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ON THE REPRESENTATION OF CERTAIN FUNCTIONALS BY MEASURES ON THE CHOQUET BOUNDARY

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1. Introduction.

M. HERVÉ [6] has recently published a simple proof of Choquet's theorem on the representation of the points of a compact convex metrizable subset of a locally convex real linear topological space as barycentres of measures carried by the extreme points of the set. F. F. BONSALL [5] has shown that, by a use of the Hahn-Banach theorem, the discussion can be made still more simple and that a restatement of the problem then allows the convexity condition to be dropped. The present paper shows that further pursuit of these ideas provides new information about the *Choquet boundary*, as defined by BISHOP and DE LEEUW [4]. It is then possible to give a simple direct proof of a result of these authors: that, in the presence of a separability condition (stated in § 4), the Choquet boundary is a G_δ set and every probability Radon measure admits *balayage* onto it. These methods are also shown, in § 5, to lead to a proof of one of Bauer's main theorems in his theory [3] of an abstract Dirichlet problem. The effect of an additional equicontinuity condition is also considered in § 5.

We consider here only real-valued functions, remarking that the passage to the complex case is known [4] to be a straightforward matter.

I am indebted to Professor F. F. BONSALL for showing me his work before publication.

2. Construction of functionals; the Choquet theorem.

We consider a compact Hausdorff space X, the set C of all real continuous functions on X, and a linear manifold L of C that contains the constant functions. We denote by M, M^+, and P respectively the spaces of Radon measures, non-negative Radon measures, and probability Radon measures on X, and by R the set of real numbers. For each $x \in X$ we define the set

$$M_x \equiv M_x(L) = \{\mu \in M^+ | \mu(g) = g(x) \text{ for all } g \in L\}.$$

The unit atomic measure at x, denoted by ε_x, belongs to M_x. Also since $1 \in L$ we have $\mu(1) = 1$ for all $\mu \in M_x$, so that $M_x \subseteq P$.

For each $f \in C$ and each $\sigma \in P$ we define

$$f^*(\sigma) = \inf \{\sigma(g) | g \in L, \quad g \geqslant f\}.$$

Each $f \in C$ is bounded, L contains the constants, and so $f^*(\sigma)$ is well-defined and finite. Evidently

$$(1) \qquad f^*(\sigma) \leqslant \max_{y \in X} f(y) \leqslant \|f\|,$$

and

$$(2) \qquad g^*(\sigma) = \sigma(g) \text{ whenever } g \in L.$$

We adopt the convenient abuse of notation of writing $f^*(x)$ for $f^*(\varepsilon_x)$, so that $x \to f^*(x)$ is precisely the upper semi-continuous real function on X defined by

$$f^*(x) = \inf \{g(x) | g \in L, \quad g \geqslant f\}.$$

LEMMA. — *For each $\sigma \in P$ the map $f \to f^*(\sigma)$ of C into* R *is a sublinear functional on* C.

Let $p(f) = f^*(\sigma)$ for all $f \in C$. Suppose $f_1, f_2 \in C$, let $\varepsilon > 0$, and choose $g_1, g_2 \in L$ such that

$$g_r \geqslant f_r, \qquad \sigma(g_r) < p(f_r) + \varepsilon \quad (r = 1, 2).$$

Then $f_1 + f_2 \leqslant g_1 + g_2 \in L$ and so

$$p(f_1 + f_2) \leqslant \sigma(g_1 + g_2) = \sigma(g_1) + \sigma(g_2) < p(f_1) + p(f_2) + 2\varepsilon.$$

But $\varepsilon > 0$ was arbitrary and so

$$p(f_1 + f_2) \leqslant p(f_1) + p(f_2)$$

for all f_1, $f_2 \in C$. One proves similary that

$$p(\lambda f) = \lambda p(f)$$

for all real $\lambda \geqslant 0$ and $f \in C$.

Now choose $h \in C$ and let W_h be the set of all points $x \in X$ that satisfy the condition that for at least one $\nu \in M_x(L)$ we have

$$\nu(h) > h(x).$$

THEOREM 1. — *For each* $h \in C$.

(3) $$W_h = \{x \in X | h^*(x) > h(x)\};$$

the set W_h *is consequently an* F_σ *set. Moreover, given* $\tau \in P$, *we can find* $\mu \in P$ *(depending on h and* τ*) such that*

(i) $\mu(g) = \tau(g)$ *for all* $g \in L$;
(ii) $\mu(W_h) = 0$.

Let $x \in W_h$ and let $\nu \in M_x$ be such that $\nu(h) > h(x)$. Then if $g \in L$, $g \geqslant h$ we have

$$g(x) = \nu(g) \geqslant \nu(h) > h(x),$$

so that $g(x) - h(x) \geqslant \nu(h) - h(x) > 0$ and hence $h^*(x) > h(x)$. This proves that

(4) $$W_h \subseteq \{x \in X | h^*(x) > h(x)\}.$$

Now take a measure $\sigma \in P$ and write $p(f) = f^*(\sigma)$, for all $f \in C$. Then by the lemma and the Hahn-Banach theorem there exists a linear functional $\nu \equiv \nu_\sigma$ on C that satisfies

$$\nu(f) \leqslant p(f) \quad \text{for all } f \in C,$$

and

$$\nu(h) = p(h).$$

By (1) the functional ν is continuous. For $g \in L$ we have, by (2), $\nu(g) \leqslant p(g) = \sigma(g)$ and also $-g \in L$, so that $-\nu(g) = \nu(-g) \leqslant \sigma(-g) = -\sigma(g)$, whence in fact

(5) $$\nu(g) = \sigma(g) \quad \text{for all } g \in L.$$

Next, (1) implies that for $f \in C$ with $f \leqslant 0$ we have $\nu(f) \leqslant 0$ and hence $\nu(-f) \geqslant 0$, so that $\nu \geqslant 0$, and thus $\nu \in M^+$.

Now take $x \in X$ with $h^*(x) > h(x)$ and let $\sigma = \varepsilon_x$ in the above construction, so that now $\nu \in M_x$, and $p(f) = f^*(x)$ for all $f \in C$. Then $\nu(h) = p(h) = h^*(x) > h(x)$ and therefore $x \in W_h$. So we have

$$\{x \in X | h^*(x) > h(x)\} \subseteq W_h,$$

which with (4) establishes (3).

Next h^*, and hence $(h^* - h)$, is upper semi-continuous and hence

$$F_n = \left\{x \in X | h^*(x) - h(x) \geqslant \frac{1}{n}\right\}$$

is closed. Therefore $W_h = \bigcup_{n=1}^{\infty} F_n$ is an F_σ set.

For the last part let $\mu = \nu_\tau$ as above. Then $\mu \geqslant 0$, and (5) provides the proof of relation (i) of theorem 1 and in particular the fact that $\mu(1) = \tau(1)$, so that $\mu \in P$.

To prove that $\mu(W_h) = 0$ it is enough to show that $\mu(F_n) = 0$ for all $n \geqslant 1$. Suppose there is an exceptional n with

$$\mu(F_n) = \delta > 0.$$

Then if $g \geqslant h$, $g \in L$ we have $g \geqslant h^*$ and consequently

$$(6) \quad \tau(g) - \mu(h) = \mu(g) - \mu(h) \geqslant \int_{F_n} (g - h)\, d\mu \geqslant \frac{\delta}{n}.$$

On the other hand

$$\mu(h) = h^*(\tau) = \inf \{\tau(g) | g \geqslant h, \ g \in L\},$$

which contradicts (6) and completes the proof that $\mu(W_h) = 0$.

Corollary (Choquet). — *Let* X *be a compact convex metrizable set in a locally convex real linear topological space. Then the set* E *of extreme points of* X *is a* G_δ *set. Moreover, for each* $a \in X$ *there exists a probability Radon measure* μ *on* X *such that*

$$\text{(j)} \qquad \mu(g) = g(a), \qquad \textit{for all} \qquad g \in L,$$

where L *is now the set of restrictions to* X *of real continuous affine functions, and,*

(jj) $$\mu(\complement E) = 0.$$

For the proof we take h to be the strictly convex real continuous function on X constructed by HERVÉ [6]. Then it is clear that $W_h \cap E = \emptyset$ (see § 3). But HERVÉ shows that if $x \in E$ then $h^*(x) = h(x)$ so that, by (3), we have $W_h = \complement E$ for this h. On taking $\tau = \varepsilon_a$ in theorem 1 we obtain therefore a $\mu \in P$ satisfying (j) and (jj). In § 4 we present a generalization of this argument.

3. Characterizations of the Choquet boundary.

Now let A(L) denote the smallest uniformly closed subalgebra of C that contains L. Evidently

$$M_x(L) \supseteq M_x(A(L)) \quad \text{for all } x \in X.$$

The *Choquet boundary* of the space X for the class of functions L is by definition the set

$$\partial_L X = \{x \in X | M_x(L) = M_x(A(L))\}.$$

The Weierstrass-Stone theorem, together with a simple measure-theoretic argument like that used to prove proposition 1 below, implies that this definition is equivalent to the slightly different one given by BISHOP and DE LEEUW [4]. If L separates the points of X then A(L) = C and so, in this case,

$$\partial_L X = \{x \in X | M_x(L) = (\varepsilon_x)\}.$$

PROPOSITION 1. — *For each linear subspace* L *of* C *that contains the constants, we have*

(7) $$\partial_L X = \bigcap_{h \in A(L)} \complement W_h = \bigcap_{g \in L} \complement W_{|g|}.$$

We emphasize here that W_f, for $f \in C$, depends on f *and* on L.

Suppose $h \in A(L)$, $x \in W_h$. Then there is a $\nu \in M_x(L)$ with $\nu(h) > h(x)$, so that $\nu \notin M_x(A(L))$ and hence $x \notin \partial_L X$. This shows that

(8) $$W_h \cap \partial_L X = \emptyset \quad \text{for all } h \in A(L).$$

Conversely suppose that $a \notin \partial_L X$, let $\nu \in M_a(L) \setminus M_a(A(L))$, and let supp ν denote the support of ν. Then we can find $b \in \operatorname{supp} \nu$, with $b \neq a$, together with a function $g_1 \in L$ such that $g_1(b) \neq g_1(a)$. For otherwise we should have

$$g(x) = g(a) \quad \text{for all} \quad x \in \operatorname{supp} \nu, \quad g \in L,$$

which would imply

$$h(x) = h(a) \quad \text{for all} \quad x \in \operatorname{supp} \nu, \quad h \in A(L),$$

and hence $\nu \in M_a(A(L))$, contrary to hypothesis.

Now define

$$g(x) = g_1(x) - g_1(a) \quad (x \in X),$$

so that $g \in L$. Then the continuous non-negative function $h = |g|$ is strictly positive at the point $b \in \operatorname{supp} \nu$ and so

$$\nu(h) > 0 = h(a),$$

so that $a \in W_h = W_{|g|}$. So we have proved that

$$\complement \partial_L X \subseteq \bigcup_{g \in L} W_{|g|}$$

which with (8) yields the desired formula (7).

By theorem 1 we now have.

Corollary 1. — *Under the same conditions*

$$(9) \qquad \begin{aligned} \partial_L X &= \{x \in X \mid h^*(x) = h(x) \text{ for all } h \in A(L)\} \\ &= \{x \in X \mid |g|^*(x) = |g(x)| \text{ for all } g \in L\}. \end{aligned}$$

Now write $F = \overline{\partial_L X}$ and consider the restriction map

$$g \to \tilde{g} \equiv g|F$$

from L into the space R(F) of real continuous functions on F, letting $\tilde{L} = \{\tilde{g} \mid g \in L\}$.

Corollary 2. — *If* L *separates the point of* X *then for each* $u \in R(F)$, $x \in \partial_L X$, *we have*

$$(10) \qquad \begin{aligned} u(x) &= \inf\{v(x) \mid v \in \tilde{L}, \quad v \geqslant u\} \\ &= \sup\{w(x) \mid w \in \tilde{L}, \ w \leqslant u\}. \end{aligned}$$

The first equality follows from the proof of (9), applied to the pair $(F, \tilde{L})$ in place of (X, L), and the obvious fact that $\partial_{\tilde{L}}F \supseteq \partial_L X$. The same reasoning applied to $-u$ then yields the second part.

4. Measures on the boundary for separable L.

In this section we suppose that L is separable.

PROPOSITION 2. — *If* L *is a separable linear subspace of* C *that contains the constants then there exists a function* $h \in A(L)$ *such that*

$$\partial_L X = \complement W_h. \tag{11}$$

Let $(g_m)_{m\geqslant 1}$ be a countable dense set in L, and let $(r_n)_{n\geqslant 1}$ be an enumeration of the rationals, and let

$$h = \sum_{m,n\geqslant 1} \frac{1}{2^{m+n}} \frac{h_{mn}}{1+\|h_{mn}\|},$$

where $\quad h_{mn}(x) = |g_m(x) - r_n| \quad (m, n \geqslant 1;\ x \in X),$

so that $h \in A(L)$. We show that this h satisfies (11).

First if $a \in X$, $\nu \in M_a(L)$, $g \in L$, $r \in \mathbf{R}$ then

$$\nu(|g-r|) = \int |g(x)-r|\nu(dx) \geqslant \left|\int (g(x)-r)\nu(dx)\right| = |g(a)-r|,$$

and hence in particular

$$\nu(h_{mn}) \geqslant h_{mn}(a) \quad (m, n \geqslant 1). \tag{12}$$

Now suppose $a \in \partial_L X$ and let $\nu \in M_a(L) \setminus M_a(A(L))$. Then as in the proof of proposition 1 we can find $b \in \operatorname{supp} \nu$, with $b \neq a$, and $p \geqslant 1$ such that $g_p(b) \neq g_p(a)$. We therefore have

$$\int |g_p(x) - g_p(a)|\nu(dx) = \delta > 0.$$

But we can find a rational r_q such that

$$h_{pq}(a) = |g_p(a) - r_q| < \frac{1}{2}\delta.$$

Then

$$\begin{aligned}\nu(h_{pq}) &= \int |g_p(x) - r_q|\nu(dx)\\ &\geqslant \int (|g_p(x)-g_p(a)| - |g_p(a)-r_q|)\nu(dx) > \delta - \frac{1}{2}\delta = \frac{1}{2}\delta.\end{aligned}$$

Hence $\nu(h_{pq}) > h_{pq}(a)$ which together with (12) shows that $\nu(h) > h(a)$, so that $a \in W_h$. We have thus shown that $\complement \partial_L X \subseteq W_h$. But $W_h \cap \partial_L X = \emptyset$ and so (11) is proved.

By theorem 1 we now have the

COROLLARY (BISHOP *and* DE LEEUW). — *If* L *is a separable linear subspace of* C *that contains the constants then the Choquet boundary* $\partial_L X$ *is a* G_δ *set. Moreover, for each* $\tau \in P$ *we can find* $\mu \in P$ *such that*

(i) $\mu(g) = \tau(g)$ *for all* $g \in L$;

(ii) $\mu(\complement \partial_L X)$

5. The boundary when is lattice.

We shall not require L to be separable in this section.

In his paper [3] Bauer has shown that the theory of the Choquet boundary becomes specially satisfactory when L is a lattice. We show here that corollary 2 to proposition 1 makes possible a direct proof of one of Bauer's results, and then consider the effect of an additional equicontinuity condition.

THEOREM 2 (BAUER). — *If* L *is a linear subspace of* C *that contains the constants, separates the points of* X, *and is a lattice for the natural partial ordering, then* $\partial_L X$ *is a closed set and the restriction map* $f \to \tilde{f} \equiv f|\partial_L X$ *from* L *into* $R(\partial_L X)$ *is an isometric linear and lattice isomorphism onto a dense subset of* $R(\partial_L X)$ (*and hence actually onto* $R(\partial_L X)$ *if* L *is complete*). *Moreover, given* $\tau \in M$, *we can find a unique* $\mu \equiv \mu_\tau \in M$ *satisfying*

(i) $\mu(g) = \tau(g)$ *for all* $g \in L$;

(ii) $\operatorname{supp} \mu \subseteq \partial_L X$.

The map $\tau \to \mu_\tau$ *in* M *is linear and it maps* M^+ *isometrically into itself.*

For this we use Bauer's maximum principle [2], which we need only in the following weak form : *if* L *is a linear subspace of*

C that contains the constants and separates the points of X then for each $f \in L$ there is a point $a \in \partial_L X$ such that

$$f(a) = \max_{x \in X} f(x).$$

Now let $F = \overline{\partial_L X}$ and consider the restriction map $f \to \tilde{f} \equiv f|F$ from L into R(F). This is linear and order-preserving. The maximum principle applied to f and to $-f$ shows that it is also an isometry. Now if also L is a lattice for the natural partial ordering then the restriction map preserves the lattice structure. For let $f, g \in L$, $h = f \wedge g$, and let $\tilde{h}$ and $u \in R(F)$ be compared, where

$$u(x) \equiv \min(f(x),\ g(x)) \quad (x \in F).$$

Following e.g. Kadison [7], we have $h \leqslant f$, $h \leqslant g$ and hence $\tilde{h} \leqslant u$. If for some $x \in \partial_L X$ we have $h(x) < u(x)$ then by corollary 2 to proposition 1 we can find $k \in L$ such that $\tilde{k} \leqslant u$ and $h(x) < k(x) \leqslant u(x)$. Then $\tilde{k} \leqslant \tilde{f}$, $\tilde{k} \leqslant \tilde{g}$ and the maximum principle implies that $k \leqslant f$, $k \leqslant g$; whence $k \leqslant f \wedge g = h$, which contradicts the inequality $h(x) < k(x)$. Since $\overline{\partial_L X} = F$ we must therefore have $\tilde{h} = u$; that is, the restriction of $f \wedge g$ to F is equal to $\min(\tilde{f}, \tilde{g})$. Likewise the restriction of $f \vee g$ to F is $\max(\tilde{f}, \tilde{g})$.

The set $\tilde{L}$ is thus a linear sublattice of R(F) that contains the constants and separates points and hence, by the Weierstrass-Stone theorem, it lies densely in R(F). Any continuous linear functional on $\tilde{L}$ is therefore representable by a unique Radon measure on F. The map $\tilde{f} \to \tau(f)$ is such a functional, and so we find $\mu \equiv \mu_\tau \in M$ to satisfy (i) and (ii). The remaining properties of the map $\tau \to \mu_\tau$ are immediate, if we assume that $F = \partial_L X$.

We complete the proof by showing that $F = \partial_L X$. For this let $x \in F$, $\nu \in M_x(L)$, $g \in L$ and let $H = \{y \in X | g(y) \leqslant g(x)\}$. Adapting a construction of Bishop and de Leeuw we write, for any Borel set E, $\tau(E) = \nu(E \cap H)$, $\sigma(E) = \nu(E \setminus H)$, so that $\tau, \sigma \in M^+$, $\tau + \sigma = \nu$. Then $\mu_\tau + \mu_\sigma = \mu_\nu$, and we have: $\mu_\nu = \varepsilon_x$ because L is dense in R(F), $\mu_\tau \geqslant 0$, $\mu_\sigma \geqslant 0$, and consequently $\mu_\tau = \tau(1)\varepsilon_x$, $\mu_\sigma = \sigma(1)\varepsilon_x$. Therefore

$$\tau(g) = \tau(1)g(x), \qquad \sigma(g) = \sigma(1)g(x),$$

which implies that $g(y) = g(x)$ in $\operatorname{supp}\tau \cup \operatorname{supp}\sigma = \operatorname{supp}\nu$. Thus every $g \in L$ takes the constant value $g(x)$ on $\operatorname{supp}\nu$; but L separates points, and hence $\operatorname{supp}\nu = x$, $\nu = \varepsilon_x$, $x \in \partial_L X$, and the proof is complete.

Now suppose that L is complete and meets the conditions of theorem 2 and let μ_x denote the measure constructed in that theorem for the special case $\tau = \varepsilon_x$, where $x \in X$. Suppose further that the functions $f \in L$ with $\|f\| \leqslant 1$ are equicontinuous at each point of $\complement \partial_L X$ and let $\emptyset \neq K \subseteq \complement \partial_L X$ with K compact. For each $u \in \mathbf{R}(\partial_L X)$ the map $x \to \mu_x(u)$ from X into **R** is, by theorem 2, the unique function $\bar{u}$ in L whose restriction to $\partial_L X$ is u. If $\mu^K(u)$ denotes the restriction of $\bar{u}$ to K then, by the maximum principle,

$$\|\mu^K(u)\|_{\mathbf{R}(K)} \leqslant \|u\|_{\mathbf{R}(\partial_L X)}$$

and hence by Ascoli's theorem the map $u \to \mu^K(u)$ from $\mathbf{R}(\partial_L X)$ into $\mathbf{R}(K)$ is a compact linear operator. If now $E \in B$ (= the class of Borel subsets of $\partial_L X$) then by a theorem of Bartle, Dunford and Schwartz [1] the map $x \to \mu_x(E)$ restricted to K is an element $\mu^K(E)$, of $\mathbf{R}(K)$. Moreover the map

$$\mu^K : B \to \mathbf{R}(K)$$

is a vector-valued regular Borel measure with conditionally compact range and we have

$$\mu^K(u) = \int_{\partial_L X} u(x)\mu^K(dx) \quad \text{for all} \quad u \in \mathbf{R}(\partial_L X)$$

where the integral exists as a strong integral in the sense of [1].

Note added in proof, *7 December* 1962.

Mokobodzki and Choquet (see Séminaire Brelot-Choquet-Deny (Théorie du Potentiel) 6e année, 1962, nº 12) have shown that further improvements in the use of the Hahn-Banach theorem to study barycentres are possible: If in the present context L separates points and $\hat{L}$ denotes the set of all $\varphi \in C$ of the form

$$\varphi = \inf(g_1, g_2, \ldots, g_n),$$

where all the g_r are in L, and if for $\sigma, \tau \in P$ we write $\sigma \preccurlyeq \tau$ whenever $\sigma(\varphi) \geqslant \tau(\varphi)$ for all $\varphi \in \hat{L}$ then $\sigma \preccurlyeq \tau$ implies that

$\sigma(g) = \tau(g)$ for all $g \in L$. The relation $\preccurlyeq$ is a partial ordering, and by Zorn's lemma each element of P is dominated by a maximal element of P. A modification of the construction in theorem 1 that uses $\widehat{f}(\sigma) \equiv \inf \{\varphi(\sigma) | \varphi \in \widehat{L}, \varphi \geqslant f\}$ in place of $f^*(\sigma)$ provides for each $\tau \in P$ and $h \in C$ a $\mu \succcurlyeq \tau$ with $\mu(W_h) = 0$. It follows that the maximal elements of P are precisely those $\mu \in P$ for which $\mu(W_h) = 0$ for all $h \in C$.

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