JEAN-PIERRE SERRE
FERNANDO Q. GOUVÊA (réd.)

Rational Points on Curves over Finite Fields

Cours de Jean-Pierre Serre, tome 6 (1985)

<http://www.numdam.org/item?id=CJPS_1985__6_>

F

Rational Points on Curves over Finite Fields

Part I "q large"

Harvard, 1985

RATIONAL POINTS ON CURVES
OVER FINITE FIELDS

PART I: "q LARGE"

Jean-Pierre Serre

Lectures given at Harvard University, September to December 1985.

Notes by Fernando Q. Gouvêa

Curves over Finite Fields

Jean-Pierre Serre

Let C be a complete non singular curve of genus g over a finite field F_q. Let $N(C)$ be the number of rational points of C. By a well-known theorem of Weil, we have

$$|N(C) - (q+1)| \leq 2g\sqrt{q} \ .$$

For a given pair (q,g), let $N_q(g) = \operatorname{Sup}_C N(C)$. Weil's inequality implies :

$$N_q(g) \leq q + 1 + 2g\sqrt{q} \ .$$

I shall discuss several improvements of this bound. Namely :

(a) $N_q(g) \leq q + 1 + g[2\sqrt{q}]$.

(b) (Asymptotic result - due to Drinfeld-Vladuţ) : for fixed q (and $g \to \infty$) $\limsup N_q(g)/g \leq \sqrt{q} - 1$, with equality when q is a square (Ihara-Zink).

(c) Explicit computation of $N_q(g)$ when $g = 1$ or 2.

(d) Numerical results :

Table of values of $N_q(g)$

$g \backslash q$	2	3	4	5	7	8	9	11	13	16	17	19	23	25
1	5	7	9	10	13	14	16	18	21	25	26	28	33	36
2	6	8	10	12	16	18	20	24	26	33	32	36	42	46
3	7	10	14	16	20	24	28	28	32	38	40	44		56
4	8	12	15	18										66
5	9													
6	10									65				
7	10													

Contents - Part I

The case $q=3$	Se 64 a
Voloch's bound	Se64a
Constructing curves: some examples	Se66
Conjectures	Se 71

Curves over $\mathbb{F}_q$

$q = p^e$, p prime, $e \geq 1$

X = curve (smooth, complete, abs. irreducible).

genus of $X = g$ is well-defined

$N(X)$ = nber of pts of X rat'l $/\mathbb{F}_q$

$= \# X(\mathbb{F}_q)$

One knows:

Weil bound: $\boxed{|N(X) - (q+1)| \leq 2g\sqrt{q}}$

If one wants $|\# X(\mathbb{F}_{q^n}) - (q^n+1)| \leq 2g\, q^{n/2}$ as a fct. of n, then $2g$ is optimal.

But for any given n, one doesn't know.

Want: upper bound, so curves with "many" pts.

Def: $N_q(g) = \sup_X N(X)$ (given g, q).

So Weil is: $\boxed{N_q(g) \leq 1 + q + 2g\sqrt{q}}$ ..

E.g., $q=2$, $g=50$; then Weil is $N \le 1+2+100\sqrt{2} = 144.\dots$

so $N \le 144$ (Weil's bound).

It's easy to see ≤ 103 (later: ≤ 40; then we'll show ≥ 40.).

So in this case $N_q(g) = 40$.

If $g = 1, 2, 3$ (small):

$g=1$ is in the literature

$g=2$ known

$g=3$ known only for $q<23$] Tuesday

Ab. Varieties

---

g large, q small [assymptotic results]

(Thursday)

(see Part II)

---

analogy:

$$\begin{tikzcd}
\text{curves w/ many pts} \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow, "\text{rel. to}"'] & & \text{fields w/ small discriminant} \arrow[d, leftrightarrow, "\text{rel. to}"] \\
\text{coding theory} \arrow[rr, leftrightarrow] & & \text{geom. of \#s}
\end{tikzcd}$$

Y. Ihara: some modular curves have lots of pts, over $\mathbb{F}_{p^2}$.

Coding theory: Goppa: curves w/ many pts

$$\begin{tikzcd}
\text{curves w/ many pts} \arrow[d, leftrightarrow, "\text{connection}"] \\
\text{codes,}
\end{tikzcd}$$

Connection w/ codes

A Linear Code over $\mathbb{F}_q$ is a vector subspace $V \subset \mathbb{F}_q^n$

(Think as $V \subset W$ together w/ a basis of W (basis must be fixed, except by scalars).)

Element $(x_1, -, x_n) \in \mathbb{F}_q^n$ is a word of length n (letters ↔ $\mathbb{F}_q$).

If $q=2$, this is a sequence of 0's and 1's.

$V \subset \mathbb{F}_q^n$ are the code-words.

message is 001001111 ∈ V

might be sent as 011001110 $\Big\}$ H-d = 2

two errors.

So we want an error-correcting code.

H-distance of two words = nber of coords where they differ.

Suppose H-d ≥ 5 (for words in V)

then the wrong word ∉ V, and the correct one is the unique closest word in V (as long as ≤ 2 errors)

Parameters are : $\begin{cases} n = \text{length of word} \\ v = \dim V \\ d = \text{min nber of non-zero coords in an element of } V, \neq 0 \text{ (since } V \text{ is subspace).} \end{cases}$

Want d large, but also v large.

[Sloane, Coding Theory]

Dual point of view

(*) Assume : for every i, $1 \leq i \leq n$, there is an $x = (x_1, -, x_n) \in V$ with $x_i \neq 0$.

Then consider the fct.

$$x = (x_1, -, x_n) \longmapsto x_i \in \mathbb{F}_q$$

non-zero linear form.

This defines an elt. $P_i \in \mathbb{P}(V^*)$

(have $V \xrightarrow[inj]{} \mathbb{F}_q^n$; by duality $\mathbb{F}_q^n \xrightarrow[surj]{} V^*$).

So find $P_1, \dots, P_n \in \mathbb{P}(V^*)$ which generate.

$v - 1 = \dim \mathbb{P}(V^*)$
$n = \#$ of pts

What is H-d ? Let $m =$ max. # of P_i lying on a hyperplane.

<u>Claim</u>: m determines d; in fact $m = n - d$.

So we want many pts in proj. space, but not too many on a hyperplane.

Now suppose $X \hookrightarrow \mathbb{P}_{r-1}$, and take $P_1, —, P_n$ = rat'l pts of X.

Then $m \leq \deg X$.

For $g = 0$, have $\mathbb{P}_1 \hookrightarrow \mathbb{P}_{r-1}$ by standard embedding, and this gives Reed-Solomon code.

Suppose X, L line bundle, $P_1, —, P_n$ rat'l pts of X.

Then take $V = H^0(X, L) = \Gamma(L)$.

And map

$$\begin{tikzcd}
V \arrow[r] \arrow[dr] & \mathbb{F}_q^n \arrow[d, phantom, "\cong"] \\
s \arrow[dr, mapsto] & L_{p_1} \times \cdots \times L_{p_n} \\
& (s(P_1), \ldots, s(P_n))
\end{tikzcd}$$

If injective, have a code, and $d \geq \deg(L)$ since sections cannot vanish at more than $\deg(L)$ pts.

For $q = p^2$, $p \geq 7$ modular curves give better codes than the previously known ones.

Refined Weil bound

Claim: $|N-(q+1)| \leq g\,[2\sqrt{q}]$, [ ] = integral part of.

Example: 1) If q is a square, ref. Weil = Weil

2) If $q=2$, $[2\sqrt{2}]=[2.8]=2$

so $|N-3| \leq 2g$

When $g=50$, this gives $|N-3| \leq 100$, so $N \leq 103$.

Pf: Weil comes from:

N = nber of fixed pts of Frob $= \pi : X \longrightarrow X$

$(x_0, -, x_d) = (x_0^p, -, x_d^p)$.

So shd have N = Trace π on "$H^0(X)$" $\rightsquigarrow 1$

− Trace π on $H^1(X)$ $\rightsquigarrow \pi_1, \dots, \pi_{2g}$

+ Trace π on $H^2(X)$. $\rightsquigarrow q$

So we get $N = 1+q-\sum_{i=1}^{2g}\pi_i$

Then he proved
- $|\pi_i| = q^{1/2}$ (R.H. for X)
- π_i is an alg. integer
- family of π_i (mult. included) is stable under Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$, i.e.,

$\prod(T-\pi_i) \in \mathbb{Z}[T]$.

Finally,

The eigenvalues of π^n (Frob. rel to $\mathbb{F}_{q^n}$) are the π_i^n.

Then we have $\# X(\mathbb{F}_{q^n}) = 1 + q^n - \sum_{i=1}^{2g} \pi_i^n$

And then this determines the π_i.

We know $\pi_i \bar{\pi}_i = q$ and $\bar{\pi}_i = \pi_j$, some j.

Claim: One can write the π_i in such a way that $\pi_{g+1}, \dots, \pi_{2g}$ are $\bar{\pi}_1, \dots, \bar{\pi}_g$.

Note:
[Jac(X) = $A/\mathbb{F}_q$ ab. var.
Then π_i = eigenvals of the Frob. on Jac(X)
And the same properties hold for any ab. variety.

Claim is equiv to : if $q = q_0^2$ is a square, then q_0 and $-q_0$ occur both with even multiplicities as eigenvalues.

(Note : it's clear for all the [illegible] other cases.)

Pf : ① (for curves only) if these mult. were odd, the const. in the fct'l eqn of Z would be -1, but it is $+1$ ↯.

② Can assume the ab. variety is simple over $\mathbb{F}_q$.

Suppose q_0 is eigenv. with mult ≥ 1.

Then the endom. $\pi - q_0$ has a kernel and

$$\dim(\mathrm{Ker}(\pi - q_0)) \geq 1$$

simplicity $\Rightarrow$ $\pi = q_0$ on ab. var., hence mult. q_0 is even. $(= 2 \dim A)$.

③ Symplectic proof

Take $V_\ell(A)$ = dual of H^1 (vect. sp/$\mathbb{Q}_\ell$).

& $\exists$ non-deg alt. form B on H^1 (def. by a polariz. $/\mathbb{F}_q$)

π viewed as endom. of V is a similitude, i.e.,

$$B(\pi x, \pi y) = q\, B(x,y).$$

Now the eigenvalues of any symplectic similitude can be paired as $\lambda_1, \lambda_1', \ldots, \lambda_g, \lambda_g'$ s.t. $\lambda_i \lambda_i' = q$.

☑

The same proof ③ shows our claim is still true for any cohomology in odd dimension.

Example in even dimension:

2-dim'l quadric over $\mathbb{F}_q$ Coh is $H^0 \oplus H^2 \oplus H^4$

H^2 is 2-dim'l (basis corresps. to lines of the two rulings of the quadric)

Se 5

- split quadric ($\overset{\text{say}}{x_1 x_2 + x_3 x_4 = 0}$), lines are def / $\mathbb{F}_q$ say e_1, e_2.

Then $\pi^* e_1 = q e_1$, $\pi^* e_2 = q e_2$, and the claim holds.

- non-split quadric : then $\pi^* e_1 = q e_2$ $\pi^* e_2 = q e_1$
  eigenvectors are $e_1 + e_2$, $e_1 - e_2$ w/ eigenval $q, -q$
  so claim is false.

Problem: why not $(\sqrt{q}, -\sqrt{q})$? So proof #2 is wrong.

So pair the π_i as given above. Set $a_i = \pi_i + \bar{\pi}_i$ $i = 1, \dots, g$.
So a_i real, family still Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$-stable, since $a_i = \pi_i + \frac{q}{\pi_i}$.
And $|a_i| \le 2\sqrt{q}$.

Then $N(X) - (1-q) = -\sum_{i=1}^{2g} \pi_i = -(a_1 + \dots + a_g)$.

For an ab. variety dim g, want: $\begin{array}{c} |Tr(\pi)| \\ \| \\ |\sum_{i=1}^{g} a_i| \end{array} \le g\, [2q^{1/2}]$

Let $m = [2q^{1/2}]$. $|a_i| < m+1$

If we take $x_i = m + 1 + a_i$, then $x_i > 0$.

The x_i are $\left\{ \begin{array}{l} \text{stable under Galois w/ multiplicities.} \\ \text{alg integers (since } \pi_i \text{ are).} \end{array} \right.$

$\therefore$ $x_1, \dots x_g \in \mathbb{Z}$ and is positive.

Then $x_i > 0$, so:

$$\frac{x_1 + \cdots + x_g}{g} \geq (x_1 \cdots x_g)^{1/g}$$ with equality only if all x_i are equal.

so
$$m + 1 + \frac{\sum a_i}{g} \geq 1$$

so $\sum a_i \geq -mg$ with equality only if $a_1 = \cdots = a_g$.

For the other ineq., apply same pf to −Frob.

So get $|\sum a_i| \leq mg$

If we have equality $\sum a_i = \pm mg$, a_i all equal, hence $a_i = \pm m$ each i.

So we have

$$\left.\begin{array}{ll} & -gm \leq Tr(\pi) \leq gm \\ \text{and if} & Tr(\pi) = gm, \text{ then } a_1 = \cdots = a_g = m \\ \text{if} & Tr(\pi) = -gm, \text{ then } a_1 = \cdots = a_g = -m. \end{array}\right] \text{Thm!}$$

[This is general: can replace Deligne's $\underset{\text{Betti number}}{B} \cdot q^{i/2}$ by $\frac{B}{2}[2q^{i/2}]$.]

A ab var / $\mathbb{F}_q$, π Frob. endom.

Thm 2 : (1) If $\mathrm{Tr}(\pi) = gm - 1$ ("down by 1"), then

$$(a_1, —, a_g) = \begin{cases} (\underbrace{m, m, \ldots, m}_{g-1}, m-1) & (g \geq 1) \\ (\underbrace{m, m, \ldots, m}_{g-2}, m + \frac{-1+\sqrt{5}}{2}, m + \frac{-1-\sqrt{5}}{2}) & (g \geq 2) \end{cases}$$

(2) If $\mathrm{Tr}(\pi) = gm - 2$ ("down by 2"), then one of the seven following possib. occurs:

$$(a_1, —, a_g) = \begin{cases} (m, m, —, m, m-2) & g \geq 1 \\ (m, —, m, m-1, m-1) & g \geq 2 \\ (m, —, m, m+\sqrt{2}-1, m-\sqrt{2}_2-1) & \\ (\text{same w/ } \sqrt{3} \qquad) & \\ (m, —, m, m-1, m+\frac{-1+\sqrt{5}}{2}, m+\frac{-1-\sqrt{5}}{2}) & \\ (m, —, m, m+\frac{-1+\sqrt{5}}{2}, m+\frac{-1+\sqrt{5}}{2}, m+\frac{-1-\sqrt{5}}{2}, m+\frac{-1-\sqrt{5}}{2}) & \\ (m, —, m, m+1-4\cos^2\frac{\pi}{7}, m+1-4\cos^2\frac{2\pi}{7}, m+1-4\cos^2\frac{3\pi}{7}). & \end{cases}$$

Smith has done computations which will allow us to extend this, in principle. Cont. next Tues.

10/1 defect 0 : $m, \dots, m$ $\qquad g \geq 0$

— 1 : $\begin{cases} m, \dots, m, \ m-1 & g \geq 1 \\ m, \dots, m, \ m+\frac{-1+\sqrt{5}}{2}, \ m+\frac{1-\sqrt{5}}{2} & g \geq 2 \end{cases}$

— 2 : $\begin{cases} m, \text{—}, m, \ m-2 & g \geq 1 \\ m, \text{—}, m, \ m-1, m-1 & g \geq 2 \\ m, \text{—}, m, \ m+\sqrt{2}-1, \ m-\sqrt{2}-1 & g \geq 2 \\ \quad '' \quad , m+\sqrt{3}-1, \ m-\sqrt{3}-1 & g \geq 2 \\ \quad '' \quad , m-1, \ m+\frac{-1+\sqrt{5}}{2}, \ m+\frac{-1-\sqrt{5}}{2} & g \geq 3 \\ \quad '' \quad , \underbrace{m+\frac{-1+\sqrt{5}}{2}, \ m+\frac{-1-\sqrt{5}}{2}}_{\text{twice}} & g \geq 4 \\ m, \text{—}, m, \ m+1-4\cos^2\frac{\pi}{7}, \dots & g \geq 3 \end{cases}$

A ab variety $/\mathbb{F}_q$, $\dim A = g$

$\pi : A \longrightarrow A$ Frobenius endom.

$\pi_\alpha, \bar{\pi}_\alpha$ eigenvalues $\qquad x_\alpha = \pi_\alpha + \bar{\pi}_\alpha$

$Tr(\pi) = \sum_{\alpha=1}^{g} (\pi_\alpha + \bar{\pi}_\alpha)$, $\quad m = [2\sqrt{q}]$.

Recall: $Tr(\pi) \leq gm$ $\qquad$ [for ab. vars. can twist to $\pi \longrightarrow -\pi$ so need not study the other ineq. $-gm \leq Tr(\pi$

If $Tr(\pi) = gm$, $\quad (x_1, \text{—}, x_g) = (m, \text{--}, m)$ $\quad$ ("defect 0")

"Defect 1" $\quad Tr(\pi) = gm - 1$
"Defect 2" $\quad Tr(\pi) \ gm - 2$
$\Big\}$ possibilities for $(x_1, \text{—}, x_g)$ are as above.

We are interested in alg. integers α, tot > 0, of deg. $d(\alpha)$, with "small" trace (w.r. to d).

Thm (Siegel, M.A. vol III, first paper) If α is as above and $\alpha \neq 1, \frac{3 \pm \sqrt{5}}{2}$, then $Tr(\alpha) > \frac{3}{2} d(\alpha)$.

(If $\alpha = 1$, $Tr(\alpha) = d(\alpha)$, $\alpha = \frac{3 \pm \sqrt{5}}{2}$, $Tr(\alpha) = 3 = \frac{3}{2} d(\alpha)$.)

[Best constant: instead of $\frac{3}{2}$, $\frac{5}{3}$].

Assume Siegel's thm: Separate out exceptional cases.

If k is a given integer ≥ 0, the number of tot > 0 α's with $Tr(\alpha) = d(\alpha) + k$ is finite for each k, and these α's can be found effectively.

Pf: By Siegel, $d(\alpha) + k > \frac{3}{2} d(\alpha)$

so $d(\alpha) < 2k$ is bounded.

But α satisfies

$$X^{d} - (d+k) X^{d-1} + \dots\dots$$

Conjugates $\alpha_1, \dots, \alpha_d$ are all positive, and all $< d(\alpha) + k$.

Hence coeffs are effectively bounded, and we can list the possible α's. ☒

For $k = 0$, get $\alpha = 1$

For $k = 1$, can take $\alpha = \begin{cases} \frac{3 \pm \sqrt{5}}{2} \\ 2 \end{cases}$

and no others, since $d(\alpha) < 2$.

For $k=2$, $d(\alpha) < 4$ so $d(\alpha)=1,2$ or 3

$d(\alpha) = 1 \longrightarrow \alpha = 3$

$d(\alpha) = 2 \longrightarrow \alpha$ satisfies $X^2 - 4X + q = 0$

roots real, hence $16 - 4q > 0$

so $q < 4$, so $q = 1, 2$ or 3

$X^2 - 4X + 1 = 0 \rightsquigarrow 2 \pm \sqrt{3}$

$X^2 - 4X + 2 = 0 \rightsquigarrow 2 \pm \sqrt{2}$

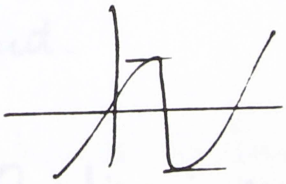
$X^2 - 4X + 3 = 0$

||

$(X-1)(X-3)$ not deg 2

$d(\alpha) = 3 \longrightarrow P(X) = X^3 - 5X^2 + pX - q = 0$

three real positive roots



∴ $3X^2 - 10X + p = 0$

has 2 real roots

so

$$\frac{\Delta}{4} = 25 - 3p > 0$$

So $p = 1, 2, \ldots, 8$.

Fix p: find roots of P': $x = \dfrac{5 - \sqrt{25-3p}}{3}$

$y = \dfrac{5 + \sqrt{25-3p}}{3}$

Compute values of P at x, y, say f, g

Want max positive, min negative, etc., Get $g < q < f$.

Find only one irred. polynomial of this form! $p=6$, $f=1$

$$X^3 - 5X^2 + 6X - 1$$

Roots are $4\cos^2\frac{\pi}{7} = 2 + \omega + \bar{\omega}$ $\qquad \omega = e^{2\pi i/7}$

and its three conjugates

So $k=2$ $\begin{cases} 3 \\ 2 \pm \sqrt{3} \\ 2 \pm \sqrt{2} \\ \text{conjs. of } 4\cos^2\frac{\pi}{7}. \end{cases}$

Smyth, Annals Inst. Fourier, 1984 : up to $k=6$

Note: $\exists\ \infty$ many α with $\mathrm{Tr}(\alpha) < 2\deg(\alpha)$

Smyth: $\exists$ only finitely many α with $\mathrm{Tr}(\alpha) < 1.7719 \deg(\alpha)$.

Open question: what is the correct constant.

Consider map $\alpha \longmapsto \dfrac{\mathrm{Tr}(\alpha)}{\deg \alpha}$. Question is equiv. to : what is first accum. pt. of $\mathrm{Im}(\alpha)$!

Can look for polynomials (not nec. irred.) $X^d - a_1 X^{d-1} + \dots$, s.t. coeffs $\in \mathbb{Z}$, all roots are real > 0.

Let F_k = set of all such pol. with $a_1 = d + k$.

Wanted: For a given degree d, list of polyn. in F_k.

Write $P = Q_1 \cdots Q_r$ $\quad Q_i$ irred / $\mathbb{Q}$

Q_i's have same property, and also

$$a_1(P) - \deg(P) = \sum_i \underbrace{a_1(Q_i) - \deg(Q_i)}_{\geq 0} .$$

So suppose $k = 1$: $P = Q_1 \cdots Q_r$

one Q_i with defect 1, others defect 0.

So $\quad (X-1) \cdots (X-1) \cdot (X-2)$

or $\quad (X-1) \cdots (X-1) \cdot (X^2 - 3X + 1)$.

Now if π is s.t. $\mathrm{Tr}(\pi) = mg - k$, $x_\alpha = \pi_\alpha + \bar{\pi}_\alpha$

take $\quad P = \prod (X - (m + 1 - \pi_\alpha - \bar{\pi}_\alpha))$.

This has tot. positive roots, defect k, hence is part of my list, etc..

Set $\quad x = [x] + \{x\}$
int part $\quad$ fract'l part.

$$2\sqrt{q} = m + \{2\sqrt{q}\}.$$

So claim : second defect 1 case is possible only if $\{2\sqrt{q}\} \geq \frac{\sqrt{5}-1}{2} = 0.6$

Since $\quad m + \frac{-1+\sqrt{5}}{2} \leq 2\sqrt{q}$ by Weil

so $\quad \{2\sqrt{q}\} \geq \frac{\sqrt{5}-1}{2}$.

And similarly, all the defect 2 cases except the first case has an inequality of this kind attached:

defect 0 —

defect 1 $\left\{ \begin{array}{l} \text{—} \\ \{2\sqrt{q}\} > \frac{\sqrt{5}-1}{2} \end{array} \right.$

defect 2 $\left\{ \begin{array}{l} \text{—} \\ \text{—} \\ \{2\sqrt{q}\} > \sqrt{2}-1 = 0.4\ldots \\ \{2\sqrt{q}\} > \sqrt{3}-1 = 0.7\ldots \\ \{2\sqrt{q}\} > \frac{\sqrt{5}-1}{2} = 0.6\ldots \\ \text{— same —} \\ \{2\sqrt{q}\} > 1-4\cos^2\frac{3\pi}{7} = 0.8\ldots \end{array} \right.$

<u>Ex</u>: The last case is possible for $q=2$:

$$\{2\sqrt{2}\} = 0.828\ldots \quad > 1 - 4\cos^2\frac{3\pi}{7}.$$

<u>We will see</u>: $\exists$ curve$/\mathbb{F}_2$, $g=3$, 7 pts, $m=2$

$1+2+6=9$, so <u>down by 2</u>,

and is of the last kind. $\boxtimes$

The possibilities all occur (one thinks) for abelian varieties — not nec. for curves.

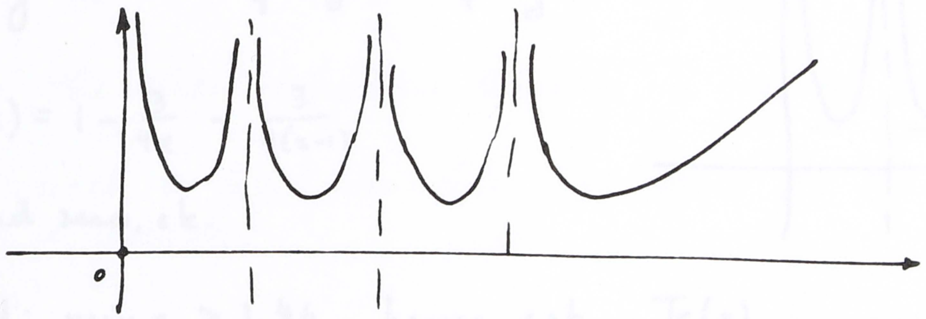
Smyth's proof of Siegel's thm :

Let $P_\lambda(X)$ be a finite family of polynomials which are monic, all roots real and positive $\geqslant 0$, coeffs in $\mathbb{Z}$. Let c_λ be $\geqslant 0$ real numbers.

Let $g(x) = x - \sum_\lambda c_\lambda \log |P_\lambda(x)|$ $\quad x > 0$, $x \neq$ root of P_λ.

and let $\min(g) =$ minimum of g on $[0, +\infty]$.

Graph is



, hence a min exists.

Let α be a totally positive alg integer $\neq$ roots of the P_λ.

Then $\boxed{\dfrac{\mathrm{Tr}(\alpha)}{\deg(\alpha)} \geqslant \min g}$.

Proof : Let $d = \deg(\alpha)$, $\overset{\alpha}{\alpha_1''}, \ldots, \alpha_d$ the conjugates, $\alpha_i > 0$.

$$|P_\lambda(\alpha_1) . P_\lambda(\alpha_2) \ldots P_\lambda(\alpha_d)| \geqslant 1$$

resultant of P_λ and irred polyn of α, so $\in \mathbb{Z}^*$

so $\sum \log |P_\lambda(\alpha_i)| \geqslant 0$.

$$\frac{\mathrm{Tr}(\alpha)}{\deg \alpha} = \frac{1}{d} \sum \alpha_i = \frac{1}{d} \sum g(\alpha_i) + \underbrace{\frac{1}{d} \sum_{i,\lambda} c_\lambda \log |P_\lambda(\alpha_i)|}_{\geqslant 0}$$

$$\geqslant \frac{1}{d} \sum g(\alpha_i) \geqslant \min g \qquad ▨$$

!

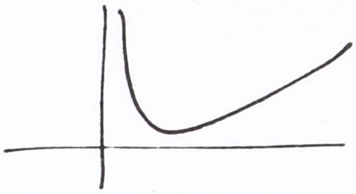
and equality only if all α_i are min pts

Examples: • $g(x) = x - \log|x|$

$\min(g) = 1$

so get $\mathrm{Tr}(\alpha) \geq \deg(\alpha)$

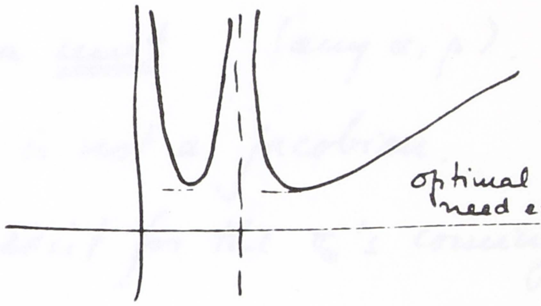
And get equality only for $\alpha = 1$.



• $g(x) = x - \frac{3}{4}\log|x| - \frac{3}{4}\log|x-1|$

$g'(x) = 1 - \frac{3}{4x} - \frac{3}{4(x-1)}$

find zeros, etc.



optimal case will need equal minim

Find: $\min g > 1.46$, hence get $\frac{\mathrm{Tr}(\alpha)}{\deg(\alpha)} > 1.46$ for $\alpha \neq 1$.

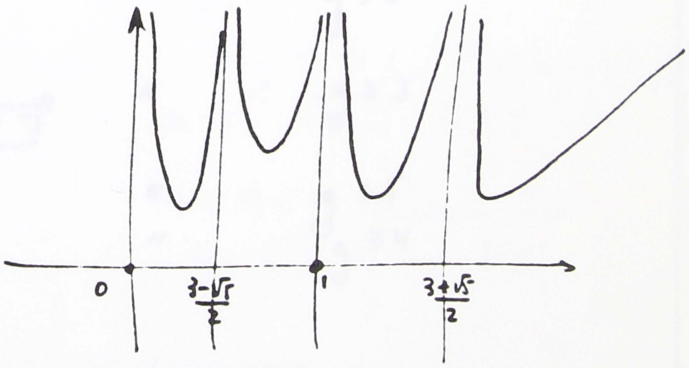
• $g(x) = x - a\log|x| - b\log|x-1| - c\log|x^2 - 3x + 1|$

Taking $a = 0.574$
$b = 0.879$
$c = 0.374$

one gets $\min g > 1.591$,

hence $\frac{\mathrm{Tr}(\alpha)}{\deg(\alpha)} > 1.59$

$\alpha \neq 1$, $\alpha \neq \frac{3 \pm \sqrt{5}}{2}$ This improves Siegel.



• Smyth gets 1.7719

$x_\alpha = \pi_\alpha + \bar{\pi}_\alpha \qquad \alpha = 1, \dots, g$

Theorem: Suppose $\{1, \dots, g\}$ can be partitioned in two non-empty subsets I and J s.t.:

a) The x_α ($\alpha \in I$) are permuted by $Gal(\bar{Q}/Q)$
(Same for x_α ($\alpha \in J$)).

b) $x_\alpha - x_\beta \quad \alpha \in I, \beta \in J$ is a <u>unit</u> (any α, β).

Then the given abelian variety is not a jacobian.

(i.e., such a partition does not exit for the π_α's coming from a curve).

[Eg.: $m, m, \dots, m, m-1$ is impossible for curves if $g \geqslant 2$]
second defect 1 case imp. " " " $g \geqslant 3$

defect 2 $\}$
——— ?
$\longrightarrow$ imp for curves if $g \geqslant 3$
$\longrightarrow$ " $g \geqslant 3$
$\longrightarrow$?
$\longrightarrow$ " $g \geqslant 3$
$\underbrace{m, \dots, m, m-1}, \underbrace{m + \frac{-1+\sqrt{5}}{2}, m + \frac{-1-\sqrt{5}}{2}}$
$\longrightarrow$ " $g \geqslant 5$
$\longrightarrow$ " $g \geqslant 4$

$\therefore$ For curves, many of the condns on g are <u>equalities</u>.

So table <u>for curves</u> is

defect 0	——	$g \geq 0$
defect 1	——	$g = 1$
	——	$g = 2$
defect 2	——	$g \geq 1$
	——	$g = 2$
	——	$g = 2$
	——	$g \geq 2$
	——	$g = 3$
	——	$g = 4$
	——	$g = 3$

Reformulate conditions:

$$P(X) = \prod_{\alpha=1}^{g} (X - x_\alpha) \qquad \text{monic, coeffs} \in \mathbb{Z}.$$

So (a) & (b) $\Longleftrightarrow$ $P = P_1 . P_2$ where $\deg(P_i) \geq 1$, P_i monic coeffs $\mathbb{Z}$ and $\operatorname{Res}(P_1, P_2) = \pm 1$.

$\Longleftrightarrow$ P_1 & P_2 generate the unit ideal in $\mathbb{Z}[X]$.

<u>Pf</u> A ab. variety $/\mathbb{F}_q$, $\underset{\substack{\shortparallel \\ F}}{\pi} : A \longrightarrow A$ (Frobenius)

Have $V : A \longrightarrow A$ s.t. $FV = VF = q$.

(c. 1970, <u>Waterhouse</u>)

$x_\alpha = \pi_\alpha + \frac{q}{\pi_\alpha}$ F + V is so on the $V_\ell(A)$.

So eigenvalues of F + V are the x_α (each twice).

Now $P(X) = \prod (X - x_\alpha)$, so $P(X)^2$ is char polyn of F + V.

This gives $P(F+V)^2 = 0$. In fact $P(F+V) = 0 \in \mathrm{End}(A)$.

So look at

$$\begin{tikzcd}
\mathbb{Z}[X] \arrow[r] & \mathrm{End}(A) \\
f \arrow[r, mapsto] & f(F+V)
\end{tikzcd}$$

get $\mathbb{Z}[X]/P \longrightarrow \mathrm{End}(A)$

If $P = P_1 \cdot P_2$, P_1 & P_2 strongly rel. prime, so $1 = Q_1P_1 + Q_2P_2$, get

$$\begin{tikzcd}
\mathbb{Z}[X]/P \arrow[r] \arrow[d, phantom, "\wr\wr"] & \mathrm{End}\, A \\
\mathbb{Z}[X]/P_1 \times \mathbb{Z}[X]/P_2 &
\end{tikzcd}$$

$$\underbrace{\varepsilon_1 = Q_1P_1 \qquad \varepsilon_2 = Q_2P_2}_{\text{are orthogonal idempotents}}$$

So define $A_1 = \ker \varepsilon_2$, $A_2 = \ker \varepsilon_1$

Then one has $A_1 \times A_2 \hookrightarrow A$

And inverse is $A \xrightarrow[(\varepsilon_1, \varepsilon_2)]{} A_1 \times A_2$

So $A \cong A_1 \times A_2$, hence A_1, A_2 are ab. varieties.

And eigenvalues of $F+V$ on A_1 give P_1.
on A_2 give P_2.

But $\mathrm{Hom}_{\mathbb{F}_q}(A_1, A_2) = 0$ (Since $\mathrm{Hom}_{\mathbb{F}_q}(\ ) \neq 0$ iff there is a common root of $F+V$.)

Polarization on ab. variety is a hom

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & \hat{A} \arrow[d, equal] \\ A_1 \times A_2 \arrow[r] & \hat{A}_1 \times \hat{A}_2 \end{tikzcd}$$

Since $A_1 \sim \hat{A}_1$, $A_2 \sim \hat{A}_2$, must have $A_1 \longrightarrow \hat{A}_1$
$A_2 \longrightarrow \hat{A}_2$

So every polarization on A is decomposable.

So Θ divisor on A is $\Theta_1 \times A_2 + A_1 \times \Theta_2$

But: on a Jacobian, the Θ-divisor is irreducible. ▨

10/8 Theorem (Beauville) If q $(=p^e, e \geq 1)$ is either of the form x^2+1 $(x \in \mathbb{Z})$ or x^2+x+1 $(x \in \mathbb{Z})$, then, if X is a curve of genus $g \geq 2$ on $\mathbb{F}_q$, $N(X) \neq q+1-gm$, $N(X) \neq q+1+gm$, where $m = [2q^{1/2}] = 2x$ or $2x+1$, resp.

Corollary: $|N(X) - (q+1)| < gm$

Take $x > 0$.

Special case (Stark): $q = 13 = 3^2+3+1$, $g = 2$ so $m = 7$.

So $|N - 14| < 14$

When $g = 2, 3$, $\exists$ different proof involving classes of Hermitian forms.

Note

1) If $q = x^2+1$, $4q = 4x^2+4 < (2x+1)^2$ (clear!)

$$\text{so} \quad 4x^2 < 4q < (2x+1)^2$$

So $[\sqrt{4q}] = 2x = m$.

If $q = x^2+x+1$ $4q = (2x+1)^2+3$ etc., so $m = 2x+1$.

$$\text{so} \quad m^2 - 4q = \begin{cases} -4 & \text{first case} \\ -3 & \text{second case} \end{cases}.$$

2) If $q = p$ prime, it is open whether there are ∞ many such primes. One thinks there should be.

In fact $\#\{p \leq P : p = x^2+1\} \sim c\,\frac{P}{(\log P)^2}$ is a conj.

3) If $q = p^e$, e odd ≥ 3

Only sol'n: $7^3 = 18^2+18+1$

Point is $y^e = x^2+1$ no soln if e odd ≥ 3 (Lebesgue, 1850)

$y^e = x^2+x+1$ only one soln (non-trivial) e odd ≥ 3
Nagel, Ljundgren

Both cases: p-adic method of Skolem.

Proof: Consider the case $q = x^2+1$, so $m = 2x$; assume X is genus ≥ 2 and $N(X) = q+1-gm$.

Then I can arrange the eigenvalues of Frob π_α s.t.

$$\begin{cases} \pi_1 + \bar{\pi}_1 = m & \pi_1\bar{\pi}_1 = q = x^2+1 \\ \vdots & \\ \pi_g + \bar{\pi}_g = m & \pi_g\bar{\pi}_g = q \end{cases}$$

So $\begin{cases} \pi_1 = \cdots = \pi_g \\ \bar{\pi}_1 = \cdots = \bar{\pi}_g \end{cases}$

(If I assumed $N(X) = q+1+gm$, would get $\pi_1 = -x+i$;

If $q = x^2+x+1$, get $\mathbb{Z}[\omega]$, $\omega^3=1$

So $\pi_1 = x+i$, $\bar{\pi}_1 = x-i$

So $\mathbb{Z}[\pi_1] = \mathbb{Z}[i]$

Let F = Frob.; then eigenvalues are $x+i$ g times, $x-i$ g times.

Put $\sigma = F - x \in \mathrm{End}(\mathrm{Jac}(X))$

$\therefore$ eigenvalues of σ are i (g times), $-i$ (g times).

$\therefore$ $\sigma^4 = 1$, $\sigma^2 = -1$.

Polariz are $\mathrm{Jac}(X) \xrightarrow{\sim} \mathrm{Jac}(X)^*$ (dual) Se 14

or equiv. classes of ample divisors

Recall :

Parenthesis on Torelli's theorem

X genus g (≥ 2) $\longrightarrow$ $\mathrm{Jac}(X)$, ab. variety dim g w/ a polarization given by image of X to power $g-1$ (which is ample divisor !).

f: Weil, in l 2 of coll. papers.

Thm : Let X, X' be two curves over a perfect field k.

Let $\varphi : \mathrm{Jac}(X) \xrightarrow{\simeq} \mathrm{Jac}(X')$ be an isom. compatible with polarizations.

Then a) if X is hyperelliptic, there exists a unique isom. $f : X \xrightarrow{\simeq} X'$ which gives φ.

b) if X is not hyperelliptic, there exists a unique isom $f : X \xrightarrow{\sim} X'$ and a unique $\varepsilon \in \{\pm 1\}$ s.t. f gives $\varepsilon\varphi$.

Corollary : If σ is an automorphism of $\mathrm{Jac}(X)$ preserving the polarization, then either σ or $-\sigma$ comes from an autom. of X.

Finit parentheses

Now we want to prove our σ is compatible with the polarization

Compatibility of σ w/ polarization

Can view polarization as giving an alternating form on V_ℓ.

V_ℓ = Tate-module attached to some ab. var.

olariz $\longleftrightarrow E: V_\ell \times V_\ell \longrightarrow V_\ell(\mathbb{G}_m) \underset{\text{not can.}}{\simeq} \mathbb{Q}_\ell$ non-deg. alternating form.

and we have $E(Fx, Fy) = q\,E(x,y)$ F = Frob. is a similitude

(using V: $FV = q$, get $E(Fx, y) = E(x, Vy)$)

$\longrightarrow$ so adjoint of F w.r.t. E is V.

In our case F is like $x+i$, so V is like $x-i$ $\Longrightarrow$

$\Longrightarrow$ in our case adj. on $\mathbb{Z}[i]$ = ring gen by F is cx conjugation.

i.e., for $\lambda \in \mathbb{Z}[F] \simeq \mathbb{Z}[i]$, $\text{adj}(\lambda) = \bar{\lambda}$.

For our σ, get $E(\sigma x, y) = E(x, \bar{\sigma} y) = E(x, \sigma^{-1} y)$

$\therefore E(\sigma x, \sigma y) = E(x,y)$.

So σ preserves E $\therefore$ the polarization. ▨

$\therefore$ σ or $-\sigma$ comes from an autom of X, so both are ($\sigma^3 = -\sigma$, et

(if $q = x^2+x+1$ get either $\overset{\text{order 3}}{\sigma}$ or $\overset{\text{order 6}}{-\sigma}$)

So σ comes from an autom. of X.

I claim: if $\omega_1, \dots, \omega_g$ are a basis of dfk's on X, then

$$\sigma^*\omega_i = \lambda\,\omega_i \quad , \ \lambda \text{ indep. of } i \text{ fixed.}$$

Pf: dfk's come from Jac(X), so we want to prove this on Tgt space to Jac(X). But $F = 0$ on tgt space, $\sigma = F - x$, hence σ acts by $-x$ on tgt space. So $\lambda = -x$ or x. ☒

Canonical map: $\text{can}: X \longrightarrow \mathbb{P}^{g-1}$

defined by taking $\omega_1, \dots, \omega_g$ as homog. coords.

Non-homogeneously: $Q \longmapsto \left(1, \frac{\omega_2}{\omega_1}(Q), \dots, \frac{\omega_g}{\omega_1}(Q)\right)$

for $g \geq 2$,
- if X is not hyperelliptic, can is an embedding
- if X is hyperelliptic, image has genus zero & can has degree 2

(So gives $X \xrightarrow{\text{order } 2} \mathbb{P}^1$ covering).

Then σ acts trivially on image(can); in the first case, this implies $\sigma = 1$; in the second case, this implies σ is of order 2, $\sigma = \pm 1$ ↯. ☒

[Original proof (w/o can) used "Woods Hole fixed pt formula".]

Review of $g = 1$:

Elliptic Curves

$\pi, \bar{\pi}$ eigenvalues of F, $\pi\bar{\pi} = q$. $a = \pi + \bar{\pi}$ trace.

a determine curve up to $\mathbb{F}_q$-isog

$|a| \leq 2q^{1/2}$. $q = p^e$ $e \geq 1$, p prime

For a given q, what are the possibilities for a?

Answer is implicit in Deuring (~1942); Waterhouse (Ann. ENS, 1969):

Answer is: Suppose $a \in \mathbb{Z}$, $|a| \leq 2q^{1/2}$

Theorem (i) if a is prime to p, a is OK (i.e., is $\operatorname{tr} F$ for some elliptic curve $/\mathbb{F}_q$) ("ordinary case").

(ii) if $p | a$, then a is OK if and only if either:

$q = p^e$, e even, $a = \pm 2p^{e/2}$

$q = p^e$, e even, $a = \pm p^{e/2}$, $p \not\equiv 1 \pmod 3$

$q = p^e$, e even, $a = 0$ $p \not\equiv 1 \pmod 4$

$$q = p^e,\ e \text{ odd} \left\{ \begin{array}{ll} a = 0 & \\ a = \pm p^{\frac{e+1}{2}} & p = 2 \text{ or } 3 \end{array} \right.$$

[Note: $p^{\frac{e+1}{2}} \leq 2p^{e/2} \Rightarrow p^{1/2} \leq 2 \Rightarrow p \leq 4$]

<u>Proof</u> (i) Start in char = 0 and reduce.

$\pi^2 - a\pi + q = 0 \implies \pi$ generates a ring $R \subset$ imag. q. field.

can prove $\implies \exists$ curve/$\mathbb{C}$ with End $\cong R$.

Write it over $\bar{\mathbb{Q}}$, then some number field K.

Prove: good redn at p for K large enough, so reduce at p.

Get an "ordinary" curve because $p \nmid a$. So End $\subset$ imag. quad. field.

<u>Prove</u>: this is def over some $\mathbb{F}_{q^N}$, and the Frob π' is π^N.

Now use <u>descent</u>. This gives the desired curve. ☒

(ii) <u>supersingular curves</u>

End = max'l order in the quat. algebra $H_{p,\infty}$ ramified at p and ∞ (and not elsewhere).

π = Frob $\in$ End has a power F^f $(f \geq 1)$ which is a scalar (i.e., $\in$ center$(H_{p,\infty})$).

So look for $\pi \in H_{p,\infty}$, π integer, $\pi\bar{\pi} = q$ s.t. some power of π is an element of $\mathbb{Q} \subset H_{p,\infty}$.

Such a π gives an ell. curve (take max'l order containing it, get ell. curve, descend.)

(Know $\displaystyle\sum_{E \text{ s.s.}} \frac{1}{\#\mathrm{Aut}(E)} = \frac{p-1}{24}$ → this shows that some ss curve exists.)

Suppose $q = p^e$, e even.

Look at $x = \frac{\pi}{p^{e/2}}$; this is still an integer (look at val'ns)

So have $x\bar{x} = 1$, x integer, $x \in$ quad. field

So $a = \pm 2p^{e/2} \longleftrightarrow$ roots of 1 : ± 1

$a = \pm p^{e/2} \longleftrightarrow$ roots of order 3, 6 or 4

(then $\mathbb{Q}(\sqrt[3]{1}) \subset \mathbb{H}_{p,\infty}$, say, then p cannot be split).

etc.

Suppose $q = p^e$, e odd; let $x = \frac{\pi}{p^{e-1/2}}$

then $x^2 - \lambda x + p = 0$ and λ is div. by p (since $\lambda = x + \bar{x}$ look at valuat'n

But $\lambda \leq 2p^{1/2}$.

So either $\lambda = 0$ or $p = 2$ $\lambda = \pm 2$, $p = 3$, $\lambda = \pm 3$.

This gives the result. ▨

Let $N_q(1)$ = max'l number of pts on ell. curve/$\mathbb{F}_q$, $m = [2q^{1/2}]$.

Theorem: $N_q(1) = q + 1 + m$, except when $q = p^e$, e odd, $e \geq 5$ and $m \equiv 0 \pmod{p}$, in which case $N_q(1) = q + m$.

smallest exceptional case: $q = 128 = 2^7$

Se 17

Proof: When can we have $a = -m$?

a) OK when $p \nmid m$

b) OK when q is a square, since $a = -2q^{1/2}$ is allowed

Remains: $q = p^e$, e odd, $p \mid m$.

If $e = 1$, OK: $p \mid a \rightarrow \begin{cases} a = 0 & \text{if } p \geq 5 \\ a = \pm p & \text{if } p = 2, 3. \end{cases}$ But $m \neq 0 \rightarrow$ cannot have $p \mid m$

If $e = 3$, have $4p^3 = m^2 + \varepsilon$ $\quad 1 \leq \varepsilon \leq 2m < 2^{3/2} p^{3/2}$

Now suppose $m = p\mu$: $4p^3 = p^2\mu^2 + \varepsilon \Longrightarrow p^2 \mid \varepsilon$

$$\text{so } p^2 < 2^{3/2} p^{3/2}$$

$$\Longrightarrow p^{1/2} < 2^{3/2} \rightarrow p < 8$$

$\rightarrow p = 2, 3, 5, 7$ and check these. $\rightarrow$ cannot have $p \mid m$.

Finally, to get $N_q(1)$ for q exceptional, note $p \mid m \Rightarrow p \nmid (m-1)$. $\square$

Exceptional: $q = 2^7$, $q = 7^5$

Se 17a

Take $q = 2^*$. When is this exceptional?

$q = 2^7$: $\quad 2\sqrt{q} = 2.2^{7/2} = 2^4 . \sqrt{2}$

$\sqrt{2} = 1.0110101000001\ldots$ in binary

$2^4 \cdot \sqrt{2} = 10110.10\ldots$

$m = [2^4\sqrt{2}] = 1011\underset{\text{even!}}{0} \quad \therefore \; p \mid m$

Therefore 2^7 is exceptional, because $\sqrt{2} = 1.011\underset{2^7}{0}1\underset{2^{11}}{0}1\underset{2^{15}}{0}\underset{2^{17}}{0}0001\ldots$

$\therefore \exists$ infinitely many exceptional 2^*.

$2\sqrt{3} = 3.11\overset{3^7 \text{ exceptional}}{0}112022\ldots$ (3-adic)

$2\sqrt{7} = 5.2\underset{7^5 \text{ exceptional.}}{0}166\ldots$ (7-adic)

So cannot know how many exceptional p^e's for $p > 2$.

10/15

last time : $\underline{g=1}$

Then $N_q(1)$ = max number of points of a curve of genus 1 over $\mathbb{F}_q$

[Get : $N_q(1) = 1 + q + m$, $m = [2\sqrt{q}]$,

except, when $q = p^e$, e odd ≥ 3, and $p \mid m$, in which case

$N_q(1) = q + m$

To find exceptional q, look at p-adic expansion of $2\sqrt{q}$.

Problem : Ell. curve over $\mathbb{Q}$, reduce mod p. For what p's does it have maximal (or minimal) number of points?

Should find an infinite number, and should have to distinguish CM and non-CM. (Very hard to handle).

Still $g=1$

ref. Weil $= q + 1 + [2\sqrt{q}]$

So have $|\text{ref. Weil} - N_q(1)| \leq 1$ for all q.

(Will see : in $g = 2$

$|\text{ref. Weil} - N_q(2)| \leq 3$.)

Se 18a

In $g=3$:

$$\left| \text{ref. Weil} - N_q(3) \right| \le ?$$

<u>Conjecture</u>: For $g = 3, 4, 5$ and not many more,

$$\left| \text{ref. Weil} - N_q(g) \right| \le C(g),$$

$C(g)$ depending only on g.

For $g=1,2$ (more generally, for hyperelliptic curves), have curve C, Frob. $\pi \in \operatorname{End}(J(C))$.

C hyperelliptic $\Rightarrow$ $\exists \sigma$ autom. of order 2 of C which acts by -1 on $\operatorname{Jac}(X)$

$$\begin{tikzcd} k' \arrow[d, dash, "2"'] & \text{quad. twist;} & \mathbb{F}_{q^2} \arrow[d, dash] \\ k & & \mathbb{F}_q \end{tikzcd}$$

then Frob of C_{twisted} = − Frob of C

And if $N(C) = q+1-a$, $N(C_{\text{twisted}}) = q+1+a$.

So $N(C)$ max $\Rightarrow$ $N(C_{\text{twisted}})$ minimum (etc.).

If char $\neq 2$, C is $y^2 = f(x)$

C_{twist} is $y^2 = u f(x)$ $u \in \mathbb{F}_q$ not a square.

char $= 2$ C is $y^2 + y = \varphi(x)$

i.e., $\operatorname{Tr}(a)=1$

C_{twist} is $y^2 + y = \varphi(x) + a$, $a \in \mathbb{F}_q$, a not of the form $b^2 + b$.

For $g=3$, not hyperelliptic, the "min" and "max" problems become separate. But for large g and fixed q, "min" will be zero.

Results of Tate and Honda

(Tate, Inventiones 2 (1966), 134-144
Tate, Sem. Bourb., exp 352
Milne & Waterhouse, Symp Pure Math AMS, ___)

"Weil number"/q is an alg. integer π s.t. all conj. of π have (arch.) abs. value $q^{1/2}$.

one-one corresp:

$$\left(\begin{array}{l}A \text{ abelian variety } /\mathbb{F}_q \\ \text{simple } /\mathbb{F}_q\end{array}\right)\Big/(\text{isogenies}) \longleftrightarrow (\text{Weil numbers})\Big/(\text{conjugation}$$

$$A \longmapsto \text{roots of Frobenius on } A$$

$$\begin{array}{l}A \text{ s.t. eigenvalues of Frob} \\ \text{are } \pi_1, \dots, \pi_d \text{ repeated a} \\ \text{certain number of times}\end{array} \longleftarrow \pi_1, \dots \pi_d, \quad d = \deg \pi$$

Tate's theorem

π Weil number, A the corresp. simple abelian variety $/\mathbb{F}_q$. Let $\Lambda = \mathrm{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q}$.

Then we know:

(1) Λ is a division algebra.

(2) $\Lambda \supset \mathbb{Q}(\pi)$, $\mathbb{Q}(\pi) =$ center of Λ.

The local invariant of Λ as element of $Br(\mathbb{Q}(\pi))$ is (i_v), v place of $\mathbb{Q}(\pi)$, $i_v \in \mathbb{Q}/\mathbb{Z}$, $\sum i_v = 0$.

(3) If v is a real place, $i_v = \frac{1}{2}$
——— complex ——, $i_v = 0$

——— l-adic place, $l \neq p$, $i_v = 0$

If v has residue char p, $i_v = \dfrac{v(\pi)}{v(q)} \left[\mathbb{Q}(\pi)_v : \mathbb{Q}_p\right]$.

(4) Call r the smallest common denominator of the i_v (i.e., the order of Λ in $Br(\mathbb{Q}(\pi))$).

Then $[\Lambda : \mathbb{Q}(\pi)] = r^2$

(5) $\dim A = \frac{r}{2}[\mathbb{Q}(\pi) : \mathbb{Q}] = d_\pi$

Corollary: The multiplicity of π as eigenvalue of Frob. in any ab. variety over $\mathbb{F}_q$ is a multiple of r.

(Since it occurs for A w/ multiplicity $r = 2d_\pi/$no. of conj's)

$q = p^2$, $\pi = p$

So $\mathbb{Q}(\pi) = \mathbb{Q}$ so $\begin{cases} i_\infty = 1/2 \\ i_l = 0 \\ i_p = \frac{1}{2}[\mathbb{Q}:\mathbb{Q}] = \frac{1}{2} \end{cases}$

So $\Lambda = H_{p,\infty}$ quat alg. ramified at p, ∞.

So $r = 2$, dim $A = \frac{r}{2}[\mathbb{Q}:\mathbb{Q}] = 1$, so A is ell. curve.

$\pi = \sqrt{p}$, $\mathbb{Q}(\pi) = \mathbb{Q}(\sqrt{p})$

$(q = p = \text{prime})$ two real places $i_{\infty_1} = \frac{1}{2}$, $i_{\infty_2} = \frac{1}{2}$

one p-adic place $i_p = 0$ (mod $\mathbb{Z}$).

(Since sum is zero mod $\mathbb{Z}$).

So get quat field in $\mathbb{Q}(\sqrt{p})$ ramif. at ∞, $= H_{p,\infty} \otimes \mathbb{Q}(\sqrt{p})$.

$r = 2$, $[\,\mathbb{Q}(\sqrt{p}/_p : \mathbb{Q}\,] = 2$ so dim $A = 2$.

$\therefore$ multiplicity of $\sqrt{p}$ as eigenvalue of Frob occurs always with even multiplicity (as desired way back when).

<u>Ord. Ell. Curves</u>: $\pi\bar{\pi} = p$, $\pi + \bar{\pi} = a$ $p \nmid a$

(over $\mathbb{F}_p$)

$[\mathbb{Q}(\pi) : \mathbb{Q}] = 2$, p splits

(look at Newton polygon) v_1, v_2 div p so $v_1(\pi) = 0$ $v_2(\pi) = 1$

So $i_{v_1} = 0$, $i_{v_2} = 1.1 = 0 \in \mathbb{Q}/\mathbb{Z}$.

$\rightarrow$ no real places.

So $\Lambda = \mathbb{Q}(\pi)$.

Recall: $\exists$ quadratic π's "forbidden" for ell. curves

In particular:

$m = [2q^{1/2}]$ is not trace Frob if $q = p^e$ e odd ≥ 3, and $p | m$.

If $\pi = \dfrac{m + \sqrt{m^2 - 4q}}{2}$, π cannot be Frob on ell. curve.

In fact: In that case the "r" attached to π is odd ≥ 5.

In particular, a curve of genus 2, 3, 4 over $\mathbb{F}_q$ cannot have $q + 1 \pm gm$ points.

For that we need $\pi_1, \bar{\pi}_1, - \cdot \pi_g, \bar{\pi}_g$

$$\text{with} \begin{cases} \pi_1 + \bar{\pi}_1 = -m \\ \\ \pi_g + \bar{\pi}_g = -m \end{cases} \Longrightarrow (\pi, \bar{\pi}) \text{ repeated } g \text{ times.}$$

But Claim $\Rightarrow$ repeated at least $r \geq 5$ times.

$\therefore g \geq 5$. ▨

Proof: Set $q = p^{2f+1}$, $f \geq 1$ (in fact $f \geq 2$).

Claim: p^{f+1} does not divide m

In fact $4q = m^2 + k$ $\qquad 1 \leq k \leq 2m$

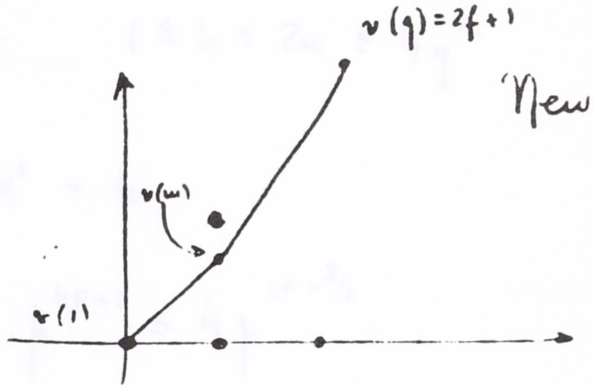
$p^{f+1} | m \Rightarrow p^{2f+1} | 4q$ and $| m^2 \Rightarrow p^{2f+1} | k$

But $2m \leq 4q^{1/2}$, so $p^{2f+1} \leq k \leq 2m \leq 4p^{f+\frac{1}{2}}$

So $p^{f+\frac{1}{2}} \leq 4 \underset{\boxed{f \geq 2}}{\Longrightarrow} p^{5/2} \leq 4 \Rightarrow p^5 \leq 16$ ↯

Hence $v_p(m) \leq f$.

π satisfies $\pi^2 - m\pi + q = 0$



Newton Polygon

I've proved $v(m)$ is below the line.

Hence p splits in $\mathbb{Q}(\pi)$ and the val of π ~~of~~ at v_1, v_2 dividing p are the slopes: $v(m)$, $2f+1-v(m)$.

i_v of Tate's theorem

At $\left\{ \begin{matrix} \infty \\ \ell \end{matrix} \right\} \longrightarrow 0$

At v_1 : $i_{v_1} = \dfrac{v(m)}{2f+1} = \alpha$ Know: $0 < \alpha < \frac{1}{2}$

At v_2 : $i_{v_2} = \dfrac{2f+1-v(m)}{2f+1}$

Denominator?

$$r = \frac{2f+1}{\gcd(2f+1, v(m))}.$$

so r is odd., ≥ 3

So we need to show $r \neq 3$ $(\therefore r \geqslant 5)$.

Assume $r = 3$, so $\alpha = \frac{1}{3}$, $v(m) = \frac{1}{3}(2f+1)$ so $f = 1+3F$

So $q = p^{6F+3}$, $m = p^{2F+1} \cdot M$, $p \nmid M$

Write $4q = m^2 + k$ $\quad 1 \leq k \leq 2m \leq 4q^{1/2}$

$$\text{So} \quad 4p^{6F+3} = p^{4F+2}M^2 + k$$

$$\text{So} \quad p^{4F+2} \mid k \implies p^{4F+2} \leq 4p^{3F+3/2}$$

$$\implies p^{F+\frac{1}{2}} \leq 4$$

$$\implies \begin{cases} p \leq 16 & \text{if } F = 0 \\ p^3 \leq 16 & \text{if } F = 1 \end{cases} \implies p = 2$$

And check the possible cases. ☒

For realizing the Weil bound, need $\pi = \dfrac{-m \pm \sqrt{m^2 - 4q}}{2}$, $p \nmid m$.
So have ell. curve E_π.

Want C s.t. $\quad \text{Jac} \sim E_\pi \times \cdots \times E_\pi$

To go down by 2, find E' w/ $Tr = -(m-2)$, and look for

$$\text{Jac} \sim E_\pi \times \cdots \times E_\pi \times E'.$$

Problem

FIND: Curves with Jacobian $\left\{\begin{matrix}\text{isogenous}\\ \text{or}\\ \text{isomorphic}\end{matrix}\right\}$ to a product of ell. curves

"Reduction of abelian integrals to elliptic integrals"
" diff forms

$$C \xrightarrow[\iota]{} \mathrm{Jac} \xrightarrow{\sim} E_1 \times \cdots \times E_g$$

find a basis of diff'l forms of first kind on C by taking $C \to E_i$ and pulling back the diff. first kind on E_i.

~1830, Legendre + Jacobi

α, β numbers

Take C genus 2 ramif. at $0, 1, \alpha, \beta, \alpha\beta, \infty$, so assume
$\alpha \neq 0, 1$
$\beta \neq 0, 1, \alpha, \alpha^{-1}$
i.e., all distinct.

Consider $y^2 = x(x-1)(x-\alpha)(x-\beta)(x-\alpha\beta)$

Basis of dfk : $\left\{ \frac{dx}{y}, \frac{x\,dx}{y} \right\}$

Define $\omega_1 = \frac{x+\sqrt{\alpha\beta}}{y}\,dx$

$\omega_2 = \frac{x-\sqrt{\alpha\beta}}{y}\,dx$

Change variables $(x,y) \longrightarrow (X, Y_1)$ $\quad X = x + \frac{\alpha\beta}{x}$

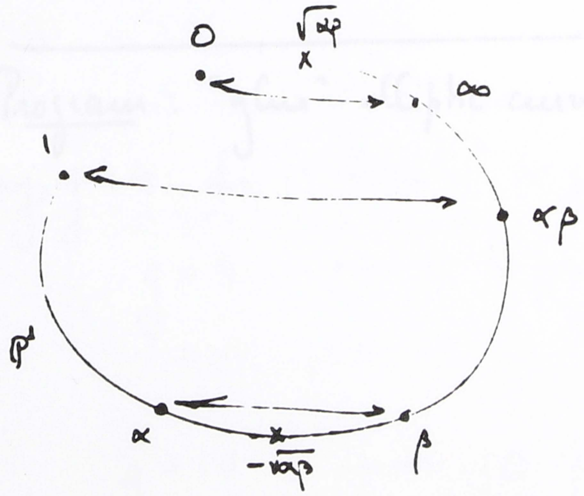
$$Y_1 = y\,\frac{x-\sqrt{\alpha\beta}}{x^2}$$

$$Y_1^{\,2} = (X-2\sqrt{\alpha\beta})(X-(\alpha+\beta))(X-(1+\alpha\beta))$$

and pullback of $\frac{dX}{Y_1}$ is ω_1 : $\omega_1 = \frac{dX}{Y_1}$

Same for Y_2 w/ change of sign, $\omega_2 = \frac{dX}{Y_2}$.

Geometrically

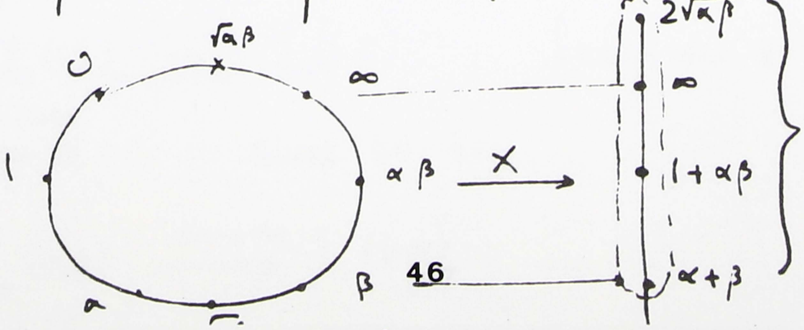


$x \xmapsto{\sigma} \frac{\alpha\beta}{x}$

involution

fixed points: $x = \frac{\alpha\beta}{x} \Longrightarrow x = \sqrt{\alpha\beta}$, $x = -\sqrt{\alpha\beta}$

mod out by σ : X = quotient by $\sigma = x + \sigma(x)$



make the ell. curve ramified here

$$\begin{tikzcd}
C \arrow[rr, dashed, "\text{check ramification}"] \arrow[d, "y"'] & & E_1 \arrow[d, "Y_1"] \\
x \text{ line} \arrow[rr] & & X\text{-line}
\end{tikzcd}$$

Y_1 ramif at those four pts

And the map $C \longrightarrow E_1$ is as given above.

On:

$$\begin{tikzcd}
& C \arrow[dl] \arrow[d] \arrow[dr] & \\
E_1 \arrow[dr, "2"'] & \mathbb{P}_{1,x} \arrow[d, "2"] & E_2 \arrow[dl, "2"] \\
& \mathbb{P}_{1,x} &
\end{tikzcd}$$

Program: "glue" elliptic curves to get C of genus 2

10/22:

Last time: reduction of abelian integrals to elliptic ones.

Legendre-Jacobi: $g=2$, example

Kuhn: information on general question: curves of genus g with Jacobian isogenous (or even isomorphic) to a product of elliptic curves.

Thesis

Does that exist for
- any g?
- an infinite seq. of g's? | char zero
- (in char p, $x^{p^a+1}+y^{p^a+1}+z^{p^a+1}=0$ has jac. which splits into s.s. elliptic curves).

Say, $g=4$. Is it "likely" to get such curves?

$g=4 \longrightarrow$ 4 parameters (to split into ell. curves)
1 eqn $\rightarrow$ ok

$g=10 \longrightarrow$ 10 parameters
$\frac{10.11}{2} - 27 >> 0$
$\Big\rangle$ rather surprising that examples exist.

Example: $g=7$, $X_0(60)$; let $J_0(60) = \mathrm{Jac}\, X_0(60)$

$J_0(15) \Rrightarrow J_0(60)$ (maps are $z \to z$, $z \to 2z$, $z \to 4z$).
ell. curve over $\mathbb{Q}$ since $60 = 4 \times 15$

$J_0(20) \rightrightarrows J_0(60)$
ell. curve

finally $\underset{\text{ell. curve}}{J_0(30)^{\text{new}}} \Longrightarrow J_0(60)$

$J_0(60) \sim \underset{J_0(15)^3 \times J_0(20)^2 \times J_0(30)''}{\text{product}}$

<u>Antwerp</u>:
$X_0(180)$; $g = 25$, splits
$X_0(198)$, $g = 29$, splits
$X_0(288)$, $g = 33$, splits
$X_0(300)$, $g = 43$, splits

Kuhn gets infinitely many examples w/ $g = 37$.

$J_0(300)$ includes 2 copies of $J_0(150)^{\text{new}}$, which splits into three elliptic curves. So have some "accidental" splitting.

<u>Question</u>: why so often?

<u>Theorem for $g = 2$</u> : (Computes $N_q(2)$) Set $m = [2q^{1/2}]$

ⓐ If <u>q is a square</u>, then:

· if $q \neq 4, 9$, then $N = 1 + q + 4q^{1/2} = 1 + q + 2m$

· if $q = 4$, $N = 10$ (Weil : 13) (down by 3)

· if $q = 9$, $N = 20$ (Weil : 22) (down by 2)

⑥ If q is not a square, define q to be special if either $p | m$ or q is represented by one of the quadratic polynomials x^2+1, x^2+x+1, x^2+x+2. Then:

- if q is not special, $N = 1 + q + 2m$ (Weil)
- if q is special, $N = \begin{cases} q+2m & \text{if } \{2\sqrt{q}\} > \frac{\sqrt{5}-1}{2} = 0.618\ldots \\ q+2m-1 & \text{if not.} \end{cases}$

Remark on the special q's : $q = p^{2e+1}$, $e \geq 0$

If $e = 0$, i.e., $q = p$, then p is special iff representable by x^2+1, x^2+x+1.

[$p | m$ only for $p = 2, 3$, and they are repres. by polyn.]
[x^2+x+2 is even, so only for $2 = 1^2+1$.

If $e \geq 1$, then $p | m$ should occur for infinitely many e (question is zeros in p-expansion of $2\sqrt{p}$), for a given p.

Both $\{2\sqrt{q}\} > \frac{\sqrt{5}-1}{2}$ and $< \frac{\sqrt{5}-1}{2}$ should occur infinitely often.

Write :

$$2\sqrt{p} = *.a_1 a_2 \ldots |0| \underbrace{1 \xrightarrow{\{2\sqrt{q}\}}}_{}$$

exceptional q

should be < slightly more often than >. (0.6 > 0.5

LIST OF $q = p^{2e+1}$ $e \geq 1$ $q \leq 10^9$ s.t. plm

There are only 14 : $\begin{cases} 2^{7,11,15,17,19,21,23,27,29} \\ 3^{7,15} \\ 5^{9,11} \\ 7^5 \end{cases}$ $\leq 10^9$

These are special by the first condition. What about second one?

<u>Thm</u> (Lebesgue, Nagell, Ljunggren)

If $q = p^{2e+1}$, $e \geq 1$ is representable by x^2+1, x^2+x+1 or x^2+x+2, then

$$q = 2^3, 2^5, 2^{13} \quad (\text{rep. by } x^2+x+2)$$

$$q = 7^3 \quad (\text{rep. by } x^2+x+1)$$

$$\begin{cases} 2^3 = 2^2 + 2 + 2 \\ 2^5 = 5^2 + 5 + 2 \\ 2^{13} = 90^2 + 90 + 2 \\ 7^3 = 18^2 + 18 + 1 \end{cases}$$

The Theorem is about

$y^n = x^2+1$ (x^2+x+1, x^2+x+2)

E.g. $y^n = x^2+1$, $n \geq 2$, $x, y \in \mathbb{Z}$, $x \neq 0$

has no solutions.

2^3 : $\{2\sqrt{8}\} = \{4\sqrt{2}\} = 0.65\ldots > \frac{\sqrt{5}-1}{2}$ so $q = 8 \to$ down by 1

$2^5, 2^{13}, 7^3 \longrightarrow$ down by 2

$\{2\sqrt{7^3}\} = 0.01\ldots$ $\quad 4 \cdot 7^3 = 37^2 + 3$

So $2\sqrt{7^3} \simeq 37$

Recall $$\sqrt{2} = 1.\underbrace{0110101}\underbrace{000001\ldots}_{\text{small}}$$

$$2\sqrt{2}^{13} = 181.\text{small}$$

Ideas for proof: 1) elementary construction of curves of genus 2 starting from ell. curves (Legendre)

2) using hermitian forms will give curves.

(In both of these, Jac $\sim E_1 \times E_2$)

1 – 3 gives existence proofs

3) Case "$\mathbb{Q}\sqrt{5}$" : using a thm. of Shimura.

4) Non-existence proofs → hermitian forms.

In the genus 1 case, our proof wasn't effective in the following sense:

we proved max of pts $= \begin{cases} 1+q+m \\ \text{or} \\ q+m \end{cases}$.

But not how to construct the corresp. ell. curve.

Only idea: Take all ell. curves (about $2q$ of them), compute number of pts on each. Stop when get the desired no. of pts.

This takes $q^2(\log q)^\alpha$ steps by the stupidest method.

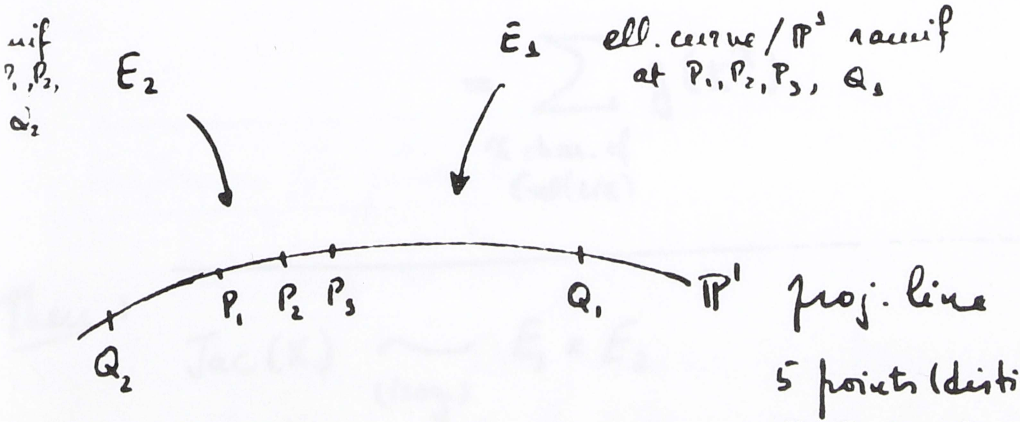
Schoof: one can compute # of pts in $(\log q)^\alpha$ steps.
(But ' implementable).

Can compute $E_{p-1}(c_4, c_6) \equiv a_p \pmod{p}$

So have to solve it in c_4, c_6. But can use it to count # of pts. No great gain.

Our construction in 1) is effective if we know the eqn. for $g=1$.

Legendre construction $\quad$ char $\neq 2$.



5 points (distinct) P_1, P_2, P_3, Q_1, Q_2

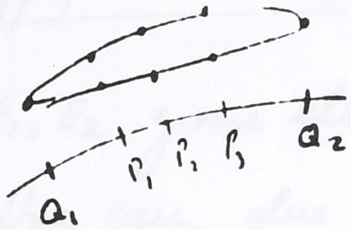
In terms of fields:

$$\begin{tikzcd}
 & K_{12} & \\
K_2 \arrow[ur, dash, "2"] & K_0 \arrow[u, dash, "2"] & K_1 \arrow[ul, dash, "2"'] \\
 & K \arrow[ul, dash, "2"] \arrow[u, dash, "2"] \arrow[ur, dash, "2"'] &
\end{tikzcd}$$

$K = K(\mathbb{P}^1)$

Se 27

$K_0 \xrightarrow{2} E_0$ ramif. only at Q_1, Q_2 ∴ genus zero



$g(K_0) = 0$

P_i split into P_i', P_i''

$g(K_{12}) = 2$ ram. at the pts $P_1', P_1'', P_2'', P_2', P_3', P_3''$

$X_{12} = X \longleftrightarrow K_{12}$ Have a group of type (2,2) acting on X

Exercise: In general, L/K, $g(K) = 0$, $Gal(L/K)$ of type $(2,2,\dots,2)$. Then

$$g(L) = \sum_{\substack{K' \subset L \\ [K':K]=2}} g(K') .$$

$$= \sum_{\substack{\chi \text{ char. of} \\ Gal(L/K)}} g(K^{\chi})$$

Thm:

$$\mathrm{Jac}(X) \underset{(\text{isog.})}{\sim} E_1 \times E_2$$

Have
$$\begin{tikzcd} & X_{12} \arrow[dl] \arrow[dr] & \\ E_1 \arrow[dr] & & E_2 \arrow[dl] \\ & \mathbb{P}_1 & \end{tikzcd}$$
; these give a map $\mathrm{Jac}(X) \to E_1 \times E_2$; use the group action to get the splitting.

In fact, kernel has type (2,2).

We apply this over $\mathbb{F}_q$, char $\neq 2$:

E_1, E_2 given ell. curves.

" One can glue them " $\iff$ $\exists X$ of genus 2 (over $\mathbb{F}_q$) s.t.

$$\text{Jac}\, X \underset{\substack{\text{isog} \\ \text{over } \mathbb{F}_q}}{\sim} E_1 \times E_2$$

$\overset{\uparrow}{\iff}$ eigenv. of Frob. on X = {those on E_1} ∪ {those of E_2}

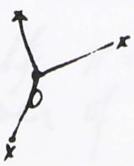
Tate-Honda

$$\Longrightarrow \text{Then} \quad N(X) = 1 + q - \overbrace{(\pi_1 + \bar{\pi}_1)}^{\text{eigenv. for } E_1} - \overbrace{(\pi_2 + \bar{\pi}_2)}^{\text{for } E_2}$$

so $\boxed{N(X) = N(E_1) + N(E_2) - q - 1.}$

Criterion for "gluing"

Let $(E_1)_2$ be the group of 2-div. pts of E_1 (of type (2,2)



Frob acts on the three non-zero pts : perm. of order 1, 2, or 3.

[Ord] : **Assume** : that order is the same for E_1 and E_2.

($y^2 = f_1(x)$; three pts $\longleftrightarrow$ zeros of $f_1(x)$

and order 1, 2, 3 corresp to)

$\updownarrow$ all three pts are rat'l

$\searrow$ one rat'l, other two conj.

$\longleftrightarrow$ zero pts rat'l.

Thm (char ≠ 2) Under this assumption, one can glue E_1 to E_2, except maybe if:

$$\begin{cases} \text{order of Frob} = 1,\ p = 3\ , & j(E_1) = j(E_2) = 0 \\ \text{order of Frob} = 2,\ \text{any } p\ , & j(E_1) = j(E_2) = 1728 \\ \text{———————} = 3,\ \text{any } p\ , & j(E_1) = j(E_2) = 0\ . \end{cases}$$

(So if $\mathrm{Aut}\, E_1 = \mathrm{Aut}\, E_2 = \{\pm 1\}$, then assumption ⟹ one can glue E_1 & E_2).

Example of non-glueing

$$\left[\begin{array}{l} q = 9, \quad E_1 \text{ s. sing.}\ , \quad \pi = +3 \\ E_2 = E_1 \end{array}\right.$$

$3x = x$ since $2x = 0$ so [Ord] is true

If X exists, then $N(X) = N(E_1) + N(E_2) - q - 1$

$$= \underbrace{9+1-6} + \underbrace{9+1-6} - 9 - 1 = -2. \ ↯$$

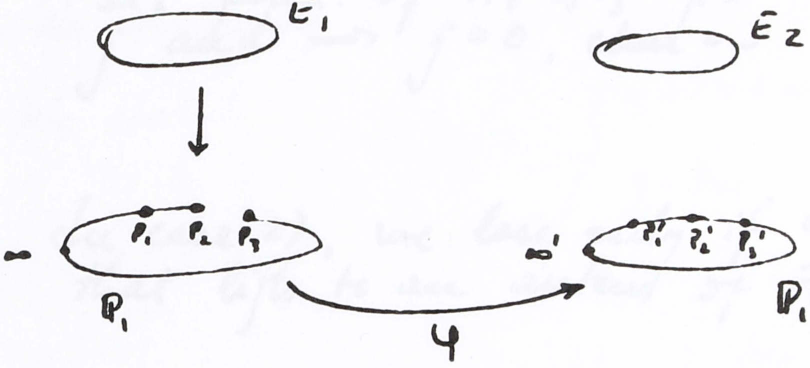
(If $\pi = -3$, $N(E_1) = 16$, so $N(X) = 22$ but X is a two-fold cover of $\mathbb{P}_1$ which has 10 pts, so $N(X) \le 20$ ↯.)

Proof: Write E_1 as quad. covering of $\mathbb{P}^1$ ramif. at ∞ and at P_1, P_2, P_3 (corresp to the 2-div. pts!)

r = order of Frob

So r acts on P_1, P_2, P_3.

Write E_2 as quad extn of some other $\mathbb{P}_1'$ ramif at ∞' and $P_1' / P_2', P_3'$



Want: $\varphi : \mathbb{P}_1 \longrightarrow \mathbb{P}_1'$ s.t. $\begin{cases} \varphi\infty \neq \infty' \\ \varphi\{P_1, P_2, P_3\} = \{P_1', P_2', P_3'\} \\ \varphi \circ \text{Frob} = \text{Frob} \circ \varphi \end{cases}$

Claim: φ exists (unless we are in the 3 exceptional cases).

Choose an isom of the set $\{P_1, P_2, P_3\}$, viewed as a set w/ Galois action, onto $\{P_1', P_2', P_3'\}$.

1) If order of Frob = 1, trivial action, 6 possible φ (up to an elem. of S_3)

2) If order of Frob = 2

two possible φ (φ_1 and $\pi \circ \varphi_1$)

3) If order of Frob = 3, have cyclic order on Pts

three possible φ (φ, $\pi\varphi$, $\pi^2\varphi$) .

Se 29

Solving this finds a φ satisfying 2nd & 3rd cond.; remain to check if $\varphi\infty \neq \infty'$.

In case 1), we lose (no φ works) only if the six perm. of P_1, P_2, P_3 fix ∞, and this $\Rightarrow$ same j and $\Rightarrow$ $j = 0$, char $= 3$ (because a large gp of autom.!)

In case 2), we lose only if ∞ is fixed by $(P_1\ P_2\ P_3) \to (P_1\ P_3\ P$ that lifts to an autom of E_1 of order 4, so $j = 1728$.

Similarly in case 3). ▨

10/29 $\boxed{g=2}$, cont.

We are trying to compute $N_q(2)$.

We stated:

$$m = [2q^{1/2}] \qquad N_q(2) = \begin{cases} q+1+2m & \text{nonspecial} \\ q+2m & \\ q+2m-1 & \end{cases} \Big\}\ \text{special}$$

in one case $(q=4)$, 3 less: $q+2m-2$

q square : special $\Leftrightarrow$ $q = 4$ or 9

q nonsquare : special $\Longleftrightarrow$ $p \mid m$ or $q = x^2+1, x^2+x+1, x^2+x+2$

Elementary glueing

E, E' over $\mathbb{F}_q$

"can glue them" $\Longleftrightarrow$ there is a curve C of genus 2 over $\mathbb{F}_q$, $Jac(C) \sim E \times E'$

$\xleftrightarrow[\text{Tate-Honda}]{}$ eigenv. of Frob. for C =

= (eigen. for E) $\cup$ (eigenv. for E')

We gave a construction which will give a glueing in many cases.

Let q be a square : $q = p^{2e}$

ⓐ Suppose, first, $p \neq 2, 3$.

Choose a supersingular curve E over $\mathbb{F}_{p^2}$ with Frob $= p$. (Known: such exist).

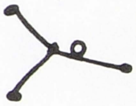
To be proved: one can glue E to E over $\mathbb{F}_{p^2}$, hence over $\mathbb{F}_q$.

Assume that; then we get C of genus 2, w/ eigenv. of Frob over $\mathbb{F}_q$ all equal to p^e. So $N = 1 + q - 4p^e$.

Making a quad. twist changes it to a curve with maximum no. of points, $N = 1 + q + 4p^e$.

(alternatively start w/ s.s. $E/\mathbb{F}_q$ with Frob $= -q^{1/2}$).

To prove glueing: look at the 2-division pts and action of Frob. on these pts.



Here Frob $= 1$ (identity) on 2-division pts, since Frob $(x) = \pm q^{1/2} x = x$ iff $2x = 0$. $\xRightarrow{(q \text{ odd})}$ no trouble (except if $p = 3$, $j = 0$, which is not the case).

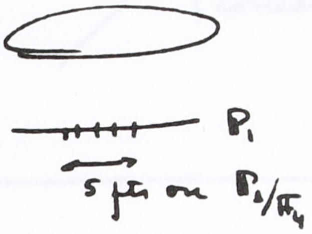
ⓑ $q = 4$ or 9

$q = 4$, Weil $= 1 + q + 2m = 1 + 4 + 24 = 13$

and covering argument $\Rightarrow N \leq 2(q+1) = 10$

10 is the exact bound: eqn is $y^2 + y = \frac{x}{x^3 + x + 1}$ over $\mathbb{F}_4$.

This gives covering



3 simple poles $\Longrightarrow$

$\Longrightarrow g = 2$.

So to prove: every pt splits completely

So : $x = 0, 1, \infty$ $(\in \mathbb{F}_2)$

$x = \rho, \rho^2$ $(\rho^3 = 1, \rho \neq 1)$

$$\left|\begin{array}{lll} x = 0 & \longrightarrow & \text{RHS} = 0 \\ x = 1 & \longrightarrow & \text{RHS} = 1 \\ x = \infty & \longrightarrow & = 0 \\ x = \rho & \longrightarrow & = 1 \\ x = \rho^2 & \longrightarrow & = 1 \end{array}\right.$$

LHS is a trace so splits completely $\Longleftrightarrow$ RHS is 0 or 1.

Over $\mathbb{F}_2$, $N_1 = 4$, $N_2 = 10$ write $a_1 = \pi_1 + \bar{\pi}_1$
$a_2 = \pi_2 + \bar{\pi}_2$

then
$$\left\{\begin{array}{l} 1 + 2a_1 - a_2 = 4 \\ 1 + 4 - (a_1^2 - 4) - (a_2^2 - 4) = 10 \end{array}\right.$$

$\pi \longrightarrow \pi^2$ $a_1 \longrightarrow a_1^2 - 2\pi_1\bar{\pi}_1$
$= a_1^2 - 4$

So
$$\left\{\begin{array}{l} a_1 + a_2 = -1 \\ a_1^2 + a_2^2 = 3 \end{array}\right.$$

So $a_1 = \dfrac{-1 \pm \sqrt{5}}{2}$

$a_2 = \dfrac{-1 - \sqrt{5}}{2}$

$\Big\rangle$ ab. var. of dim 2, irred. (comes from $\mathbb{Q}(\sqrt{5})$).

For $\mathbb{F}_9$: $N \leq 20$

eqn: $y^2 = (x^3 - x$ [illegible]$)^2 - 1 = (x^3-x+1)(x^3-x-1)$

Clearly $g=2$, to show: can be solved in $\mathbb{F}_9$ for every value of x.

$x = \infty$ — OK (look at coeff.)

$x \in \mathbb{F}_3$, $x^3 - x = 0 \longrightarrow y^2 = -1$ (but -1 is a square in $\mathbb{F}_9$)

$x \in \mathbb{F}_9 - \mathbb{F}_3$, $x^3 - x$ is antisymm. under $x \mapsto x^3$ so $(x^3-x)^2$ is symm, hence $\in \mathbb{F}_3^\times$, in fact $= -1$, so works.

$$y^2 = (x^3 - x)^2 - 1$$
$$= x^6 + x^4 + x^2 - 1$$

and map to $y^2 = x^3 + x^2 + x - 1$

so $\overset{\text{Jac}}{V}$ is isog to product $E_1 \times E_2$.

Ⓒ $q = 2^{2e}$, $e \geqslant 2$

To construct : curve with Weil bound

[ⓓ For $q = 3^{2e}$, $e \geqslant 2$ ⟶ assigned to Bob Kuhn !?]

Start w/ $y^2 + y = x^3$ $/\mathbb{F}_2$, s.s.

This has 3 points, so $\pi = \sqrt{-2}$, since $1 + 2 - (\pi + \bar{\pi}) = 3$

<u>Over</u> $\mathbb{F}_4$, $\pi = -2$, Frob $= -2$.

Over $\mathbb{F}_q$, $q = 4^e$, Frob $= (-2)^e$

<u>To be proved</u>: E can be glued to itself over $\mathbb{F}_{4^e}$, $e \geqslant 2$.

Consider

$$\begin{tikzcd}[column sep=small]
\mathbb{F}_q(x,y) \arrow[d, dash, "2"'] & & \mathbb{F}_q(x,y') \arrow[dll, dash, "2"] \\
\mathbb{F}_q(x) & &
\end{tikzcd}$$

Where $(y')^2 + y' = (x+c)^3$, $c \in \mathbb{F}_q - \mathbb{F}_4$.

(Clearly isom. over $\overline{\mathbb{F}}_q$)

Want to make composition :

$$\begin{tikzcd}[column sep=small]
& & {} \arrow[dll, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
\text{genus 1}\ \mathbb{F}_q(x,y) \arrow[drr, dash] & & \mathbb{F}_q(x,z) \arrow[d, dash] \arrow[drrr, bend left] & & \mathbb{F}_q(x,y')\ \text{genus 1} \arrow[dll, dash] & \\
& & \mathbb{F}_q(x) & & & \text{Want: genus 0}
\end{tikzcd}$$

And don't want constant field extn.

Middle field is $\mathbb{F}_q(x,z)$, $z = y + y'$ (Artin-Schreier extn).

$$z^2 + z = \cancel{x^3} + \cancel{x^3} + cx^2 + c^2x + c^3$$

$$= \underbrace{(c^{1/2}x)^2 + c^{1/2}x} + (c^2 + c^{1/2})x + c^3$$

$t = z + c^{1/2}x$

$$t^2 + t = (c^2 + c^{1/2})x + c^3$$ Artin-Schreier, pole of order 1 (odd) at ∞

$c^2 + c^{1/2} \neq 0$, because $c^4 \neq c$ (since $c \notin \mathbb{F}_4$).

This is a conic, hence genus zero.

So composite has genus 2, and Jac ~ product $E \times E$

This gives the ~~[illegible]~~ desired construction. $\boxed{//}$

ⓓ For $q = 3^{2e}$, $e \geq 2$

We use a different method.

Intermezzo: On polarizations

A abelian variety.

(Mumford) : a polarization is a homomorphism $\varphi : A \longrightarrow A^*$
(A^* = dual of A) such that:

If D is a divisor of A and if $a \in A$, let D_a = translate of D by a, i.e., let $\tau_a : x \longmapsto x + a$, $D_a = \tau_a(D)$.

Then $D_a - D$ represents a point in $A^* = \mathrm{Pic}^\circ(A)$.

This map $a \longmapsto D_a - D \in A^*$ is a homomorphism $\varphi_D : A \longrightarrow A^*$.

Then a *polarization* is $\varphi : A \longrightarrow A^*$ s.t. there exists (over some extn of $\mathbb{F}_q$) an *ample* divisor D s.t. $\varphi_D = \varphi$. This defines Class$(D) \in NS(A)$ uniquely.
(φ is an isogeny, hence has a degree/deg φ.)

$$\deg_{pol}(\varphi) = \sqrt{\deg \varphi} = \frac{D^g}{g!} = \chi(A, \underline{\mathcal{L}}(D)) = \dim H^0(A, \underline{\mathcal{L}}(D))$$

(cf. Mumford, Ab. *Varieties*)

$$D^g = \underbrace{D \cdot D \cdot \ldots \cdot D}_{\substack{\text{intersection}\\ \text{multiplicity}}}$$

Criterion: $\varphi : A \longrightarrow A^*$ comes from an element (ample or not) of $NS(A) \Longleftrightarrow \varphi^* = \varphi$

$[\ \varphi^* : A^{**} = A \longrightarrow A^*$, so this makes sense$\ ] \longleftarrow$ cf. Mumford.

Example: Let E be an elliptic curve, $End(E) = \mathbb{Z}$.

Consider $A = E \times \cdots \times E$ n times.

$$A^* = E^* \times \text{—} \times E^* = E \times \text{—} \times E$$

$$\varphi : E \times \text{—} \times E \longrightarrow E \times \text{—} \times E \qquad End(E) = \mathbb{Z}$$

So $\varphi = \begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix}$ $a_{ij} \in \mathbb{Z}$.

$\varphi = \varphi_0$, $D \in NS(A) \Longleftrightarrow \varphi = \varphi^*$.

So comes from $NS(A) \Longleftrightarrow \varphi = {}^t\varphi$

φ polarization $\Longleftrightarrow \varphi = {}^t\varphi$ (φ symm) and φ is the matrix of a pos. definite quad. form.

E.g., use the E_8 quad form, to put an interesting polarization of deg 1 on $\underbrace{E \times \cdots \times E}_{8}$.

Loojenga, Inventiones ~1974,75 (careful: don't believe the statements).

polarization of degree 1 = principal polarization.

If C is a curve of genus g, its jacobian J has a natural polarization of degree 1 whose D is "Θ".

Have a map $C \longrightarrow J$, then define

$$\Theta = \underbrace{C + \cdots + C}_{g-1 \text{ times}} \underset{\text{birat'l}}{\simeq} C^{(g-1)}$$

So this gives $\varphi : J \xrightarrow{\approx} J^*$.

φ principal $\Rightarrow \deg_{\text{pol}} \varphi = 1 \Rightarrow \dim H^0(A, \underline{L}(D)) = 1 \Rightarrow$ the divisor class (for lin. equiv.) contains a unique positive divisor

So up to translation, can speak of the divisor of φ.

<u>Theorem</u>: Let A be an ab. variety of dim 2 with principal polarization and let C be its theta divisor. ∴ Then <u>either</u> C is nonsingular irred of genus 2 and
$A = \mathrm{Jac}(C)$

<u>or</u> C is $E_1 \cup E_2$ intersecting at 1 pt and
$A = E_1 \times E_2$ (as polarized variety).

If everything is over k and k perfect, then, in the indecomposable case everything is over k; in the decomposable case the ell. curves might be permuted by Gal(k'/k) for some quad. extn. k'/k, and then A will be indec. over k and dec. over k'.

[A similar theorem is true in dim 3, but that is harder]

<u>Proof</u>: principal polarization $\Rightarrow C \cdot C = C^2 = 2$

$\left(\deg \varphi = \frac{C^g}{g!} \text{ and } g = 2\right)$

Write $C = \sum m_i C_i$, C_i irreducible.

So $\sum m_i m_j C_i \cdot C_j = 2$

(on ab. variety, $C_i \cdot C_j \geq 0$ and even > 0 except when $C_i = C_j =$ ell. curve
→ use translation on A → find argument).

So either we have $2 = 2$
or
$2 = 1 + 1$

so either $\left\{ \begin{array}{l} C \text{ is irred} \\ \text{or} \quad C = E_1 + E_2 \quad 2 \text{ ell. curves } w/\ E_1 \cdot E_2 = 1 \end{array} \right.$

$$\hookrightarrow C^2 = \underbrace{E_1^2 + E_2^2}_{=0} + 2E_1E_2$$

If C irred, need still prove nonsing of genus 2.

Cannot be genus 0 or 1. Let g_a = arith. genus of C = $= \dim H^1(C, \mathcal{O}_C)$.

Then $g_a = g + \sum$ local contrib. at sing. pts.

Adjunction formula : $\boxed{2g_a - 2 = C \cdot C + C \cdot K}$

K = can. divisor

Here ab. variety so $K = 0$, and $C \cdot C = 2$, hence $2g_a = 4$ so $\boxed{g_a = 2}$.

Since $g = 0$ is impossible (can't embed in ab. var.)

If $g = 1$, map is homom., so can't be.

So $g = 2$ and nonsingular.

(Proof of exercise : adj. formula $\rightarrow$ $C.C \geq 0$ and $C \cdot C = 0$ only if $g_a = 1$.)

Dictionary for $g=2$ Over a perfect field k

Abelian varieties/k
w/ principal polariz.
which are indecomposable
over any quad. extn of k
$\Longleftrightarrow$
curves of genus 2 /k
(up to isom.)

Statement of Torelli's theorem in general (over a perfect field k)

Assume $g \geq 2$.

Let C and C' be curves of genus g over k, J, J' their jacobians as polarized abelian varieties.

(*) If J and J' are isomorphic/k then C and C' are isom./k.

More precisely:

i) if C is hyperelliptic and $F: J \longrightarrow J'$ is an isom. of polarized ab. varieties, then $\exists! f: C \xrightarrow[\cong]{} C'$ giving F by functoriality.
In particular, $\mathrm{Aut}(C) \xrightarrow{\sim} \mathrm{Aut}(J, \text{pol.})$

ii) if C is not hyperelliptic, for every $F: J \xrightarrow{\cong} J'$, there exists a unique isom. $f: C \xrightarrow{\cong} C'$ and a unique $\varepsilon \in \{\pm 1\}$ s.t. f gives εF by functoriality.

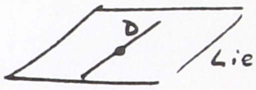
In particular. $\mathrm{Aut}(C) \times \{\pm 1\} = \mathrm{Aut}(J, \text{pol})$

Fine de l'Intermezzo

Back to $\mathbb{F}_{q^e}$, $e \geq 2$

Take E w/ Frob $= -q^{1/2}$

Will consider $E \times E \xrightarrow[\text{isogeny}]{\text{insep.}} J \longleftrightarrow$ classified by



J = same pts, but only fcts whose deriv. along the given line = 0

will choose D not rat'l over $\mathbb{F}_{p^2}$

Put a polarization on E×E by $\begin{pmatrix} p & \alpha \\ \bar{\alpha} & p \end{pmatrix}$, $\alpha\bar{\alpha} = p(p-1)$,

Prove: this descends to J + becomes principal there, and J is indec., which gives a curve!

Moret-Bailly → this kind of construction w/ varying D. Find

$$\begin{tikzcd} J \arrow[d, "\text{ab. scheme}"] \\ \mathbb{P}_1 \end{tikzcd}$$

corresp to a surface
$$\begin{tikzcd} S \arrow[d] \\ \mathbb{P}_1 \end{tikzcd}$$

s.t. generic fiber is curve of genus 2

5p - 5 exceptional fibers

For $p = 2$, $5p-5 = 5$ → rat'l pts / $\mathbb{F}_4$

$p = 3$ $5p-5 = 10$ → ——— / $\mathbb{F}_9$

$p \geq 5$ pts left over!

So this picks out the exceptional pts.

11/5 (g=2 cont.)

q square $\Longrightarrow$ Weil bound is attained except for $q = 4, 9$

We have proved this except:
Missing case : $q = 9^e$, $e \geq 2$.

Start with E ell. curve, supersingular, over $\mathbb{F}_{p^2}$, s.t. Frob on $E = p$.

For such E : Well-known : End(E) is a max'l order in the quaternion algebra $H_{p,\infty}$
(and this max'l order can be imposed on E).

Choose E such that End $E \ni \alpha$ with $\alpha\bar{\alpha} = p(p-1)$

This is possible : Consider $\mathbb{Q}(\sqrt{-p(p-1)})$; this is imag. quad., ramif. at p, so splits $H_{p,\infty}$.

So $\alpha = \sqrt{-p(p-1)}$ is an integer of this field, and can be included in $H_{p,\infty}$.

So choose a max'l order containing the image of α.

Now let $A = E \times E$

polarization : $\varphi : A \longrightarrow A^* = E \times E$

$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad a,b,c,d \in$ End(E)

φ is a polarization $\longleftrightarrow$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ hermitian $\gg 0$

("between the lines" in Mumford, Ab. Vars., toward the end.)

[Mumford starts w/ $E \times E$, say, w/ a given polarization, say the obvious one. Identify $\varphi : A \longrightarrow A^*$ as End, and asks which φ are polarizations.
So says, take End $\otimes \mathbb{R}$, and then φ polar. corresp to positive definite symmetric matrices.

φ is of degree 1 $\longleftrightarrow$ "det" $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$

(Note : hermitian $\Rightarrow$ $a, d \in \mathbb{Z}$
$b = \bar{c}$ so we have only
$ad - b\bar{b} = 1$).

Choose $\varphi = \begin{pmatrix} p & \alpha \\ \bar{\alpha} & p \end{pmatrix}$

$$\det \varphi = p^2 - \alpha\bar{\alpha} = p^2 - p(p-1) = p.$$

We find : Ker φ has order p^2 (say, look at ℓ-adic representations.)

tgt space at E is 1-dim'l, so End E $\longrightarrow \mathbb{F}_{p^2}$ (action on tgt space)
residue field at p.

$\alpha \longrightarrow 0$ ($\alpha\bar{\alpha} = p(p-1) \longrightarrow 0$!)

$p \longrightarrow 0$

So φ is 0 on the tgt space of A.

[A ab variety in char p., dim. g, there is a subgroup of A t_A = "kernel of Fr"; this is one point with nilpotents. If $x_1, \dots, x_g$ are local coords around 0 and k is the ground field, the group t_A is def. by $x_i^p = 0$

the order of t_A (as gp scheme) is p^g, $g = \dim A$.

(the algebra is $k[x_1, \dots, x_g]/(x_i^p)$).

So counting orders gives $\operatorname{Ker} \varphi = t_A$ in our case.

Take $\mathbb{F}_q \supsetneq \mathbb{F}_{p^2}$, so $q = p^{2e}$, $e \geq 2$, and now view E and A over $\mathbb{F}_q$.

Choose a line $D \subset$ tgt space of A whose slope is <u>not</u> in $\mathbb{F}_{p^2}$. [This means something since $A = E \times E$.]

It makes ~~sense~~ to "divide" A by D; this means

$$\begin{tikzcd} \text{``tangent space''} \arrow[r, leftrightarrow] \arrow[d, phantom, "\cup"] & t_A \\ D \arrow[r, leftrightarrow] & {[D] \subset t_A} \end{tikzcd}$$

(subgroup scheme)

(stable under p-th power map)

(which is zero here) → curve is s.s.!

So "dividing A by D" is just taking $A/[D]$.

Direct def'n of $A/[D]$:

- same pts as A
- less rat'l functions: those having derivative by D equal to 0.

(So, e.g., $A/t_A = A^{(p)}$).

E.g. $A: \; y^2 = x^3 + ax + b$

$$\begin{tikzcd} A: y^2 = x^3 + ax + b \arrow[d, "Fr"'] \\ A^{(p)} \; Y^2 = X^3 + a^p X + b^p \end{tikzcd}$$

Fr is $Y = y^p$, $X = x^p$

take $t = \frac{x}{y}$ param. near zero; so kernel is given by $t^p = 0$.

So set $J = A/[D]$.

So ask whether the polarisation descends to J:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi"'] & A^* \\ J \arrow[r, dashed, "\exists \varphi_J ?"'] & J^* \arrow[u, "\pi^*"'] \end{tikzcd}$$

For this, a nec. condn is $\varphi([D]) = 0$; in our case, this holds, since $\ker \varphi = t_A$.

(du Mumford): If D has dim. 1, this nec. condn. is also sufficient.

[<u>General fact</u> is <u>If</u> $\left\{ \begin{array}{l} \varphi : A \longrightarrow A^* \quad \text{polariz.} \\ N \text{ finite subgp of } A \\ \text{Assume } N \subset \ker \varphi, \ |N| \text{ is prime.} \end{array} \right.$

Then φ descends to a polarization of A/N.]

Deg φ_J = ? Count degrees of various kernels $\Rightarrow \deg \varphi_J = 1$.

J has a polarization of deg. 1 (def. over $\mathbb{F}_q$).

[D as an alg. group / k

Affine algebra is $\Lambda : k \oplus kx \oplus \cdots \oplus kx^{p-1}$; $x^p = 0$

$$\begin{array}{c} \| \\ k[x]/(x^p) \end{array}$$

Comultiplication is $x \longmapsto x \otimes 1 + 1 \otimes x$

order (D) = $\dim_k \Lambda = p$

t_A is a group of order p^2 which, over $\overline{\mathbb{F}}_p$, has infinitely many subgroups of order p.]

<u>Claim</u>: J is not isomorphic over k (as abelian variety) to a product of two elliptic curve. (k any extn. of $\mathbb{F}_q$).

If E is any ss elliptic curve / $k \supset \mathbb{F}_{p^2}$, look at tgt(E).

<u>Claim</u>: tgt(E) has a natural "$\mathbb{F}_{p^2}$-structure" which is functorial.

This is so because E comes by scalar extn from a curve over $\mathbb{F}_{p^2}$ with Frob $= p$ (for any such E). Then tgt(E) comes from the tgt space over $\mathbb{F}_{p^2}$.

Suppose now $J = E_1 \times E_2$, E_i s.s.

Then $A = E \times E \longrightarrow J \xrightarrow{\sim} E_1 \times E_2$.

Look at tgt map to $E \times E \longrightarrow E_1 \times E_2$, which is $\mathbb{F}_{p^2}$-<u>rational</u>.

But ker $= D$ has ir<u>rational</u> slope, so contrad. §.

So J is not $E_1 \times E_2$.

$\therefore$ This J is the jacobian of a $C/\mathbb{F}_q$ of genus 2, and Frob will be p^e (since that is invariant under isogeny), so this realizes the Weil minimum.

So q = square is done. ☑

---

q <u>non-square</u>

$$q \text{ special} \Longleftrightarrow \begin{cases} p \mid m & m = [2q^{1/2}] \\ q = x^2+1 \\ \quad\; x^2+x+1 \\ \quad\; x^2+x+2 \end{cases}$$

<u>Thm</u>: If q is not special, then $N_q(2) = 1 + q + 2m$.

We will choose an ell. curve E with Trace(Frob) = $-m$.
This exists because q is not special (seen before!)

To be proved: E can be glued to itself, i.e., $\exists\, C$ s.t. $\mathrm{Jac}(C) \underset{\text{isogenous}}{\sim} E \times E$.

We use the "elementary glueing":

ⓐ $p \neq 2$

<u>Lemma</u>: Let $a \in \mathbb{Z}$, $p \nmid a$ and $|a| \leq 2\sqrt{q}$, $a^2 - 4q \neq -3, -4$, q not a square.
Then there exists an elliptic curve $E/\mathbb{F}_q$ with $\mathrm{Tr}(\mathrm{Frob}) = a$ and $j(E) \neq 0, 1728$.

<u>Proof</u>: An E exists with $\mathrm{Tr}(\mathrm{Frob}) = a$, and this is an <u>ordinary</u> curve since $p \nmid a$.

If $\mathrm{End}(E) \neq \mathbb{Z}[i]$ or $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$, then $j(E) \neq 0, 1728$, so ok.

<u>Assume</u> for instance that $\mathrm{End}\, E = \mathbb{Z}[i]$, and let $\pi = \mathrm{Frob}$, [redacted] $\pi = x + yi$,

and
$$\begin{cases} \mathrm{Tr}\,\pi = a = 2x \\ x^2 + y^2 = \pi\bar{\pi} = q \end{cases}$$

<u>Claim</u>: $y \neq \pm 1$

(if $y^2 = 1$, $q = x^2+1$ so $4q = 4x^2+4 = a^2+4$ and $a^2-4q = -4$ against hypothesis).

So $\mathbb{Z}[\pi] \subsetneqq \mathbb{Z}[i]$

[Can use: $\exists$ another curve isog to E w/ $\mathrm{End} = \mathbb{Z}[\pi]$.]

[Instead:]

Choose ℓ prime, $\ell | y$; look at the action of π on ℓ-division points ($\ell \neq p$; otherwise $p|x$ so $p|a$)

So $\pi \equiv x \bmod (\ell \,\mathrm{End}\, E)$,

so π acts by homothety on ℓ-div. pts

have $\ell+1$ subgps of order ℓ, all stable by π, of these, at most 2 are stable by $\mathbb{Z}[i]$.

So choose a subgp D which is <u>not</u> stable by $\mathbb{Z}[i]$, and replace E by E/D.

Then $\mathrm{End}(E/D) = \{x+iy \mid \ell | y\} \neq \mathbb{Z}[i]$, so choose E/D now.

In the other case, proceed similarly. $\boxslash$

Now choose E acc. to lemma for $a = m$, since $m^2 - 4q \neq -3, -4$

(if $m^2 - 4q = -4 \Rightarrow 4q = 4 + m^2 \Rightarrow 2|m \Rightarrow q = 1 + (\frac{m}{2})^2$ ↯)
$ = -3$, similar

Now E can be glued to itself (because exceptions had $j = 0$ or 1728), and we are done. $\boxtimes$

④ $\underline{\underline{p=2}}$ 2^e (e odd) is nonspecial iff $\begin{cases} 2 \nmid m \\ 2^e \text{ is not } x^2+x+2 \end{cases}$

[redacted]

Choose E_1 and E_2 with $\mathrm{Tr}(\mathrm{Frob}) = -m$ and which are not isom., even after quadratic field extension.

(1) <u>Lemma</u>: This is possible — later.

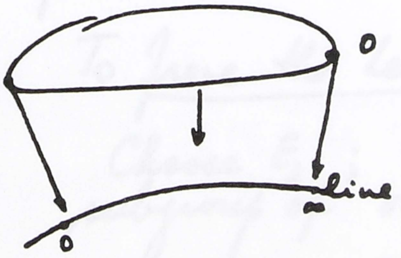
(2) Then one can glue E_1 to E_2.

<u>Proof of (2)</u>

Write E as quad covering of $\mathbb{P}_1$ ram at ∞.

Covering is $E/_\sim$ where $x \sim -x$.

Other ramif pt is the pt of order 2, rat'l, so can map to zero.



So Artin-Schreier eqn is

$$\boxed{y^2 + y = \lambda x + \mu/x + \nu} \quad \lambda, \mu \neq 0$$

replace x by λx, so can write $y^2+y = x + \frac{\mu}{x} + \nu$

(over $\overline{\mathbb{F}}_2$, write μ as x^2+x, and then ν can be taken $= 0$)

So have $E_1 : \left\{ \begin{array}{l} y_1^2 + y_1 = x + \frac{\mu_1}{x} + \nu_1 \\ y_2^2 + y_2 = x + \frac{\mu_2}{x} + \nu_2 \end{array} \right.$

$\mu_1 + \mu_2$

$$\begin{tikzcd}
& K \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{F}_q(x, y_1) \arrow[dr, dash] & \mathbb{F}_q(x, y_1 + y_2) \arrow[d, dash] & \mathbb{F}_q(x, y_2) \arrow[dl, dash] \\
& \mathbb{F}_q(x) &
\end{tikzcd}$$

$$(y_1 + y_2)^2 - (y_1 + y_2) = \frac{\mu_1 + \mu_2}{x} + \nu_1 + \nu_2$$

genus zero

[if $\mu_1 = \mu_2$, $\mu_1 + \mu_2 = 0$ so get const field extn.]

So K is the field of a curve of genus 2. This proves glueing ☒

To prove the Lemma (1):

Choose E_1; E_1 ordinary $\Rightarrow E_1$ has a unique subgroup of order 2, say N.

Put $E_2 = E_1/_N$. Is $E_2 \cong E_1$?

If not, we are done.

If $E_1 \cong E_2$, then $\exists\, \varphi : E_1 \longrightarrow E_1$ with kernel N, hence of degree 2.

Now E_1 ordinary, so $\mathrm{End}(E_1)$ is an order in an imag. quad. field, and $\varphi \in \mathrm{End}(E_1)$, $\varphi\bar{\varphi} = 2$.

Then $|\mathrm{Tr}\,\varphi| \leq 2\sqrt{2}$ (since $|\varphi| = \sqrt{2}$)

So $\mathrm{Tr}\,\varphi = -2, -1, 0, 1, 2$

$\mathrm{Tr}\,\varphi = -2 \implies \varphi = -\cancel{1 \pm i}$

$-1 \implies \varphi = \frac{-1+\sqrt{-7}}{2}$

$0 \implies \varphi = \pm\cancel{\sqrt{-2}}$

$1 \implies \cancel{1 \pm i}$

impossible, since 2 splits in End E by general facts.

$2 \implies \frac{1 \pm \sqrt{-7}}{2}$

So $\varphi = \pm\frac{1 \pm \sqrt{-7}}{2}$, and $\mathrm{End}\,E_1 = \mathbb{Z}\left[\frac{1 \pm \sqrt{-7}}{2}\right]$

So remains to show that I can choose E_1 with $\mathrm{End}(E_1) \neq \mathbb{Z}\left[\frac{1 \pm \sqrt{-7}}{2}\right]$.

Assume $\mathrm{End}\,E_1 = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$

$\mathrm{Frob} = \pi = x + \frac{1+\sqrt{-7}}{2}y \qquad x, y \in \mathbb{Z}$

Again: claim that $y^2 \neq 1$

$$\pi\bar{\pi} = q = (x+\tfrac{1}{2})^2 + 7\frac{y^2}{4} \bullet$$

if $y^2 = 1$, get $q = (x+\frac{1}{2})^2 + \frac{7}{4} = x^2 + x + 2$

Now $\mathbb{Z}[\pi] \subsetneq \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$, and by the argument above I can change E_1 s.t. $\mathrm{End}(E_1) = \mathbb{Z}[\pi]$, for instance (or choose ℓ/y as before, etc.). ▨

11/12

$\boxed{g=2}$

We were reduced to q not a square; q is non-special is done (we get $q+1+2m$, $m=[2\sqrt{q}]$.)

Need to show: if q is special:

- a) $q+1+2m$ is impossible
- b) $q+2m$ is possible iff $\{2\sqrt{q}\} > \frac{\sqrt{5}-1}{2}$
- c) if q special and $\{2\sqrt{q}\} < \frac{\sqrt{5}-1}{2}$, then $q+2m-1$ is possible.

---

a) Proof: q is special $\Longleftrightarrow$

a1) q is represented by the quad. polyn. x^2+1, x^2+x+1 (special case of Beauville)

OR

a2) q is represented by x^2+x+2 (so $q=2^e$, e odd) One finds $e=1,3,5,13$. ⟶

⟶ see this later

OR

a3) $p \mid m$

$2^e = x^2+x+2$

$2^{e+2} = (2x+1)^2+7 = y^2+7$

Ramanujan's equation

Case a2: We show that $q+1+2m$ is impossible.

→ If $q=2$, $[2\sqrt{2}]=2$, so this is special also because $p|m$ (a3).

$$1+q+2m = 1+2+4 = 7 > 2(q+1)$$, so of course we are done.

→ If $q=2^3$, $[2\sqrt{8}] = [\sqrt{32}] = 5$, so $1+q+2m = 1+8+10 = 19 > 2(q+1)$, again impossible.

→ Recall that $1+q+2m$ is possible only if eigenvalues of Frob are $\pi, \bar{\pi}, \pi, \bar{\pi}$, $\pi+\bar{\pi} = -m$, $\pi\bar{\pi} = q$.

If $q = 2^5$, $m = [2\sqrt{32}] = [\sqrt{128}] = 11$, so

$$m^2 - 4q = 121 - 128 = -7$$

So $\boxed{\pi = \frac{-11+\sqrt{-7}}{2}}$.

We must show this is impossible.

in $\mathbb{Q}(\sqrt{-7})$, 2 splits as $(\alpha)(\bar{\alpha})$, $\alpha = \frac{1+\sqrt{-7}}{2}$, $\alpha\bar{\alpha} = 2$.

Since $\pi\bar{\pi} = 2^5$, so $(\pi) = (\alpha)^i(\bar{\alpha})^j$ $i+j=5$,
but $2 \nmid \pi$, so must have $(\pi) = (\alpha)^5$ or $(\pi) = (\bar{\alpha})^5$

So $\pi = \pm\alpha^5$ or $\pm\bar{\alpha}^5$. In fact, $\pi = -\alpha^5 = (-\alpha)^5$

So Frob. over $\mathbb{F}_{2^5}$ is a fifth power.

Take $\mathrm{Jac}(C)/_{\mathbb{F}_{2^5}}$.

$\mathrm{End}(\mathrm{Jac}(C)) \supset \mathbb{Z}[\pi]$ (in fact $= M_2(\mathbb{Z}[\alpha])$)

So $\operatorname{End}(\operatorname{Jac}(C)) \supset \mathbb{Z}[\pi] = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$

define $\pi_1 = -\alpha \in \operatorname{End}(\operatorname{Jac}(C))$, and then $\pi_1^5 = \pi$ = Frobenius on $\operatorname{Jac}(C)$.

- So have ab. var. $A/_{\mathbb{F}_{p^e}}$, $\pi = \text{Frob}/_{\mathbb{F}_{p^e}}$, and $\pi = \pi_1^e$ for some $\pi_1 \in \operatorname{End}(A)$. Would like to conclude: exists a structure $/_{\mathbb{F}_p}$ for A with Frobenius π_1.

Need <u>more</u>: π_1 acts by 0 on the tangent space of A.

(Above $\alpha = \frac{1+\sqrt{-7}}{2}$, $\pi = \frac{-11+\sqrt{-7}}{2}$, so $\alpha = \pi + 6$, and both π and 6 act by 0 on tangent space, hence so do α and $\pi_1 = -\alpha$!)

Now define $A^{(p)}$ as usual. Then we have

$$\begin{tikzcd} A \arrow[r, "F_p"] \arrow[dr, "\pi_1"'] & A^{(p)} \arrow[d, dashed] \\ & A \end{tikzcd}$$

(The map $\downarrow$ exists if and only if π_1 kills the tgt space, so OK.)

$A^{(p)} \xrightarrow{\cong} A$ is an isom. by degree computation!

So $A \xrightarrow{\pi_1} A$ gives an isom. of $A^{(p)}$ to A, so Galois descent is OK.

Hence we have an $\mathbb{F}_2$-structure on $\operatorname{Jac}(C)$ with Frobenius $-\alpha$

Polariz. is invariant by $-\alpha$ ($\alpha = \pi + 6$!) (π respects the polarization)

So we get (Torelli): there is an $\mathbb{F}_2$-structure on C with Frob $-\alpha$.

Now go to $\mathbb{F}_8 = \mathbb{F}_{2^3}$, and get Frobenius $= -\alpha^3 = \frac{5+\sqrt{-7}}{2}$

Now nber of pts$/\mathbb{F}_8$ $= 1 + 8 - 2\cdot 5 = -1$ ↯. (contrad).

→ If $q = 2^{13}$, we find $m = 191$, and use

$$\left(\frac{1+\sqrt{-7}}{2}\right)^{13} = -\frac{181+\sqrt{-7}}{2}$$

$$\boxed{\left(\frac{1+\sqrt{-7}}{2}\right)^{e} = \frac{a_e + b_e\sqrt{-7}}{2}}$$

When is $b_e = \pm 1$?

e odd; $e = 3, 5, 13$

→ analytic method!

we'll come back to this.

Case a3: $p | m$; I can assume $q = p^e$, e odd ≥ 3

(when $e = 1$, $p | m \Rightarrow p = 2$ or 3 and these have been done already).

have $1 + q + 2m$ iff $\pi, \bar\pi, \pi, \bar\pi$, $\begin{cases} \pi + \bar\pi = -m \\ \pi\bar\pi = q \end{cases}$

Now, $p \mid m$, $q = p^e$ e odd $\geqslant 3$.

By Tate's theorem, $\exists f_\pi$ (we saw f_π odd $\geqslant 5$) s.t. the multiplicity of π as root of Frob. in any abelian variety is divisible by f_π.

So we cannot have $f_\pi \mid 2$! So the jacobian cannot exist in this case, done. ▨

b) $\left\{ \begin{array}{l} \text{If } \{2\sqrt{q}\} < \frac{\sqrt{5}-1}{2}, \text{ then } q+2m \text{ is impossible} \\ \\ \text{If } \{2\sqrt{q}\} > \frac{\sqrt{5}-1}{2}, \text{ then } q+2m \text{ \underline{is} possible.} \end{array} \right.$

We've shown "down by one" (for a curve of genus 2) is possible <u>only</u> if

$$\text{Frob}: \quad \pi_1, \overline{\pi}_1, \overline{\pi}_2, \pi_2$$

where $\left\{ \begin{array}{l} \pi_1 + \overline{\pi}_1 = -m + \dfrac{1+\sqrt{5}}{2} \\ \\ \pi_2 + \overline{\pi}_2 = -m + \dfrac{1-\sqrt{5}}{2} \end{array} \right.$

and this is possible only when $m + \frac{-1+\sqrt{5}}{2} \leq 2\sqrt{q}$, i.e., only when $\{2\sqrt{q}\} = 2\sqrt{q} - m \geqslant \frac{\sqrt{5}-1}{2}$

So first statement is OK.

Now assume $\{2\sqrt{q}\} \geqslant \frac{\sqrt{5}-1}{2}$

First, make an ab. variety of dim. 2 with $\pi_1, \bar{\pi}_1, \pi_2, \bar{\pi}_2$ as required above, i.e.,

$$\begin{cases} \pi_1 + \bar{\pi}_1 = -m + \frac{1+\sqrt{5}}{2} \\ \\ \pi_2 + \bar{\pi}_2 = -m + \frac{1-\sqrt{5}}{2} \end{cases}$$

We want this ab. variety to be ordinary:

- $\pi_1 + \bar{\pi}_1$ and $\pi_2 + \bar{\pi}_2$ are prime to p

× if $p | m$, obvious, since $\frac{1 \pm \sqrt{5}}{2}$ is a unit

× if $p \nmid m$, q is repres. by x^2+1, x^2+x+1, x^2+x+2,

so
$$4q = \overset{=(2x)^2+4}{4x^2+4} \text{ so } m = 2x$$
$$4q = \overset{=(2x+1)^2+3}{4x^2+4x+4} \text{ so } m = 2x+1$$
$$4q = 4x^2+4x+8 = (2x+1)^2+7 \text{, so } m = 2x+1.$$

so $\{2\sqrt{q}\}$ is small, usually $< \frac{\sqrt{5}-1}{2}$. (at least for $q > 8$)

Only exceptions are $q = 2, 8$ (2 is covered: $p | m$)

$\{2\sqrt{8}\} = 0.656\ldots > \frac{\sqrt{5}-1}{2}$

[For $q=2$, want $g=2$, 6 pts. Take
$$y^2 + y = \frac{x^2+x}{x^3+x+1}$$
]

Construction of a curve w/ 18 (= 2(1+8)) points over $\mathbb{F}_8$

Choose an irred cubic polyn $f(x)$ on $\mathbb{F}_8$, e.g. x^3+x+c for suitable c.

Write

$$y^2+y=\frac{a+bx+cx^2}{x^3+x+c} \qquad a,b,c\in\mathbb{F}_8.$$

$$\text{(i.e.,)} \quad y^2+y=\frac{a+bx+cx^2}{f(x)} \qquad a,b,c\in\mathbb{F}_8.$$

We want to choose a,b,c s.t.

$$\mathrm{Tr}_{/\mathbb{F}_2}\left(\frac{a+bx+cx^2}{f(x)}\right)=0 \qquad \text{for all } x\in\mathbb{F}_8.$$

$a,b,c\in\mathbb{F}_8=\mathbb{F}_2^3$ (as $\mathbb{F}_2$-vector space)

So $(a,b,c)\in\mathbb{F}_2^9$, and get 8 homog. eqns in 9 unknowns, so choose a solution which is not trivial!

Now Tate-Honda says such an ab. variety exists.

Note $\mathbb{Q}(\pi)\supset\mathbb{Q}(\sqrt{5})$..

Have

$$\begin{tikzcd} \mathbb{Q}(\pi) \arrow[d, dash, "2\ \text{imag. quad. extn.}"] & \text{CM-field of deg. 4 w/ assoc. real field } \mathbb{Q}(\sqrt{5}). \arrow[l] \\ \mathbb{Q}(\sqrt{5}) \arrow[d, dash, "2"] & \\ \mathbb{Q} & \end{tikzcd}$$

So $\mathrm{End}_{\mathbb{F}_q}(A) \otimes \mathbb{Q} = \mathbb{Q}(\pi)$ (Tate).

We have A <u>up to isogeny</u>.

Next step: existence of such an A <u>having a polarization of degree 1</u>.

(<u>Shimura</u>: A in char 0, CM by $\begin{array}{c} K \\ 2\,| \\ K_0 \end{array}$ $\longrightarrow$ OK)

<u>Shimura's Thm (char 0)</u>: If CM type is K/K_0 with K ramified over K_0, then $\exists$ A in the isogeny class having polariz. compatible w/ CM-type, of degree 1. [Proc. London Math. Soc., <u>34</u> (1977), p. 67, remark.]

Here OK, since $\mathbb{Q}(\sqrt{5})$ has class number 1. $\to$ no quadratic unramified extn.

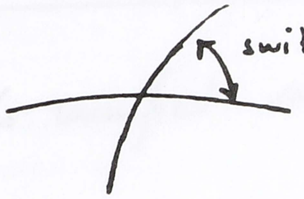
<u>Claim</u>: Sh's thm is OK in char p for ordinary abelian varieties

(proof later)

Then we get A w/ the right Frob + polarization of degree 1. It remains to check that this is indec.. Over $\mathbb{F}_q$, it is indec. because $\mathrm{End}_{\mathbb{F}_q}(A) = \mathbb{Q}(\pi)$ is a field.

It could decompose over $\mathbb{F}_{q^2}$. This doesn't happen:

if indec over $\mathbb{F}_q$, dec over $\mathbb{F}_{q^2}$, it's



So on V_ℓ, matrix of Frob is $\left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$

so trace = 0.

But our trace is $-2m + 1 \neq 0$.

This proves that our curve exists. ☑

Proof of Claim above

char 0 Shimura $\Longrightarrow$ char p Shimura for <u>ord</u> ab. vars.

Use "canonical lifting": lift $A/_{\mathbb{F}_q}$ into $\mathcal{A}/_{W(\mathbb{F}_q)}$, and End does not change.

By Shimura, $\mathcal{A} \underset{\text{isog}}{\sim} \mathcal{A}'$, on $\mathcal{A}'$ have a polariz. of degree 1.

ker of $\mathcal{A} \longrightarrow \mathcal{A}'$ is stable under max'l order

So $A \to A'$ exits over the field of fractions of $W(\mathbb{F}_q)$.

Polarization is $A' \longrightarrow (A')^*$.

So reduce everything mod p.

Another way: take Shimura's proof and show it works in char p.

Translation of K ramified over K_0:

$$K/K_0 \text{ ramified} \Longleftrightarrow Cl(K) \xrightarrow{\text{norm}} Cl^{\text{strict}}(K_0) \text{ is onto}$$

[Cl^{strict} : $\mathfrak{a} \sim 1$ if $\mathfrak{a} = (\alpha)$, $\alpha \gg 0$.

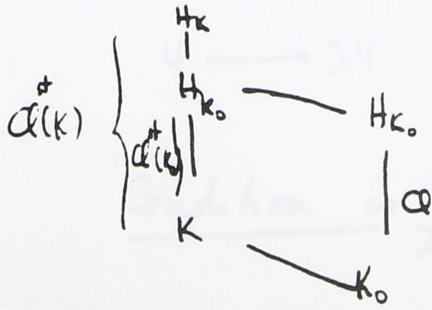
For tot. imag. field, $Cl(K) = Cl^{\text{strict}}(K)$, and map is just $\mathfrak{a} \longmapsto \mathfrak{a}\bar{\mathfrak{a}}$.]

Pf of $\Longleftrightarrow$ CFT!

$$\begin{tikzcd} H_{K_0} \arrow[d, dash] \\ K_0 \end{tikzcd}$$

H_{K_0} = Hilbert class field
max. ab. unr extn, except maybe at ∞

Then $Gal(H_{K_0}/K_0) \cong Cl^{\text{strict}}(K_0)$



and res: $Gal(H_K/K) \longrightarrow Gal(H_{K_0}/K_0)$ is norm map.

If K/K_0 unr, we'd have,

$$\begin{tikzcd} & H_{K_0} \arrow[d, dash] \\ H_K \arrow[r, dash] & K \arrow[d, dash] \\ & K_0 \end{tikzcd}$$

and then coker $Cl(K) \longrightarrow Cl^{st}(K_0)$ has order 2. $\boxed{V}$

Shimura's proof

Choose A s.t. $\mathrm{End}(A)$ is a max'l order in K, and choose some polarization φ. $\varphi : A \longrightarrow A^*$.

Look at $\mathfrak{a}$ ideal of $\mathcal{O}_K$ s.t. $\ker \varphi \cong \mathcal{O}_K/\mathfrak{a}$

Shimura-Taniyama : $\mathfrak{a}$ "is" an ideal of $\mathcal{O}_{K_0}$.

Replacing φ by $\lambda\varphi$ gives a polarization if $\lambda \in \mathcal{O}_{K_0}$, $\lambda >> 0$.

May replace A by some $A/_{\left(\substack{\text{finite subgp} \\ \text{stable by } \mathcal{O}_K}\right)}$. $\longleftrightarrow$ some ideal $\mathfrak{v} \subset \mathcal{O}_K$

Want $\lambda\varphi$ to descend to $A/_{(\quad)}$ w/ degree 1

$\varphi \longmapsto \lambda\varphi$ is $\mathfrak{a} \longrightarrow \lambda\mathfrak{a}$

Condition is : if $\lambda\mathfrak{a} = \mathfrak{v}\bar{\mathfrak{v}}$, then on $A/_{(\,)}$, $\lambda\varphi$ has degree 1.

This is equiv to: the class of $\mathfrak{a}$ in $Cl^{st}(K_0)$ is an image of $\mathfrak{v} \in \mathcal{O}_K$. So OK if $Cl(K) \longrightarrow Cl^{st}(K_0)$ is onto.

Se 47a

This proves part ⓑ. ☒

Remains to prove ⓒ

11/19: Remains: If q is not a square and "special", down by 2 is possible, i.e., there exists a curve with $q+2m-1$ points.

In fact, one can find a curve [redacted] of type $(m-1, m-1)$ or $(m, m-2)$.

For $q=2,3$ just write the equation.

more precisely, for $q=2$, $(m, m-2)$ is possible, $(m-1, m-1)$ is not)

for $q=3$, both are possible

Assume $q \geq 5$.

Cases:
1) $p \mid m$ and $p \neq 2 \Longrightarrow (m-1, m-1)$ is possible
2) $p \mid m$ and $p = 2 \Longrightarrow$ " "
3) $p \nmid m$ and q special, $p \neq 2 \Longrightarrow$ " "
4) $p \nmid m$, q special, $p = 2 \Longrightarrow (m, m-2)$ is possible

$\longrightarrow$ only three cases: $q = 2^3, 2^5, 2^{13}$

This means

$(m-1, m-1)$ possible : Can glue E_1, E_2 with $\text{Trace}(\text{Frob}) = -(m-1)$

$(m, m-2)$ possible : Can glue E_1, E_2 with $\text{Trace}(\text{Frob}) = \begin{cases} -m \\ -(m-2) \end{cases}$

Case 1 : $p \mid m$, hence $p \nmid (m-1)$, so E_1 and E_2 exist (and are ordinary!).

Also, $\mathbb{Z}[\pi]$ is an order in an imag. quad. field w/ disc. $-4q + \text{Tr}(\pi)^2 = -4q + (m-1)^2 = -\left(4q - (m-1)^2\right)^{8}$

$$4q = m^2 + k \qquad 1 \leq k \leq 2m$$

$$4q - (m-1)^2 = m^2 + k - m^2 + 2m - 1 = k + 2m - 1 \geq 8$$

$k \equiv -m^2 \equiv 0$ or $3 \pmod 4$

so $k \equiv 3 \pmod 4$

$k = 3, 7, \ldots.$

$p \mid m,\ p \neq 2 \implies 2m - 1 \geq 5$

so ≥ 8.

One can choose E_1, E_2 such that $\text{End}(E_i) = \mathbb{Z}[\pi]$.

So $\text{End} \not\supset \mathbb{Q}\left\{\begin{matrix}\text{cube roots of } 1 \\ \text{or } i\end{matrix}\right\}$

So $j \neq 0, 1728$

so elementary glueing is OK.

Case 2 $\overset{p=2}{\checkmark}$: Choose E, $\mathrm{End}\,E = \mathbb{Z}[\pi]$, and, [redacted] disc < -8.

Now take E, $E' = E/_{E[2]}$ (in char 2 !)

Not isomorphic (otherwise $\exists$ isogeny $E \to E$ deg 2 $\longrightarrow \mathbb{Q}(\sqrt{-7})$ ζ).

Write these as $y^2 + y = x + \frac{a}{x} + b$

$$y^2 + y = x + \frac{a'}{x} + b'$$ $a \neq a'$

And we get the usual diagram

$$\begin{tikzcd}
 & \text{gluing.} & \\
\mathbb{F}_q(E) \arrow[ur, dash] & & \mathbb{F}_q(E') \arrow[ul, dash] \\
 & \mathbb{F}_q(x) \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

This does ②.

<u>Case 3</u>: $p | m$, q special, $p \neq 2$:

Same proof, but problem is whether $p | m-1$.

q special, $p = 2 \longrightarrow 4q = m^2 + 3$ or $4q = m^2 + 4$.

So $4q = m^2 + k$, $k = 3$ or 4

$p | m-1 \Rightarrow m \equiv 1 \pmod{p} \Rightarrow k \equiv -1 \bmod p$

so 4 or $5 \equiv 0 \bmod p$

$\longrightarrow p = 5$, $k = 4$, m even

- So if $p \neq 5$, same proof above works since $p \nmid (m-1)$.

- If $\dfrac{p = 5}{(\Rightarrow k = 4)}$ a) $q = 5$; then $m = [2\sqrt{5}] = 4$, $m - 1 = 3$ and $5 \nmid 3$. So OK.

  b) $q = 5^e$, e odd $\geqslant 3$; would have $5^e = \left(\frac{m}{2}\right)^2 + 1$

  But the eqn: $y^e = x^2 + 1$ $x \neq 0$ $e \geqslant 2$ has no $\mathbb{Z}$-solutions [proof later].

So ⓑ is empty, and ③ is done.

Case 4: $q = 2^3, 2^5, 2^{13}$

Claim is : $(m, m-2)$ is possible.

$p = 2$, $2 \nmid m \Rightarrow 2 \nmid m-2$

So we find E_1, E_2 ord. elliptic curves with $Tr(Frob) = \Big\} \begin{matrix} -m \\ -(m- \end{matrix}$

and we glue them as in ② (and we know they are not geometrically isomorphic ~~because~~ $m \neq \pm(m-2)$.
If $m = -m+2$, $m = 1$; bad case is $k = 7$, so $4q = m + 7$ so $m = 1 \Rightarrow q = 2$.) ▨

Glueing and Hermitian Forms

Idea: E, $R = End_{\mathbb{F}_q}(E)$

We want to construct the jacobian of a curve

$$J = E \times E \quad w/ \text{ a map } \quad E \times E \xrightarrow{\varphi} E \times E$$

$$\text{given by } \quad \Phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$$

φ polarization $\Longleftrightarrow$ Φ hermitian $\gg 0$

deg 1 $\Longleftrightarrow$ $ad - bc = 1$

If not decomposable, then $\exists ! C$ of genus 2 with

$$Jac(C) = (E \times E, \varphi).$$

So E has been glued to E.

$\underbrace{E\times\cdots\times E}_{n \text{ times}} = \text{"}E\otimes_R L\text{"}$ L free of rk n over R

So try for projective.

We want to develop this first, then.

Tensor Product (and Hom) in abelian categories

$\mathcal{C}$ an abelian category
R a ring w/ 1.
Let $E\in\mathcal{C}$ with a morphism $R\longrightarrow \mathrm{End}(E)$ given

To the pair (M a right R-module finitely presented, E w/ R-action)

I want to attach an object of $\mathcal{C}$ called "$M\otimes_R E$"

Properties shd be:
- $R\otimes_R E = E$
- compatible w/ direct sum
- right exact.

Write $$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

$$\text{Set}\left\{\begin{array}{l} R^m \otimes_R E = E^m \\ \\ R^n \otimes_R E = E^n \end{array}\right.$$

And define $M \otimes_R E = \mathrm{Coker}\,(E^m \longrightarrow E^n)$.

Indep. of resolution is easy, and also follows from alternate definition:

Want $\mathrm{Hom}\,(M \otimes_R E,\ F) = \mathrm{Hom}_R(M,\ \mathrm{Hom}\,(E,F))$

(for $F \in \mathcal{C}$) where $\mathrm{Hom}\,(E,F)$ is a right R-module via the action on E.

So $M \otimes_R E$ represents the functor on the right.

Another construction is "$\mathrm{Hom}_R(M,E)$" where M is a finitely presented left R-module.

Do the same: choose a resolution

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

$$E^m \underset{\text{transpose of the matrix}}{\longleftarrow} E^n \longleftarrow \underbrace{\mathrm{Hom}_R(M,E)}_{\text{define this as the kernel.}} \longleftarrow 0$$

Se 51

This represents a functor too:

$$\mathrm{Hom}(F, \mathrm{Hom}_R(M,E)) = \mathrm{Hom}_R(M, \underbrace{\mathrm{Hom}(F,E)}_{\text{viewed as a left } R\text{-module}})$$

I'll apply this to: $\mathcal{C}$ category of comm. alg. groups $/k$
(or the subcat. of *proper* such).

And R will be Noeth., M finitely generated.

In this case, $\mathrm{Hom}_R(M,E)$ can also be described by:

$$\begin{tikzcd} S' \arrow[d, dash] \\ k = S \end{tikzcd} \qquad (\mathrm{Hom}_R(M,E))(S') = \mathrm{Hom}_R(M, E(S'))$$

(This defines it as a functor on k-schemes, so determines it.)

Example: E ell. curve, $R = \mathbb{Z}$, $M = \mathbb{Z}/n\mathbb{Z}$

Then $\mathrm{Hom}_{\mathbb{Z}}(M,E) = E[n]$

and the fla above says $E[n](S') = E(S')[n]$.

But "$\otimes$" is zero!

Now apply this to $\left\{ \begin{array}{l} E \text{ an elliptic curve} \\ R = \mathrm{End}(E), \quad M \text{ fin. generated.} \end{array} \right.$

Then $M \otimes_R E$ is an ab. variety of dimension $rk_R M$ (*)
and $\mathrm{Hom}_R(M, E)$ comm. group of dim $= rk_R M$.

(*) since it is $\mathrm{Coker}(E^m \longrightarrow E^n)$!
Note: can also define Tor_n, Ext^n as before...

M projective:

Dual of $M \otimes_R E$ is $M^* \otimes_R E$, where M^* is the dual of M, $M^* = \mathrm{Hom}(M, R)$ a *left* R-module in a natural way, but use the *involution* $r \mapsto \bar{r}$ of R corresp. to the polarization (= cx conj. or quat. conj.) and use this to make M^* a *right* R-module.

i.e., $\lambda \in M^*$, $r \in R$, define $\boxed{(\lambda r)(m) = \lambda(m \bar{r})}$.

(Note that M projective $\Rightarrow$ have $M \underset{pr}{\rightleftarrows} R^n$)
So define $M \otimes_R E \rightleftarrows E^n$ using the induced projector.)

(For an ab. variety $(M \otimes_R A)^* = M^* \otimes_R A^*$.)

[Can extend for M not projective?]

Claim: $\mathrm{Hom}(M \otimes_R E, N \otimes_R E) = \mathrm{Hom}_R(M,N)$

An arrow $\mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}(M\otimes_R E, N \otimes_R E)$ is obvious.

Since M is projective, it's enough to check for free modules, hence for $M=N=R$, and then

$$\mathrm{Hom}(E,E) = \mathrm{Hom}_R(R,R) = R$$

(by our assumption on R)

Claim: If $M \otimes_R E = B \oplus C$ B, C ab. varieties, then $M = M_B \oplus M_C$ as R-modules and $B = M_B \otimes E$, $C = M_C \otimes E$.

Pf: Corresponds to $p \in \mathrm{End}(M \otimes_R E)$ s.t. $p^2 = p$; but this comes from a projector of M, which gives M_B & M_C. ☒

<u>E, R, M projective</u>

Define $A = M \otimes_R E$. [Note that this is isog. to $\underbrace{E \times \cdots \times E}_{\text{rk } M \text{ times}}$.]

What is a principal polarization on A?

Claim: Corresps. to an R-hermitian $\gg 0$ form on M with discriminant 1.

Herm. form on M is $\Phi\ M\times M \longrightarrow R$ w/ the usual properties; can also view as $\varphi: M \longrightarrow M^*$ homs s.t. $\varphi^* = \varphi$.

(via $\Phi(m_1, m_2) = \varphi(m_1)(m_2)$. antilinear in m_1, linear in m_2.)

Φ pos. def. $\longrightarrow$ $\Phi(m,m) > 0$ for all $m \neq 0$

disc 1 : φ is an isomorphism.

[Note $\phi(m,m) \in R$, $\overline{\phi(m,m)} = \phi(m,m)$

and (R=End E !) $\{r \in R \mid \bar{r} = r\} = \mathbb{Z}$

So $\phi(m,m) > 0$ makes sense.]

Now the equivalence

principal polariz on A $\longleftrightarrow$ R-herm. forms on M, pos. def., disc 1.] call such "hermitian modules"

is clear, except maybe that pos. def $\Longleftrightarrow$ φ polarization

If M is free, we have $E\times - \times E \xrightarrow{\varphi} E\times - - \times E$ and then it's in Mumford.

Also

Indec of polariz. $\longleftrightarrow$ Indec of the herm. module M.

(OK)

This gives a method for getting curves of genus 2 out of E and an indec. hermitian module of rk 2, but need to check indec. over quad. extn.

For instance, if Tr(Frob E) $\neq 0$, this is automatic. (saw this last time).

Set $J = M \otimes_R E$. Indec $\Longrightarrow J = J(C)$

: Example

E CM by $\sqrt{-2}$

$R = \mathbb{Z}[\sqrt{-2}]$

$M = R \times R$

$\Phi = \begin{pmatrix} 2 & 1+\sqrt{-2} \\ 1-\sqrt{-2} & 2 \end{pmatrix}$, $\det = 4 - (1+2) = 1$.

pos def ✓

indec: values of Φ are even $\not\cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

So this gives a curve C

$\tilde{S}_4 = \mathrm{Aut}(M) = \mathrm{Aut}(C)$

$\tilde{S}_4$ = 2-sheeted cover of S_4

genus(C) = 2, $\Longrightarrow$ C unique up to quad. twist.

eqn: $y^2 = x^5 - x$

Will show: indec. hermitian H exist, except when R has disc. $-3, -4, -7$.

Also: under some condns on R, this gives every ab. variety isog. to a product of E's.

11/26 ($g=2$)

We connected

$$\text{curves} \longleftrightarrow \text{"binary" hermitian forms, pos. def. of disc. } 1.$$

E ell. curve, $R = \mathrm{End}_k(E)$ $\overset{\text{assume:}}{= \mathrm{End}_{\bar k}(E)}$; assume $\mathrm{rk}\, R = 2$. (so R = order in an imag. quad. field).

Choose an R-module P proj. of rank 2, with a Hermitian form $H: P \times P \longrightarrow R$, positive definite (i.e., $H(x,x) > 0$ if $x \neq 0$), disc 1, (i.e, H define an isom. $P \longrightarrow P^*$ (twisted dual)).

Then $E \otimes P = A$ is an ab. variety of dim 2 on which H gives a polariz. of deg 1.

$$H \text{ indec} \longrightarrow A \text{ indec}/_k \Rightarrow A = \mathrm{Jac}(C)$$

indecomposability is no problem because we assume $\mathrm{End}_k(E) = \mathrm{End}_{\bar k}(E)$.

for a well-defined C of genus 2.

[k field of defn]

$-d = \operatorname{disc} R$

Write $R = R_{-d}$, so $R_{-4} = \mathbb{Z}[i]$

$$R_{-3} = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$$

Theorem (Hayashida-Nishi)

Such an indecomposable binary form exists if and only if $d \neq 3, 4, 7$.

First: note: if $A = E \otimes P$, then $P = \operatorname{Hom}_R(E, A)$

for $x \in P$, $H(x) = ?$

interpret x as $E \xrightarrow{x} A$

$H(x) =$ degree of x^* (polarization)

Proof that every P is trivial on R_{-3}, R_{-4}, R_{-7}

Here $P = R \oplus R$ since the class no. is 1.

So $H = \begin{pmatrix} \lambda & \alpha \\ \bar{\alpha} & \mu \end{pmatrix}$, $\lambda, \mu \in \mathbb{Z}$, > 0

$\lambda\mu - \alpha\bar{\alpha} = 1$, $\alpha \in R$

Want $H \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

i.e., [redacted] have e_1, e_2 basis of P s.t. $e_1 \cdot e_1 = \lambda$, etc.

e_1: any primitive vector (i.e., $e_1 \neq 0$, Re_1 direct factor).

Assume e_1 has been chosen with smallest $e_1 \cdot e_1$, i.e., λ is minimal.

So $\lambda \leq \mu$ (since $\mu \in \mathbb{Z}$ and $\mu = e_2 \cdot e_2$!)

and α can be changed into any $\alpha + \lambda r$ for $r \in R$.

Claim: by suitable choice of r, I can make α ~~to~~ be such that $\alpha\bar{\alpha} < \frac{3}{4}\lambda^2$.

(Proof later)

Then $\lambda\mu - \alpha\bar{\alpha} = 1 \implies \alpha\bar{\alpha} = \lambda\mu - 1 < \frac{3}{4}\lambda^2$

$\lambda\mu < \frac{3}{4}\lambda^2 + 1$

$\lambda \leq \mu \implies \lambda^2 < \frac{3}{4}\lambda^2 + 1 \implies \frac{1}{4}\lambda^2 < 1$

$\implies \boxed{\lambda = 1}$.

Then take $r = -\alpha \implies \alpha$ can be made 0

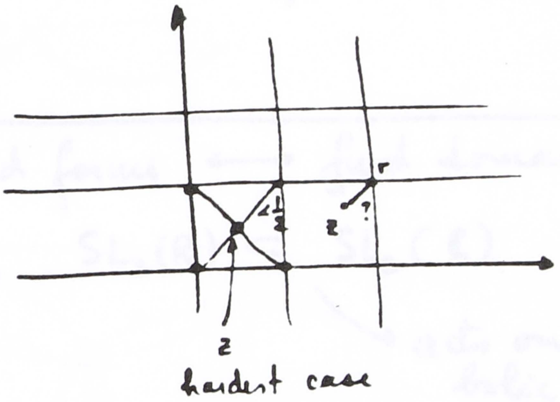
and $\lambda\mu - \alpha\bar{\alpha} = 1$ forces $\underline{\underline{\mu = 1}}$. ∎

Proof of claim

I want to find r s.t. $\left|\frac{\alpha}{\lambda} + r\right|^2 < \frac{3}{4}$.

Lemma: For every $z \in \mathbb{C}$, $\exists r \in R$ s.t. $|z - r|^2 < \frac{3}{4}$.

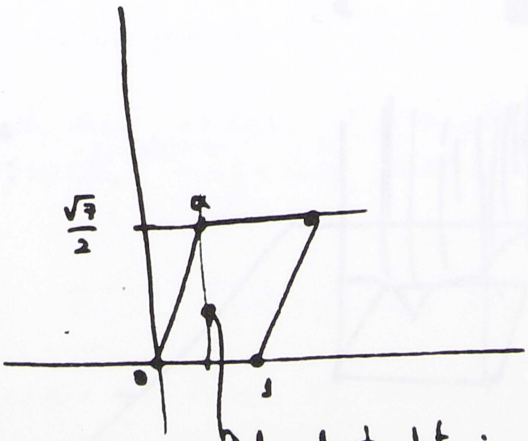
For $\mathbb{Z}[i]$:



so for $\mathbb{Z}[i]$ can get $|z - r| < \frac{1}{2}$.

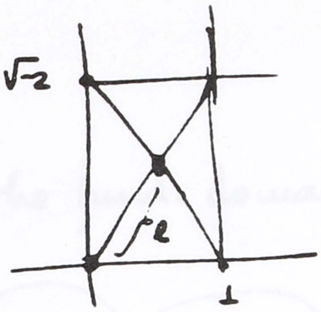
For $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ find can get $|z - r| < \frac{1}{3}$

For R_7:



hardest pt : center of circle through 0, 1, α — so check.

For R_{-8}



length $= \frac{3}{4}$

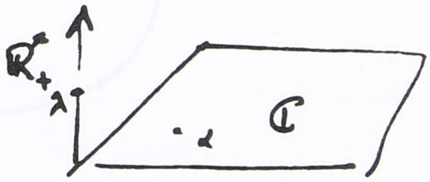
so can't work!

In fact, exists a unique indec. form: $\begin{pmatrix} 2 & 1+\sqrt{-2} \\ 1-\sqrt{-2} & 2 \end{pmatrix}$.

Ordinary quad forms ⟷ fund domain for $SL_2(\mathbb{Z})$

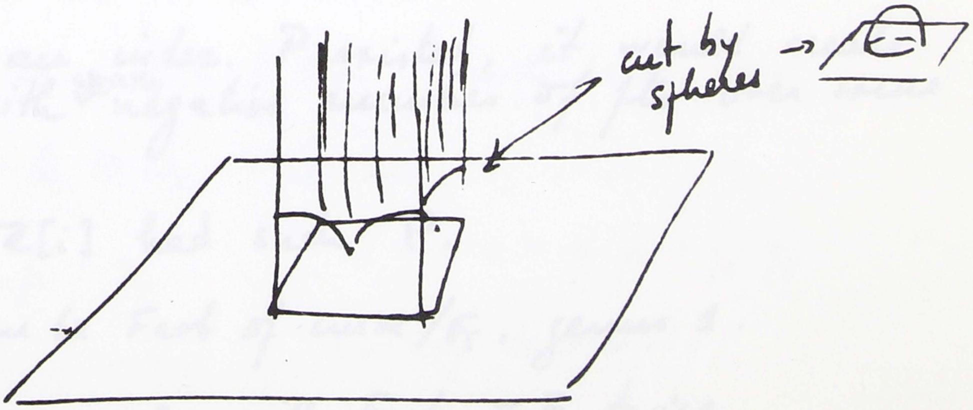
In our case, $SL_2(R) \subset SL_2(\mathbb{C})$

$\searrow$ acts on 3-dim'l hyperbolic space

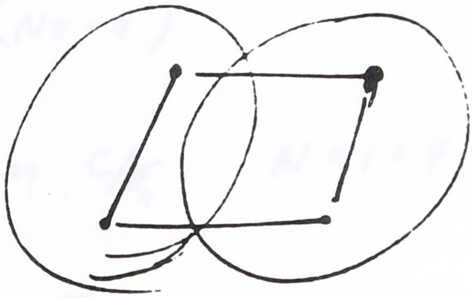


$$|z_1 + \underset{\underset{\mathbb{C}}{\cap}}{\alpha} z_2|^2 + \underset{\underset{\mathbb{R}}{\cap}}{\lambda}|z_2|^2$$

Fund. domain looks like:

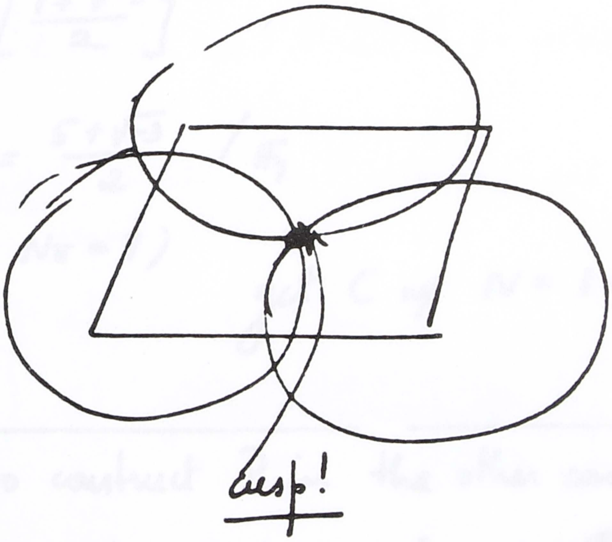


So draw the fund. domain + the equators of the spheres



would have cl # = 1

(only cusp = ∞)



Bianchi has such pictures for low discriminant.

2nd proof (of nonexistence if $d = -3, -4, -7$)

Idea: if such an indec. P existed, it would create a curve with strictly negative number of pts over some $\mathbb{F}_q$.

E.g., suppose $\mathbb{Z}[i]$ had such P.

$\pi = 2+i$ can be Frob of curve$/\mathbb{F}_5$, genus 1.

$P \longrightarrow \exists C/\mathbb{F}_5$ $g = 2$, with Frob $\pi, \bar{\pi}$ twice.

So $N = 1+5-2(\pi+\bar{\pi}) = 6-8 = -2$ ↯.

• $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$

Take $\pi = \frac{3+\sqrt{-7}}{2}$ $/\mathbb{F}_4$

$(N\pi = 4)$

P gives $C/\mathbb{F}_4$ $\quad N = 1+4 - 2(\pi+\bar\pi) = 5-6 = -1.$

• $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$

$\pi = \frac{5+\sqrt{-3}}{2}$ $/\mathbb{F}_7$

$(N\pi = 7)$

get C w/ $N = 1+7-2(\cdot 5) = -2$. ∎

How to construct P in the other cases:

Lemma: Let P be such a [redacted] module : $P \in \mathbb{P}$.
Let $e \in P$, primitive and with $e\cdot e = 2$.

$\mathbb{P}$ = category of modules as above...

Then either P is indec.

or $P \cong R \oplus R$ with herm. form $\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

Pf: Suppose $P = Q_1 \oplus Q_2$, Q_i rk 1.

Then $e = q_1 + q_2$ $\qquad 2 = e\cdot e = q_1\cdot q_1 + q_2\cdot q_2$

(orthog. decomp!)

∴ either $q_1 \cdot q_1 = q_2 \cdot q_2 = 1$ and then q_1, q_2 gen Q_1, Q_2 and we have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

or $e = q_1$ & $q_2 = 0$. This cannot occur, since e is primitive, and would generate Q_1 so disc $Q_1 = 2$ and disc P = disc $Q_1 \cdot$ disc $Q_2 \neq 1$.

$e \cdot (\lambda e + \mu) = 2\lambda$ is also a contradiction. ☒

<u>Lemma 2</u>: let e, P be as in Lemma 1, and $f \in P$ such that

a) $e \cdot f \notin \mathbb{Z}$

b) [redacted] $f \cdot f = z_1 \bar{z}_1 + z_2 \bar{z}_2$ with $z_1, z_2 \in R$

$$\Downarrow$$

$$z_1, z_2 \in \mathbb{Z}.$$

Then P is indecomposable.

<u>Pf</u>: P dec $\longrightarrow$ $P = R \oplus R$ with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by Lemma 1 and $e = (1, 1)$ (by the proof)

Say $f = (z_1, z_2)$.

Then $f \cdot f = z_1 \bar{z}_1 + z_2 \bar{z}_2 \longrightarrow z_1, z_2 \in \mathbb{Z}$ by (b).

But $e \cdot f = z_1 + z_2 \in \mathbb{Z}$ contradicting (a). ☒

Now it remains to write down some matrices $\begin{pmatrix} 2 & \alpha \\ \bar{\alpha} & \mu \end{pmatrix}$ under these conditions (μ will give the conditions on f, of course).

12/3 $g=2$ (the end)

We were looking at binary hermitian forms, (pos. def. disc. 1) on $R = R_{-d}$, order in an imag. quad. field.

(We know $d \equiv 0$ or $3 \pmod 4$, $d>0$,

and R_{-d} has basis over $\mathbb{Z}$: 1 and either $\begin{cases} \sqrt{-d/4} & \text{if } d \equiv 0\,(4) \\ \frac{1+\sqrt{-d}}{2} & \text{if } d \equiv 3\,(4) \end{cases}$

We want to prove: if $d \neq 3,4,7 \Rightarrow \exists$ indec. binary herm form

Lemmas above: P proj hermitian module

1) $e \in P$, $e.e = 2$, primitive

$\Rightarrow$ either P indec

or $P = R_{-d} \oplus R_{-d}$, $e \mapsto (1,1)$

2) $\exists f$, $f \in P$, s.t. $e.f \notin \mathbb{Z}$ and $f.f = z_1\bar{z}_1 + z_2\bar{z}_2$, $z_i \in R$, $\Rightarrow z_1, z_2 \in \mathbb{Z}$ $\Longrightarrow$ P indec.

Se 58

In cases ① – ③, $P = R \oplus R$, matrix in $\begin{pmatrix} 2 & \alpha \\ \bar{\alpha} & \mu \end{pmatrix}$ $e = (1,0)$ $f = (0,1)$

$2\mu - \alpha\bar{\alpha} = 1$

Cases

① $d \equiv 0 \pmod 8$, $P = R \oplus R$

$$\alpha = 1 + \sqrt{-d/4}$$

so $\alpha\bar{\alpha} = d/4 + 1 \equiv 1 \bmod 2$

so $\mu = \frac{1+\alpha\bar{\alpha}}{2} = 1 + \frac{d}{8}$.

Then $e \cdot e = 2$, primitive

$e \cdot f = \alpha \notin \mathbb{Z}$

$f \cdot f = \mu = 1 + \frac{d}{8}$

$$N\left(x + y\sqrt{\tfrac{-d}{4}}\right) = x^2 + \frac{d}{4}y^2 \geq \frac{d}{4} \text{ if } y \neq 0$$

But $\frac{d}{4} > 1 + \frac{d}{8}$ unless $d = 8$.

So if $d > 8$, P is indec.

If $d = 8$, $\begin{pmatrix} 2 & 1+\sqrt{-2} \\ 1-\sqrt{-2} & 2 \end{pmatrix}$

is indec, since $z \cdot z$ is even for any z.

② $d \equiv 4 \pmod 8$

take $\begin{cases} \alpha = \sqrt{\frac{-d}{4}} \\ \mu = \frac{1+\alpha\bar{\alpha}}{2} = \frac{1}{2} + \frac{d}{8} \end{cases}$ and check as before.

③ $d \equiv 3 \pmod 8$

take $\begin{cases} \alpha = \frac{1+\sqrt{-d}}{2}, \text{ so } \alpha\bar{\alpha} = \frac{1+d}{4}, \\ \mu = \frac{5+d}{8} \end{cases}$ again as before.

$[d \neq 7]$

④ $d \equiv 7 \pmod 8$

This implies $h(-d) > 1$: 2 splits as $\mathfrak{p}\bar{\mathfrak{p}}$, i.e.:

$$R_{-d} \otimes \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad (*)$$

(*): if $x = \frac{1+\sqrt{-d}}{2}$, x satisfies $x^2 - x + \frac{1+d}{4} = 0$.

Mod 2: $x^2 - x \equiv 0 \pmod 2$

and this splits mod 2

And then one checks that $\mathfrak{p}$ is not principal.

If it were, $\exists \alpha \in R$ s.t. $\alpha\bar{\alpha} = 2$; but $d > 7$ so this can't be.

Choose an inv. ideal $\mathfrak{a}$, not principal (e.g., $\mathfrak{p}$).

Now take $P = R + \mathfrak{a}$ (not free).

A hermitian form on P is given by a matrix

$$\begin{pmatrix} \lambda & \alpha \\ \bar{\alpha} & \mu \end{pmatrix} \quad \left[\begin{array}{l} \lambda \in \mathbb{Z} \quad , \ \alpha \in \frac{1}{N\mathfrak{a}} \mathfrak{a} \ , \ (\bar{\alpha} \in \frac{1}{N\mathfrak{a}} \bar{\mathfrak{a}}) \\ \mu \in \frac{1}{N\mathfrak{a}} \mathbb{Z} \end{array} \right.$$

[herm form is $\lambda z_1 \bar{z}_1 + \alpha \underbrace{z_1 \bar{z}_2}_{\in \bar{\mathfrak{a}}} + \bar{\alpha} \bar{z}_1 z_2 + \mu z_2 \bar{z}_2$]

with $z_1 \in R$, $z_2 \in \mathfrak{a}$ $\Big\}$ $\bar{\mathfrak{a}} \mathfrak{a} = N\mathfrak{a}$.

So if $\alpha \in \frac{1}{N\mathfrak{a}} \mathfrak{a}$, ok!

To check: form is $R \oplus \mathfrak{a} \longrightarrow R \oplus \bar{\mathfrak{a}}$, so get conditions

"det = 1" means $(\lambda\mu - \alpha\bar{\alpha}) N\mathfrak{a} = 1$

Take $\lambda = 2$ to have a vector of length 2.

Claim: $\exists\, \alpha \in \frac{1}{N\mathfrak{a}} \mathfrak{a}$, $\exists\, \mu \in \frac{1}{N\mathfrak{a}} \mathbb{Z}$ s.t. $(2\mu - \alpha\bar{\alpha}) N\mathfrak{a} = 1$

Write $\alpha = \frac{1}{N\mathfrak{a}} z$, $z \in \mathfrak{a}$, $\mu = \frac{m}{N\mathfrak{a}}$, and then we want

$$2m - \frac{z\bar{z}}{N\mathfrak{a}} = 1 .$$

So we want: $\exists\, z \in \mathfrak{a}$ s.t. $\frac{z\bar{z}}{N\mathfrak{a}} \equiv 1 \pmod{2}$.

Choose z a local generator at $\mathfrak{a}$ (since locally principal!).

Then by Lemma 1, we are done, since P is <u>not</u> free. $\blacksquare$

[For ternary herm. forms, bad rings should be d = -3, -4, -8, -11 ; still can't prove those are all the bad rings.]

E ell. curve, $R = \mathrm{End}(E)$

* We associated $A = P \otimes_R E$.

Then

form indec $\Longrightarrow$ A indec, pol. deg 1

For $g = 2 \Longrightarrow$ <u>curve</u>.

* Which A's are of the form $P \otimes_R E$?

[Suppose we want to prove:

For $q = 13$, $g = 2$, $N \neq 28$.

If $N = 28$, should have $\pi = \dfrac{-7 \pm \sqrt{-3}}{2}$ twice,

which corresps to E with $R = R_{-3}$.

Then $\mathrm{Jac}(C) \cong P \otimes E$ for some P

polariz on Jac $\longrightarrow$ herm. $\underbrace{\text{form on } P, \text{ indec.}}_{\text{contradiction!}}$]

by what we'll prove!

<u>Assume</u>: E ordinary, $R = \mathrm{End}\, E$, R maximal order.

[if R not max. order, should use R is a <u>Gorenstein</u> ring]

Let A be an abelian variety on which R acts, isogenous over gd. field to $\underbrace{E \times E \times - - - \times E}_{g}$.

<u>Thm</u>: Then $A \cong P \otimes_R E$, with $P = \mathrm{Hom}_R(E, A)$, and P is projective.

<u>Pf</u>: A is isogenous to $L \otimes E$ L free R-module.

$A \longrightarrow L \otimes E$ isogeny.

primes dividing order of kernel?

a) $\ell \neq$ char.

Use
$$\begin{tikzcd} V_\ell(A) \arrow[r, phantom, "="] & V_\ell(L \otimes E) \\ T_\ell A \arrow[u, phantom, "\cup" rotate=90] \arrow[r, hook] & T_\ell(L \otimes E) \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

and $\ell \mid$ order of kernel $\rightarrow$ lattices are different

View $L \subset K^g$

$L \subset P \subset K^g$ corresps to: for finitely many l
$L \otimes R_l \subset P \otimes R_l$

Same for $P \subset L \subset K^g$.

So choose $P \subset L$ s.t. $P_l = P \otimes \mathbb{Z}_l = T_l A$

Then we'll have $T_l A = T_l(P \otimes E)$ so ok at l.

Since we have A ordinary, it has a $T_p A$, free of rank $2g$ over $\mathbb{Z}_p$

$$T_pA = (\text{naïve } T_p) \oplus (\mathbb{Z}_p\text{-dual of } T_p^{(1)}(\text{dual of } A))$$

$$T_p^{(1)}(A) \text{''} \underbrace{\varprojlim A^{\text{ét}}[p^n]}_{\text{rk } g}$$

One proves that this T_pA satisfies the same formalism as before.

[For general A, must add $\oplus$ piece coming from Witt-vector cohomology]

Note: for ordinary E, $R_p = \mathbb{Z}_p \oplus \mathbb{Z}_p$

This gives a proof. $\boxed{\checkmark}$

Skolem Method

We needed to consider $y^n = \begin{cases} x^2+1 \\ x^2+x+1 \\ x^2+x+2 \end{cases}$

n odd ≥ 3.

List: $\left\{\begin{array}{l} \text{none for } x^2+1 \\ 7^3 = 18^2+18+1 \\ 2^3 = 2^2+2+2 \\ 2^5 = 5^2+5+2 \\ 2^{13} = 90^2+90+2 \end{array}\right.$

So look at equations $\begin{cases} y^n = x^2+1 \\ y^n = x^2+x+1 \\ 2^n = x^2+x+2 \end{cases}$ $\begin{cases} y > 1 \\ n \text{ odd} \geq 3 \end{cases}$

Claim: no other solutions than listed above.

① $y^n = x^2+1$: Lebesgue

② $y^n = x^2+x+1$: Nagel + Ljunggren

③ $2^n = x^2+x+2$: Nagel

Cf. Mordell, Diophantine Equations.

(Proofs for ① + ③, refs. for ②.)

②: Nagel: $y^n = x^2 + x + 1 \implies n \equiv 0 \pmod 3$ (easy; Nagel reduces to $y = 13$. Then must show $13^n \neq x^2 + x +$ gives wrong argument, can be corrected.)

Once $3 | n$, look at $y^3 = x^2 + x + 1$, and show integral points are $\begin{cases} y = 1, & x = 0 \text{ or } -1 \\ y = 7, & x = 90 \text{ or } -91 \end{cases}$ (Ljunggren)

Simpler proof: Tzanakis, J. Numb. Th 18 (1984))

We give a proof for case ③: $2^n = x^2 + x + 2$.

Note: $$4 \cdot 2^n = 4 \cdot x^2 + 4 \cdot x + 1 + 7$$
$$= (2x + 1)^2 + 7$$

So work in R_{-7}, set $\omega = \frac{1 + \sqrt{-7}}{2}$, so $\begin{cases} \omega + \bar{\omega} = 1 \\ \omega \bar{\omega} = 2 \end{cases}$

(So $\omega^2 - \omega + 2 = 0$).

And our equation is $\boxed{\omega^n \bar{\omega}^n = (x + \omega)(x + \bar{\omega})}$

Se 62

Can replace x by $-1-x$, so choose x even.

$(\omega) = \mathfrak{p}$, $2 = \mathfrak{p}\bar{\mathfrak{p}}$.

So we must have $x+\omega = \mathfrak{p}^i \bar{\mathfrak{p}}^j$ $\quad i+j=n$

And $2 \nmid x+\omega$. If $i,j \geq 1$, $x+\omega$ div by $\mathfrak{p}\bar{\mathfrak{p}} = 2$; no!

So $x+\omega = \pm\omega^n$ or $\pm\bar{\omega}^n$

x even $\Longrightarrow$ $x+\omega = \pm\bar{\omega}^n$ is impossible.

So $x+\omega = \pm\omega^n$

Now $\omega^n - \bar{\omega}^n = \pm(\omega - \bar{\omega})$

sign is $-$: in $R/\omega^2 R = \mathbb{Z}/4\mathbb{Z}$

$$\bar{\omega} \longrightarrow -1$$
$$\omega \longrightarrow 2$$
$$\omega^2 \longrightarrow 0$$

So image of eqn is

$$0 - (-1)^n = \pm(2+1)$$

n odd : $1 = \pm(2+1)$

so have $-$.

So $\omega^n - \bar{\omega}^n = -(\omega - \bar{\omega})$

So $\boxed{\omega^n = -x-\omega}$

In $R = \mathbb{Z} + \mathbb{Z}\omega$

$\omega^n = a_n + b_n\omega$

Question is : do we get $b_n = -1$?

We do for $n = 3, 5, 13$. Claim: no others.

<u>Skolem's method</u>:

Look at $\omega^n - \bar{\omega}^n = -(\omega - \bar{\omega})$ as an eqn for n, and interpret p-adically.

[Works for $p = 29, 37, 43, 71, 79, 109, 191 \cdots$]

We take $p=7$, but

Take $K = \mathbb{Q}(\sqrt{-7})$, $\hat{K}$ = completion at the prime $|7$

$$v : \hat{K}^* \longrightarrow \mathbb{Z}$$

$\pi = \sqrt{-7}$, $v(\pi) = 1$, $v(7) = 2$, so $e = 2$.

We want to think of the eqn. with $n \in \mathbb{Z}_7$, if possible.

$\omega \in U_{\hat{K}}$, but is not $\equiv 1 \bmod \pi$

Residue field is $\mathbb{F}_7^*$, and ω has order 6.

So we must work with <u>n mod 6</u>.

Cases: $\left\{\begin{array}{l} n \equiv 1 \pmod 6 \\ n \equiv 3 \;(\;\text{"}\;) \\ n \equiv 5 \;(\;\text{"}\;) \end{array}\right.$

[our examples are 3, 5, 13 ; one in each case!]

To be proved: in each case, there exists at most one value n s.t.

$$\omega^n - \bar{\omega}^n = -(\omega - \bar{\omega})$$

Let n_0 = one solution, so $n = n_0 + 6t$

$$\omega^{n_0}\omega^{6t} - \bar{\omega}^{n_0}\bar{\omega}^{6t} = -(\omega - \bar{\omega})$$

Now $\omega^6 \equiv 1 \pmod{\pi}$

$\omega = \frac{1+\pi}{2}$, so $\alpha = \omega^6 = \frac{(1+\pi)^6}{2^6} = \frac{1+6\pi+\dots}{2^6}$

mod 7, $2^6 \equiv 1$, $6 \equiv -1$

So get $\alpha \equiv 1 - \pi \pmod{\pi^2}$ (note $7 \sim \pi^2$)

So our equation is $\boxed{\underbrace{\omega^{n_0}\alpha^t - \bar{\omega}^{n_0}\alpha^t}_{\|} = -(\omega - \bar{\omega})}$

Now take $t \in \mathbb{Z}_7$ $f(t)$ analytic.

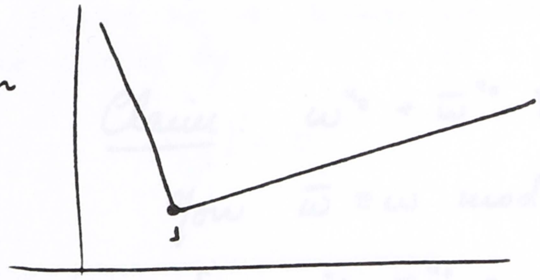
We want to solve $f(t) = \text{const.}$

$\therefore$ only finitely many solutions.

So expand $f(t)$ in a power-series

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots \qquad v(a_0) \geqslant 0$$

if we can show $v(a_1) = 1$, $v(a_n) \geqslant 2$ for $n \geqslant 2$ then we have at most one solution.



Or: know one solution, so choose n_0 s.t. $a_0 = 0$; divide by t

$$f(t) = t(1 + \underbrace{- - - -}_{v(\ ) \geqslant 0}).$$

$$\alpha^t = e^{t \log \alpha}, \text{ so } a_n = \omega^{n_0} \cdot \frac{(\log \alpha)^n}{n!} - \bar{\omega}^{n_0} \frac{(\log \bar{\alpha})^n}{n!}$$

$\alpha = 1 - \pi \mod \pi^2$

$\therefore \log(\alpha) \equiv -\pi \mod \pi^2$ so $v(\log \alpha) = v(\log \bar{\alpha}) = 1.$

So $v(a_1) \geqslant 1$, $v(a_2) \geqslant 2$, . . . , $v(a_6) \geqslant 6$

For $n=7$ $\quad v(a_7) \geq 7-2 = 5$, etc.

So $v(a_n) \geq 2$, $n \geq 2$

So only thing to check is $v(a_1) = 1$ (i.e., no cancellations).

$$a_1 = \omega^{n_0}(-\pi) + \bar{\omega}^{n_0}(\bar{\pi}) \mod \pi^2 \qquad \bar{\pi} = -\pi$$

$$= -\pi(\omega^{n_0} + \bar{\omega}^{n_0}) \mod \pi^2$$

<u>Claim</u>: $\omega^{n_0} + \bar{\omega}^{n_0} \not\equiv 0 \mod \pi$

Now $\bar{\omega} \equiv \omega \mod \pi$

So $\omega^{n_0} + \bar{\omega}^{n_0} \equiv 2\omega^{n_0} \pmod{\pi}$, done!

<u>Remarks</u>:
- * don't really need analytic fcts
- * could use binomial expansion.
- * we were helped by there being no unit in the field; if there were a unit, would get
$$\omega^n - \bar{\omega}^n = (\text{unit})^m(\omega - \bar{\omega})$$
and then we'd need more equations.

Other primes: take p, $\left(\frac{p}{7}\right)=1$.

* We need, mod p, $\omega^n - \bar{\omega}^n = -(\omega - \bar{\omega})$ only for 3, 5, 13

* need $v(a_1) = 1$.

(eg., 23 works for first *, not for second.)

12/5 $\underline{g=3}$ — see table of (q, N) (p. Se64b)

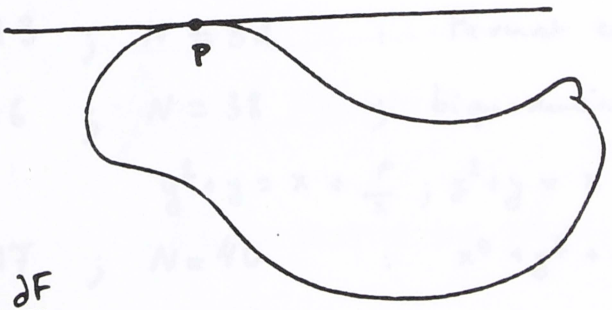
<u>Voloch's bound</u> (for $g=3$)

$N \leq 2q + 6$, except for "special cases"

<u>Assume not hyperelliptic</u> (if hyperelliptic, $N \leq 2q+2$).

∴ curve is a nonsingular quartic in plane, so given by

$F(x,y)=0$ or $F_4(x,y,z)=0$



Write equation "Frob of P ∈ tgt at P" in homog. coords:

$$G = X^q F'_X + Y^q F'_Y + Z^q F'_Z = 0$$

$F'_x = \frac{\partial F}{\partial x}$

"<u>Special" if F divides</u> G (i.e., Frob P ∈ tgt at P for every P).

If not special, the number of inters. of $F=0$ & $G=0$ is $4(q+3)$.

If $P=(x,y,z)$ is a <u>rational</u> point, it clearly is in the intersection; moreover, the two curves are <u>tangent</u> at such a point, so each rat'l point counts at least twice.

Se 64 b

$\boxed{g=3}$ Maximum number of points

2q+6

$q=2$; $N=7$: twisted Klein curve, $\Sigma x^4+\Sigma x^2y^2+x^2yz+xy^2z=0$

$q=3$; $N=10$: $y^3-y=x^4-x^2$

$q=4$; $N=14$: Klein curve $\Sigma x^4+\Sigma x^2y^2+\Sigma x^2yz=0$ 14

$q=5$; $N=16$: $x^4+y^4=2z^4$ 16

$q=7$; $N=20$: cubic covering $t^3=y-x^2+xy$ of the elliptic curve $y^2-y=x^3-x^2$ 20

㉒

$q=8$; $N=24$: Klein curve false

$q=9$; $N=28$: Klein curve = Fermat curve $(x^4+y^4+z^4=0)$ ㉔

$q=11$; $N=28$: $x^4+y^4+z^4+2(3x^2y^2+4y^2z^2+4z^2x^2)=0$ 28

$q=13$; $N=32$: Fermat curve 32

$q=16$; $N=38$; biquadratic extension of $\mathbb{F}_{16}(x)$ defined by 38

$y^2+y=x+\frac{\rho}{x}$; $y^2+y=x+\frac{\rho^2}{1+x}$ where $\rho\in\mathbb{F}_4-\mathbb{F}_2$

$q=17$; $N=40$: $x^4+y^4+z^4+4y^2z^2=0$ 40

$q=19$; $N=44$ 44

$q=23$; $N=48$: $x^4+y^4+z^4-5(x^2y^2+y^2z^2+z^2x^2)=0$

$q=25$; $N=56$: Klein curve

Voloch's bound.

$q=27$; $N=$?

$q=29$; $N=$?

?

?

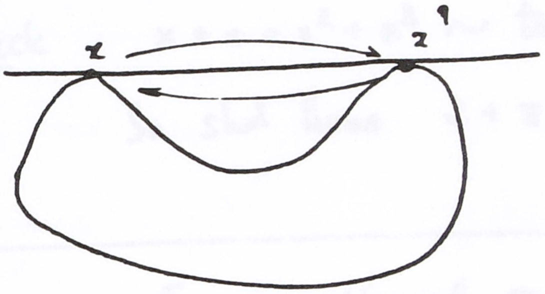
(This list is not entirely garanteed...)

Harvard, dec. 85

J-P.S.

Therefore $N \leq 2(g+3) = 2g+6$.

Example:



x of deg 2 on a bitangent then $x \in$ intersection

Similarly, could have a triangle.

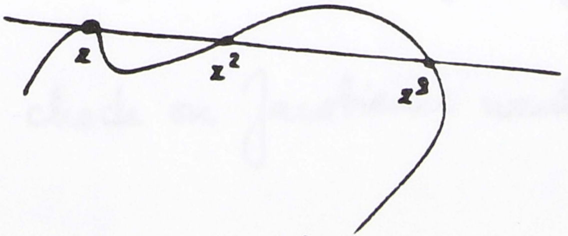
Special curves:

Klein curve in char. 2 / $\mathbb{F}_2$ or $/\mathbb{F}_8$

$$\Sigma x^4 + \Sigma x^2y^2 + \Sigma x^2yz = 0$$

($\Sigma x^4 = x^4+y^4+z^4$, etc.)

Check:



for any x

Can write the polynomial as
$$\frac{\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^8 & y^8 & z^8 \end{vmatrix}}{\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix}}$$

(and then it's obvious that x, x^2, x^8 are collinear)

Eqn of tgt at [redacted] (x,y,z) is:

$$X \begin{vmatrix} y^2 & z^2 \\ y^8 & z^8 \end{vmatrix} + Y \cdot \begin{vmatrix} & \end{vmatrix} + Z \cdot \begin{vmatrix} & \end{vmatrix} = 0$$

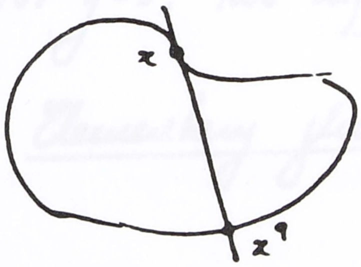
<u>Another check</u> $x + x + x^2 + x^8 \sim$ trivial divisor

So shd have $2 + \pi + \pi^8 = 0$ on Jac.

Klein curve over $\mathbb{F}_3$ is special over $\mathbb{F}_9$.

$\shortparallel$

Fermat $x^4 + y^4 = z^4$



Then: every tgt is inflection tangent, and other inters. is x^9.

Write equations to check.

Tangent is $Xx^3 + Yy^3 + Zz^3 = 0$

at x^9: $x^{12} + y^{12} + z^{12} \overset{?}{=} 0$, yes since this is $(x^4 + y^4 + z^4)^3 = 0$

(char 3 !)

On check on Jacobian: want $3 + \pi_g = 0$

$\pi_g = -3$, which is true.

This is why Voloch's bound fails at 8, 9.

Can prove char $\geq 5 \Rightarrow$ non-special

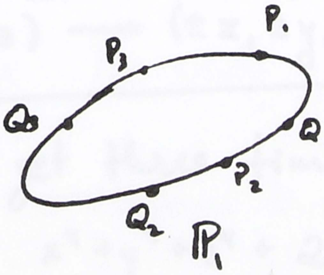
Guess: these are the only special curves.

For $g=2$, we used glueing of elliptic curves, either

direct method "2-glueing"
or
hermitian forms

For $g=3$, two difficulties:

Elementary glueing



Choose 3 pairs of pts

Make $E_1/\mathbb{P}_1$ ramified at $P_2\,Q_2\,P_3\,Q_3$

$E_2/\mathbb{P}_1$ " " $P_1\,Q_1\,P_3\,Q_3$

Get:

$$\begin{tikzcd} & k(C) \arrow[dl, dash] \arrow[d, dash] \arrow[dr] & \\ & k(E_3) \arrow[dd] & \\ k(E_1) \arrow[dr, dash] & & k(E_2) \arrow[dl, dash] \\ & k(x) & \end{tikzcd}$$

E_3 ramified only at $P_1\,Q_1\,P_2\,Q_2$

So get C of genus 3, $\mathrm{Jac}(C) \sim E_1 \times E_2 \times E_3$

Consider $\boxed{ax^4 + by^4 + cz^4 + dx^2y^2 + ey^2z^2 + fz^2x^2 = 0}$

Assume $\Delta \neq 0$

Set $z=1$: $ax^4 + by^4 + c + dx^2y^2 + ey^2 + fx^2 = 0$

Set $Y = y^2$ $\quad ax^4 + bY^2 + c + dx^2Y + eY + fx^2 = 0$

$$\Big\downarrow$$

$Y^2 = \lambda X^4 + \mu X^2 + \nu$ which is an elliptic curve.

doing w/ each variable, get 3 curves.

Or look at it as a group of type (2,2) acting by $(x,y,z) \longrightarrow (\pm x, \pm y, \pm z)$!

E.g., to get three times the same curve E, we find

$$x^4 + y^4 + z^4 + 2a(x^2y^2 + y^2z^2 + z^2x^2) = 0$$

So choose a to get the curve E.

E.g. for 11, 17 $\quad$ 2 of one kind, 1 different
for 23 $\quad$ 3 of the same

<u>For q=23</u>

Voloch bound doesn't work. (gives 52)

$m = [2\sqrt{23}] = [\sqrt{92}] = 9$

$m^2 - 4q = -11.$

Weil gives $N \le 1 + q + 3m = 51$

This is impossible:

N=51 only if Jac $\sim E \times E \times E$ with $m = 9$, so Frob should be $\pi = \frac{-9 \pm \sqrt{-11}}{2}$.

So $R = \mathrm{End}(E) = \mathbb{Z}[\pi] = R_{-11}$.

$\therefore$ Jac $\cong E \times E \times E$ + polariz. $\longleftrightarrow$ indec. herm. form as before, of rk 3.

Kneser: No such form over R_{-11}.

$1+q+3m-1 \longrightarrow$ down by 1 is impossible for $g=3$

$1+q+3m-2 = 49$ pts? $\underbrace{\text{down by 2, genus 3}}_{\text{only case is } m, m, m-2}$

that would mean Jac $\sim E \times E \times E'$

$$\begin{cases} E \text{ w/ } \pi = \frac{-9 \pm \sqrt{-11}}{2} \\ E' \text{ w/ } \pi' = \frac{-7 \pm \sqrt{-43}}{2} \end{cases}$$

Claim: no such thing exits.

(Problem: Jac is only _isogenous_ to $E\times E\times E'$).

Let $\varphi = F + V = \pi + \bar{\pi}$

J_9 = conn comp of $\mathrm{Ker}(\varphi + 9)$

J_7 = ———— $\mathrm{Ker}(\varphi + 7)$

$J_9 \cong E\times E$ _isomorphic_

$J_7 \cong E'$

$J_9 \cap J_7$ _killed_ by 2

So $J = J_9\times J_7 / \Delta$ $\Delta \subset E'[2]$

so Δ of type 1, (2) or (2,2)

1 → Jac = product → polariz. splits → no.

(2) → also can't be

(2,2)

Polariz of deg 1 on J gives a polariz on $J_9\times J_7$, of degree 4.

Again, this splits; only interesting case is deg 2 on J_9, deg 2 on J_7.

Se 69

So need polariz of deg 2 on J_q.

J_q has CM by R_{11}, so want

$$\begin{pmatrix} \lambda & \alpha \\ \bar{\alpha} & \mu \end{pmatrix} \quad \left\{ \begin{array}{l} \lambda\mu - \alpha\bar{\alpha} = 2 \\ \lambda > 0 \\ \text{coeff} \in R_{-11} \end{array} \right.$$

<u>Theorem</u>: only case is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

But then J_q splits: $J = E \times E \times E'/\Delta$, and the whole J splits, which is not possible. ☑

<u>$q = 19$</u> Voloch bound gives 44.

$$\left[\begin{array}{l} m = [2\sqrt{19}] = [\sqrt{76}] = 8 \\ m^2 - 4q = -12 \end{array} \right.$$

$1 + 9 + 3 \cdot 8 = 44$

So I want to prove m, m, m is possible.

So take E w/ $\pi = -4 \pm \sqrt{-3}$

Want Jac $\sim E \times E \times E$.

Look for a hermitian module for $\overset{R =}{\mathbb{Z}[\pi]} = \mathbb{Z}[\sqrt{-3}] \subsetneqq$ max.

Form is $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1+\sqrt{-3} \\ 1 & 1-\sqrt{-3} & 3 \end{pmatrix}$ on $R \oplus R \oplus R$.

To show indec., go to $\tilde{R} = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$. There it is decomposable, so look at the lattice in $\tilde{R} \times \tilde{R} \times \tilde{R}$, and check: there is no vector of length 1.

Take $A = P \otimes_R E$.

Use:

Thm (Oort + ________):

A principally polarised ab. variety of dim 3, indec., is a jacobian over a quad. extension.

1) If C hyperelliptic, then C can be chosen $/k$ s.t. $\operatorname{Jac} C \cong A/k$

2) If C not hyperelliptic, then $\exists$ C/k unique + a quad twist $\varepsilon : \operatorname{Gal}(\bar{k}/k) \longrightarrow \pm 1$ s.t. $\operatorname{Jac} C \cong A$ (twisted by ε).

---

over a finite field, Frob of C is either $\begin{cases} \text{Frob of } A \\ \text{or} \\ -\text{Frob of } A \end{cases}$

or put $\varphi = c\prod_{36} \otimes \vartheta$.

Suppose $c=1$; then φ "is" the twist.

Reformulation:

the twist A_ε (A twisted by ε) can be written

$$A_\varepsilon = P \otimes E_\varepsilon$$

Start with P ; look at all E_ε deduced from E by a quad. twist.

Then P determines a well-defined E_ε among all those, the unique one s.t. $P \otimes E_\varepsilon = \mathrm{Jac}(C)$.

Example for Gross:

E w/ CM by $\frac{1+\sqrt{-7}}{2}$ $\quad (j = -3^3 5^3)$

Look for P over this R :

$$P \text{ given by } \begin{pmatrix} 2 & \alpha & \bar{\alpha} \\ \bar{\alpha} & 2 & -1 \\ \alpha & -1 & 2 \end{pmatrix}, \quad \alpha = \frac{1+\sqrt{-7}}{2}$$

$$\text{Automorphism} = \{\pm 1\} \times \underbrace{G_{168}}_{\text{simple of order 168}} \quad / \mathbb{Q}(\sqrt{-7})$$

$P \otimes E = \mathrm{Jac}$ (Klein)

So curve will have either 44 or 1+19 − 38 = −4 points, so it has 44 pts.

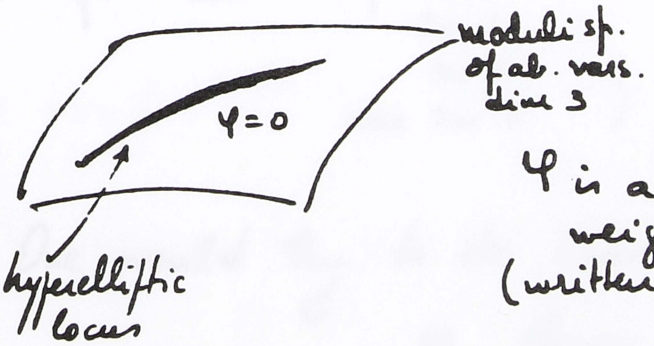
So E ell. curve, R = End(E), take P herm. module as usual.

Make $A = E \otimes_R P$

Cont + — = this is a jacobian.

① Is C hyperelliptic?

Given g_2, g_3 (P herm. module) ⟶ hyperelliptic ?



Ψ is a mod form on Siegel space, weight 18 (written in Igusa on genus 3)

mod form on Siegel space (dim 3)$\Big|_{CM \otimes_R \text{hermitian modules}}$ = ?

② If C not hyperelliptic ($\Psi \neq 0$), find ε = "$\sqrt{\Psi}$".

Igusa says: Klein noticed:
take Δ of a plane quartic

Check $\Delta_9^2 = c \cdot \Psi_{18}$

$\boxed{\Psi = \prod_{36} \vartheta}$.

So P selects the E w/ good reduction outside 7.
Klein has potentially good redn. everywhere, but in char 7, Klein becomes hyperelliptic.

In general, have no way to determine the sign when C is not hyperelliptic.

Define $N_q(3)$ = max nber of points for $g=3$ over $\mathbb{F}_q$.

<u>Conjecture</u>: $|N_q(3) - \text{Weil bound}|$ is bounded when q varies. (say ≤ 6)

(For $N_q(2)$ had $1+q+2m$
$2m-1$
$2m-2$
once $2m-3$).

One would try to go down by 3, $m-1$, $m-1$, $m-1$
or down by 6, $m-2$, $m-2$, $m-2$.

Take E w/ one of those, $R = R_{-d}$.

$$d = 4q - (m-1)^2$$

$4q \geq m^2$ so $d \geq 2m-1$

So look at indec. herm. modules of rk 3 on R_{-d}.
Number of such goes to ∞ like d^{γ}, γ large.

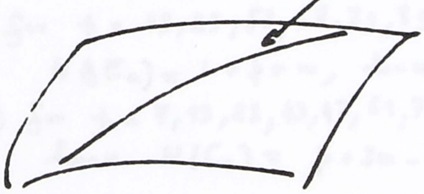
$d >> q^{1/4}$

For each such herm. module, should compute the sign.

Getting + gives curves w/ many pts.

For $q=4$, would still make same conjecture (in fact for 5, 6, 7, or 8 too)

$A_p = P \otimes E$ pp. $\Psi=0$ hypersurface of jacobians



So want $\Psi(A_p) = 0$

So chance is $\frac{1}{q}$. But then quad twist, so chance is $\frac{1}{2q}$ of winning.

But number of P is large.

For very large g, say $g = 100$, jacobians have too large a codimension.

So expect a breaking point between 3 + 10.

For $g = 3$, using $x^4 + y^4 + z^4 + 2a(x^2y^2 + y^2z^2 + z^2x^2) = 0$. (C_a)

this has Jac $\sim E_a \times E_a \times E_a$, E_a: $y^2 = (a^2-1)x^4 + \dots$

<u>Question</u>: can I choose E for $m-1$, say? I.e., how many pts can this have?

[Added dec. 9, 1985] - A machine computation, made by M. Nitzberg, shows that:

1) for $p = 19, 29, 53, 67, 71, 89$, one may choose a $(a = 3, 13, 0, 14, 36, 20)$ such that $N(E_a) = 1 + p + m$, hence $N(C_a) = 1 + p + 3m$: optimal bound!

2) for $p = 7, 13, 23, 43, 47, 61, 79, 97$, one may choose a such that $N(E_a) = 1 + p + (m-1)$, hence $N(C_a) = p + 3m - 2$, which is optimal at least for $p = 7, 13, 23$

Rational Points on Curves
over Finite Fields

Part II – "g large"

Harvard, 1985

RATIONAL POINTS ON CURVES
OVER FINITE FIELDS

PART II: "g LARGE"

Jean-Pierre Serre

Lectures given at Harvard University, September to December 1985

Notes by Fernando Q. Gouvêa

Contents - Part II

9/26 Thursday: g large.

Ihara: X genus g / $\mathbb{F}_q$, $g \geq 1$

Suppose $N(X) = q + 1 + 2g q^{1/2}$ (so q is a square).

Then : $g \leq \frac{1}{2}(q - q^{1/2})$

Proof: $N(X) = 1 + q - \sum_{\alpha=1}^{2g} \pi_\alpha$ $\qquad |\pi_\alpha| = q^{1/2}$

So we must have all $\pi_\alpha = -q^{1/2}$

So $\pi_\alpha^2 = q$, and if $N_2 = N_2(X) = \#(\mathbb{F}_{q^2})$,

$$N_2 = 1 + q^2 - \sum_{\alpha=1}^{2g} \pi_\alpha^2 = 1 + q^2 - 2gq$$

But $N_2 \geq N_1$, so $1 + q^2 - 2gq \geq 1 + q + 2gq^{1/2}$

$$q^2 - q \geq 2g(q + q^{1/2})$$

so $g \leq \frac{1}{2}(q - q^{1/2})$. ☑

B. Segre : Curve in $\mathbb{P}_3$ given by $x^{q^{1/2}+1} + y^{q^{1/2}+1} + z^{q^{1/2}+1} = 0$

has genus $g = \frac{1}{2}(q^{1/2})(q^{1/2} - 1) = \frac{1}{2}(q - q^{1/2})$.

and has $N = q^{3/2} + 1 = q + 1 + 2g q^{1/2}$.

So the bound above is exact.

Proof of $N = q^{3/2}+1$ for this curve.

Consider

$$\begin{tikzcd} \mathbb{F}_q \arrow[d, dash, "2"'] \\ \mathbb{F}_{q^{1/2}} \end{tikzcd} \qquad x \mapsto \bar{x} = x^{q^{1/2}}$$

Then the eqn is $x\bar{x} + y\bar{y} + z\bar{z} = 0 \quad x,y,z \in \mathbb{F}_q$.

Hermitian form! Want: how many isotropic vectors?

E.g.: given $y, z \in \mathbb{F}_q$, want to solve

$$x\bar{x} = -(\underbrace{y\bar{y} + z\bar{z}}_{\in \mathbb{F}_{q^{1/2}}}) \text{ has (in } x\text{) } \begin{cases} 1 \text{ soln. } x=0 \text{ if } y\bar{y}+z\bar{z}=0 \\ q_0+1 \text{ solns if not} \end{cases}$$

$q_0 = q^{1/2}$

$$\begin{tikzcd} \mathbb{F}_q^* \arrow[r, "N"] & \mathbb{F}_{q^{1/2}}^* \\ x \arrow[r, mapsto] & x\bar{x} \end{tikzcd}$$

$$y\bar{y} + z\bar{z} = 0 \longrightarrow \begin{cases} y = 0 = z \\ \text{or} \\ z \neq 0 \ \binom{q-1}{\text{poss.}}, \ q_0+1 \text{ solns in } y. \end{cases}$$

so $\underline{1 + (q_0^2-1)(q_0+1)}$ solns

So solns in $\mathbb{F}_q^3$ is $\underbrace{1 + (q_0^2-1)(q_0+1) + (q_0+1)(q_0^4 - 1 - (q_0^2-1)(q_0+1))}_{\text{this}}$

so $N = (\text{this}) - 1 \Big/ (q_0^2-1) = (q_0+1)(1+q_0^2+1-q_0+1) = (q_0+1)(q_0^2-q_0+1)$

$\uparrow$ in $\mathbb{P}^3$!

$= q_0^3 + 1 \quad \square$

				group acts	
	$g=0$,	$q+1$ pts,	$\mathbb{P}_1$,	PGL_2	(type A_1)
	$g=\frac{1}{2}(q-q^{1/2})$,	$q^{3/2}+1$,	Fermat-type curve,	PU_3	(type A_2^2) ($^2A_2$?
can add		q^2+1		Sz	$^2B_2$
		q^3+1		Rec	$^2G_2$

When is the Weil bound attained?

q square.

Then $N = q+1+2gq^{1/2} \Rightarrow g = ?$

$g = \frac{1}{2}(q-q^{1/2})$ is a possibility, and is the maximum

For $0 \leq g \leq \frac{1}{2}(q-q^{1/2})$?

Examples: 1) $q=4$, so $q^{1/2}=2$

The Fermat curve is $x^3+y^3+z^3=0$, so $g=1$, has 9 points

2) $q=9$, so $q^{1/2}=3$, get $x^4+y^4+z^4=0$, $g=3$.

Have curves for $g=1$, $q=9$.

For $g=2$, no such curve!

Would give $N = 1+9+4\cdot 3 = 22$;

but $g=2$ is 2-sheeted cov. of $\mathbb{P}_1$, and $\mathbb{P}_1$ has
$q=9 \to 10$ points, so $N \le 20$.

(In fact, correct bd is 20).

---

3) $q=16$, $q^{1/2}=4$ $\quad x^5+y^5+z^5=0$.

Weil bound is $1+16+2g\cdot 4 = 17+8g$

$$\left.\begin{array}{l} g=1 \longrightarrow 25 \text{ OK} \\ g=2 \longrightarrow 33 \text{ OK} \end{array}\right\} \text{see table}$$

$g=3 \to 41$ not the bound (bound is 38).

$g=4,5$?

$g=6 \to$ yes, and the last one.

Suppose X has Weil upper bound $= N(X)$ (or Weil lower bound).

And suppose
$$\begin{tikzcd} X \arrow[d, "\text{non-const. morphism.}"] \\ X' \end{tikzcd}$$

Then same holds for X'.

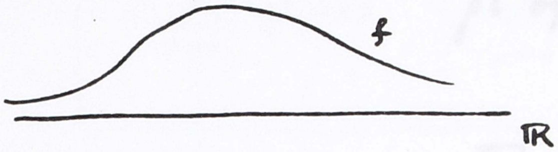
Think of Jac's. In X, every eigenvalue is $-q^{1/2}$ or $q^{1/2}$

and $J(X') \hookrightarrow J(X)$ so eigenvalues on $J(X')$ must be a subset of those on $J(X)$, but those are all equal!

finite ker?

Explicit Formulas

Number Fields



and take $\sum f(\log p)$

And we want a fla: $* + \sum f(\log p) = -\sum_{\substack{\tau \text{ zero} \\ \text{of zeta}}} \Phi(\tau) + \Phi(0) + \Phi(1)$

$\Phi = \begin{matrix}\text{Fourier} \\ \text{Mellin}\end{matrix}$ transform of f

Stark: $*$ includes $\log d$, and choose f, Φ well

Get inequality for d → Stark, Odlyzko, Poitou.

Notations

X, g, q, π_α eigenvalues of Frob arranged as $\pi_1, \dots, \pi_g, \bar{\pi}_1, \dots \bar{\pi}_g$

$$\pi_\alpha = q^{1/2} e^{i\varphi_\alpha} \quad , \quad 0 \le \varphi_\alpha \le \pi$$

$$N_n = \# X(\mathbb{F}_{q^n}) = 1 + q^n - \sum_{\alpha=1}^{g} (\pi_\alpha^n + \bar{\pi}_\alpha^n)$$
$$= 1 + q^n - 2\sum_{\alpha=1}^{g} q^{n/2} \cos n\varphi_\alpha$$

$Z(T)$ = zeta-fct of X

$$= \exp\left\{ \sum_1^\infty \frac{N_n T^n}{n} \right\} = \frac{P(T)}{(1-T)(1-qT)}$$

where

$$P(T) = \prod_{\alpha=1}^{g} (1-\pi_\alpha T)(1-\bar{\pi}_\alpha T$$

$$a_1 = N_1 = N$$

Also if a_d = number of "pts of degree d" of the scheme X.

pt of degree d = orbit of Frob of order d in $X(\overline{\mathbb{F}_q})$.

Have $N_n = \sum_{d|n} d a_d$ (clear!)

And $Z(T) = \prod_{\substack{P\in X \\ \text{closed pt}}} \frac{1}{1-T^{\deg P}}$

$$= \prod_{d\geq 1} \frac{1}{(1-T^d)^{a_d}}$$

Let $f(\theta)$ be a trigonometric polynomial of the form

$$f(\theta) = 1 + 2\sum_{n\geq 1} c_n \cos n\theta \qquad (\text{finite sum}).$$

$$= \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \qquad (c_0 = 1,\ c_{-n} = c_n).$$

To f I attach polynomials in t:

$$\Psi_d(t) = \sum_{\substack{n\geq 1 \\ d|n}} c_n t^n$$

$$\Psi(t) = \Psi_1(t) = \sum_{n\geq 1} c_n t^n .$$

"Explicit formula"

$$\boxed{\sum_{\alpha=1}^{g} f(\varphi_\alpha) + \sum_{d \geq 1} d a_d \Psi_d(q^{-1/2}) = g + \Psi(q^{-1/2}) + \Psi(q^{1/2})}$$

$$\sum_{\alpha=1}^{g} f(\varphi_\alpha) = g + 2 \sum_{n,\alpha} c_n \cos n\varphi_\alpha$$

$$= g + 2 \sum_{n} c_n \sum_{\alpha} \cos n\varphi_\alpha$$

$N_n = q^n + 1 - q^{n/2} \sum 2\cos n$

so $\sum 2\cos n\varphi_\alpha = \dfrac{q^n + 1 - N_n}{q^{n/2}}$

$$= \underbrace{g}_{\checkmark} + \sum_{n \geq 1} c_n (\underbrace{q^{n/2}}_{\Psi(q^{1/2})} + \underbrace{q^{-n/2}}_{\Psi(q^{-1/2})} - q^{-n/2} N_n).$$

Need only show that

$$\sum d a_d \Psi_d(q^{-1/2}) \overset{?}{=} \sum c_n q^{-n/2} N_n$$

$$\overset{?}{=} \sum c_n q^{-n/2} \sum_{d \mid n} d a_d$$

$$\sum d a_d \sum_{d \mid n} c_n q^{-n/2}$$

so OK. ▨

<u>Examples</u>:

i) $f=1$: $c_n = 0 \quad n \geq 1$, $\Psi_d = 0$

Get $q = q$.

ii) $f = 1 + \cos\theta$ $\quad c_1 = \frac{1}{2}$, $c_n = 0 \quad n \geq 2$.

$$\Psi_1 = \Psi = \tfrac{1}{2} t$$

$$\Psi_n = 0 \quad n \geq 2$$

Then:

$$q + \sum_\alpha \cos \varphi_\alpha + N \tfrac{1}{2} q^{-1/2} = q + \tfrac{1}{2} q^{-1/2} + \tfrac{1}{2} q^{1/2}$$

$\times 2q^{1/2}$:

$$q^{1/2} \sum_\alpha 2\cos\varphi_\alpha + N = 1 + q$$

so $\boxed{N = q + 1 - q^{1/2} \sum_\alpha 2\cos\varphi_\alpha}$ is Weil's formula.

<u>Assumptions</u> : (1) $f(\theta) \geq 0$ for all θ

(2) $c_n \geq 0$ for all n

Abbreviate: "f is doubly positive", $f \gg 0$.

Examples: $f = 1$, $f = 1 + \cos\theta$

Now if $f \gg 0$,

$$\underset{\geq 0}{\sum f(\varphi_a)} + \sum_{d \geq 1} \underset{\geq 0}{d a_d} \underbrace{\Psi_d(\;)}_{\geq 0}$$

So in that case one gets

$$\sum_{d \geq 1} d a_d \Psi_d(q^{-1/2}) \leq g + \Psi(q^{-1/2}) + \Psi(q^{1/2})$$

Taking only $d=1$ $(a_1 = N)$, get $N \Psi(q^{-1/2}) \leq g + \Psi(q^{-1/2}) + \Psi(q^{1/2})$

so $\boxed{N - 1 \leq \frac{g + \Psi(q^{1/2})}{\Psi(q^{-1/2})}}$

Now: we want to choose f so that this is optimal.

Also get $\boxed{g \geq (N-1)\Psi(q^{-1/2}) - \Psi(q^{1/2})}$

One can then do:

① determine (for a given N, q), the "best" bound on g.

Solved by Oesterlé (at least for $q \geq 3$)

② Assymptotic results as $g \to \infty$, q fixed.

③ Nice special cases (Suzuki & Ree curves)

④ Numerical bounds, say, for $q = 2$.

For $N = q+1$, $N = q^{3/2}+1$ we know

$N = q+1$ $\quad g = 0$

$N = q^{3/2}+1$ $\quad g = \frac{1}{2}(q - q^{1/2})$ q a square

---

For $N = q^2+1$

Choose $f = 1 + \sqrt{2}\cos\theta + \frac{1}{2}\cos 2\theta$

$= \frac{1}{2}(1 + \sqrt{2}\cos\theta)^2$ Then $f >> 0$.

Then $\begin{cases} \Psi(t) = \frac{1}{2}(\sqrt{2}t + \frac{1}{2}t^2) \\ \Psi_2(t) = \frac{1}{4}t^2 \end{cases}$ $\qquad N-1 = q^2$

Find $g \geq \left[q^2\left(\sqrt{2}q^{-1/2} + \frac{1}{2}q^{-1}\right) - \left(\sqrt{2}q^{1/2} + \frac{1}{2}q\right) \right]\frac{1}{2}$

so $\boxed{g \geq \frac{\sqrt{2}}{2}\left(q^{3/2} - q^{1/2}\right)}$

Is there such a curve?

$$g \overset{?}{=} \frac{\sqrt{2}}{2}\left(q^{3/2} - q^{1/2}\right) = (2q)^{1/2}(q-1) \Longrightarrow q = 2^{2f+1}$$

$\Rightarrow$ 2q = square

Sz(q) ⟷ Suzuki groups?

<u>Deligne-Lusztig varieties</u> connected to semi-simple groups over $\mathbb{F}_q$ and their <u>twisted</u> forms.

Sz ⟷ "$^2B_2$"-groups

Take, say, SL_n and Frobenius $x \longmapsto x^{(q)}$

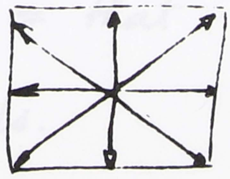
fixed pts give $SL_n(\mathbb{F}_q)$

Now use $x \longmapsto \sigma(x^{(q)})$ σ some outer autom. of the group.

If σ is $u \longmapsto {}^t u^{-1}$, get a different group.

char = 2, B_2 has such a "bad" autom. (not algebraic)

Root system is



(See Tits in Sem. Bourbaki)

Suzuki gps are gps acting on q^2+1 elements, $q = 2^{2f+1}$
Simple if $f \geq 1$. Called [redacted] Sz(q).

$Sz(2) = C_4 \cdot C_5$

Sz(3) is simple.

Deligne & Luzstig

Affine curve with no rat'l pt w/ action of Sz(q).

Pts at ∞ are q^2+1 and give the original representation of Sz(q)

Lusztig in *Inventiones*: genus as given, etc.

Next case: q^3+1

$f = \cos^2\varphi\,(1 + \frac{2}{\sqrt{3}}\cos\varphi)^2 \qquad g \geq \frac{\sqrt{3}}{2}(q^{5/2} - q^{1/2}) + \frac{1}{2}(q^2 - q)$

g integer, = that $\Rightarrow q = 3^{2f+1}$

Have Ree groups.

<u>We had</u>:

10/3 X curve $/\mathbb{F}_q$, $\pi_\alpha = q^{1/2} e^{i\varphi_\alpha}$

genus g

$f(\theta) = 1 + \sum_{n \geq 1} 2 c_n \cos n\theta$ (finite sum)

$\Psi_d(t) = \sum_{\substack{n \geq 1 \\ n \equiv 0 (\text{mod } d)}} c_n t^n$, $\Psi = \Psi_1$

Then $$\sum_{\alpha=1}^{g} f(\varphi_\alpha) + \sum_{d \geq 1} d a_d \Psi_d(q^{-1/2}) = g + \Psi(q^{-1/2}) + \Psi(q^{1/2})$$

a_d = # closed pts. of degree d

$N_n = \sum_{d | n} d a_d$ $N_1 = N = a_1$

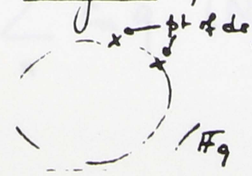
If $f >> 0$ (i.e., $f(\theta) \geq 0$ for all θ and $c_n \geq 0$ for all n):

then $$\boxed{\sum d a_d \Psi_d(q^{-1/2}) \leq g + \Psi(q^{-1/2}) + \Psi(q^{1/2})}$$

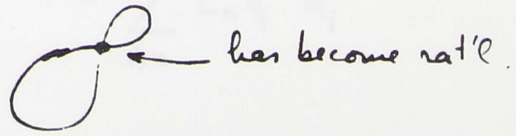
$$\Rightarrow \boxed{(N-1)\Psi(q^{-1/2}) \leq g + \Psi(q^{1/2})}$$

(<u>Today</u>: let $g \to \infty$)

<u>Remark</u>: We proved these inequalities for X projective nonsingular. For X <u>singular</u>, the bound as stated doesn't work.



collapse:



So should replace q by sth which "sees" the topology.

X (singular) projective, abs. irred., let $B = 1^{\text{st}}$ Betti number (for l-adic cohomology, for instance).

Thm: $(N-1)\Psi(q^{-1/2}) \le B/2 + \Psi(q^{1/2})$

Define the "arithmetic genus" $p_a(X) = \dim H^1(X, \mathcal{O}_X)$

Known: $B/2 \le p_a(X)$.

So we get $(N-1)\Psi(q^{-1/2}) \le p_a(X) + \Psi(q^{1/2})$.

Eg: $X \subset \mathbb{P}^2$ plane curve, abs. irred, of degree n.

Then $p_a(X) = \frac{1}{2}(n-1)(n-2)$

Pf (originally was given in terms of $N_n = \cdots$)

Claim: $N_n = 1 + q^n - \sum_{i=1}^{B} \alpha_i^n$ $\quad |\alpha_i| = 1$ or $q^{1/2}$

Groth.-Deligne: (Still the Lefschetz formula, except for the wrong size of eigenvalues)

Or: from Weil. $\boxed{}$

Take $N_n - 1 = q^n - \sum_{i=1}^{B} \alpha_i^n \ge N - 1$

$$\sum_n c_n q^{-n/2}(N_n - 1) = \sum_n c_n q^{n/2} - \sum_{i,n} c_n q^{-n/2} \alpha_i^n$$

So $$(N-1)\,\Psi(q^{-1/2}) \le \Psi(q^{1/2}) = \sum_{i,n} c_n q^{-n/2} \alpha_i^n$$

∴ <u>To be proved</u>: $$-\sum c_n q^{-n/2} \alpha_i^n \le \frac{B}{2}.$$

or $$\mathrm{Re}\left(-\sum c_n q^{-n/2} \alpha_i^n\right) \le \frac{B}{2}$$

Since the LHS is real anyway

Consider $\frac{1}{2} + \Psi(t) = F(t)$ polynomial in t

If $t = e^{i\varphi}$, $\mathrm{Re}(F(e^{i\varphi})) = \mathrm{Re}\left(\frac{1}{2} + \sum c_n e^{in\varphi}\right) = \frac{1}{2} f(\varphi) \ge 0.$

So: <u>$\mathrm{Re}(F(t)) \ge 0$ for all t with $|t| = 1$</u>

$\Downarrow$ cx analysis

$\mathrm{Re}(F(t)) \ge 0$ for all t with $|t| \le 1$.

(apply max. principle to $\exp(F(t))$. Get $|\exp(-F(t))| \le 1$ for $|t| = 1$, hence for $|t| \le 1$.)

Now

$$\mathrm{Re}\left(-\sum c_n q^{-n/2} \alpha_i^n\right) = \sum_i \mathrm{Re}\left(\frac{1}{2} - F(q^{-1/2}\alpha_i)\right) \quad \text{and} \quad |q^{-1/2}\alpha_i| \le 1$$

$$\le \sum_{i=1}^{B} \mathrm{Re}\left(\frac{1}{2}\right) = \frac{B}{2}$$ so QED. ☒

One can do similar things for higher dim'l varieties, in <u>odd</u> dimension.

<u>Example</u>: X proj non-sing variety, dim 3, abs. irred.
Assume: $B_1 = 0$, $B_2 = 1$, B_3 "large"
(E.g., any <u>complete intersection</u>)

If X has N points, one can prove

$$\frac{B_3}{2} \geq N\Psi(q^{-3/2}) - \left(\Psi(q^{-3/2}) + \Psi(q^{-1/2}) + \Psi(q^{1/2}) + \Psi(q^{3/2})\right)$$

(Exercise) (use Deligne – for large N this gives better bounds)

<u>Now</u>: q fixed, $g \to \infty$
we have:

Let $k \geq 1$ be a fixed integer.

Let X^λ be curves of genus $g_\lambda \longrightarrow \infty$.

$a_d(X^\lambda) =: a_d^\lambda$

<u>Theorem</u>: $$\limsup_{g_\lambda \to \infty} \frac{1}{g_\lambda} \sum_{d=1}^{k} \frac{d\, a_d^\lambda}{q^{d/2} - 1} \leq 1.$$

<u>Corollary</u>: For $k=1$, $a_1^\lambda = N^\lambda = \# X^\lambda(\mathbb{F}_q)$, and we have

$$\limsup_{g_\lambda \to \infty} \frac{N^\lambda}{g_\lambda} \leq q^{1/2} - 1.$$

(Thm. of Drinfeld–Vladut).

Se Th 9

Weil gives $N^\lambda \leq 1 + q + 2g^\lambda q^{1/2}$

so $\frac{N^\lambda}{g^\lambda} \leq 2q^{1/2} + o(1)$

If $q = 2$, Weil $2q^{1/2} = 2.828$, $[2q^{1/2}] = 2$, But $q^{1/2} - 1 = 0.414$.

Proof of Thm:

We have $\sum_{d=1}^{k} d a_d^\lambda \psi_d(q^{-1/2}) \leq g^\lambda + \Psi(q^{1/2}) + \psi(q^{-1/2})$

$$\frac{1}{g^\lambda} \sum_{d=1}^{k} d a_d^\lambda \psi_d(q^{-1/2}) \leq 1 + \frac{1}{g^\lambda}(\qquad\qquad)$$

$g_\lambda \to \infty$: $$\boxed{\limsup \frac{1}{g^\lambda} \sum_{d=1}^{k} d a_d^\lambda \psi_d(q^{-1/2}) \leq 1.}$$

true for every ψ_d coming from an $f \gg 0$.

Lemma: ① If $f \gg 0$, $f = 1 + \sum 2c_n \cos n\theta$, then $c_n \leq 1$.

② For any P, any $\varepsilon > 0$, $\exists f$ s.t. $c_n \geq 1 - \varepsilon$ for all $n = 1, \dots, P$.

(So can get f like $1 + 2\cos\theta + 2\cos 2\theta + \cdots =$ "$\sum_{n \in \mathbb{Z}} e^{in\theta}$" = Dirac measure at 0 on the circle

If $c_n = 1$ for all n, then OK, because $\psi(t) = t + t^2 + \cdots$,
$\psi_d(t) = t^d + t^{2d} + \cdots = \frac{t^d}{t^d - 1} = \frac{1}{t^{-d} - 1}$

and $\psi_d(q^{-1/2}) = \frac{1}{q^{d/2} - 1}$, which gives the theorem.

Then we need only do a convergence argument using the lemma. ▨

Pf of Lemma :

① $$\left\{\begin{array}{l} c_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \\ \text{and} \qquad 1 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(\theta)\, d\theta \end{array}\right.$$

and since $|\cos n\theta| \leq 1$ we get

$$c_n = \left|\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,|\cos n\theta|\, d\theta \leq 1 .$$

② Given P, ε

P integer ≥ 1, let $t = e^{i\theta}$

Write $$f_P = \frac{1}{2P+1}\left(t^{-P} + t^{-P+1} + \dots + 1 + \dots + t^{P}\right)^2$$

$$f_P(\theta) = \frac{1}{2P+1}\left(1 + 2\cos\theta + \dots + 2\cos P\theta\right)^2 .$$

$$f_P = \frac{1}{2P+1}\left(t^{-2P} + 2t^{-2P+1} + \dots + (2P+1)\cdot 1 + 2Pt + \dots + t^{2P}\right)$$

So $$c_n(f_P) = \frac{2P-n+1}{2P+1}$$

for fixed n, $P \to \infty$, $c_n \to 1$. ▨

Ihara's tower theorem

X as usual $/\mathbb{F}_q$, g

S = finite non-empty set of closed points of X

Assumption: There exists a sequence $X^\lambda \to X$ of unramified finite coverings of X in which every element of S splits completely and $\deg(X^\lambda \to X) \to \infty$.

Then:

$$\boxed{\sum_{P \in S} \frac{\deg P}{q^{\deg(P)/2} - 1} \leq g - 1}.$$

Special case: If all points in S are rat'l ($\deg = 1$) we get

$$|S| \leq (g-1)(q^{1/2}-1)$$

If $X = \mathbb{P}_1$, $S = \emptyset$, $X^\lambda = X/_{\mathbb{F}_{q^\lambda}}$ constant field extn.

then $\Sigma = 0 \leq 0 - 1$ is false.

So $S \neq \emptyset$ is necessary.

Proof: 1st case: the field of constants of the X^λ is just $\mathbb{F}_q$.

If $n^\lambda = [X^\lambda : X]$ = degree of covering, then $g^\lambda - 1 = n^\lambda (g-1)$.
(Cover is unramified!)

$$a_d^\lambda \geq n^\lambda a_d(S) \qquad a_d(S) = \# \text{ of } P \in S \text{ of degree } d.$$

So get $$\limsup_{n^\lambda \to \infty} \frac{1}{1+n^\lambda(g-1)} \sum \frac{d\, n^\lambda a_d(S)}{q^{d/2}-1} \leq 1$$

$n^\lambda \to \infty$

Now $$\frac{n^\lambda}{1+n^\lambda(g-1)} \longrightarrow \frac{1}{g-1},$$

so $$\limsup_{n_\lambda \to \infty} \underbrace{\frac{1}{g-1} \sum \frac{d\, a_d(S)}{q^{d/2}-1}}_{\text{constant!}} \leq 1$$

So we get the inequality we want. ▨

2nd case : general case.

Note: degree of a const. field extn. has a bound:

Indeed, if $P \in S$ has degree d, the degree of the constant field extn divides d.

Now (take a subsequence) we can assume that the constant field extn is $\mathbb{F}_{q^d}$ (for some d) for all λ.

Then
$$\begin{tikzcd} X^\lambda \arrow[d] \\ X/\mathbb{F}_{q^d} \arrow[d] \\ X/\mathbb{F}_q \end{tikzcd}$$
; apply case 1 to the top layer

Every $P \in S$ gives d points in $X/\mathbb{F}_{q^d}$ of degree $\deg(P)/d$, and the new "q" is q^d.

So this gives exactly the same formula!

Get $d \sum_{P \in S} \frac{\deg(P)/d}{(q^d)^{\frac{\deg(P)}{2d}} - 1} \leq g - 1$ which is the fla we want. ▨

Ihara, Journ. Math Soc. Japan?

Let $A(q) = \limsup_{g \to \infty} \frac{N_q(g)}{g}$

Drinfeld-Vladut Thm $\Longrightarrow$ $\boxed{A(q) \leq q^{1/2} - 1}$

Theorem (Ihara, Zink)

If q is a square, then $A(q) \geq q^{1/2} - 1$.

Corollary: If q is a square, $A(q) = q^{1/2} - 1$.

$A(q)$ is not known for other q.

Known: $A(q) > 0$; $A(q) > c \log q$ (some $c > 0$) [Use Golod-Shafarevich!] → next time

For $q = 2$:

$A(2) \leq 0.414\ldots$ $(= \sqrt{2} - 1)$ (D-V)

$A(2) \geq 0.205.$ $\left(= \frac{8}{39}\right)$

Proof when $q = p^2$: Use modular curves $X_0(N)$, etc.

(For $q = p^{2e}$, $e \geq 2$, use Shimura curves)

$\Gamma_0(N)$, $X_0(N)$ mod. curve

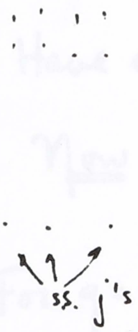
Choose $N = \ell$ prime, $\ell \equiv -1 \pmod{12}$, $\ell \neq p$.

Let $X = X_0(\ell)$, genus $g = \frac{\ell+1}{12}$.

Supersingular points are rat'l / $\mathbb{F}_{p^2}$, and their number N^{ss} is given by

$$\boxed{N^{ss} = \frac{p-1}{12}(\ell+1)}$$

$$\begin{tikzcd} X_0(\ell) \arrow[d] \\ \mathbb{P}^1 \text{ (param. by } j) \end{tikzcd}$$



so $\frac{N}{g} \geq \frac{N^{ss}}{g} = p-1 = q^{1/2}-1$

Now take $\ell \to \infty$, and the bound is obtained. ▨

$$A(q) = \limsup_{g \to \infty} \frac{N_q(g)}{g}$$

Have seen: 1) $A(q) \leq q^{1/2} - 1$ (Drinfeld - Vladut)

Now 2) If q is a square, $A(q) = q^{1/2} - 1$.

For $q = p^2$, modular curves → enough s.s. points.

$\{\pm 1\} \subset G \subset GL_2(\mathbb{Z}/\ell\mathbb{Z})$, ℓ prime, $\ell \geq 3$
[illegible] subgp

There the modular curve w.r.t. to G, affine, but add pts at ∞. Corresp to moduli problem:

E ell. curve + "G-structure on its ℓ-div. pts"

G-structure: a family of $E_\ell \xrightarrow[\varphi]{\sim} \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$

s.t. 1) $\varphi, \varphi' \in$ family $\Rightarrow \varphi' = s\varphi$ for some $s \in G$

2) $\varphi' \in$ family, $s \in G \Rightarrow s\varphi' \in$ family.

X_G is the moduli space (completed).

Can view in terms of X_ℓ = mod. curve of level ℓ (corresp to $G = \{\pm 1\}$)

$$\begin{tikzcd} X_\ell \arrow[d, dash] \\ \mathbb{Q} \end{tikzcd}$$

$$GL_2/\{\pm1\} \xrightarrow{\det} \mathbb{F}_\ell^\times$$

gives an extension

$$\begin{tikzcd} X_\ell \arrow[dd, dash] & \text{abs. irred. here} \\ & \mathbb{Q}(\zeta_\ell) \\ \mathbb{Q} \arrow[ur, dash] & \end{tikzcd}$$

then: $X_G = X_\ell/G$

If $G \twoheadrightarrow \mathbb{F}_\ell^\times$, then X_G is defined over $\mathbb{Q}$.

Assume $G \supset \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$, $\lambda \in \mathbb{F}_\ell^\times$

so $\det G = \begin{cases}\mathbb{F}_\ell^\times \\ \text{or} \\ (\mathbb{F}_\ell^\times)^2\end{cases}$

So ground field is either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{\pm\ell})$, $\pm\ell \equiv 1\ (4)$

Can do this over any k, char $k \neq \ell$ (for char $= \ell$, Katz + Mazur).

In char p, $p \neq \ell$, X_G is defined over either $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$.

Theorem: Every s.s. point $((E,\varphi),\ E\ \text{s.s.})$ is rational over $\mathbb{F}_{p^2}$ (on X_G).

Pf: E ss. can be written on $\mathbb{F}_{p^2}$, its Frob. being $-p$. So $-p \in G$ by our assumption, and Frob. stabilizes (E, φ). ☒

a) $\Gamma_0(\ell)$-curve : $G = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ gives our result above.

b) interesting case: $G = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{F}_\ell^\times \right\}$ $\quad X_G = X(\ell)$

$$\begin{cases} \text{genus of } X(\ell) : & 2g - 2 = \frac{1}{12}(\ell^2 - 1)(\ell - 6) \\ \#\text{ s.s. points} : /\mathbb{F}_{p^2} & N^{ss} = \text{[redacted]} \dfrac{(p-1)(g-1)}{1 - \frac{6}{\ell}} . \end{cases}$$

For low values :

$\ell = 7$ Klein curve $\qquad PSL_2(\mathbb{Z}/\ell\mathbb{Z})$ acts on $X(\ell)$

$|PSL_2(\mathbb{Z}/7\mathbb{Z})| = 168$

$g = 3$, $N^{ss} = 14(p-1)$

($PGL_2(\ )$ acts on curve over $\mathbb{Q}$, PSL_2 on curve over $\mathbb{Q}(\sqrt{-7})$).

Example : $\begin{cases} p = 2, \text{ so } & N^{ss} = 14 & (\text{over } \mathbb{F}_4) \\ p = 3, & N^{ss} = 28 & /\mathbb{F}_9 \\ p = 5 & N^{ss} = 56 & /\mathbb{F}_{25} \end{cases}$

Weil bound is $\quad 1 + p^2 + 6p \quad$ = resp, 17, 28, 56

($2g\sqrt{p^2} = 6p$.)

So for $p = 3, 5$ we have curves w/ max. no. of pts

$\Rightarrow$ for $p = 3, 5$, no rat'l cusps, etc.

Can prove: best for $p=2$ also.

$\therefore$ Klein curve gives us the best for $q = 4, 9, 25$ (also 8, but need: cusps are rational).

<u>alternate approach</u>

$\mathrm{Jac}(X(7)) = E \times E \times E$ $\qquad$ E has CM by $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$,

unique such def $/\mathbb{Q}$,
good red. outside 7,
grossencharacter

<u>for p inert</u>: $\left(\frac{p}{7}\right) = -1 \Rightarrow$ eigenv. of $F_v / \mathbb{F}_p$ in E is $\pm\sqrt{-p}$

$\Rightarrow /\mathbb{F}_{p^2}$ it is $-p$ (twice).

So on $\mathrm{Jac}(X(7))$, get $-p$ (six times), so get the Weil bound for p inert.

$\ell = 11$, $g = 26$, $N^{ss} = 55(p-1)$

$p=2$: $/\mathbb{F}_4$, $N^{ss} = 55$ $\quad$ which is best possible (exp. fla.)

$p=3$: $/\mathbb{F}_9$, $N^{ss} = 110$ $\quad$ (?) (exp. fla gives ≤ 111).

$\qquad \overset{\shortparallel}{\#}\, X(11)(\mathbb{F}_9)$

Use Jac(X(11)) is isogenous to the product of 11 times E_1, 10 times E_2, 5 times E_3, E_i ell curves

	Ant. IV
E_1 ell. curve cond 11	11
E_2 " " cond 11^2, no CM	121_F
E_3 " " cond 11^2, CM	121_D

Studied by Ligozat (Mod. Fcts V or VI)

So can check that the $N^{ss} = N$ (Ant. IV gives eigenvalues of Frob. on these curves).

Theorem (Golod-Šafarevič)

Thm on Artin local rings:

- R ring, I two-sided ideal, $R/I = k$ comm. field.
- Every $r \in R$, $r \notin I$ is invertible (i.e., R is local w/ max'l ideal I).
- R is "Artin" $\iff \begin{cases} I^n = 0 \text{ for large } n \\ \text{and} \\ I^m/I^{m+1} \text{ is a finite dim'l } k\text{-vector space} \\ (\text{for } m = 0, 1, \dots) \end{cases}$

(Example: G finite l-group, $R = \mathbb{F}_l[G]$, $I = \Delta(G)$, $k = \mathbb{F}_l$.)

Take M = f.g. R-module (left).

Then M/IM is a k-vector space of finite dim.

$$d_0(M) = \dim_k(M/IM) = \text{min. no. of generators of } M.$$

(cf $x_1, -, x_d \in M$ give a basis for $M/_{IM}$, NAK => they generate M

M $\qquad\qquad d = d_0(M)$

Choose $x_1, -, x_d \in M$ generating M. This gives

$$0 \longrightarrow M_1 \longrightarrow R^d \longrightarrow M \longrightarrow 0$$

$\parallel$

kernel = module of relations between the x_i

$M_1 \subset IR^d$

Up to isom., M_1 depends only on M. Define

$d_1(M) := d_0(M_1) =$ "nber of relations between x_i"

Also: exact seq. gives: $\otimes k$

$$0 = \mathrm{Tor}_1(R^d, k) \longrightarrow \mathrm{Tor}_1(M,k) \longrightarrow M_1/_{IM_1} \longrightarrow R_d/_{IR_d} \underset{\cong}{\longrightarrow} M/_{IM} \longrightarrow 0$$

So find $\mathrm{Tor}_1(M,k) \cong M_1/IM_1$,

so $d_1(M) = \dim_k \mathrm{Tor}_1(M,k)$

$d_i(M) = d_0(M_i)$ $\qquad M_i = i^{\text{th}}$ term of a minimal resolution of M

$\parallel$ $\dim_k \mathrm{Tor}_i(M,k)$.

Take $M = k$: $d_0(k) = 1$

$$0 \to I \to R \to k \to 0$$

So $M_1 = I$; $d_1(k) = \dim_k(I/I^2) = \textcircled{d}$

$d_2(k) = \textcircled{r}$ by defn.

<u>Theorem</u> (Golod-Šafarevič, refined by Vinberg and Gaschütz)

Assume $d \geq 1$, i.e., that $I \neq 0$, i.e., that R is not a field. Then

$$r > \frac{d^2}{4}.$$

[One has examples of $r \sim \frac{d^2}{3}$ or $\frac{3d^2}{8}$; $\frac{d^2}{2}$ is easy; Best is unknown.

[<u>Example</u>: if $R = \mathbb{Z}/\ell^2\mathbb{Z}$, $k = \mathbb{Z}/\ell\mathbb{Z}$, get $M_1 \cong \mathbb{Z}/\ell\mathbb{Z}$, so all $d_i = 1$.

<u>Pf</u>: Have $0 \to J \to R^d \to I \to 0$

$$R^r \to J \to 0$$

So get $R^r \xrightarrow{\varepsilon} R^d \to I \to 0$ $\qquad \varepsilon(R^r) \subset IR^d$

since $R^d/IR^d \xrightarrow{\sim} I/I^2$

Tensor w/ R/I^nR:

$$\begin{tikzcd}
R^r/I^nR^r \arrow[r] \arrow[dr] & R^d/I^nR^d \arrow[r] & I/I^{n+1} \arrow[r] & 0 \\
 & R^r/I^{n-1}R^r \arrow[u] & &
\end{tikzcd}$$

$\therefore \varepsilon(I^{n-1}R^r) \subset I^nR^d$

so factors

So $\frac{R^r}{I^{n-1}R^r} \longrightarrow \frac{R^d}{I^n R^d} \longrightarrow \frac{I}{I^{n+1}} \longrightarrow 0$ $\boxed{n \geq 1}$

define $a(n) = \ell\left(R/I^n R\right) = \sum_{i=0}^{n-1} \dim_k \left(I^i/I^{i+1}\right)$

$a(0) = 0$

$a(1) = 1$

$a(2) = 1 + d$ $\qquad R/I^2 \; : \; \underset{1}{R/I}, \; \underset{d}{I^2/I}$

$\vdots$

$a(n)$ ultimately constant.

Now exact seq. above gives:

$$d\,a(n) \leq r\,a(n-1) + \underbrace{a(n+1) - 1}_{\ell(I/I^{n+1})} \qquad n \geq 1$$

Claim: this implies $r > \frac{d^2}{4}$.

For this introduce $\sum_{n=0}^{\infty} a(n) t^n = t + (1+d)t^2 + \cdots = t f(t)$

so $\underline{f(t) = \sum_{n=0}^{\infty} a(n+1) t^n}$

Write $f \succ 0$ for : all coeffs of f are ≥ 0.

multiply ineq by t^n and add:

$$\sum_{n\geq 1} d\,a(n)t^n < r\sum_{n\geq 1} a(n-1)t^n + \sum_{n\geq 1} a(n+1)t^n - \sum_{n\geq 1} t^n$$

$$d\,f(t)t < r\,t^2 f(t) + f(t) - 1 - \sum_{n\geq 1} t^n$$

so $$d\,f(t)t < r\,t^2 f(t) + f(t) - \frac{1}{1-t}$$

so $$f(t)(rt^2 - dt + 1) > \frac{1}{1-t}$$

Assume $r \leq \frac{d^2}{4}$; then $rt^2 - dt + 1 = (1-\lambda t)(1-\mu t)$ $\lambda\mu$ real ≥ 0

Now, $\frac{1}{1-\lambda t}$ has pos. coeffs.

λ & μ can't both be zero, since $d \neq 0$.

Multiply!

$$f(t) > \frac{1}{(1-t)(1-\lambda t)(1-\mu t)}$$

and either λ or μ is ≥ 1 : $\lambda + \mu = d \geq 1$, $\lambda\mu = r$

But the coeffs of $f(t)$ are bounded, since the $a(n)$ are.

So it's enough to show the coeffs of RHS are not bded.

RHS $> \frac{1}{(1-t)^2}$ coeffs not bded. ▨

G finite ℓ-group, $R = \mathbb{F}_\ell[G]$, $I = \Delta(G)$

Why is I = radical of G?

Up to isom., an ℓ-group has only one irred. repres. in ch. ℓ, namely the trivial repres. (any non-triv. repres. has a fixed vector). then radical $= \bigcap \ker(\text{repres}) = I$.

$$d = \dim \mathrm{Tor}_1^R(k, k) = \dim H_1(G, \mathbb{Z}/\ell\mathbb{Z})$$

$$r = \dim \mathrm{Tor}_2^R(k, k) = \dim H_2(G, \mathbb{Z}/\ell\mathbb{Z})$$

dual of $H_i = H^i$

$$\text{so} \begin{cases} d = \dim H^1(G, \mathbb{Z}/\ell\mathbb{Z}) \\ r = \dim H^2(G, \mathbb{Z}/\ell\mathbb{Z}) \end{cases}$$

And: d = min. no. of generators of G

assume $x_1, \ldots, x_d \in G$ generate G;

then r = min. nber of rels between x's which define G as an ℓ-group. (or pro-ℓ-group)

Theorem (Golod-Šaf.) If G is a finite non-triv. ℓ-group then d and r as above satisfy $r > d^2/4$.

$d = 1$, $r > \frac{1}{4}$ i.e., $r \geq 1$; x gen, $x^\ell = 1$ cyclic order ℓ (sharp)

$d = 2$, $r > \frac{4}{4}$ i.e., $r \geq 2$; x, y gen., $yxy^{-1} = x^{1+\ell}$, $xyx^{-1} = y^{1+\ell}$ (order is ℓ^3)

Define $z = xyx^{-1}y^{-1} = y^\ell = x^{-\ell}$, so $z \in Z(G)$

so $x^{\ell^2} = 1$ so $z^\ell = 1$. So order ℓ^3.

$y x^\ell y^{-1} = x^{\ell+\ell^2}$

since $x^\ell = y^{-\ell}$]

Seth 17

$d=3$, $r > \frac{9}{4}$ i.e., $r \geq 3$ (ℓ odd): 3 gen. z, y, x

$$\begin{cases} yxy^{-1} = x^{1+\ell} \\ zyz^{-1} = y^{1+\ell} \\ xzx^{-1} = z^{1+\ell} \end{cases}$$

Mennicke → finite group. $d=3, r=$

$d=4$, $r > \frac{16}{4}$ i.e., $r \geq 5$ → is $d=4$, $r=5$ possible?

(think: $d=4$, $r=6$ is.)

$x_1, \dots, x_d$ with $(x_i, x_j)=1$, $x_i^\ell = 1$ give $\frac{d(d-1)}{2}+d$ rels.

10/17

Class Field Towers

C curve, genus g, over $\mathbb{F}_q$, p = char.

K its function field.

ℓ a prime number ($\ell = p$ is ok)

S finite non-empty set of "primes" of K i.e., of closed points of C.

Look at

$$\begin{tikzcd} K_1 \arrow[d, dash] \\ K \end{tikzcd}$$

K_1 = max. abelian ℓ-extension of K, (whose Galois group is an ℓ-group) unramified (everywhere) in which the elements of S split completely.

K_1/K is a finite extension of K

Cond. on S makes it finite (otherwise if $S = \emptyset$, have the const. field extn.)

If ℓ divides the $\deg(P)$ for every $P \in S$, then K_1 contains $\mathbb{F}_{q^\ell} \cdot K$. If not, it does not.

$$\begin{tikzcd} K_1 \arrow[d, dash] \arrow[rr, squiggly] & & C_1 \arrow[d] \\ K & & C \end{tikzcd}$$

Define $K_2 = (K_1)_1$ with respect to S_1 = inverse image of S.

So have $K \subset K_1 \subset K_2 \subset \cdots \subset K_\infty = \bigcup_n K_n$

K_∞ = max. Galois extension of K where S splits completely, whose Galois group is pro-ℓ, and unramified.

Question: Is K_∞ / K finite?

Let $G_S = Gal(K_\infty/K) = \varprojlim Gal(K_n/K)$

Assume G_S is finite; it is an ℓ-group. Then we have

$d = \dim H_1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ (min. nber of generators)

$r = \dim H_2(G_S, \mathbb{Z}/\ell\mathbb{Z})$ (min. nber of "relations" (as ℓ-group).)

Theorem: Assume G_S is finite (i.e., the tower stops). Then

$$r - d \leq \begin{cases} |S| - 1 & \text{if } \ell \nmid q-1 \\ |S| & \text{if } \ell \mid q-1 . \end{cases}$$

(We know: $r > \frac{d^2}{4}$ if $d \geq 1$ Golod-Šaf., which will give a contradiction for suitable S.)

Proof (Same as Iwasawa's in No. field case):

First: Using class field theory to find $Gal(K_1/K)$:

C_K = idèle class group

Have

$$1 \longrightarrow E_S \longrightarrow \prod_{p\in S} K_p^* \times \prod_{v\notin S} U_v \longrightarrow C_K \longrightarrow \overset{\text{(quotient)}}{Ab_S} \longrightarrow 1$$

E_S = S-units (= unit outside S)

Then $Gal(K_1/K) = (Ab_S)_\ell$ (ℓ-part)

Next:

Now: K_∞/K finite, so $(K_\infty)_1 = K_\infty$

Write seq. for K_∞:

$$1 \longrightarrow \underset{\substack{\parallel \\ S_\infty\text{-units of } K_\infty}}{E_{S_\infty}} \longrightarrow \prod_{\tilde{p}\in S_\infty} K^*_{\infty,\tilde{p}} \times \prod_{\tilde{v}\notin S_\infty} U_{\tilde{v}} \longrightarrow C_{K_\infty} \longrightarrow Ab_\infty \longrightarrow 1$$

$$(K_\infty)_1 = K_\infty \iff (Ab_\infty)_\ell = \{1\}$$

Let $G = G_S$; G acts on everything.

- Ab_∞ has trivial cohomology (order prime to ℓ, G ℓ-group)

Also. For the product

K_p^*-part: $\tilde{P}$ $\circlearrowleft G$ permutes $\quad \prod_{\tilde{P}\to P} K^*_{\infty,P}$ is induced (trivial coh.)

$$\begin{tikzcd}\tilde{P} \arrow[d] \\ P \in S\end{tikzcd}$$

U_v-part: $\tilde{v}$ $\quad G_{\tilde v}$ -- stab $\tilde{v}$

$$\begin{tikzcd}\tilde{v} \arrow[d] \\ v\end{tikzcd}$$

Shapiro's lemma:

coh $= H^q(G_{\tilde v}, U_{\tilde v}) =$ trivial (because everything is unramified).

So LES of coh gives a map

$$\hat{H}^q(G, C_{K_\infty}) \xrightarrow[\cong]{\delta} \hat{H}^{q+1}(G, E_{S_\infty})$$ for each $q \in \mathbb{Z}$.

δ is an isom. for all q because middle term has trivial cohom.

$\boxed{\hat{H}^q \text{ is Tate cohomology}}$

Know: $$\hat{H}^q(G, C_{K_\infty}) \xleftarrow[\wedge u_{K_\infty}]{\cong} \hat{H}^{q-2}(G, \mathbb{Z})$$

$$u_{K_\infty} \in H^2(G, C_{K_\infty})$$

Choose $q+1=0$ so $q=-1$: Get $\hat{H}^{-3}(G, \mathbb{Z}) \cong \hat{H}^0(G, E_{S_\infty}) =$

$$= E_S / \mathrm{Norm}(E_{S_\infty})$$

Know $E_S \cong \mathbb{Z}^{|S|-1} \times \mathbb{F}_q^{\times}$.

We want a quotient of $E_S/_{\text{Norms}}$ which is an ℓ-group.

So

$$rk_\ell\,(\hat{H}^{-3}(G,\mathbb{Z})) \leq \begin{cases} |S|-1 & \text{if } \ell \nmid (q-1) \\ |S| & \text{if } \ell \mid (q-1). \end{cases}$$

It remains to show: $rk_\ell\,(\hat{H}^{-3}(G,\mathbb{Z})) = r-d$.

$$\hat{H}^{-3}(G,\mathbb{Z}) = H_2(G,\mathbb{Z})$$

[Group of coeffs A, trivial action

$$H_q(G,A) = H_q(G,\mathbb{Z}) \otimes A \;\oplus\; \mathrm{Tor}_1(H_{q-1}(G,\mathbb{Z}), A)$$

$q=2$, $A = \mathbb{Z}/\ell\mathbb{Z}$:

$$H_2(G,\mathbb{Z}/\ell\mathbb{Z}) = H_2(G,\mathbb{Z}) \otimes \mathbb{Z}/\ell\mathbb{Z} \quad \oplus \quad \mathrm{Tor}_1(H_1(G,\mathbb{Z}), A)$$

$$= H_2(G,\mathbb{Z})/_{\ell H_2(G,\mathbb{Z})} \quad \oplus \quad \ell\text{-part of } H_1(G,\mathbb{Z})$$

So take ranks: $r = rk_\ell\,(H_2(G,\mathbb{Z})) + d$.

Since: $H_1(G,\mathbb{Z}/\ell\mathbb{Z}) = H_1(G,\mathbb{Z})/_{\ell H_1(G,\mathbb{Z})}$ since H_0 is free

So $rk_\ell\,\hat{H}^{-3}(G,\mathbb{Z}) = r-d$.]

Or: look at

$$0 \to \mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z} \to 0$$

$$H_2(G,\mathbb{Z}) \xrightarrow{\ell} H_2(G,\mathbb{Z}) \to H_2(G,\mathbb{Z}/\ell\mathbb{Z}) \to H_1(\ ) \xrightarrow{\ell} H_1(\ )$$

So $\quad 0 \to H_2(G,\mathbb{Z})/\ell H_2(G,\mathbb{Z}) \to H_2(G,\mathbb{Z}/\ell\mathbb{Z}) \to \mathrm{Ker}(\ell \text{ in } H_1(G,\mathbb{Z})) \to$

So dimensions add. So QED! ☑

Theorem: The $(S-\ell)$ class field tower of K is infinite if $|S| \gtreqless \frac{d^2}{4} - d + \left\{\begin{matrix} 1 & \text{if } \ell \mid (q-1) \\ 0 & \text{if not} \end{matrix}\right\}$, and $d \geq 2$.

Pf: Otherwise

$$r - d \leq \begin{cases} |S| - 1 \\ |S| \end{cases}$$

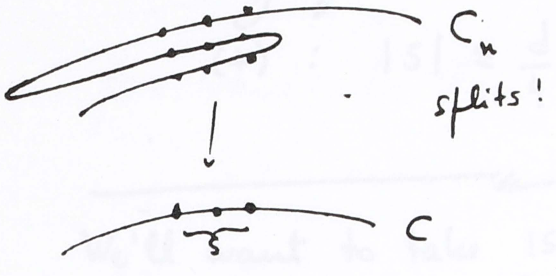
$$r > \frac{d^2}{4} \quad \text{if } d \geq 1$$

And of course $d=1$ doesn't work (get $|S| \leq \frac{1}{4}$) ☑

Suppose that all elements in S have degree 1, i.e., S is made up of rational points, and assume that the corresp. class field tower is infinite.

Then $\quad A(q) \geq \dfrac{|S|}{g_c - 1}$, where $A(q) = \limsup\limits_{g\to\infty} \dfrac{N_q(g)}{g}$

g_c = genus of C.



So nber of rat'l pts of $C_n \geq [C_n : C] \cdot |S$

genus of $C_n = g_{C_n}$; $g_{C_n} - 1 = [C_n : C](g - 1$

Notice that $g_c \geq 2$ (For $g_c = 0$, no unr. coverings; for $g_c = 1$, covering would have $g_{C_n} = 1$ and nber pts $\to \infty$!)

So $$\frac{N_q(g_n)}{g_n} \geq |S| \frac{[C_n : C]}{1 + [C_n : C](g_c - 1)}$$

$$\geq |S| \frac{1}{\frac{1}{[C_n : C]} + g_c - 1}$$

$n \to \infty$ so $$\boxed{A(q) \geq \frac{|S|}{g_c - 1}}.$$

Corollary: If (S, ℓ) satisfy $|S| \leq \frac{d^2}{4} - d + \left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}$, $d \geq 2$, then
$$A(q) \geq \frac{|S|}{g_c - 1}$$

(and in particular $A(q) > 0$).

- For every q, we want to find K, S, ℓ satisfying

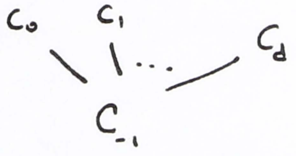
$$(*) : \quad |S| \leq \frac{d^2}{4} - d + \begin{cases} 1 & \text{if } \ell \mid q-1 \\ 0 & \text{if not.} \end{cases} \quad d \geq 2.$$

We'll want to take $|S| = 1$, $d \geq 5$.
We'll choose $\ell = 2$ (So for q odd, enough to find $d \geq 4$).

<u>Construction</u> (char $= p \neq 2$)

Choose $\ell = 2$; K will be some quadratic extension of $K_0 = \mathbb{F}_q(T)$ corresp. to curve $/C_{-1}$ of genus 0.

Want



point at ∞ should split completely.

Let $\Phi_0, \ldots, \Phi_d$ be irred. monic polyn. of even degree distinct

Then define C_i by $y_i^2 = \Phi_i(T)$.

$$y_i^2 = T^{\text{even}} + \cdots \Rightarrow \infty \text{ is split in each } C_i$$

So set $K = \mathbb{F}_q(T, \sqrt{\Phi_0 \cdots \Phi_d})$

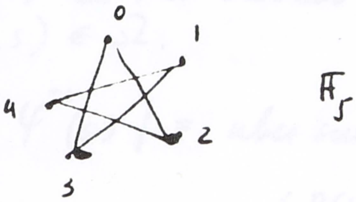
Over K, I have d indep. quadratic extens, unramified, ∞ splits completely.

Choose $d = 5$ for instance, and the "d" in the theorem is then at least 5, so done. $\boxtimes$

For char = 2, do the same with Artin-Schreier extensions.

Now want: In $\mathbb{F}_q$, $A, B \subset \mathbb{F}_q$ as large as possible s.t. every $a - b$, for $a \in A$, $b \in B$, is a non-square.

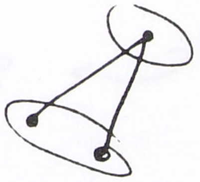
E.g. $q = 5$



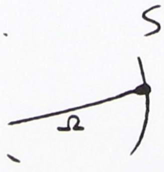
join pts whose difference is not a square

Want: a maximal complete bipartite graph embedded in this graph

Call this a B-subgraph

E.g.  in our case.

<u>Graph theorem</u>



Let R, S be two finite sets, $\Omega \subset R \times S$.

Let $m \geq 1$ be such that every $s \in S$ is Ω-related to at least m points of R.

Let a, b be integers ≥ 1 such that $b\binom{|R|}{a} \leq |S|\binom{m}{a}$. (*)

Then $\exists\, A \subset R$ and $B \subset S$ with $\begin{cases} |A| = a, \ |B| = b \\ A \times B \subset \Omega. \end{cases}$

Let X = set of pairs (A, s) with $|A| = a$, $s \in S$ and $A \times \{s\} \subset \Omega$.

Then $|X| = ?$

Project $X \xrightarrow{\varphi} S$

$(A, s) \longmapsto s$

Let $R(s)$ be the subset of R made of the elements r s.t. $(r, s) \in \Omega$.

So $|\varphi^{-1}(s)|$ = nber subsets of $R(s)$ w/ a elements

$= \binom{R(s)}{a} \geq \binom{m}{a}$ since $R(s) \geq m$

Then $|X| \geq |S| \binom{m}{a}$

Then $X \xrightarrow{\psi}$ Set of subsets of R with a elements $\longleftarrow$ has $\binom{|R|}{a}$ elements

$(A, s) \longmapsto A$

Hence some fiber of ψ has at least $|X| / \binom{|R|}{a}$ elements

But $\dfrac{|X|}{\binom{|R|}{a}} \geq \dfrac{|S| \binom{m}{a}}{\binom{|R|}{a}} \geq b$.

So choose A whose fiber has $\geq b$ elements, and choose B in the fiber with $|B| = b$. Done! ▨

10/23 **Class Field Towers (cont.)**

We showed: If $\begin{cases} \Omega \subset R \times S \\ \text{every } s \in S \text{ is } \Omega\text{-related to at least } m \text{ elements of } R \end{cases}$

then $\exists A \subset R$, $B \subset S$, $|A| = a$, $|B| = b$ given, s.t. $A \times B \subset \Omega$

provided that $b\binom{|R|}{a} \le |S| \binom{m}{a}$.

For $\mathbb{F}_q$, $q = p^e$, $p \neq 2$:

Take $R = S = \mathbb{F}_q$, $\Omega = \{(r,s) \mid r - s \text{ is a nonzero square in } \mathbb{F}_q\}$, so $m = \frac{q-1}{2}$.

Let $q \longmapsto a(q), b(q)$ be two functions of a variable q, with integral values ≥ 1 for $q = p^e$, p prime $\neq 2$, with

$$\begin{cases} a(q) \le c_1 \log q \\ b(q) \le q^{c_2} \end{cases}$$

where $c_2 + c_1 \log 2 < 1$.

Claim: Then, for q large enough, there exists $A, B \subset \mathbb{F}_q$ with $|A| = a(q)$, $|B| = b(q)$, and $A \times B \subset \Omega$.

[This is approx. what the naïve approach would give

b_1 $\frac{\qquad\qquad}{q/2}$

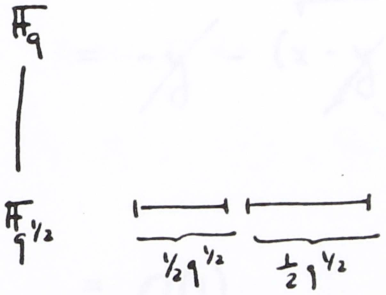
b. b. $q \cdot \left(\frac{1}{2}\right)^2$

$\dot{b}_1 \quad \dot{b}_2 \cdots\cdots \dot{b}_a$ $\qquad\qquad \overline{q\left(\tfrac{1}{2}\right)^{\alpha}}$?

$\alpha \sim \log q \Rightarrow \left(\tfrac{1}{2}\right)^{\alpha} = \frac{1}{q^{\varepsilon}}$

which suggests $a(q) \le c_1 \log q$ is reasonable]

If q is a square,



every difference $\in \mathbb{F}_{q^{1/2}} \subset (\mathbb{F}_q^{\times})^2$

So $a(q) \le q^{1/2}$, $b(q) \le q^{1/2}$. Much larger!

Proof of Claim

To be checked: for q large enough,

$$b(q)\binom{q}{a(q)} \overset{?}{\le} q\binom{\frac{q-1}{2}}{a(q)}$$

i.e., $$b(q)\frac{q!}{(q-a(q))!} \overset{?}{\le} q\,\frac{\left(\frac{q-1}{2}\right)!}{\left(\frac{q-1}{2}-a(q)\right)!}$$

Stirling: $\log(x!) = (x+\tfrac{1}{2})\log x - x + O(1)$

Suppose $1 \leq y \leq x^{1/2}$. Then

$$\log\left(\frac{x!}{(x-y)!}\right) = (x+\tfrac{1}{2})\log x - x - (x-y+\tfrac{1}{2})\left(\log x + \log(1-\tfrac{y}{x})\right) +$$

$$+ x - y + O(1)$$

$$= y\log x + O(1)$$

Since $-y - (x-y+\tfrac{1}{2})\log(1-\tfrac{y}{x}) =$

$$= \cancel{-y} - \overbrace{(x - \cancel{y} + \tfrac{1}{2})(-\tfrac{y}{x}} + O(\tfrac{y^2}{x^2}))$$

$$\boxed{\frac{y^2}{x} \leq 1}$$

$$= O(1)$$

So check that

$$\log b(q) + a(q)\log q + O(1) \overset{?}{\leq} \log q + a(q)\log\frac{q-1}{2}$$

$a(q) \leq c_1 \log q$
$b(q) \leq q^{c_2}$] $\log b(q) \leq c_2 \log q$.

So $\log q - \log b(q) \overset{?}{\geq} a(q)\left(\log q - \log\frac{q-1}{2}\right) + O(1)$

$$\log\frac{q-1}{2} = \log(q-1) - \log 2 = \underbrace{\log q + \log(1-\tfrac{1}{q})}_{o(1)} - \log$$

So $a(q)\left(\log q - \log \frac{q-1}{2}\right) + O(1) \le c_1 \log q \,(\log 2 + o(1))$

$$\log q - \log b(q) \geqslant (\log q)(1 - c_2)$$

So want $\boxed{1 - c_2 > c_1 \log 2}$, which is our condition. ▨

Starting from such $A, B \subset \mathbb{F}_q$, we make a 2-class field tower starting from $\mathbb{P}_1$ and making quadratic extensions.

Assume a = even = 2α

Write $\{a_1, a_1', \dots, a_\alpha, a_\alpha'\} = A$.

If t is the variable in $\mathbb{P}_1$, take the quad. extn. given by $\sqrt{(t-a_i)(t-a_i')}$ $i = 1, \dots, \alpha$.

(So Fct. field is $\mathbb{F}_q(t, \sqrt{(t-a_i)(t-a_i')})$)

In that ext., the points of B split completely (by our choice of Ω).

Now go to $C \longleftrightarrow \mathbb{F}_q(t, \sqrt{\prod (t-a_i)(t-a_i')})$

Elements of B give B_C = subset of C coming from B,

$$|B_C| = 2|B| = 2b(q)$$

and C has $\alpha - 1$ independent quadratic extns which are unramified and where B_C splits completely.

(Namely, given by $\sqrt{(t-a_i)(t-a_i')}$ $i = 1, \dots, \alpha - 1$.

The 2-class field tower is infinite if $|B_c| \leq \frac{(\alpha-1)^2}{4} - (\alpha-1)$

So choose $\alpha \sim c \log q$

$$|B_c| \sim c'(\log q)^2$$

if $c' < \frac{c}{4}$, the condn. is satisfied.

Hence, infinite class field tower, so

$$A(q) \geq \frac{|B_c|}{g_c - 1} \asymp \frac{(\log q)^2}{\log q} \asymp \log q .$$

C is hyperelliptic, ramified at $2\alpha = a(q)$ pts, and so $g_c = \alpha - 1 \asymp \log q$.

So $\boxed{A(q) \geq c \log q}$ for some c. (For q odd).

When q is a square, we can take $|B_c| \asymp q^{1/2}$, $g_c \asymp q^{1/4}$, so get $A(q) \geq c q^{1/4}$.

(But modular tower gives $q^{1/2}$).

For $q = 2^e$, can use $\left\{ \begin{array}{l} \text{Artin-Schreier extns} \\ \text{or} \\ \text{a 3-tower} \end{array} \right.$

Construction for $g=2$ **Claim**: $A(2) \geqslant \frac{2}{9} = 0.222\ldots$

First: Simple construction for $A(2) \geqslant \frac{1}{5} = 0.2$.

Take
$$\begin{tikzcd} C \arrow[d, "2"'] & \\ \mathbb{P}_1 & t \end{tikzcd}$$

$$y^2+y = t^3+t+\sum_{\substack{\text{irred polyn } \varphi(t)\\ \text{of deg } 2,3,4}} \frac{t^2+t}{\varphi(t)}$$

So: sum is over
$$\left\{\begin{array}{l} \deg 2 \\ \deg 3 \\ \deg 4 \end{array}\right. \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \left\{\begin{array}{l} t^2+t+1 \\ t^3+t+1, \quad t^3+t^2+1 \\ t^4+t+1, \quad t^4+t^3+1, \quad t^4+t^3+t^2+t+1. \end{array}\right.$$

We'll see $g_C = 21$.

Take $\{0,1\} \subset \mathbb{P}_1$; these split completely in C to give a set S of 4 pts.

$$\begin{tikzcd} S \arrow[d] & C \arrow[d, "2"] \\ \{0,1\} & \mathbb{P}_1 \end{tikzcd}$$

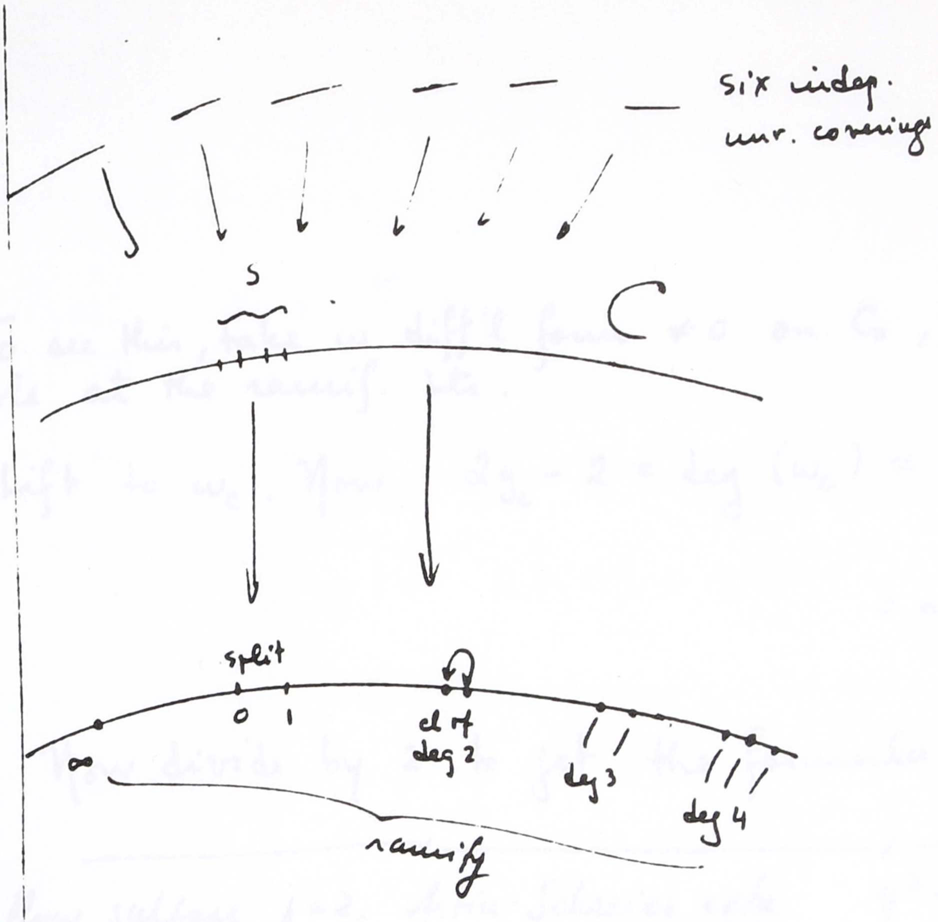
Have 6 indep. unramified quad. extns, given by
$$y^2+y = \frac{t^2+t}{\varphi(t)} \quad \text{for the six } \varphi(t) \text{ above}$$

and $\{0,1\}$ split completely. (Same argument as before.)

/ C is ramif at roots of $\varphi(t)$ and at ∞.

Picture →

Se Th 25



Assume : $g_C = 21$, class field tower is infinite.

Then $A(2) \geq \frac{|S|}{g_C - 1} = \frac{4}{21-1} = \frac{4}{20} = \frac{1}{5}$.

Class Field Tower : condition is $|S| \leq \frac{d^2}{4} - d + 1$

$|S| = 4$, $d \geq 6$, so $\frac{d^2}{4} - d + 1 \geq \frac{36}{4} - 6 + 1 \geq 4$.

So class field tower is infinite.

Genus : Know: $g_C - 1 = n(g_{C_0} - 1) + \underbrace{\sum \text{contrib. of ramification}}_{?}$.

$$\begin{tikzcd} C \arrow[d, "n"'] \\ P \in C_0 \end{tikzcd}$$

Go to alg. closure: take a local param t at P, look at dt. Lift it to C. Ramif ⇒ dt will have zeros on the fiber

and contrib of ramif $= \frac{1}{2}$(degree of dt on the fiber)

To see this, take ω diff'l form $\neq 0$ on C_0, w/o zero or pole at the ramif. pts.

Lift to ω_c. Now $2g_c - 2 = \deg(\omega_c) = n\deg(\omega)$ + extra zeros

$$= n \deg(\omega) + \sum_{\text{on fiber}} \text{deg of } dt$$

Now divide by 2 to get the formula.

Now suppose $p=2$, Artin-Schreier extn $y^2 + y = \psi(t)$

local computation, so work in field of power-series

$$\psi(t) = \frac{c_0}{t^n} + \cdots \qquad n \text{ odd} \geq 1.$$

[If ψ is hol., no ramif.; if $\frac{1}{t^2} + \cdots$, remove $\psi - \left(\frac{1}{t^2} + \frac{1}{t}\right)$ until we get n odd].

Claim: local contrib at that place is $\frac{1}{2}(n+1)$,

i.e., "deg of dt in extension" $= n+1$ (even)

$$\begin{tikzcd} K \arrow[d, dash, "2"'] \\ K_0 \quad t \end{tikzcd}$$

$$v_K(y) = \frac{1}{2} v_K\left(\frac{1}{t^n}\right) = -n \quad \text{since } v(t) = 2 \text{ (ramification)}$$

$$dy = \frac{c_0}{t^{n+1}} dt + \cdots \qquad \text{(char 2)}$$

Pf θ local parameter of K, $y = \theta^{-n} \cdot$ unit, so $dy = \dfrac{d\theta}{\theta^{n+1}} + \cdots$

so $v_K(dy) = -(n+1)$

So $v_K(dy) = -2(n+1) + v_K(dt) \implies v_K(dt) = n+1$.

In our case, $$y^2+y = \underbrace{t^3+t}_{\substack{\text{pole of order } 3 = n \\ \text{at } \infty \\ \frac{1}{2}(n+1) = 2}} + \underbrace{\sum_{\deg 2,3,4} \frac{t^2+t}{\varphi(t)}}_{\substack{\text{simple poles at these} \\ \deg \varphi = 2 \to \text{two simple poles} \\ \to \frac{1}{2}(2) + \frac{1}{2}(2) = 2}}$$

Same for $\deg \varphi = 3$ or 4.

So $$g_C - 1 = 2(g_0 - 1) + \sum \text{local contrib}$$
$$= -2 + 2 + \sum \deg \varphi$$
$$= -2 + 2 + 20$$

So $\boxed{g_C = 21}$. ▨

Lemma Let C be a curve over $\mathbb{F}_2$; let S be a set of closed points of C, let $\underline{m}$ be a positive divisor of C disjoint from S. Assume that $\deg(\underline{m}) \geq |S|$. Then there is a quadratic extension of C which is unramified outside $\underline{m}$, where S splits completely and where the contribution of ramification is at most $\deg \underline{m}$.

Proof – later (done by CFT)

Assuming the lemma, we can improve the result above:

Choose C = an elliptic curve with 2 rational points (and no more).

$N_1 = \# C(\mathbb{F}_2) = 2$

Now $N_n = \# C(\mathbb{F}_n) = 1 + 2^n - \underbrace{(\pi^n + \bar{\pi}^n)}_{t_n}$

and $t_n = t_{n-1} t_1 - q t_{n-2}$ since $\pi\bar{\pi} = q$!

Here $t_0 = 2$, $t_1 = 1$, $q = 2$ $(N_1 = 2 = 1 + 2 - t_1$

So $t_n = t_{n-1} - 2 t_{n-2}$

So $t_2 = -3$, $t_3 = -5$, $t_4 = 1$

$\therefore N_1 = 2$, $N_2 = 8$, $N_3 = 14$, $N_4 = 16$

If a_i = nber of closed pts of $C/\mathbb{F}_2$ of degree i (so $N_n = \sum_{i|n} i a_i$)

So $\underbrace{a_2 = 3, \ a_3 = 4}_{\text{lots of these}}$, $a_4 = 2$.

s.t. two rat'l pts split completely

Make 7 quadratic extensions of C ramified each at a different closed pt of degree 2 or 3; contrib. to ramification will be just 2, 2, 2, 3, 3, 3, 3.

Vith same onstruction n before } $d = 6$ as before, and $g - 1 = 2(1-1) + \sum$ contrib

$g - 1 = 18 \implies g = 19$

So get $A(2) \geq \frac{4}{18} = \frac{2}{9}$. ☑

10/31 Something from CFT

X curve $/\mathbb{F}_q$, genus g, K fct. field.

(modulus) $\underline{m} = \sum_{P \in S} n_P P$ be a positive divisor, S finite, $n_P \geq 1$, P closed.

$$Cl_{\underline{m}}(X) = \{\text{divisors of } X \text{ prime to } S\} \Big/ \{(f)\}$$

where $f \in K^\times$, and $f \equiv 1 \bmod^\times \underline{m}$ (this means that $f \in 1 + \mathfrak{m}_P^{n_P}$ locally at each $P \in S$, or. equiv., $v_P(f-1) \geq n_P$ $\forall P \in S$).

If $\underline{m} = 0$, $Cl_{\underline{0}}(X) = Pic(X)$
$\qquad\qquad\qquad\qquad \overset{\shortparallel}{Cl(X)}$

Recall
$$\boxed{0 \to J(\mathbb{F}_q) \to Cl(X) \xrightarrow{\deg} \mathbb{Z} \to 0}$$
exact

and

$$0 \to \text{kernel} \to Cl_{\underline{m}}(X) \to Cl(X) \to 0$$

obvious almost obvious.

assume $\underline{m} \neq 0$; [redacted] if $U_P =$ local units at P, $U_P^{(n_P)} = \{u \in U_P \mid v(u-1) \geq n_P\}$.

We have $$0 \to \mathbb{F}_q^\times \to \prod_{P \in S} U_P / U_P^{(n_P)} \to Cl_{\underline{m}}(X) \to Cl(X) \to 0$$

<u>CFT</u>:

Let G be a finite group and let

$$\alpha : \mathrm{Cl}_m(X) \longrightarrow G \quad \text{be onto.}$$

Then CFT <u>constructs</u> an abelian extension K_α/K with Galois group G s.t.

1) K_α/K unramified outside S

2) If $P \notin S$, the Frob of P in G is the <u>image</u> by α of "P", viewed as a divisor.

3) If $P \in S$, the map

$$\begin{tikzcd} U_P \arrow[r] \arrow[rr, bend right] & \mathrm{Cl}_m(X) \arrow[r, "\alpha"] & G \end{tikzcd}$$

is the one attached by <u>local</u> class field theory, and the image is the inertia group.

I also have $K_P^* \longrightarrow G$, image is decomp. group.

____________, ____________, ____________

Let $P_1, \ldots, P_s$ be closed points disjoint from S. If I want extns in which these split completely, I have:

To have P_i split in K_α/K it is nec. & suff. that $\alpha((P_i)) = 0$, $(P_i) \in Cl_{\mathfrak{m}}(X)$.

Let ℓ be a prime number, let $d_P = \ell$-rank of $U_P/U_P^{(n_P)}$ (this is a group of order $(q^{\deg(P)}-1)q^{(n_P-1)\deg(P)}$)

Let $\varepsilon = \ell$-rank of $\mathbb{F}_q^\times$ $\begin{pmatrix} 0 \text{ if } \ell \nmid q-1 \\ 1 \text{ if } \ell \mid q-1 \end{pmatrix}$

Claim Assume that $s \leq \sum d_P - \varepsilon$.
Then : $\exists$ a cyclic extension of deg ℓ obtained through an α, where the P_i's *split* *completely*.

Have $0 \to \text{Local} \to Cl_{\mathfrak{m}} \to J(\mathbb{F}_q) \times \mathbb{Z} \to 0$,
so $Cl_{\mathfrak{m}} \cong \mathbb{Z} \times \Phi$ $\Phi \supset \text{Local}$, *finite*.

So ℓ-rank of $Cl_{\mathfrak{m}}/\ell Cl_{\mathfrak{m}}$ is $\geq 1 + \sum d_P - \varepsilon$.

There is a *hyperplane* $Cl_{\mathfrak{m}}/\ell Cl_{\mathfrak{m}}$ (as $\mathbb{F}_\ell$-vector sp.) containing all the P_i; this gives the desired extension.

Given K_α/K, assuming no constant-field extn. corresps to the condn:

$$\begin{tikzcd} Cl_{\mathfrak{m}} \arrow[r, "\deg"] \arrow[dr, "\alpha"'] & \mathbb{Z} \\ & G \end{tikzcd}$$

$\deg : \operatorname{Ker}\alpha \to \mathbb{Z}$ is surjective.

In this case, I want the genus of K_α.

Look at characters $\chi : G \to \mathbb{C}^\times$. This gives maps

$$U_P \longrightarrow Cl_m \longrightarrow G \xrightarrow{\chi} \mathbb{C}^\times .$$

So exp. of cond. of χ at $P = f_P(\chi) = \begin{cases} 0 & \text{if } U_P \to \mathbb{C}^\times \text{ is trivial} \\ \text{smallest } e \text{ s.t. } U_P \to \mathbb{C}^\times \text{ is trivial on } U_P^{(e)} & \end{cases}.$

So Formula for the genus g_α of K_α:

$$2g_\alpha - 2 = [K_\alpha : K](2g-2) + \deg(\text{discriminant ideal})$$

and

$$\text{disc} = \sum_{P,\chi} f_P(\chi) \cdot P ,$$

so we get

$$\boxed{2g_\alpha - 2 = |G|(2g-2) + \sum f_P(\chi) \, \underline{\deg P}}$$

——— 〃 ——— 〃 ———

In our situation: $g=2$, $\ell=2$, $|G|=2$

At each P we get <u>one</u> number f_P (only one character!).

On the other hand, we've noted that writing

$$y^2 + y = \frac{a_0}{t^m} + \cdots\cdots = \varphi \quad m \text{ odd} \geq 1,$$

the local contrib was measured by m.

Have: $\boxed{f_p = m + 1}$

"Proof": I get $U_p \longrightarrow \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$

explicitly: $u \in U_p$ can be viewed as $u(t) \in \mathbb{F}_{2^e}[[t]]$; then the map is

$$u(t) \longmapsto \mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}\left(\mathrm{Res}\left[\varphi(t)\frac{du(t)}{u(t)}\right]\right)$$

(For an Artin-Schreier extn as above!)

Then, if $u \equiv 1 \bmod t^{m+1}$

$$\frac{du}{u} = c\,t^m\,dt + \cdots\cdot$$

$$\varphi\frac{du}{u} \text{ hol., so } \mathrm{Res} = 0$$

But if $u \equiv 1 \bmod t^m$, not $\bmod t^{m+1}$, will get simple pole, hence Res $\neq 0$.

2-rank of $U_p/U_p^{(n_p)}$ $\qquad$ n_p even ≥ 2

res field $\mathbb{F}_{2^e}$

$U_p/U_p^{(n_p)} = \{ \alpha_0 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1} \mod t^n \}$ $\qquad$ $\alpha_0 \neq 0 \Rightarrow \alpha_0 \in \mathbb{F}_{2^e}^*$

$\alpha_i \in \mathbb{F}_{2^e}$

for 2-rank, think only

$1 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1}$ $\qquad$ $\alpha_i \in \mathbb{F}_q = \mathbb{F}_{2^e}$, so

order $= q^{n-1}$

squares : $\underbrace{1 + \alpha_1^2 t^2 + \cdots}$ = those where $\alpha_1 = \alpha_3 = \cdots = 0$)

there are $\frac{n}{2}$ odd indices, so order $= q^{n/2 - 1}$

so : $|G/G^2| = q^{n/2}$, so 2-rank $= e \cdot \frac{n}{2}$.

This proves the statement made last time about constructing extensions where certain points split. (Just check that #points $< \sum d_p - \varepsilon$)

Se Th 29

Optimal functions for number of points

$\mathbb{F}_q$, genus g, N points

(*) **If** $c_n \geq 0$, and $f = 1 + \sum 2c_n \cos n\theta \geq 0$ for all θ

$= 1 + \sum c_n(t^n + t^{-n})$, $t \in S^1$ unit circle.

then $\boxed{g \geq (N-1)\sum c_n q^{-n/2} - \sum c_n q^{n/2}}$

Problem: Knowing N and q, what is (if any) the **best choice** of (c_n)? (I.e., the one that maximizes the expression.)

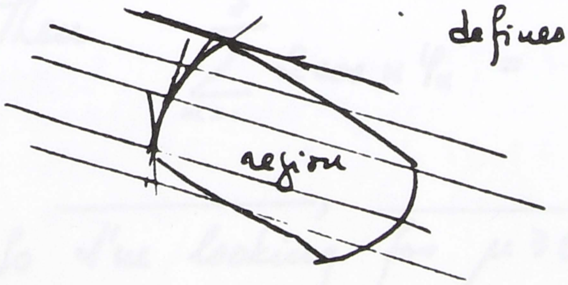
If $(c_n) \in (*)$, what is the max of $\sum c_n((N-1)q^{-n/2} - q^{n/2})$

Call the **max** $g(N,q)$ $\overset{\wedge}{-\sum c_n \gamma_n}$

This is a **linear programming** question:

given variables, $\underbrace{\text{linear inequalities on them}}$

defines ~~some~~ convex region



given linear form ⟵ cut region by lines

look for max. value.

Se Th 29a

In linear programming, every problem has a dual problem. So let's introduce the dual problem, but in a more natural way.

Suppose we want $\geq N$ pts, q $\to$ genus?

We must have then $N_n \geq N, \forall n$. $\left(N_n = \# X(\mathbb{F}_{q^n})\right)$

If $\varphi_1, \dots, \varphi_g$ are the angles of Frob.

Know: $N_n = q^n + 1 - q^{n/2} \sum_{\alpha=1}^{g} 2 \cos n \varphi_\alpha$

So $q^n + 1 - q^{n/2} \sum_{\alpha=1}^{g} 2 \cos n \varphi_\alpha \geq N$

Introduce the measure on S^1 $\mu = \sum \left(\delta_{e^{i\varphi_\alpha}} + \delta_{e^{-i\varphi_\alpha}} \right)$ δ = Dirac measure.

$\mu \geq 0$, $\mu(S^1) = 2g$.

Then $\sum_{\alpha=1}^{g} 2 \cos n \varphi_\alpha = \int t^n \mu(t) \quad \left[= \int \frac{1}{2}(t^n + t^{-n}) \mu(t) \right.$

So I'm looking for $\mu \geq 0$ on S^1 with

$$\int t^n \mu(t) \leq \underbrace{q^{n/2} - (N-1) q^{-n/2}}_{\gamma_n} \qquad n = 1, 2, \dots$$

(and μ symmetric with respect to $t \mapsto \bar{t}$)

So look at all

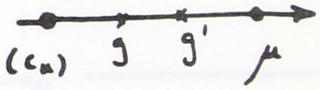
μ symmetric
$$\boxed{\mu \geq 0, \quad \underset{(**)_n}{\int t^n \mu(t) \leq \gamma_n} \quad n = 1, 2, \dots} \qquad (**),$$

and ask what is the lowerbound of $\frac{1}{2}\int 1\,\mu(t)$?
Call it $g'(N,q)$

This is the dual problem.

Lemma 1: If $\mu \in (**)$, $(c_n) \in (*)$, then

$$-\sum c_n \gamma_n \leq \int \tfrac{1}{2}\mu(t).$$



In particular, $g(N,q) \leq g'(N,q)$

Lemma 2: We have equality above iff μ has support contained in the set of zeros of the fct $1 + \sum c_n (t^n + t^{-n})$ on S^1 and $(**)_n$ is an equality for every n s.t. $c_n \neq 0$.

Theorem: Let μ and (c_n) be such as in Lemma 2. Then $g(N,q) = \frac{1}{2}\int \mu(t) = -\sum c_n \gamma_n$.

Proof: Let $f = 1 + \sum c_n (t^n + t^{-n}) \geq 0$

So $\mu(f) = \int f \mu = \mu(1) + 2 \sum c_n \mu(t^n) \geq 0$

But $\mu(t^n) = \int t^n d\mu \leq \gamma_n$, so

$$\mu(\tfrac{1}{2}) \geq -\sum c_n \mu(t^n) \geq -\sum c_n \gamma_n$$

QED, Lemma 1.

Proof (2): Want $\mu(f) = 0$ above, hence since $f \geq 0$ μ must be concentrated on the zeros.

I also want $\mu(t^n) = \gamma_n$ unless $c_n = 0$,

QED (2). ☑

Example: Take $q + 1 \leq N \leq q^{3/2} + 1$

Claim: in this range Weil is optimal.

$$\text{Weil} \longleftrightarrow 1 + \cos\theta, \quad \text{so} \quad \underbrace{c_1 = \tfrac{1}{2}, \; c_n = 0, \; n \geq 2}_{\text{Claim: this choice is optimal.}}$$

It is enough to exhibit a μ with equality!

We want

$$g = -\tfrac{1}{2}\gamma_1 = -\tfrac{1}{2}\left(q^{1/2} - (N-1)q^{-1/2}\right)$$

$$= \tfrac{1}{2}\left((N-1)q^{-1/2} - q^{1/2}\right) \geq 0$$

And take μ = Dirac at $t=-1$ (angle $\theta = \pi$) with weight $2g$, where g is given by this last eqn.

To check: $\begin{cases} 1+\cos\theta \geq 0 \quad \text{OK}, \quad c_n \geq 0 \quad \text{OK} \\ \text{so } (*) \text{ OK}. \end{cases}$

$\begin{cases} \text{clearly } \mu \text{ is concentrated at the zero of } 1+\cos\theta \\ \text{to check: } \begin{cases} \mu(t^n) \leq \gamma_n \quad \text{for } n \geq 2 \\ \mu(t) = \gamma_1 \end{cases} \end{cases}$

$$\mu(t) = 2g\,(-1) = -2g$$

to check $-2g = \gamma_1$ OK by construction

$$\mu(t^2) = 2g\,(-1)^2 = -\gamma_1 \overset{?}{\leq} \gamma_2$$

$$-\gamma_1 = -q^{1/2} + (N-1)q^{-1/2} \overset{?}{\leq} q - (N-1)q^{-1} = \gamma_2$$

$$(N-1)(q^{-1/2} + q^{-1}) \overset{?}{\leq} q + q^{1/2}$$

$$(N-1)(1+q^{1/2}) \overset{?}{\leq} q^{3/2}(1+q^{1/2})$$

$$(N-1) \overset{?}{\leq} q^{3/2}$$

<u>OK</u> by condition on N.

$\mu(t^3) = -2g = \gamma_1 \overset{?}{\leq} \gamma_3$ $\quad$ $\gamma_3 \geq 0$, $-2g < 0$ OK.

$\mu(t^4) = 2g = -\gamma_1 \leq \underbrace{\gamma_2 \leq \gamma_4}$

γ_n increases for n large.

and larger n are similarly OK. ▨

__Theorem on Linear Inequalities on $\mathbb{R}^n$:__

Let f_α be finitely many additive functions $(\sum \alpha_i x_i + \beta)$ on $\mathbb{R}^n$. Then the following are equiv.:

(i) The equations $f_\alpha \geq 0$ have no common solution.

(ii) $\exists c_\alpha \geq 0$ s.t. $1 + \sum c_\alpha f_\alpha = 0$ identically in $\mathbb{R}^n$.

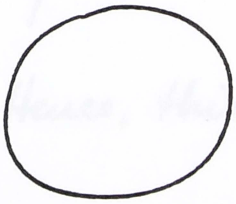
11/7 **Oesterlé's "Optimal" computation of lower bound for g** (given N, q)

On Machine:

```
ON
RUN 2000
NOMBRE DE POINTS?      65   enter
Q ?                    8     "
GENRE >= 14
```

Recall: Last time we defined a "dual" problem to what we wanted, and saw that

$$\left.\begin{array}{l}\text{admissible } \mu \text{ measure} \\ \qquad + \\ \text{admissible } (c_n)\end{array}\right\} \text{match} \rightarrow \text{optimal solution.}$$



$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

admissible μ
- μ will be a positive measure on S^1 s.t.
- symmetric w.r.t $z \mapsto \bar{z} = z^{-1}$
- $\int t^n \mu(t) = \langle t^n, \mu \rangle \geq q^{n/2} - (N-1) q^{-n/2}$ for $n \geq 1$

(Where N, q are given).

admissible (c_n)
- $c_n \geq 0$
- $1 + \sum_{n \geq 1} c_n (t^n + t^{-n}) \geq 0$ on S^1

We attach

$$\begin{cases} \mu \longmapsto \frac{1}{2}\int \mu(t) = \langle \frac{1}{2}, \mu \rangle & \Big) \approx \\ (c_n) \longmapsto \sum c_n\left((N-1)q^{-n/2} - q^{n/2} \right) \end{cases}$$

We say μ and (c_n) <u>match</u> if $\frac{1}{2}\int \mu(t) = \sum c_n(\quad)$; if this happens, the common value is the best $g(N,q)$ that the explicit formula can give.

(Usually not even rational, of course...)

Oesterlé found an explicit choice of (c_n) (working for every (q,N)), an explicit choice of μ (working for $q \geq 3$ and sometimes for $q=2$), and they match.

Hence, this gives $g(N,q)$, at least for $q \geq 3$.

- <u>Let $\lambda = N-1$, $\alpha = q^{1/2}$</u>

So condn. on μ is $\int t^n \mu(t) \geqslant \alpha^n - \lambda\alpha^{-n}$, $n \geqslant 1$

We showed
$$\begin{cases} \lambda \leq \alpha^3 \longrightarrow \text{Weil estimate is best} \\ \qquad \text{(i.e., optimal is given by } 1+\frac{1}{2}(t+t^{-1}) = 1+\cos\varphi \text{)} \\ \lambda = \alpha^4 \longrightarrow \text{Suzuki (char} = 2\text{)} \\ \lambda = \alpha^6 \longrightarrow \text{Ree} \end{cases}$$

Method:

Define m by $\boxed{\alpha^m < \lambda \leq \alpha^{m+1}}$. $\quad \left(m = \left[\frac{\log \lambda}{\log \alpha} \right] \right)$

I will assume $m \geq 2$. (if not, $g = 0$ is OK ...)

Put $u = \dfrac{\alpha^{m+1} - \lambda}{\lambda\alpha - \alpha^m}$; by the assumptions, $0 \leq u < 1$.

Consider the equation:

$$\cos \frac{m+1}{2}\varphi + u \cos \frac{m-1}{2}\varphi = 0$$

There is exactly one solution φ_0 in the range $\frac{\pi}{m+1} \leq \varphi_0 < \frac{\pi}{m}$.

Then the optimal g (for $q \geq 3$, at least) is

$$\boxed{g = \frac{(\lambda - 1)\alpha \cos \varphi_0 + \alpha^2 - \lambda}{\alpha^2 - 2\alpha \cos \varphi_0 + 1}}$$

(When $q = 2$, this is the value given by a choice of (c_n).)

We try to find μ of the following shape:

concentrated on a symmetric set $T \subset S^1$, with $|T| = m - 1$, with

(T) $$-\int t^n \mu(t) = \alpha^n - \lambda \alpha^{-n} \quad \text{for } n = 1, \dots, m-1,$$

and the mass ν_t of $t \in T$ being strictly positive. We also need $\nu_t = \nu_{\bar{t}}$, of course.

On the other hand, look for (c_n) s.t.

$$f(t) = 1 + \sum_{n=1}^{m-1} c_n (t^n + t^{-n}) \quad \text{is } \underline{\text{zero}} \text{ on } T$$

$(c_n \geq 0,\ f(t) \geq 0$ on $S^1)$.

If we can do this, we have a match (as seen before).

<u>Lemma</u> If T satisfies the condition (T), then)
T is <u>contained</u> in the set of solutions of

$$t^{m+1} + 1 + u(t^m + t) = 0 \qquad (1)$$

(This has $m+1$ solutions on S^1, which are symmetric, so to get T we need to discard one pair).

Rewrite as $t = e^{i\varphi}$. get the equation for u given above.

So we throw out the solutions $t = e^{\pm i\varphi_0}$:

T = solutions of (1) which are different from $e^{\pm i\varphi_0}$.

Proof of Lemma: Suppose T is given.

T has $m-1$ elements in S^1.

$\mu = \sum \nu_t \delta_t$ so the integral is $\sum_{t\in T} \nu_t t^n = \alpha^n - \lambda\alpha^{-n}$ $\quad n=1,—,m-1$

The system $$\boxed{\sum_{t\in T} \nu_t t^n = \alpha^n - \lambda\alpha^{-n} \qquad n=1,—,m-1} \quad (*)$$

has $(m-1)$ linear eqns, $(m-1)$ unknowns, determinant is Van der Monde t $(t\in T)$, so $\neq 0$.

So it has a unique solution $\nu_t =$ —

Now force $\nu_t = \nu_{\bar t}$ for every $t\in T$. This will imply equation (1).

Rewrite $(*)$ as follows:

$(*) \Longleftrightarrow$ for every polynomial Φ of degree $\leq m-1$, with constant term 0,

$$\boxed{\sum \nu_t \Phi(t) = \Phi(\alpha) - \lambda \Phi(\alpha^{-1})}. \quad (**)$$

Let $$P(x) = \prod_{t\in T} (x-t)$$

T symmetric $\Rightarrow$ $P(x^{-1}) = P(x)\, x^{1-m}$

$$-\frac{1}{x^2} P'(x^{-1}) = P'(x)\, x^{1-m} + (1-m) P(x) x^{-m}$$

So if $t \in T$, $\quad -\frac{1}{t^2} = \bar{t}^2$:

$$\boxed{-\bar{t}^2 P'(\bar{t}) = P'(t)\, t^{1-m}}$$

Let $t \in T$; define $\quad Q_t(x) = x \prod_{\substack{t' \in T \\ t' \neq t}} (x - t') = \dfrac{x\,P(x)}{x-t}$

Now $\left|\begin{array}{l} Q_t(t') = 0 \quad \text{for } t' \in T,\ t' \neq t. \\ Q_t(t) = t\,P'(t) \end{array}\right.$

Apply (**) to $\Phi = Q_t$:

get $\quad \nu_t\, Q_t(t) = Q_t(\alpha) - \lambda\, Q_t(\alpha^{-1})$

so $\quad \boxed{\nu_t = \dfrac{Q_t(\alpha) - \lambda\, Q_t(\alpha^{-1})}{t\,P'(t)}}$

rewrite: $\quad t\,P'(t)\,\nu_t = \dfrac{\alpha P(\alpha)}{\alpha - t} - \lambda\, \dfrac{\alpha^{-1} P(\alpha^{-1})}{\alpha^{-1} - t}$

we have $\quad P(\alpha^{-1}) = P(\alpha)\,\alpha^{1-m}$

so $\quad t\,P'(t)\,\nu_t = P(\alpha) \left\{ \dfrac{\alpha}{\alpha - t} - \lambda\, \dfrac{\alpha^{-m}}{\alpha^{-1} - t} \right\}$

$$t P'(t)\, \nu_t = P(\alpha)\, \frac{1 - \alpha t - \lambda\alpha^{1-m} + t\lambda\alpha^{-m}}{1 - \alpha t - \alpha^{-1} t + t^2}$$

Now $\nu_t = \nu_{\bar t} \longrightarrow \dfrac{1}{t P'(t)} \cdot \dfrac{1 - \alpha t - \lambda\alpha^{1-m} + t\lambda\alpha^{-m}}{1 - \alpha t - \alpha^{-1} t + t^2} = \dfrac{1}{\bar t P'(\bar t)}\, \dfrac{1 - \alpha\bar t - \lambda\alpha^{1-m} + \bar t\lambda\alpha^{-m}}{1 - \alpha\bar t - \alpha^{-1}\bar t + \bar t^2}$

Also $-P'(\bar t) = P'(t) \cdot t^{3-m}$, so

$$\frac{1}{t} \cdot \{ \text{———} \} = \frac{-1}{t^{2-m}} \cdot \{ \text{———} \}$$

so $-t^{1-m} \cdot \{ \text{———} \} = \{ \text{———} \}$

$$-t^2 \left(1 - \alpha\bar t - \alpha^{-1}\bar t + \bar t^2 \right) = 1 - \alpha t - \alpha^{-1} t + t^2$$

So end up with

$$-t^{m+1} \left(1 - \alpha\bar t - \lambda\alpha^{1-m} + \lambda\alpha^{-m}\bar t\right) = \left(1 - \alpha\bar t - \lambda\alpha^{1-m} + \lambda\alpha^{-m} t\right)$$

$$t^{m+1}\left(1 - \lambda\alpha^{1-m}\right) + t^m\left(\lambda\alpha^{-m} - \alpha\right) - t\left(\alpha - \lambda\alpha^{-m}\right) + \left(1 - \lambda\alpha^{1-m}\right) = 0$$

$$u = \frac{\alpha^{m+1} - \lambda}{\lambda\alpha - \alpha^m} = \frac{\alpha - \lambda\alpha^{-m}}{\lambda\alpha^{1-m} - 1}$$

So $\quad t^{m+1} + 1 + u\left(t^m + t\right) = 0 .$ $\blacksquare$

Note this works for any m; our choice of u is equiv to $u \in [0,1)$.

Now, study the equation $\cos\frac{m+1}{2}\varphi + u\cos\frac{m-1}{2}\varphi = 0$, where $0 \le u < 1$.

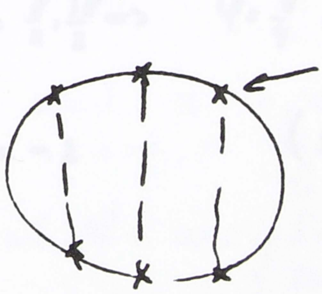
$$F(\varphi) = \frac{\cos\frac{m+1}{2}\varphi}{\cos\frac{m-1}{2}\varphi}$$

(We want to show $t^{m+1} + 1 + u(t^m + t) = 0$ has $m+1$ distinct solutions on S^1, and "locate" them.)

if m is even : -1 is a solution

$(m = 6)$

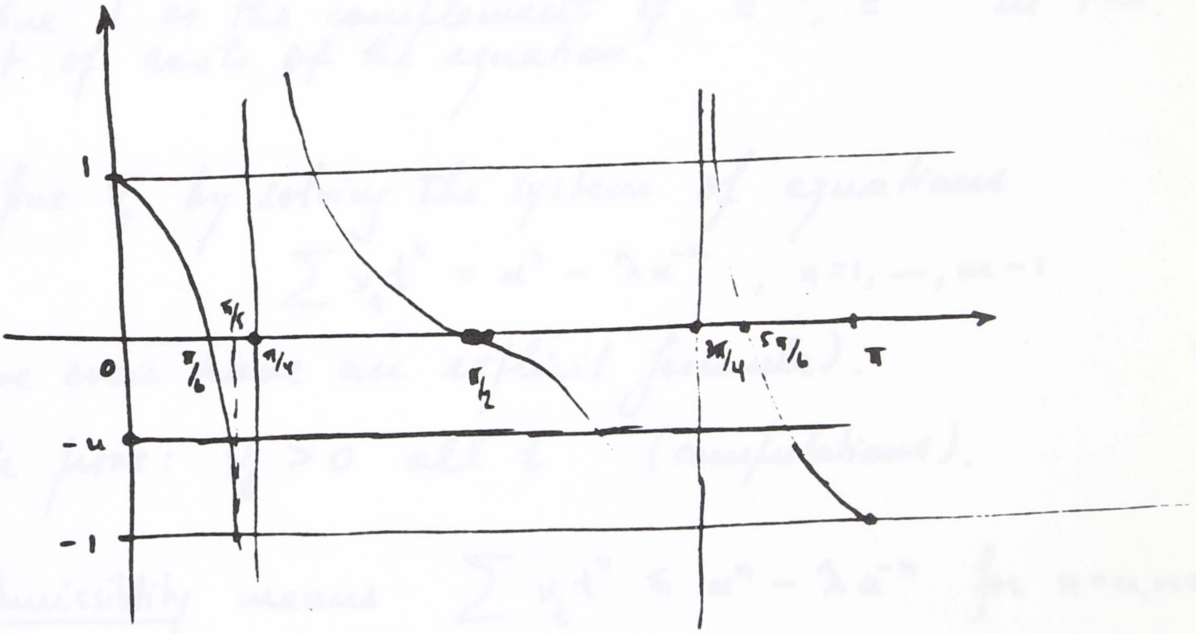
if m is odd : -1 is not a solution



$m = 5$

Take $m = 5$, for instance.

Graph on $0 \le \varphi \le \pi$ of $F(\varphi) = \frac{\cos 3\varphi}{\cos 2\varphi}$



$\cos 2\varphi = 0 \iff 2\varphi = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow \varphi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

$F(\pi/5) = \frac{\cos 3\pi/5}{\cos 2\pi/5} = -1$ (since $3\pi/5 + 2\pi/5 = \pi$)

Given $u \in [0,1)$, I want $F(\varphi) = -u$ 3 times

We know: the first soln. is φ_0, $\frac{\pi}{6} \le \varphi_0 < \frac{\pi}{5}$

and that is the only soln in that interval.

(And the same happens from any m).

So we know:

the eqn. $t^{m+1}+1+u(t^m+t)=0$ has exactly $m+1$ solns. on S^1, and exactly one of the form $e^{i\varphi_0}$, $\frac{\pi}{m+1} \leq \varphi_0 < \frac{\pi}{m}$.

Define T as the complement of $e^{i\varphi_0}$, $e^{-i\varphi_0}$ in the set of roots of the equation.

Define ν_t by solving the system of equations

$$\sum \nu_t t^n = \alpha^n - \lambda\alpha^{-n}, \quad n=1, -, m-1$$

(we even have an explicit formula).

We prove: $\nu_t > 0$ all t (computations).

<u>Admissibility</u> means $\sum_t \nu_t t^n \leq \alpha^n - \lambda\alpha^{-n}$ for $n=m, m+1$,

This is a long computation; it shows: OK if $\alpha \geq \sqrt{3}$, i.e. $g \geq 3$.

Also OK if $\alpha=\sqrt{2}$ and $n=m, m+1$ (but not always of $\alpha=\sqrt{2}$, $n=m+2$, $n \geq m+3$, OK again!)

Up to $\lambda = 130$, bad N's are
51, 52, 53
70, 71, …, 77
98, 99, .. 110
137, ….

guess, bad if: $\frac{\log\lambda}{\log\alpha} \simeq$ integer × 0.4

$N=50$, $g \geq 65$; $N=54$, $g \geq 72$

Se Th 37

(For bad N's, Oesterlé's result is not optimal.)

Now to give $f = 1 + \sum c_n(X^n + X^{-n})$ which will match.

Take $P(X) = \prod_{t\in T}(X-t)$; write

$$P(X)P(X^{-1}) = \sum_{-(m-1)}^{m-1} a_n X^n \qquad , a_n > 0$$

Defn: $f(X) = \frac{1}{a_0} P(X)P(X^{-1})$

$$\therefore \quad c_n = \frac{a_n}{a_0}$$

To compute a_n

$$\begin{cases} a_n = (m-n)\cos n\varphi_0 \sin\varphi_0 + \sin(m-n)\varphi_0 \\ a_0 = m\sin\varphi_0 + \sin m\varphi_0) \end{cases}$$

Finally

$$\frac{1}{2}\sum \nu_t = g = \sum c_n(\lambda\alpha^{-n} - \alpha^n)$$ (by the formulas!)

and we end up with $g = \dfrac{(\lambda-1)\alpha\cos\varphi_0 + \alpha^2 - \lambda}{\alpha^2 - 2\alpha\cos\varphi_0 + 1}$.

[Recall : the zeta fct of a curve is

$$\frac{\prod_{a=1}^{2g}(1-q^{1/2}e^{i\varphi_a}T)}{(1-T)(1-qT)}$$

ν_t is connected to $\prod(1-q^{1/2}tT)^{\nu_t}$

So numberfield analog: replace the zeros of ζ by a measure.]

Remark : If λ large w.r.t. α $(= q^{1/2})$

Then $\varphi \approx 0$, so get $g \approx \frac{\lambda\alpha - \lambda}{(\alpha-1)^2} = \frac{\lambda}{\alpha-1}$

So we expect $g \approx \frac{N}{\sqrt{q}-1}$.

A better approx comes from $\varphi \simeq \frac{\pi}{m}$, $m \sim \frac{\log \lambda}{\log \alpha}$.

Then: for large λ,

$$g \geq \frac{\lambda}{\alpha-1} - \frac{\pi^2}{2}\frac{\alpha(\alpha+1)}{(\alpha-1)^3}\frac{\lambda}{(\log\lambda)^2} + O\left(\frac{\lambda}{(\log\lambda)^3}\right)$$

11/21 **We look at $q=2$ with varying g**

① **Upper bounds for N**

Use the explicit formula

given $f(\theta) = 1 + \sum c_n 2\cos n\theta$, $c_n \geq 0$, $f(\theta) \geq 0$,

we have
$$g \geq (N-1)\sum c_n q^{-n/2} - \sum c_n q^{n/2}.$$

Start with the example
$$f(\theta) = \frac{1}{c}\sum\left(1 + 2x_1\cos\theta + \cdots + 2x_m\cos m\theta\right)^2$$
$$x_i \geq 0, \quad c = 1 + 2x_1^2 + \cdots + 2x_m^2.$$

1st choice: $x_1 = 1$, $x_2 = 0.7$, $x_3 = 0.2$

gives $N \leq 0.83\,g + 5.35$

$g=1$ → not good $N \leq 6$ ($N=5$ is best)

$g=2,3,\ldots,11$ ⇒ bound given (except for $g=7$, $N \leq 11$, and $N=10$ is best).

e.g., $g=5 \Rightarrow N \leq 4.15 + 5.35 \leq 9.50 \longrightarrow N \leq 9$ and we'll construct $N=9$ curves later.

<u>2nd choice</u>: $x_1 = 1.05$, $x_2 = 0.8$, $x_3 = 0.4$

$$N \le 0.766g + 5.97$$

for $g = 13, \dots, 20$ gives same as Oesterlé

<u>3rd choice</u>: 1, 0.8, 0.6, 0.4, 0.1

$$N \le 0.6272g + 9.562$$

for $g = 50$, $N \le 31.36 + 9.562 = 40.9 \cdots$

so $\underline{N \le 40}$.

This justifies all the upper bounds on the table except for $\underline{g=7}$. tables: see pp. SeTh 38b, 38c

$\underline{g=7}$ * bound given by explicit fla is 11

<u>Theorem</u>: A curve with $N = 11$ does not exist.

<u>Proof</u>: Let C be such a curve.

What is its zeta function?

Eigenvalues of Frob are $\pi_1, \bar{\pi}_1, \pi_2, \bar{\pi}_2, —, \pi_7, \bar{\pi}_7$.

We'll know ζ if we know N over $F_2, \dots, F_{2^7}$; if a_d = # closed pt's of deg d.

<u>Need</u>: $a_1, a_2, —, a_7$.

<u>Know</u>: $a_1 = 11$

Maximal number of points of a curve of genus g over the field $\mathbb{F}_2$

g	Max. nber	g	Max. nber	g	Max. nber
0	3	10	12 or 13	20	19, 20 or 21
1	5	11	13 or 14	21	21
2	6	12	14 or 15	…	
3	7	13	14 or 15	39	33
4	8	14	15 or 16	…	
5	9	15	17	50	40
6	10	16	16, 17 or 18		
7	10	17	17 or 18		
8	11	18	18 or 19		
9	12	19	20		

Bounds for N = Max. Nber

$N \leq 0.83\,g + 5.35$

$N \leq 0.766\,g + 5.97$

$N \leq 0.6272\,g + 9.562$

Upper bound for the number of points of a curve of genus g over the field F_2

This upper bound is the one obtained by the " <u>explicit formula</u> " using Oesterlé's trigonometrical polynomial.

---

g :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
N ≤	5	6	7	8	9	10	11	11	12	13	14	15	15	16	17	18
g :	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
N ≤	18	19	20	21	21	22	23	23	24	25	25	26	27	27	28	29
g :	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
N ≤	29	30	31	31	32	33	33	34	35	35	36	37	37	38	38	39
g :	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
N ≤	40	40	41	42	42	43	43	44	45	45	46	47	47	48	48	49
g :	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
N ≤	50	50	51	51	52	53	53	54	54	55	56	56	57	57	58	59
g :	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96
N ≤	59	60	60	61	62	62	63	63	64	65	65	66	66	67	68	68
g :	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112
N ≤	69	69	70	70	71	72	72	73	73	74	75	75	76	76	77	77
g :	113	114	115	116	117	118	119	120								
N ≤	78	79	79	80	80	81	82	82								

Let $$f = \frac{25}{109}(1+\cos\theta)(1+\frac{6}{5}\cos\theta + \frac{6}{5}\cos 2\theta)^2$$

$$= 1 + \sum 2c_n \cos n\theta$$

$$c_1 = \frac{98}{109}, \ \dots, \ c_5 = \frac{9}{218}$$

We have

$$\sum_{d=2}^{5} d a_d \sum_{n \equiv 0 (d)} c_n q^{-n/2} \leq g + \sum c_n q^{-n/2} - (N-1) \sum c_n q^{n/2}$$

$$0.743 a_2 + 0.408 a_3 + 0.165 a_4 + 0.036 a_5 \leq 7 + 4.577 - \\ - 11.506 \\ \leq 0.069$$

This implies $\boxed{a_2 = a_3 = a_4 = 0, \quad a_5 \leq 1.}$

Have a_6 and a_7 to consider, still:

Let $a_5 = \alpha$, $a_6 = \beta$, $a_7 = \gamma$ (so $\alpha = 0$ or 1).

Have $\pi_1, —, \pi_7$, $u_i = \pi_i + \bar{\pi}_i$

Define $f(T) = \prod_{i=1}^{7} (T - u_i) \in \mathbb{Z}[T]$.

this has real roots in the interval $[-2\sqrt{2}, 2\sqrt{2}]$.

Writing $f(T)$ in terms of α, β, γ :

$$f(T) = T^7 + 8T^6 + 21T^5 + 14T^4 - 19T^3 + (\alpha - 20)T^2 +$$
$$+ (8\alpha + \beta - 5)T + 31\alpha + 8\beta + \gamma - 106 .$$

Theorem: Such a polynomial $f_{\alpha,\beta,\gamma}$ with $\alpha, \beta, \gamma \in \mathbb{Z}$, $\alpha = 0$ or 1, has all its roots real and in $[-2\sqrt{2}, 2\sqrt{2}]$ if and only if $(\alpha, \beta, \gamma) = (0, 11, 22)$.

[Sturm → condition for $f_{\alpha\beta\gamma}$ to have roots in some interval]

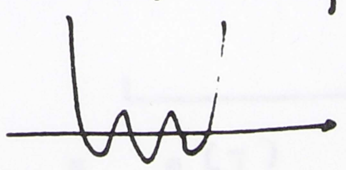
$f_{\alpha\beta\gamma}$ all roots real ⟹ derivative also has real roots

So first check that $f'''(T)$ does have four real roots.

Have $\alpha = 0$ or $\alpha = 1$.

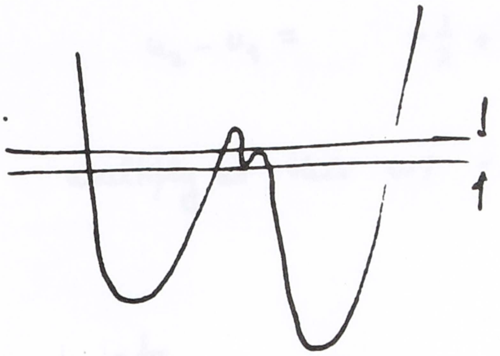
$\boxed{\alpha = 0}$ take $f''(T)$ is ok.

know $f'(T)$ up to translation



to adjust the constant, look at highest min, lowest max

E.g.



constant must make x-axis lie here

This gives : $\begin{cases} \text{if } \alpha = 1, & 5 \leq \beta \leq 10 \\ \text{if } \alpha = 0 & 9 \leq \beta \leq 13 \end{cases}$

This gives only eleven cases : look one-by-one at $f(T)$, in the same way.

Find $\alpha = 1$ not possible.

$\alpha = 0, \beta = 11, \gamma = 22$ works (by a hair). ☑

(Conditions are $\gamma \leq 22, \gamma \geq 22, \gamma \leq 22, \gamma \geq 22, \gamma \leq 25, \gamma \geq 18, \gamma \geq -9650$).

We find $f = T^7 + 8T^6 + 21T^5 + 14T^4 - 19T^3 - 20T^2 + 6T + 4$

$$= \underbrace{(T+2)(T^2+2T-2)}_{=\ g(T)} \cdot \underbrace{(T^2+T-1)(T^2+3T+1)}_{h(T)}$$

then g, h generate $\mathbb{Z}[T]$

So if u_1, u_2, u_3 roots of g, u_4, u_5, u_6, u_7 roots of h, then *every* $u_1 - u_4, \ldots, u_1 - u_7, u_2 - u_4, \ldots, u_3 - u_7$ is a *unit*.

e.g. , $u_2 = -1 + \sqrt{3}$ $u_4 = \frac{-1+\sqrt{5}}{2}$

$$u_2 - u_4 = -\frac{1}{2} + \sqrt{3} - \frac{\sqrt{5}}{2}$$

multiply be this w/ $-\sqrt{3}$ $\longrightarrow \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^2 - 3 = \frac{3\sqrt{5}}{2} - 3$

$$= \frac{-3+\sqrt{5}}{2} \text{ unit.}$$

etc.

So $f = g.h$, g & h generate $\mathbb{Z}[T]$, and this is impossible for a Jacobian, so the curve C does not exist. ▨

—————— / —————— / ——————

Next unknown case is $g=10$: analogous method on a computer gave 100 or 200 polynomials...

—————— // —————— // ——————

Construction of Examples

We want to construct curves

g	0	1	2	3	4	5	6	7	8	9	10	....
N	3	5	6	7	8	9	10	10	11	12	12 or 13	

bd = 13

∴ have curve = 12

Exercise: Prove that minimum no. of points is 3 when $g=0$, 1 when $g=1$, 0 when $g \geq 2$!

Se Th 41

$g=0$ is no fun

$g=1$ we've seen, $g=2$ too

Formulas:
$$\begin{aligned} g=1: &\quad y^2+y=x^3+x \\ g=2: &\quad y^2+y=\frac{x^2+x}{x^3+x+1} \end{aligned}$$

$g \underline{=3}$ if hyperelliptic, at most 6 pts (in general, the hyperelliptic curves have too few pts).

if not, it is a plane quartic in $\mathbb{P}^2$, which itself has 7 points!

So take $[x,y,z]$ homog. coords, find a polyn which passes through all pts:

get $x^3y+y^3z+z^3x+x^2y^2+y^2z^2+z^2x^2+x^2yz+y^2zx=0$

need to prove: <u>nonsingular</u> (hence $g=3$) and goes through all pts.

(since $x,y,z \in \mathbb{F}_2 \Rightarrow x^2=x$, etc., so just

$$\cancel{xy}+yz+zx+\cancel{xy}+yz+\cancel{zx}+\cancel{xyz}+xyz=0\,!)$$

To check it's nonsingular, need only check irreducibility. For that: exists an automorphism of $\mathbb{P}_2$ of order 7 fixing the curve.

This implies irreducibility.

This is a <u>twist</u> of the Klein curve

$G = SL_3(\mathbb{F}_2) = GL_2(\mathbb{F}_2)$ order 168

G acts on $\mathbb{P}_2$; to find inv. polynomials

do:

$$Q_4(x,y,z) = \frac{\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^8 & y^8 & z^8 \end{vmatrix}}{\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix}}$$ has degree 4

$\to$ = product of all the linear forms

$\to$ this gives the Klein curve

This has no rat'l point.

If we twist it by a $C_7 \subset G$, we get the curve above. (i.e., wrt $\begin{array}{c}\mathbb{F}_{2^7} \\ \mid C_7 \\ \mathbb{F}_2\end{array}$)

For <u>$g=4$</u>: if not hyperelliptic, can. embedding in $\mathbb{P}_3$ then curve = inters. of surfaces

$$\text{curve} = (\text{quadric } \underbrace{\text{surface}) \cap (\text{cubic}}_{\substack{\text{transversal inters,} \\ \text{but surfaces can have} \\ \text{singularities.}}} \text{ surface})$$

if quadric $= \mathbb{P}_1 \times \mathbb{P}_1$, 9 pts — so not obvious at once we can get 8.

In $\mathbb{P}_1 \times \mathbb{P}_1$, the curve we want has affine eqn:

$$x^2y^3 + x^3y^2 + xy^3 + x^3y + x^2y^2 + x^2 + y^2 + 1 = 0.$$

After $g=4$, explicit construction is not practical. Use CFT instead.

Start from C_1, known.

Try to find

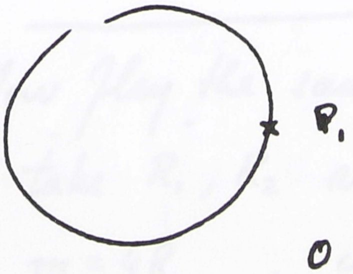
$$\begin{tikzcd} C_2 \arrow[d, "\text{abelian extn.}"] \\ C_1 \end{tikzcd}$$

With "little" ramification ($g(C_2)$ small)

and s.t. "many" rational points split.

<u>1st case</u>: $C_1 = \mathbb{P}_1$, $g = 0$

Choose $Q \in \mathbb{P}_1$ with $\deg Q = 3$, so res. field $= \mathbb{F}_8$.



$\mathbb{F}_8^\times$ = cyclic order 7.

Take $\mathfrak{m} = Q$, look at $Cl_\mathfrak{m}$.

$$0 \longrightarrow \mathbb{F}_8^\times \longrightarrow Cl_\mathfrak{m} \xrightarrow[\deg]{} \mathbb{Z} \longrightarrow 0$$

CFT: finite quotients of $Cl_\mathfrak{m}$ describe the ab. extns w/ cond. $\leq Q$.

Choose a rat'l point $R \in \mathbb{P}_1(\mathbb{F}_2)$.

$R \longmapsto Cl_m$, deg $R = 1$

So let G = quotient of Cl_m by $\langle R \rangle \cong \mathbb{F}_8^{\times}$.

So corresp to

$$\begin{tikzcd} C \arrow[d, "⑦"] \\ \mathbb{P}_1 \end{tikzcd}$$

Frob R is killed ⟹ R splits completely giving 7 pts)
(and ⟹ no const. field extn.)

$$2g(C) - 2 = 7(-2) + \underbrace{6 \cdot 3}_{\text{sum of Artin conductors}}$$

$$2g = 2 - 14 + 18 = 6$$

So $\boxed{\boxed{g = 3}, N = 7}$

Now play the same game as follows:

take R_1, R_2 rat'l points on $\mathbb{P}_1$

$m = 4.R_2$, construct G as before, $G = Cl_m / \langle R_1 \rangle$

$\cong$ local group mod 4.

get: $G \cong$ local gp mod $4R_2$

$$= \{1+\alpha_1 t + \cdots + \alpha_3 t^3\} \big/ 1+t^4$$

$$\cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

So find
$$\begin{tikzcd} C \arrow[d, "G"] \\ \mathbb{P}_1 \end{tikzcd}$$
s.t. : $\begin{cases} R_1 \text{ splits completely into 8 pts (rat'l)} \\ R_2 \text{ is completely ramified} \to 1 \text{ pt which is rat'l.} \end{cases}$

$$2g-2 = 8(-2) + \sum f_\chi$$

Characters on G

$\chi = 1$: $f = 0$

χ trivial on t^3 : $f = 2$

χ trivial on t^2 : $f = 3$ twice

——— : $f = 4$ four times

So $2g-2 = -16 + 2 + 6 + 16 = 8 \implies \boxed{g=5}$.

So this gives the curve we need for $g=5$. ☒

11/26 $(q=2$, cont.) See table, pp. SeTh 43b, 43c

Construction

Start from sth you know, say $g=0$, 3 pts, C_0

$$\begin{tikzcd} C \arrow[d, "\text{abelian}"] \\ C_0 \end{tikzcd}$$

give: conductor, group G, n pts in C_0 splitting completely.

$g=50$ $N=40$? $40=8\times 5$

So want C_0 w/ $g_0=1$, ell curve $y^2+y=x^3+x$ ...) w/ 5 rat'l pts.

Want a covering of degree 8 in which all five pts split completely.

G = of type $(2,2,2) = (2)\times(2)\times(2)$.

First make a $(2,\dots,2)$ extension of rank 8; then $P_1,\dots,P_5$ give Frob. elements.

$(2,\dots,2)=\mathbb{F}_2^8$ has a quotient of dim 3 ($(2,2,2)$) in which $P_1, —, P_5 \to 0 \mid \Longrightarrow$ pts split completely.

Choose a pt P_7, of degree 7 $(=8-1)$

(# pts over $\mathbb{F}_{2^7}$ is $\geq 2^7+1-2\cdot 2^{7/2} > 129-32 > 5$)

Choose conductor $\underline{m} = 2P_7$

TABLE

of curves of low genus over $\underset{\sim}{F}_2$ having many points

Each curve C is obtained as an abelian covering $C \rightarrow C_o$ of a curve C_o of lower genus, occuring earlier in the table (or of genus 0).

The table gives :

the genus g of C;

the number N of rational points of C (I underline N if it is maximal for the corresponding genus);

the genus g_o of C_o ;

the conductor $\underline{m}$ of $C \rightarrow C_o$ (I write $\underline{m}$ as $aP_1 + a'P_1' + bP_2 + .$ where P_1, P_1', P_2 ... are distinct closed points of C_o of degrees 1,1,2,...);

the Galois group G of the covering $C \rightarrow C_o$ (a cyclic group of order m is denoted by (m));

the number n of rational points of C_o which split completely ir

the number r of rational points of C_o which are totally ramifie in C.

For all the cases considered in the table, we have $N = r + |G|n$.

---

g	N	g_o	$\underline{m}$	G	n	r
1	$\underline{5}$	0	$4P_1$	(2)	2	1
2	$\underline{6}$	0	$2P_3$	(2)	3	0
3	$\underline{7}$	0	P_3	(7)	1	0
4	$\underline{8}$	1	$2P_1 + 4P_1'$	(2)	3	2
5	$\underline{9}$	0	$4P_1$	(2)×(4)	1	1
6	$\underline{10}$	1	$2P_5$	(2)	5	0
7	$\underline{10}$	1	$2P_6$	(2)	5	0
8	$\underline{11}$	2	$2P_1 + 2P_4$	(2)	5	1
9	$\underline{12}$	2	$2P_6$	(2)	6	0

<u>Table</u> (continued)

g	N	g_0	$\underline{m}$	G	n	r
10	12	2	$2P_7$	(2)	6	0
11	13	3	$12P_1$	(2)	6	1
12	14	0	$P_3 + P'_3$	(7)	2	0
13	14	3	$2P_3 + 2P_5$	(2)	7	0
14	15	0	P_4	$(3)\times(5)$	1	0
15	$\underline{17}$	1	$10P_1$	$(2)\times(2)$	4	1
16	16	4	$2P_9$	(2)	8	0
17	17	0	$5P_1$	$(2)\times(8)$	1	1
18	18	5	$2P_9$	(2)	9	0
19	$\underline{20}$	1	$2P_6$	$(2)\times(2)$	5	0
20	19	2	0	(19)	1	0
21	$\underline{21}$	0	$P_2 + P_3$	$(3)\times(7)$	1	0
39	$\underline{33}$	1	$12P_1$	$(2)\times(2)\times(2)$	4	1
50	$\underline{40}$	1	$2P_7$	$(2)\times(2)\times(2)$	5	0

Harvard, November 1985

J.-P. Serre

look at $Cl_{\mathfrak{m}}$:

$$0 \longrightarrow L_{\mathfrak{m}} \longrightarrow Cl_{\mathfrak{m}} \longrightarrow Cl_0 = \mathbb{Z}/(5) \times \mathbb{Z} \longrightarrow 0$$

$$L_{\mathfrak{m}} = \mathbb{F}_{2^7}[[x]]^{\times} / \{1 + (x^2)\} = \mathbb{F}_{2^7}^{*} \times \mathbb{F}_{2^7}$$

$$Cl_{\mathfrak{m}} / 2Cl_{\mathfrak{m}} = \mathbb{F}_{2^7} \times \mathbb{Z}/2$$ of type $(2, \ldots, 2)$, rank 8.

$$\begin{tikzcd} & \tilde{C} \arrow[dl] \arrow[dd, ": \text{max'l } (2,2,\ldots,2) \text{ extension w/ cond} \le 2P_7."] \\ C \arrow[dr, "(2,2,2)"'] & \\ & C_0 \end{tikzcd}$$

(this includes a const. field extn.)

where the 5 pts split. (I.e., choose a quotient of type (2,2,2) such that pts split)

$$2g - 2 = 2^3 (2g_0 - 2) + \sum_{\chi \neq 1} \deg f_\chi$$

χ = char of order 2 so cond $= 2P_7$ or 0 but can't be zero so $f_\chi = 14$.

$$2g - 2 = 8 \times 0 + (8-1) \times 14$$

$$g = 1 + 7 \times 7 = 50.$$

With $2P_6$, would get $\overset{\text{extn } (2,2) \text{ w/}}{} \; 4\cdot 5 = 20$ pts and

$$2g-2 = 0 + (4-1)\times 12$$

so $\boxed{g=19}$.

With $2P_5$, get extn. of deg 2, w/

$$2g-2 = 0 + 10 \longrightarrow \boxed{g=6,\ N=10}$$

$\boxed{g=13}$ $N=14$, $\underset{\text{w/ 7 pts}}{g_0=3}$, $|G|=2$

$$2g-2 = 2(6-2) + \deg f$$

$$24 = 8 + \deg f$$

So $\deg f = 16$. Must take $\underline{m} = 2P_8$ (contra table!)

table : $\underline{m}$: $2P_3 + 2P_5$

must check there is a pt of deg 3 (and one of deg 5)

take $L_{\underline{m}}$, $\mathbb{Z}$

so $rk = 1 + \deg \underline{m}$
$= 9$

so mod out as before.

• Recall construction :

$$\begin{tikzcd} C_3 \arrow[d, "7"'] \\ C_0 = P_1 \end{tikzcd}$$

res field = $\mathbb{F}_8$ ramif at a $P_3 \in C_0$

so C_3 has a pt of deg 3

Seth 45

$g = \underline{20 \quad N = 19}$, $g_0 = 2$, $h = 19$

$$Cl_0 = \underset{\underset{P_1}{\nearrow}}{\mathbb{Z}} \times \mathbb{Z}/(19)$$

Split so that $P_1 \to$ generator. Get an extn. by the quotient.

Then $2g - 2 = 19(22 - 2) \longrightarrow g = 1 + 19 = 20$.

For very large g (say $g \sim 10^{10}$) we only get N of about the same order (say $0.2 \cdot 10^{10}$) using the class field towers, as before.

For $10 \leq g \leq 20$, one might be able to fill some of the gaps.