi)$ , our claim is proved.

Summarizing, we can say: On  $P_1 \times E$  there is an infinite number of homotopy classes of continuous 1-subbundles of the tangent bundle, with Chern classes  $ax$ ,  $a \in \mathbf{Z}$  arbitrary and  $x + by$ ,  $b \in \mathbf{Z}$  arbitrary. The classes with Chern class  $ax$ ,  $a = 1$  and  $a \geq 3$ , and all the classes with Chern class  $x + by$  cannot be represented by holomorphic subbundles; the class  $2x$  by exactly one subbundle, the one tangent to the fibres  $P_1 \times e$ , and the classes with Chern class  $ax$ ,  $a \leq 0$  by an infinite family of subbundles, naturally parametrised by a Zariski open subset of a projective space.

Using the methods of this section, and the theorem of Borel and Remmert, it seems possible to determine all holomorphic fields of complex line elements on any compact homogeneous Kähler manifold, at least in the following restricted sense. First of all, it seems reasonable to expect a theorem valid for all rational homogeneous manifolds  $V_d$ , and extending Theorems 3.1 and 3.2. Probably, such a theorem can be proved rather easily in the same way as in the special cases just mentioned, by using the results of J. Tits in [17]. On the other hand, on a

torus  $T_e$  the situation is completely different. Since the tangent bundle of  $T_e$  is trivial, a holomorphic 1-subbundle can be seen as a holomorphic map of  $T_e$  into  $P_{e-1}$ . For a « general » torus there are no such maps but constant maps ([14], §2). However, for special tori there are many such maps and to find all of them one has to solve two problems of different nature ([15], § 2): (a) to determine all homomorphisms with a subtorus as fibre from  $T_e$  onto any lower dimensional torus  $T_f \subset P_{e-1}$ , and (b) to find all (ramified) coverings of  $T_f$  onto algebraic varieties in  $P_{e-1}$ . Modulo these problems it should be possible to find all holomorphic fields of complex line elements on a product  $V_d \times T_e$ .

The problem to find all holomorphic fields of complex line elements on a compact, complex manifold  $V_d$ ,  $d \geq 2$ , is clearly equivalent to the problem of finding all exact sequences

$$(4) \quad 0 \rightarrow \tau \rightarrow \theta_V \otimes \xi^{-1}$$

where  $\xi$  is a suitable holomorphic and  $\tau$  the trivial line bundle on  $V_d$ . Given  $\xi$ , the following conditions are obviously necessary for the existence of a sequence (4):

$$(a) \quad C_d(\theta_V \otimes \xi^{-1}) = 0$$

$$(b) \quad \dim H_0(V, \theta_V \otimes \xi^{-1}) > 0$$

However, these conditions are not sufficient. The trouble is, that all holomorphic sections of  $\theta_V \otimes \xi^{-1}$  can be « non general » in the sense that their (analytic) set of zeros has components of dimension  $> 0$ . This can happen in the simplest cases, as is shown by the following example, for which  $\xi = \tau$ . Let  $C$  be a curve of genus  $g \geq 2$ ,  $W_1 = P_1 \times C$ , and  $W_2$  obtained from  $W_1$  by blowing up  $4g - 4$  points  $a_1, \dots, a_{4g-4}$  on one and the same fibre  $p \times C$ ,  $p \in P_1$ . Since  $C_2(W_1) =$  Euler Poincaré characteristic of  $W_1 = 2(2 - 2g)$ , and blowing up a point adds one to the Euler characteristic, we have  $C_2(W_2) = (4 - 4g) + (4g - 4) = 0$ . The identity component of the group of automorphisms of  $W_1$ , leaving  $a_1, \dots, a_{4g-4}$  all fixed is naturally isomorphic to the identity component  $G$  of the group of all automorphisms of  $W_2$ . Each projectivity of  $P_1$ , leaving  $p$  fixed induces an automorphism of  $W_1$ , leaving  $p \times C$  pointwise fixed. Therefore,  $\dim H^0(W_2, \theta_{W_2}) > 0$ . On the other hand each automorphism  $g \in G$  leaves a curves

invariant, because it is lifted from an automorphism of  $W_1$ , leaving  $p \times \mathbb{C}$  pointwise fixed. Hence there are holomorphic sections of  $\theta_{W_1}$ , not identically zero, but all of them are zero on a curve.

It does not seem obvious to give a simple criterion that guarantees that  $\theta_V \otimes \xi^{-1}$  has at least one section without zero's, once the conditions (a) and (b) are satisfied. But even so, in practise sometimes it will already be difficult to find all holomorphic line-bundles  $\xi$  satisfying (a) and (b). In many cases, like in those of this section, it is possible to limit the possibilities a priori, by using special devices. For example, suppose a bundle  $\xi$  is found, satisfying condition (a). Then, if  $\dim H^0(V, \xi) > 0$  it follows that for each section  $s$  of  $\xi$  there is a 1-parameter group of automorphisms of  $V$ , leaving the zero set of  $s$  pointwise fixed, which is often seen to be impossible. Also, in many cases a bundle  $\xi$  is excluded because its restriction to a (non singular) rational curve on  $V$  would be such that it contradicts Proposition 2.4.

#### 4. Holomorphic fields of line elements with isolated singularities.

Let  $V_d$  be a complex manifold,  $\xi$  a holomorphic line bundle on  $V$ . If  $\theta_V \otimes \xi$  has a holomorphic section with only isolated zeros, this section gives a holomorphic field of complex line elements with isolated singularities on  $V$  in an obvious way. If  $V$  is compact, the index sum of this field equals  $C_d(\theta_V \otimes \xi)$ . We shall show that this is the only way of getting holomorphic fields of complex line elements with isolated singularities on any complex manifold  $V_d$ ,  $d \geq 2$ .

**LEMMA 4.1.** — *Let  $V_d$ ,  $d \geq 2$ , be a complex manifold,  $\alpha$  a holomorphic  $d$ -vector bundle on  $V$ ,  $\beta$  the associated bundle with fibre  $\mathbb{P}_{d-1}$ ,  $x$  a point on  $V$ ,  $s$  a holomorphic section of  $\beta|V - x$ . Then there is a neighbourhood  $U$  of  $x$  on  $V$ , and a holomorphic section  $s'$  of  $\alpha|U$ , zero at most at  $x$ , such that on  $U - x$ ,  $s = \pi(s')$ , where  $\pi$  is the canonical bundle map.*

**PROOF.** — Let  $W$  be a neighbourhood of  $x$  on  $V$ , such that  $\alpha|W$  is trivial, and let  $S|W$  be given by  $\xi_1(p), \dots, \xi_d(p)$ , with  $p \in W$  and  $(\xi_1, \dots, \xi_d)$  a set of homogeneous coordinates in

$\mathbb{P}_{d-1}$ . The functions  $\frac{\xi_i(p)}{\xi_d(p)}$  are meromorphic on  $W - x$ , for in each point  $p \in W$  there is at least one index  $i$ , such that  $\xi_i(p) \neq 0$  and  $\frac{\xi_i(p)}{\xi_d(p)} = \frac{\xi_i(p)}{\xi_j(p)} \left(\frac{\xi_d(p)}{\xi_j(p)}\right)^{-1}$ . According to Levi's extension theorem for meromorphic functions ([2], p. 50)  $\frac{\xi_i(p)}{\xi_d(p)}$  are meromorphic also in  $x$ . Hence there is a neighbourhood  $W_i$  of  $x$ , such that as far as  $\frac{\xi_i(p)}{\xi_d(p)}$  is defined on  $W_i - x$  this function is given by  $\frac{g_i(p)}{h_i(p)}$ ,  $g_i$  and  $h_i$  holomorphic in  $W_i$ .

If we set on  $U' = W_1 \cap \dots \cap W_{d-1}$  for  $i = 1, \dots, d - 1$   $k'_i(p) = g_i(p) \cdot h_1(p) h_2(p) \dots h_{i-1}(p) h_{i+1}(p) \dots h_d(p)$  and

$$k'_d(p) = h_1(p) \dots h_d(p),$$

then  $(k'_1(p), \dots, k'_d(p))$  is a section  $s''$  of  $\alpha|U'$  with  $\pi(s'') = s$  everywhere where  $k'_1(p), \dots, k'_d(p)$  do not vanish simultaneously. Hence  $s''$  is a section of  $s$ , considered as a subbundle of  $\alpha|U' - x$ . The zero set is an analytic set  $A$  of complex codimension 1 on  $U'$ , on which all functions  $k'_1, \dots, k'_d$  vanish. After dividing  $k'_1, \dots, k'_d$  by suitable powers of the local equations of the irreducible components of  $A$  at  $x$  and restricting to a suitable neighbourhood  $U \subset U'$  of  $x$  we get functions  $k_1, \dots, k_d$  giving a section  $s$  of  $\alpha|U$  as was required.

**THEOREM 4.2.** — *Let  $V_d, d \geq 2$ , be a complex manifold,  $\{x_i\}, i \in I$  a set of isolated points on  $V, X = \bigcup_{i \in I} x_i, \alpha$  a holomorphic  $d$ -vectorbundle on  $V, \beta$  the associated bundle with fibre  $\mathbb{P}_{d-1}, s$  a section of  $\beta|V - X$ . Then there is a holomorphic line bundle  $\xi$  on  $V$  and a holomorphic section  $s'$  of  $\alpha \otimes \xi$  such that on  $V - X$   $s = \pi(s')$ , where  $\pi$  is the canonical bundle map.*

**PROOF.** — Let  $U_i$  be an open neighbourhood of  $x_i$ , such that  $U_i \cap U_j = \emptyset$  if  $i \neq j$ .  $s$  can be considered as a holomorphic 1-subbundle  $\gamma$  of  $\alpha$ . If  $U_i$  is small enough, it follows from Lemma 4.1 that  $\gamma$  has a non vanishing section over  $U_i - x_i$ , hence  $\gamma$  is trivial on  $U_i - x_i$ . Therefore  $\gamma$  can be extended to a bundle on  $V$ . Now  $\alpha \otimes \gamma^{-1}|V - X$  has a section  $s''$  with  $\pi(s'') = s$ . By Hartogs theorem  $s''$  can be exten-

ded to a section  $s'$  of  $\alpha \otimes \gamma^{-1}$  over  $V$ . Setting  $\xi = \gamma^*$  we get the statement of the theorem.

**COROLLARY.** — *Let  $V_d$  be a complete algebraic manifold,  $s$  a holomorphic field of complex line elements with isolated singularities on  $V$ . Then  $s$  is a rational field.*

Theorem 4.2. reduces the problem of finding all holomorphic fields with isolated singularities on a complex manifold  $V_d$  to the problem to find the holomorphic sections with isolated zeros of all bundles  $\theta_V \otimes \xi$ , where  $\xi$  is a holomorphic line bundle on  $V_d$ . So, if the question is asked wheter on a compact manifold  $V_d$  there is holomorphic field with only isolated singularities and of index sum  $i$ , a necessary condition is the existence of a holomorphic line bundle  $\xi$  on  $V$  with

$$C_d(\theta_V \otimes \xi) = i \quad \text{and} \quad \dim H^0(V, \theta_V \otimes \xi) > 0.$$

However, due to the same troubles as in the case of fields without singularities, this condition is not sufficient. This follows already from the example in the last part of section 3 and also from the results on  $P_1 \times P_1$  given below. The following theorem gives a criterion wheter a holomorphic  $d$ -vector bundle  $\alpha$  on  $V_d$  with  $C_d(\alpha) > 0$  has a section with only isolated singularities.

**THEOREM 4.3.** — *Let  $V_d$  be a compact complex manifold,  $\alpha$  a holomorphic  $d$ -vector bundle on  $V$ . For each point  $x \in V_d$ , let  $g_x$  be the dimension of the linear subspace of the fibre of  $\alpha$  at  $x$ , generated by the global sections on  $V$ . If we set*

$$W(e) = \{x \in V \mid g_x \leq e\}$$

*then for  $0 \leq e \leq d$   $W(e)$  is an analytic subset of  $V$ . If for  $0 \leq e \leq d$   $\dim W(e) \leq e$ , then there exist holomorphic sections of  $\alpha$  with isolated zeros. If  $g_x$  is constant on  $V$  and  $C_d(\alpha) > 0$ , then there exist holomorphic sections of  $\alpha$  with isolated zeros if and only if  $g_x = d$ .*

**PROOF.** — It is clear that the sets  $W(e)$  are analytic subsets of  $V$ , because they are locally given as the sets where the rank of a matrix of holomorphic functions does not exceed a certain number.

Now let  $S = H^0(V, \alpha)$ ,  $g = \dim S$ ,  $T = V \times S$ ,  $\rho: T \rightarrow V$ ,

$\sigma: T \rightarrow S$  the canonical projections,  $B$  the bundle space of  $\alpha$ ,  $\pi: S \rightarrow B$  the canonical holomorphic map and  $s$  the zero section of  $\alpha$ .  $\pi^{-1}(s)$  is an analytic subset  $A$  of  $T$ .  $A$  has a finite number of irreducible components, all of which have dimension  $\leq g$  if  $\dim W(e) \leq e$  for all  $e$ . Since  $\sigma$  is a proper map, it follows from this (see [13]), p. 356) that for all points  $p \in S$  outside of a proper analytic subset of  $S$ ,  $\sigma^{-1}(p)$  consists of a finite number of points, *i.e.* the section  $p \in S$  has only isolated zeros.

Finally, if  $g_x$  is constant on  $V$ ,  $A$  is irreducible. If in addition  $C_d(\alpha) > 0$  and if there exist holomorphic sections of  $\alpha$  with isolated zeros, then  $A$  has dimension  $s$  and  $g_x = d$ .

REMARKS. 1. — In general, the condition of Theorem 4.3 is only sufficient, not necessary, for the existence of sections with isolated zeros, as can be seen from the following example :

Let  $V_d$ ,  $d \geq 2$  be a compact complex manifold, with a holomorphic line bundle  $\xi$  without holomorphic sections, and let  $\tau$  be the trivial line bundle on  $V$ . For the direct sum of  $\tau$  and  $d - 1$  copies of  $\xi$  we have  $\dim W_1 = d \geq 2$ , but obviously there is a holomorphic section of  $\alpha$  on  $V$  without singularities.

2) It follows from the proof of Theorem 4.3 that the holomorphic sections of  $\theta_V \otimes \xi$  with a zero set of strictly positive dimension form (in the case of a compact  $V$ ) an analytic subset  $U$  of the complex vector space  $H^0(V, \theta_V \otimes \xi)$  with the property that  $s \in U$  implies  $\gamma s \in U$  for all complex numbers  $\gamma$ . Therefore, the holomorphic fields of complex line elements with isolated singularities, arising from sections of  $\theta_V \otimes \xi$  are parametrised in a natural way by the points of a Zariski-open subset in the projective space of

$$H^0(V, \theta_V \otimes \xi).$$

Given any holomorphic field of line elements with isolated singularities on  $V$ , there is only *one* holomorphic line bundle  $\xi$  on  $V$ , such that the field arises from a section on  $\theta_V \otimes \xi$ . To see this, first consider the case of a field  $\mathfrak{F}$  without singularities. If  $\theta_V \otimes \eta$  has a section, projecting on  $\mathfrak{F}$ , we have an exact sequence  $0 \rightarrow \tau \rightarrow \theta_V \otimes \eta$ , where  $\tau$  is the trivial line bundle on  $V$ , and hence an exact sequence  $0 \rightarrow \eta^{-1} \rightarrow \theta$ ,



where  $\eta^{-1}$ , considered as a field of line elements, is precisely  $\mathfrak{F}$ . This shows that in this case  $\eta$  is determined in a unique way. In the general case, if a section of  $\theta_V \otimes \xi_1$  and one of  $\theta_V \otimes \xi_2$  give the same field,  $\xi_1$  and  $\xi_2$  are isomorphic outside of the singularities of the field, but such an isomorphism can be extended to all of  $V$ . (This follows for example from Hartogs theorem and the fact that a holomorphic function on a manifold of dimension at least 2 has no isolated zeros).

These remarks give a complete classification of holomorphic fields of complex line elements with isolated singularities on any compact, complex manifold. This classification can be extended in a straight forward way to the case of fields which have singularities in an analytic subset of codimension  $\geq 2$ .

We proceed by considering two special cases :

(1) a product  $V = V^{(1)} \times \dots \times V^{(k)}$ , where  $V^{(1)}, \dots, V^{(k)}$  are irreducible rational homogeneous manifolds;

(2) the surfaces  $\Sigma_n$ ,  $n \geq 1$ .

In the case (1), we use the notations of the proof of Theorem 3.1, denoting by  $\pi_i: V \rightarrow V^{(i)}$  the canonical projection, by  $\mu_i$  the line bundle with Chern class dual to the natural generator of  $H^2(V^{(i)}, \mathbb{Z})$ , and setting  $\pi_i^*(\mu_i) = \eta_i$ .

Then we have

**THEOREM 4.4.** — *The bundle  $\theta_V \otimes \eta_1^{a_1} \otimes \dots \otimes \eta_k^{a_k}$  has holomorphic sections with isolated zeros if and only if all  $a_i$  are non negative; with exception of the case  $k = 1$  and  $V = \mathbb{P}_n$  ( $n \geq 2$ ) in which case  $\theta \otimes \eta^a$  has such sections if and only if  $a \geq -1$ .*

**PROOF.** — For  $n = 1$  the statement follows from the considerations in the proof of Theorem 3.1.

In general, if all  $a_i$  are non negative, it follows from Proposition 2.2 and the fact that all bundles  $\mu_i$ ,  $i = 1, \dots, n$  are ample, that the global sections of  $\theta_V \otimes \eta_1^{a_1} \otimes \dots \otimes \eta_k^{a_k}$  generate each fibre of this bundle. Therefore, the result follows from Theorem 4.3.

Now let  $k \geq 2$ , and suppose at least one  $a_i$ , say  $a_1$ , is negative. The restriction of  $\theta_V \otimes \eta_1^{a_1} \otimes \dots \otimes \eta_k^{a_k}$  to  $V^{(1)} \times p$ ,  $p \in V^{(2)} \times \dots \times V^{(k)}$  is the direct sum of  $\theta_V \otimes \mu_1^{a_1}$  and a positive number of copies of  $\mu_1^{a_1}$ . Since this last bundle has no holomorphic section besides the zero, each section of the

restriction of  $\theta_V \otimes \eta_1^{a_1} \otimes \dots \otimes \eta_n^{a_n}$  to  $V^{(1)} \times p$  is contained in  $\theta_{V^{(1)}} \otimes \mu_1^{a_1}$ . This means that the global sections of

$$\theta_V \otimes \eta_1^{a_1} \otimes \dots \otimes \eta_n^{a_n}$$

do not generate the fibres of this bundle. It follows from the last statement of Theorem 4.3 that this bundle has no holomorphic section with isolated singularities only.

Let  $d$  be the dimension of  $V$ . Since the cohomology ring and the Chern classes of  $V$  are known, the numbers

$$C_d(\theta_V \otimes \eta_1^{a_1} \otimes \dots \otimes \eta_n^{a_n})$$

can be calculated. The same holds for

$$\dim H^0(V, \theta_V \otimes \eta_1^{a_1} \otimes \dots \otimes \eta_n^{a_n}) \quad ([7]).$$

For example, in the case  $n = 1$  and  $V = P_d$  we easily derive from the exact sequence (2.5), from  $\dim H^1(P_d, \eta^a) = 0$  for all  $a$  ([7]), and from  $\dim H^0(P_d, \eta^a) = 0$  if  $a < 0$ ,  $\binom{n+a}{n}$  if  $a \geq 0$ .

$$\dim H^0(P_d, \theta_P \otimes \eta^a) = \begin{cases} 0 & \text{if } a \leq -2 \\ n+1 & \text{if } a = -1 \\ (n+1)\binom{n+a+1}{n} - \binom{n+a}{n} & \text{if } a \geq 0. \end{cases}$$

In particular, the standard fields with one singularity of index 1 (obtained by joining the points of  $P_d$  with a fixed point) are the only ones of index 1. But they are *not* the only ones with exactly one singular point, as will be shown below.

Application of the same ideas to the dual bundle of  $\theta$  (*i.e.* the covariant tangent bundle) gives immediately the results of W. Habicht ([8]), p. 155) and generalizations of these results to quadrics, . . . .

Consider the projective plane  $P_2$  with homogeneous coordinates  $(x_1, x_2, x_3)$ . A triple

$$(f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3)) \neq (0, 0, 0)$$

of homogeneous polynomials  $x_1, x_2, x_3$  of the same degree  $d \geq 1$ , satisfying the identity  $f_1x_1 + f_2x_2 + f_3x_3 \equiv 0$  can be interpreted as a holomorphic field of complex line elements on  $P_2$ .

This field has singularities, which need not to be isolated. It is easy to prove ([9], p. 484) that all solutions  $(f_1, f_2, f_3)$  of degree  $d$  of  $\sum_{i=1}^3 f_i x_i \equiv 0$  can be written in the form

$$\begin{aligned} f_1 &= x_2 g_3 - x_3 g_2 \\ f_2 &= x_1 g_3 + x_3 g_1 \\ f_3 &= x_1 g_2 - x_2 g_1 \end{aligned}$$

where  $g_1, g_2$  and  $g_3$  are arbitrary homogeneous polynomials in  $x_1, x_2, x_3$  of degree  $d - 1$ . A triple  $(g_1, g_2, g_3)$  gives

$$(f_1, f_2, f_3) = (0, 0, 0)$$

if and only if  $(g_1, g_2, g_3) = (hx_1, hx_2, hx_3)$ , where  $h$  is an arbitrary homogeneous polynomial of degree  $d - 2$  in  $x_1, x_2, x_3$ . It follows that the fields of degree  $d$  form a projective space of dimension  $3 \binom{d+1}{2} - \binom{d}{2}$  for  $d \geq 2$  and 2 for  $d = 1$ . The fields with isolated singularities in this family will be the fields arising from sections with isolated zeros of the bundle  $\theta_{\mathbb{P}^3} \otimes \eta^{d-2}$ . The situation can also be described in a slightly different way. Attach to the point  $(x_1, x_2, x_3)$  the line element through this point determined by line through  $(x_1, x_2, x_3)$  and  $(g_1, g_2, g_3)$ . The singularities of the field are the points  $(x_1, x_2, x_3)$  for which  $(x_1, x_2, x_3) = (g_1, g_2, g_3)$ . For each  $d$  there are fields with only one singular point; for example, if we set

$$g_1(x_1, x_2, x_3) = x_2^{d-1}, g_2(x_1, x_2, x_3) = x_3^{d-1} \text{ and } g_3(x_1, x_2, x_3) = 0,$$

then the only singularity is the point  $(1, 0, 0)$ . This construction has an obvious generalization to  $\mathbb{P}_n$ ,  $n \geq 2$ : attach to each point  $(x_1, \dots, x_{n+1})$  the line element determined by the line through  $(x_1, \dots, x_{n+1})$  and  $(g_1, \dots, g_{n+1})$  where  $g_1, \dots, g_{n+1}$  are homogeneous polynomials in  $x_1, \dots, x_{n+1}$  of degree  $d - 1$ ,  $d \geq 1$ . In particular, for each  $d$  there are again fields with only one singular point, one can take for example

$$g_1 = x_2^{d-1}, \dots, g_n = x_{n+1}^{d-1}, g_{n+1} = 0.$$

This description can easily be translated in terms of Plücker coordinates.

In the topological case, if on  $V_d$  a  $d$ -vector bundle  $\alpha$  is given,

there are continuous sections of  $\alpha$  which have in prescribed points  $a_1, \dots, a_k$  on  $V_d$  zeros of prescribed index  $s_1, \dots, s_k$ , provided of course that  $\sum_{j=1}^k s_j = C_d(\alpha)$ . This is not true in the holomorphic case, even if we restrict ourselves to positive indices and to bundles with many holomorphic sections with isolated zeros. A simple example is the following one. Take for  $V$  the projective plane  $P_2$  and for  $\alpha$  the tangent bundle  $\theta_P$ . Then there is no holomorphic section of  $\theta$  having zeros (of index 1) only on three points on a line, because each projectivity of  $P_2$  leaving three points on a line fixed leaves every point of the line fixed. One can ask, however, more restrictive questions like the following one. Be given positive integers  $n_1, \dots, n_k$  with  $\sum_{i=1}^k n_i = C_d(\alpha)$ , does there exist a holomorphic section of  $\alpha$  having exactly  $k$  zeros, of index  $n_1, \dots, n_k$  respectively? In particular, does there exist a holomorphic section of  $\alpha$  with only one singularity (of index  $C_d(\alpha)$ )? It does not seem easy to give a general answer to these questions. In the case of  $P_n$  the remarks above will provide a positive answer to the last question for all bundles  $\theta_{P_n} \otimes \eta^a$  which have holomorphic sections.

As another example, consider the 2-dimensional quadric  $V = P_1 \times P_1$ . Let  $x, y \in H_2(V, Z)$  be represented by the horizontal and vertical fibres, and let  $\xi, \eta$  be the line bundles on  $V$  with  $C_1(\xi) = x$  and  $C_1(\eta) = y$ . Then each line-bundle on  $V$  is of the form  $\xi^a \otimes \eta^b$ ,  $a, b \in Z$ , and we have

$$C_2(\theta_V \otimes \xi^a \otimes \eta^b) = 2(ab + a + b + 2).$$

According to Theorem 4.4,  $\theta_V \otimes \xi^a \otimes \eta^b$  has holomorphic sections with a positive number of isolated zeros if and only if  $a \geq 0, b \geq 0$ . Therefore, in this case, as index sums of continuous fields can occur all even numbers, and as index sums of holomorphic fields only the non negative even numbers with exception of 2. All the bundles with either

$$a = -1, b \geq 0 \text{ or } a \geq 0, b = -1$$

have holomorphic sections, but never with isolated zeros.

On the surfaces  $\Sigma_n$  we have, with the notations of section 2.

**THEOREM 4.5.** — *The bundle  $\theta_{\Sigma_n} \otimes \xi^a \otimes \eta^b$  has holomorphic sections with a positive number of isolated zeros if and only if  $b \geq 0$ ,  $a \geq nb - 1$ .*

**PROOF.** — We have the exact sequence

$$0 \rightarrow \xi^{n+a} \otimes \eta^{2+b} \rightarrow \theta \otimes \xi^a \otimes \eta^b \rightarrow \xi^{2+a} \otimes \eta^b \rightarrow 0$$

with the corresponding exact cohomology sequence (\*)

$$\begin{aligned} 0 \rightarrow H^0(\Sigma_n, \xi^{n+a} \otimes \eta^{2+b}) \rightarrow H^0(\Sigma_n, \theta \otimes \xi^a \otimes \eta^b) \\ \rightarrow H^0(\Sigma_n, \xi^{2+a} \otimes \eta^b) \rightarrow H^1(\Sigma_n, \xi^{n+a} \otimes \eta^{2+b}) \rightarrow \dots \end{aligned}$$

Now if  $b < 0$ ,  $H^0(\Sigma_n, \xi^{2+a} \otimes \eta^b) = 0$  (section 2), hence in this case  $\theta \otimes \xi^a \otimes \eta^b$  has either no holomorphic sections, or sections with a zero divisor or sections with no zeros at all. Therefore, in our case we have  $b \geq 0$ . Let  $\rho$  be the line-bundle on  $K_n$  with  $C_1(\rho)$  equal to the natural generator of  $H_0(K_n, \mathbb{Z})$ . Then  $\theta \otimes \xi^a \otimes \eta^b|_{K_n} = \rho^{a-n(b+1)} \oplus \rho^{2+a-nb}$  has no non vanishing holomorphic sections if  $a < nb - 2$ , so we find  $a \geq nb - 2$ .

First we consider the case  $a \geq nb - 1$ . We claim: the global sections of  $\theta \otimes \xi^{-1}$  generate  $\theta \otimes \xi^{-1}|_p$  if  $p \in \Sigma_n - K_n$ . Let  $F$  be any fibre of  $\Sigma_n$ .  $\theta \otimes \xi^{-1}|_F = \sigma^2 \oplus \tau$ , where  $C_1(\sigma)$  equals the natural generator of  $H_0(F, \mathbb{Z})$  and  $\tau$  the trivial line-bundle on  $\Sigma_n$ . A holomorphic section of  $\theta \otimes \xi^{-1}|_F$  is either contained in the subbundle  $\sigma^2$  or has no vector in common with it, in particular not the zero vector. From (\*) and section 2, we find  $H^1(\Sigma_n, \theta \otimes \xi^{-1} \otimes \eta^{-1}) = 0$ , hence the homomorphism of  $H^0(\Sigma_n, \theta \otimes \xi^{-1})$  into  $H^0(K_n, \theta \xi^{-1}|_{K_n})$ , induced by the embedding of  $K_n$  into  $\Sigma_n$ , is surjective. It follows that there is a global section  $s$  of  $\theta \otimes \xi^{-1}$ , such that  $s|_F$  is transversal to  $\sigma^2$  (more precisely transversal to the unique subbundle of  $\theta \otimes \xi^{-1}|_F$  which is isomorphic to  $\sigma^2$ ). It remains to be shown that for each point  $q \in F$ ,  $q \notin K_n$  there is a section of  $\theta \otimes \xi^{-1}$  not zero at  $q$  and contained in  $\sigma^2$  along  $F$ . This is clear, however, because there is such a section in  $\xi^{n-1} \otimes \eta^2$  and this bundle is a subbundle of  $\theta \otimes \xi^{-1}$ , the restriction of which to  $F$  is  $\sigma^2$ . To finish the proof of the if-part of the theorem, it will be sufficient to show (Theorem 4.3) that for all  $a, b$ ,  $b \geq 0$ ,  $a \geq nb - 1$  there are global sections of  $\theta \otimes \xi^a \otimes \eta^b$  not identically zero on  $K_n$ . Since

$\dim H^0(K_n, \theta \otimes \xi^a \otimes \eta^b|_{K_n}) \neq 0$ , it is sufficient to prove that for  $b \geq 0, a \geq nb - 1, H^1(\Sigma_n, \theta \otimes \xi^a \otimes \eta^{-1}) = 0$ . This is a direct consequence of the sequence (\*) and section 2. We are left with the case  $b \geq 0, a = nb - 2$ . We shall prove that each holomorphic section of  $\theta \otimes \xi^{nb-2} \otimes \eta^b$  is identically zero on  $K_n$ , or, equivalently :

$$\dim H^0(\Sigma_n, \theta \otimes \xi^{nb-2} \otimes \eta^b) = \dim H^0(\Sigma_n, \theta \otimes \xi^{nb-2} \otimes \eta^{b-1}).$$

In the case  $b = 0$ , it follows from (\*) and Theorem 3.2 that we have an isomorphism

$$0 \rightarrow H^0(\Sigma_n, \xi^{n-2} \otimes \eta^2) \rightarrow H^0(\Sigma_n, \theta \otimes \xi^{-2}) \rightarrow 0$$

and it is clear that each section of  $\xi^{n-2} \otimes \eta^2$  is identically zero on  $K_n$ . From now on we assume  $b \geq 1$ . Since we have standard injections

$$0 \rightarrow \theta \otimes \xi^{nb-2} \otimes \eta^{b-2} \rightarrow \theta \otimes \xi^{nb-2} \otimes \eta^{b-1}$$

and

$$0 \rightarrow \theta \otimes \xi^{nb-2} \otimes \eta^{b-1} \rightarrow \theta \otimes \xi^{nb-2} \otimes \eta^b$$

we can define a coherent sheaf  $\mathfrak{S}_{n,b}$  on  $\Sigma_n$  by the exact sequence

$$0 \rightarrow \theta \otimes \xi^{nb-2} \otimes \eta^{b-2} \rightarrow \theta \otimes \xi^{nb-2} \otimes \eta^b \rightarrow \mathfrak{S}_{n,b} \rightarrow 0$$

$\mathfrak{S}_{n,b}$  is concentrated on  $K_n$ . Since

$$\dim H^1(\Sigma_n, \theta \otimes \xi^{nb-2} \otimes \eta^{b-2}) = 0$$

by (\*) and section 2, and since

$$\dim H^0(K_n, \theta \otimes \xi^{nb-2} \otimes \eta^{b-1}|_{K_n}) = n + 1,$$

it will be sufficient to show that  $\dim H^0(\Sigma_n, \mathfrak{S}_{n,b}) = n + 1$ . We consider separately the cases  $n \geq 2$  and  $n = 1$ .

Let  $n \geq 2$ . Since  $\xi^{nb} \otimes \eta^b$  is trivial in a neighbourhood of  $K_n$ ,  $\mathfrak{S}_{n,b}$  is isomorphic with

$$\begin{aligned} \mathfrak{S} &= \mathfrak{S}_{n,1} \dim H^0(\Sigma_n, \mathfrak{S}) = \dim H^0(\Sigma_n, \theta \otimes \xi^{n-1} \otimes \eta) \\ &\quad - \dim H^0(\Sigma_n, \theta \otimes \xi^{n-2} \otimes \eta^{-1}) \\ &= \dim H^0(\Sigma_n, \theta \otimes \xi^{n-2} \otimes \eta) - (3n - 2) \end{aligned}$$

Let C be a cross section of  $\Sigma_n$ , of homology class  $nx + y$ , and consider the exact sequence

$$0 \rightarrow \theta \otimes \xi^{-2} \rightarrow \theta \otimes \xi^{n-2} \otimes \eta \rightarrow \theta \otimes \xi^{n-2} \otimes \eta|_C \rightarrow 0$$

Now if we can show that the induced homomorphism  $\alpha: H^0(\Sigma_n, \theta \otimes \xi^{n-2} \otimes \eta) \rightarrow H^0(C, \theta \otimes \xi^{n-2} \otimes \eta|_C)$  is surjective, then we are ready, because then we have

$$\begin{aligned} \dim H^0(\Sigma_n, \mathcal{O}) &= \dim H^0(\Sigma_n, \theta \otimes \xi^{n-2} \otimes \eta) - (3n - 2) \\ &= \dim H^0(\Sigma_n, \theta \otimes \xi^{-2}) + 2 \\ &= \dim H^0(\Sigma_n, \xi^{n-2} \otimes \eta^2) + 2 = n + 1. \end{aligned}$$

To show that  $\alpha$  is surjective, we proceed as follows.  $(\gamma^n \oplus \gamma^2) \otimes \gamma^{n-2} = \theta \otimes \xi^{n-2} \otimes \eta|_C$ , where  $C_1(\gamma)$  is a natural generator of  $H_0(C_1, \mathbf{Z})$ .  $\alpha$  will be surjective if and only if the natural homomorphisms  $H^0(\Sigma_n, \theta) \rightarrow H^0(C, \gamma^n \oplus \gamma^2)$  and  $H^0(\Sigma_n, \xi^{n-2} \otimes \eta) \rightarrow H^0(C, \gamma^{n-2})$  are surjective. This is clear for the last one; as to the first one, its surjectivity follows from the following remarks: First, the homomorphism  $H^0(\Sigma_n, \xi^n \otimes \eta^2) \rightarrow H^0(C, \xi^n \otimes \eta^2|_C)$ , induced by the embedding of  $C$  in  $\Sigma_n$  is surjective. Second, there is an automorphism of  $\Sigma_n$ , leaving exactly two prescribed fibres invariant. This automorphism gives a section of  $\theta$ , the restriction of which to  $C$  is the sum of a field, as considered in the first remark and a field tangent to  $C$  with two prescribed zeros.

For  $n = 1$  the argument is analogous, but for the fact that instead of proving the surjectivity of the natural homomorphism  $H^0(\Sigma_1, \theta) \rightarrow H^0(C, \theta|_C)$  one proves the surjectivity of the natural homomorphism

$$H^0(\Sigma_1, \theta \otimes \xi^{-1}) \rightarrow H^0(C, \theta \otimes \xi^{-1}|_C).$$

This again follows from two facts:

1) the natural homomorphism  $H^0(\Sigma, \eta^2) \rightarrow H^0(C, \eta^2|_C)$  is surjective;

2) there is an automorphism of  $\Sigma_1$  leaving a prescribed point  $p \in C$  and a fibre outside  $p$  pointwise fixed, without transforming any other fibre onto itself. This automorphism gives a section of  $H^0(\Sigma_1, \theta)$ , leading in an obvious way to a section of  $H^0(\Sigma_1, \theta \otimes \xi^{-1})$  which gives a section of  $\theta \otimes \xi^{-1}|_C$  with a prescribed zero.

REMARKS. — 1) Another proof of Theorem 4.5 can be given by explicit calculation of  $\dim H^0(\Sigma_n, \theta \otimes \xi^a \otimes \eta^b)$  as was done for  $a = -2, b = 0$  in the proof of Theorem 3.3.

(2) We have

$$C_2(\theta \otimes \xi^a \otimes \eta^b) = 2ab - nb^2 + bn + 2a + 4b + 4,$$

which is always an even number. All even numbers occur this way, already for  $b = 0$  and  $a$  arbitrary. From the theorem it follows that  $\theta \otimes \xi^a$ ,  $a \geq -1$  has holomorphic sections with isolated zeros, hence all non negative even numbers occur index sums of holomorphic fields of complex line elements.

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