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The optimal constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group

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Abstract. We determine the best (optimal) constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group and the non-negative functions for which equality holds.

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1. Introduction

The goal of this note is to determine the best (optimal) constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group and the non-negative extremal functions, *i.e.*, the functions for which equality holds. Alternatively, this is equivalent to finding the Yamabe constant of the standard quaternionic contact structure of the sphere.

The proof is inspired by the case $\lambda = Q - 2$ of the recent remarkable paper of Frank and Lieb [11] who obtained the sharp form of the Hardy-Littlewood-Sobolev (HLS) inequalities [10] with exponent λ, $0 < \lambda < Q$, on the Heisenberg group $\mathbb{H}^n$ of homogeneous dimension $Q = 2n + 2$ and the standard CR unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, together with their limiting cases $\lambda = 0$ and $\lambda = Q$ (see [22] for the Euclidean version and [11] for other results in the CR setting). Previously, Branson, Fontana and Morpurgo [6] settled the limiting case $\lambda = 0$ and pointed out that the old idea of Szegö [26] and Hersch [20] can be used to find the sharp form of the logarithmic HLS inequality and its dual Onofri's inequality on the Heisenberg group. This center of mass technique and the conformal invariance was used earlier by Onofri [25] on the round two dimensional sphere (see also [2] and [12]) and Chang and Yang [8] who extended it to higher dimension thereby giving an alternative proof of the Beckner-Onofri's inequality, see [4]. As well known, the case

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$\lambda = Q - 2$ is dual to the L^2 Sobolev embedding inequality, whose sharp constant on the Heisenberg group and the CR sphere was found by Jerison and Lee [19]. In fact, [19] found all non-negative solutions to the CR Yamabe equation that is the Euler-Lagrange functional of the CR Yamabe functional. In comparison, [11] determines the best constant and all functions for which the minimum is achieved, by simplifying parts of [19] while answering a less general question. However, thanks to [11] we have the sharp form of the general Hardy-Littlewood-Sobolev type inequalities.

The conformal nature of the problem we consider is key to its solution. The analysis is purely analytical. In this respect, even though the quaternionic contact (qc) Yamabe functional is involved, the qc scalar curvature is used in the proof without its geometric meaning. Rather, it is the conformal sub-laplacian that plays a central role and the qc scalar curvature appears as a constant determined by the Cayley transform and the left-invariant sub-laplacian on the quaternionic Heisenberg group. Note that this method does not give all solutions of the qc Yamabe equation on the quaternionic contact sphere. The complete solution of the latter problem requires some additional very non-trivial argument and it is at this place where the geometric nature of the problem becomes even more important. In the CR setting, the solution of the CR Yamabe problem was achieved with the help of an ingenious divergence formula by Jerison and Lee [19]. The other known sub-Riemannian case is that of the qc Yamabe equation on the seven dimensional standard quaternionic contact sphere [17]. Another relevant result appeared earlier [13], where the sub-Riemannian Yamabe equation was solved in the unifying setting of groups of Iwasawa type under an additional assumption of partial symmetry of the solution. This result can be used at the final stage of all known proofs after such symmetry has been shown to exist. We recall that the groups of Iwasawa type comprise of the complex (="usual"), quaternion and octonian Heisenberg groups, which are defined by (1.4) replacing, correspondingly, the quaternions $\mathbb{H}$ with the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonians $\mathbb{O}$.

Given a compact quaternionic contact manifold M of real dimension $4n+3$ with an $\mathbb{R}^3$-valued contact form $\eta = \{\eta_1, \eta_2, \eta_3\}$, *i.e.*, a codimension three horizontal distribution H determined as the kernel of η such that $d\eta_{|_H}$ are the fundamental two forms of a quaternionic hermitian structure $(\mathbf{g}, I_1, I_2, I_3)$ on H, $(d\eta_s)_{|H} = 2\mathbf{g}(I_s., .) = 2\omega_s$, $s = 1, 2, 3$, a natural question is to determine the qc Yamabe constant of the conformal class $[\eta]$ of η defined as the infimum

$$\lambda(M, [\eta]) = \inf\left\{\Upsilon(u) : \int_M u^{2^*}\,\mathrm{Vol}_\eta = 1,\ u > 0\right\}, \tag{1.1}$$

where $\mathrm{Vol}_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (\omega_1)^{2n}$ denotes the volume form determined by η. The qc Yamabe functional of the conformal class of η is defined by

$$\Upsilon(u) = \int_M \left(4\frac{Q+2}{Q-2}\,|\nabla u|^2 + S\,u^2\right)\mathrm{Vol}_\eta, \qquad \int_M u^{2^*}\,\mathrm{Vol}_\eta = 1,\ u > 0,$$

denoting by ∇ the Biquard connection [5] of η, and S standing for the qc scalar curvature of (M, η). This is the so called *qc Yamabe constant problem*. In this paper we shall find $\lambda(S^{4n+3}, [\tilde\eta])$, where $\tilde\eta$ is the standard qc form on the unit sphere S^{4n+3}, see (2.1). The question is of course related to the solvability of the qc Yamabe equation

$$\mathcal{L}u \equiv 4\frac{Q+2}{Q-2}\,\triangle u - S\,u = -\,\overline{S}\,u^{2^*-1}, \tag{1.2}$$

where $\triangle$ is the horizontal sub-Laplacian, $\triangle u = \mathrm{tr}^g(\nabla du)$, S and $\overline{S}$ are the qc scalar curvatures correspondingly of (M, η) and $(M, \bar\eta)$, $\bar\eta = u^{4/(Q-2)}\eta$, and $2^* = \frac{2Q}{Q-2}$. Here, and throughout the paper, $Q = 4n+6$ is the homogeneous dimension. The natural question is to find all solutions of the qc Yamabe equation. This is the so called *qc Yamabe problem*, which is equivalent to finding all qc structures conformal to a given structure η (of constant qc scalar curvature) which also have constant qc scalar curvature. As usual the two problems are related by noting that on a compact quaternionic contact manifold M with a fixed conformal class $[\eta]$ the qc Yamabe equation characterizes the non-negative extremals of the qc Yamabe functional.

The $4n+3$ dimensional sphere is an important example of a locally quternionic contact conformally flat qc structure characterized locally in [18] with the vanishing of a curvature-type tensor invariant. From the point of view of the qc Yamabe problem the sphere plays a role similar to its Riemannian and CR counterparts. A solution of the qc Yamabe problem on the seven dimensional sphere equipped with its natural quaternionic contact structure was given in [17] where more details on the qc Yamabe problem can be found. The main result of [17] is the following:

Theorem ([17]). *Let $\tilde\eta = \frac{1}{2h}\eta$ be a conformal deformation of the standard qc-structure $\tilde\eta$ on the quaternionic unit sphere S^7. If η has constant qc scalar curvature, then up to a multiplicative constant η is obtained from $\tilde\eta$ by a conformal quaternionic contact automorphism. In particular, $\lambda(S^7) = 48\,(4\pi)^{1/5}$ and this minimum value is achieved only by $\tilde\eta$ and its images under conformal quaternionic contact automorphisms.*

Another motivation for studying the qc Yamabe equation and the qc Yamabe constant of the qc sphere comes from its connection with the determination of the norm and extremals in a relevant Sobolev-type embedding on the quaternionic Heisenberg group [13] and [27] and [28]. As well known, the sub-Riemannian Yamabe equation is also the Euler-Lagrange equation of the extremals for the L^2 case of such embedding results. Recall the following Theorem due to Folland and Stein [10].

Theorem (Folland and Stein). *Let $\Omega \subset G$ be an open set in a Carnot group G of homogeneous dimension Q and Haar measure dH. For any $1 < p < Q$ there exists $S_p = S_p(G) > 0$ such that for $u \in C^\infty_o(\Omega)$*

$$\left(\int_\Omega |u|^{p^*}\,dH(g)\right)^{1/p^*} \le S_p \left(\int_\Omega |Xu|^p\,dH(g)\right)^{1/p}, \tag{1.3}$$

where $|Xu| = \sum_{j=1}^{m} |X_j u|^2$ *with* $X_1, \dots, X_m$ *denoting a basis of the first layer of* G *and* $p^* = \frac{pQ}{Q-p}$.

Let S_p be the best constant in the Folland-Stein inequality, *i.e.*, the smallest constant for which (1.3) holds.

In [17] we determined *all* extremals, *i.e.*, solutions of the qc Yamabe equation, and the best constant in Folland and Stein's theorem when $p = 2$ in the case of the seven dimensional quaternionic Heisenberg group. In the case of the complex (*i.e.* "usual") Heisenberg group this was done earlier by Jerison and Lee [19] who determined *all* solutions to the CR Yamabe equation on the CR sphere.

Following the idea of [11], the main result of this paper determines the best constant in the Folland and Stein's theorem when $p = 2$ and the functions for which it is achieved in the case of the quaternionic Heisenberg group $\boldsymbol{G}$ of any dimension.

As a manifold $\boldsymbol{G} = \mathbb{H}^n \times \operatorname{Im}\mathbb{H}$ with the group law given by

$$(q', \omega') = (q_o, \omega_o) \circ (q, \omega) = (q_o + q, \omega + \omega_o + 2 \operatorname{Im} q_o \bar{q}), \tag{1.4}$$

where $q,\ q_o \in \mathbb{H}^n$ and $\omega, \omega_o \in \operatorname{Im}\mathbb{H}$. The standard quaternionic contact(qc) structure is defined by the left-invariant quaternionic contact form

$$\tilde{\Theta} = (\tilde{\Theta}_1,\ \tilde{\Theta}_2,\ \tilde{\Theta}_3) = \frac{1}{2}\,(d\omega - q' \cdot d\bar{q}' + dq' \cdot \bar{q}'),$$

where $\cdot$ denotes the quaternion multiplication. The purpose of the present note is to prove the next

Theorem 1.1. a) *Let* $\boldsymbol{G} = \mathbb{H}^n \times \operatorname{Im}\mathbb{H}$ *be the quaternionic Heisenberg group. The best constant in the* L^2 *Folland-Stein embedding inequality* (1.3) *is*

$$S_2 = \frac{\left[2^3\ \omega_{4n+3}\right]^{-1/(4n+6)}}{2\sqrt{n(n+1)}},$$

where $\omega_{4n+3} = 2\pi^{2n+2}/(2n+1)!$ *is the volume of the unit sphere* $S^{4n+3} \subset \mathbb{R}^{4n+4}$. *The non-negative functions for which* (1.3) *becomes an equality are given by the functions of the form*

$$F = \gamma \left[(1 + |q|^2)^2 + |\omega|^2\right]^{-(n+1)}, \qquad \gamma = \text{const}, \tag{1.5}$$

and all functions obtained from F *by translations* (3.2) *and dilations* (3.3).

b) *The qc Yamabe constant of the standard qc structure of the sphere is*

$$\lambda(S^{4n+3}, [\tilde{\eta}]) = 16\,n(n+2)\left[((2n)!)\,\omega_{4n+3}\right]^{1/(2n+3)}. \tag{1.6}$$

These constants are in complete agreement with the ones obtained in [17] and [13] taking into account the next Remark and the well known formulas involving the gamma function

$$\Gamma(n+1) = n!, \qquad \Gamma(z+n) = z(z+1)\dots(z+n-1)\Gamma(z), \qquad n \in \mathbb{N},$$

$$\Gamma(2z) = 2^{2z-1}\,\pi^{-1/2}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) \qquad \text{– the Legendre formula,}$$

$$\omega_m = 2\pi^{(m+1)/2}/\,\Gamma\left((m+1)/2\right) = \begin{cases} \dfrac{2^{(m+2)/2}\pi^{m/2}}{(m-1)!!}, & m\text{-even}\,, \\ \dfrac{2\pi^{(m+1)/2}}{\left(\dfrac{m-1}{2}\right)!}, & m\text{-odd}\,, \end{cases}$$

where ω_m is the volume of the unit m–dimensional sphere in $\mathbb{R}^{m+1}$. Our result partially confirms the Conjecture made after [13, Theorem 1.1]. In addition, the fact that any function of the described form is a solution of Yamabe problem was first noted in [14] in the setting of groups of Heisenberg type. Of course, this class of groups is much wider than the class of groups of Iwasawa type.

Remark 1.2. With the left-invariant basis of Theorem 1.1 the quaternionic Heisenberg group is not a group of Heisenberg type. If we consider it as a group of Heisenberg type then the best constant in the L^2 Folland-Stein embedding theorem is, *cf.* [13, Theorem 1.6],

$$S_2 \;=\; \frac{1}{\sqrt{4n(4n+4)}}\, 4^{3/(4n+6)}\, \pi^{-(4n+3)/2(4n+6)} \left(\frac{\Gamma(4n+3)}{\Gamma((4n+3)/2)}\right)^{1/(4n+6)}.$$

and extremals are given by dilations and translations of the function

$$F(q,\omega) \;=\; \gamma\,\left[(1+|q|^2)^2 \;+\; 16|\omega|^2)\right]^{-(n+1)}, \;(q,\omega)\in \boldsymbol{G}.$$

It is worth pointing that the Yamabe extremals in the sub-Riemannian setting have applications to sharp inequalities in the Euclidean setting. For example, in the paper [29] the extremals are determined of some *Euclidean* Hardy-Sobolev inequalities involving the distance to a $n-k$ dimensional coordinate subspace of $\mathbb{R}^n$. This is achieved by relating extremals on the Heisenberg groups to extremals in the Euclidean setting. In the particular case when $k=n$ one obtains the Caffarelli-Kohn-Nirenberg inequality, see [7], for which the optimal constant was found in [15].

Convention 1.3. We use the following conventions:

- the abbreviation *qc* will stand for quaternionic contact;
- $\boldsymbol{G}$ will denote the qc Heisenberg group;
- $\tilde{\eta}$ will denote the standard qc form on the unit sphere S^{4n+3}, see (2.1). Note that this form is actually twice the 3-Sasakain qc form on S^{4n+3};

- Vol_η will denote the volume form determined by the qc form η, thus $\mathrm{Vol}_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (\omega_1)^{2n}$, see [16, Chapter 8].

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2. The model quaternionic contact structures

In this section we review the standard quaternionic contact structure on the quaternionic Heisenberg group and the $4n+3$-dimensional unit sphere. We will rely heavily on [16], but prefer to repeat some key points in order to make the current paper somewhat self-contained. Besides serving as a background, this section will supply some key numerical constants - the qc scalar curvature and the first eigenvalue of the sub-laplacian of the standard qc form of the sphere. This will be achieved using the conformal sub-laplacian and the properties of the Cayley transform.

First let us recall the quaternionic Heisenberg group [16, Section 5.2]. We remind the following model of the quaternionic Heisenberg group $\boldsymbol{G}$. Define $\boldsymbol{G} = \mathbb{H}^n \times \operatorname{Im}\mathbb{H}$ with the group law given by $(q', \omega') = (q_o, \omega_o) \circ (q, \omega) = (q_o + q, \omega + \omega_o + 2\operatorname{Im} q_o\, \bar{q})$, where $q,\ q_o \in \mathbb{H}^n$ and $\omega, \omega_o \in \operatorname{Im}\mathbb{H}$. In coordinates, with $\omega = ix+jy+kz$ and $q_\alpha = t_\alpha+ix_\alpha+jy_\alpha+kz_\alpha, \alpha = 1, \dots n$, a basis of left-invariant horizontal vector fields $T_\alpha, X_\alpha = I_1T_\alpha, Y_\alpha = I_2T_\alpha, Z_\alpha = I_3T_\alpha, \alpha = 1\dots, n$ is given by

$$\begin{aligned} T_\alpha &= \partial_{t_\alpha} + 2x_\alpha\partial_x + 2y_\alpha\partial_y + 2z_\alpha\partial_z & X_\alpha &= \partial_{x_\alpha} - 2t_\alpha\partial_x - 2z_\alpha\partial_y + 2y_\alpha\partial_z \\ Y_\alpha &= \partial_{y_\alpha} + 2z_\alpha\partial_x - 2t_\alpha\partial_y - 2x_\alpha\partial_z & Z_\alpha &= \partial_{z_\alpha} - 2y_\alpha\partial_x + 2x_\alpha\partial_y - 2t_\alpha\partial_z\,. \end{aligned}$$

The above vectors generate the *horizontal space*, denoted as usual by H. In addition, by declaring them to be an orthonormal basis we obtain a metric on the horizontal space, which is the so called *horizontal metric*. The central (vertical) vector fields ξ_1, ξ_2, ξ_3 are described as follows

$$\xi_1 = 2\partial_x \quad \xi_2 = 2\partial_y \quad \xi_3 = 2\partial_z\,.$$

The standard quaternionic contact form, written as a purely imaginary quaternion valued form $\tilde{\Theta} = i\tilde{\Theta}_1 + j\tilde{\Theta}_2 + k\tilde{\Theta}_3)$, is

$$2\tilde{\Theta} = d\omega - q'\cdot d\bar{q}' + dq'\cdot\bar{q}',$$

where $\cdot$ denotes the quaternion multiplication. The Biquard connection coincides with the flat left-invariant connection on $\boldsymbol{G}$, in particular the qc scalar curvature vanishes.

Following [16], we give another model of the Heiseneberggroup, which is the one we will use in this paper. Let us identify $\boldsymbol{G}$ with the boundary Σ of a Siegel domain in $\mathbb{H}^n \times \mathbb{H}$,

$$\Sigma = \{(q', p') \in \mathbb{H}^n \times \mathbb{H} : \Re\, p' = |q'|^2\},$$

by using the map $(q', \omega') \mapsto (q', |q'|^2 - \omega')$. Since $dp' = q' \cdot d\bar{q}' + dq' \cdot \bar{q}' - d\omega'$, under the identification of $\boldsymbol{G}$ with Σ we have also $2\tilde{\Theta} = -dp' + 2dq' \cdot \bar{q}'$. Taking into account that $\tilde{\Theta}$ is purely imaginary, the last equation can be written also in the following form

$$4\,\tilde{\Theta} = (d\bar{p}' - dp') + 2dq' \cdot \bar{q}' - 2q' \cdot d\bar{q}'.$$

Now, consider the Cayley transform, see [21] and [9], as the map $\mathcal{C} : S \mapsto \Sigma$ from the sphere $S = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$ minus a point to the Heisenberg group Σ, with $\mathcal{C}$ defined by

$$(q', p') = \mathcal{C}\Big((q, p)\Big), \qquad q' = (1+p)^{-1}\, q, \qquad p' = (1+p)^{-1}\,(1-p)$$

and with an inverse map $(q, p) = \mathcal{C}^{-1}\big((q', p')\big)$ given by

$$q = 2(1+p')^{-1}\, q', \qquad p = (1+p')^{-1}\,(1-p').$$

The Cayley transform maps $S^{4n+3} \setminus \{(-1, 0)\}$, $(-1, 0) \in \mathbb{H}^n \times \mathbb{H}$, to Σ since

$$\Re\, p' = \Re \frac{(1+\bar{p})(1-p)}{|1+p\,|^2} = \Re \frac{1-|p|}{|1+p\,|^2} = \frac{|q|^2}{|1+p\,|^2} = |q'|^2.$$

Writing the Cayley transform in the form $(1+p)q' = q$, $(1+p)p' = 1-p$, gives

$$dp \cdot q' + (1+p) \cdot dq' = dq, \qquad dp \cdot p' + (1+p) \cdot dp' = -dp,$$

from where we find

$$\begin{aligned} dp' &= -2(1+p)^{-1} \cdot dp \cdot (1+p)^{-1} \\ dq' &= (1+p)^{-1} \cdot [dq - dp \cdot (1+p)^{-1} \cdot q]. \end{aligned}$$

The Cayley transform is a conformal quaternionic contact diffeomorphism between the quaternionic Heisenberg group with its standard quaternionic contact structure $\tilde{\Theta}$ and the sphere minus a point with its standard structure $\tilde{\eta}$. In fact, by [16, Section 8.3] we have

$$\Theta \stackrel{\text{def}}{=} \lambda \cdot (\mathcal{C}^{-1})^*\, \tilde{\eta} \cdot \bar{\lambda} = \frac{8}{|1+p'\,|^2}\, \tilde{\Theta}.$$

where $\lambda = |1+p\,|\,(1+p)^{-1}$ is a unit quaternion and $\tilde{\eta}$ is the standard contact form on the sphere,

$$\tilde{\eta} = dq \cdot \bar{q} + dp \cdot \bar{p} - q \cdot d\bar{q} - p \cdot d\bar{p}. \tag{2.1}$$

Lemma 2.1. *The qc scalar curvature* $\tilde S$ *of the standard qc structure* (2.1) *on* S^{4n+3} *is*

$$\tilde S = \frac{1}{2}(Q+2)(Q-6) = 8n(n+2). \tag{2.2}$$

Remark 2.2. Notice that the standard qc contact form we consider here is twice the 3-Sasakian form on S^{4n+3}, which has qc scalar curvature equal to $16n(n+2)$ [16].

Proof. Let us introduce the functions

$$\begin{aligned} h &= \frac{1}{16}|1+p'|^2 = \frac{1}{16}\Big[(1+|q'|^2)^2+|\omega'|^2\Big], \\ &\qquad (q',p')\in\Sigma\subset\mathbb{H}^n\times\mathbb{H}, \quad p'=|q'|^2+\omega', \\ &\text{and} \\ \Phi &= (2h)^{-(Q-2)/4} = 8^{(Q-2)/4}\Big[(1+|q'|^2)^2+|\omega'|^2\Big]^{-(Q-2)/4}, \end{aligned} \tag{2.3}$$

so that now we have

$$\Theta = \frac{1}{2h}\tilde\Theta = \Phi^{4/(Q-2)}\tilde\Theta.$$

With the help of [16, Section 5.2] a small calculation shows that the sub-laplacian of h w.r.t. $\tilde\Theta$ is given by $\triangle h = \frac{Q-6}{4} + \frac{Q+2}{4}|q'|^2$ and thus Φ is a solution of the qc Yamabe equation on the Heisenberg group Σ

$$\triangle\Phi = -K\,\Phi^{2^*-1}, \qquad K=(Q-2)(Q-6)/8, \tag{2.4}$$

where $\triangle$ is the sub-laplacian on the quaternionic Heisenberg group. Denoting with $\mathcal{L}$ and $\tilde{\mathcal{L}}$ the conformal sub-laplacians of Θ and $\tilde\Theta$, respectively, we have

$$\Phi^{-1}\mathcal{L}(\Phi^{-1}u) = \Phi^{-2^*}\tilde{\mathcal{L}}u.$$

We remind, *cf.* [5] and [17], that for a qc contact form Θ the conformal sublaplacian is,

$$\mathcal{L} = a\triangle_\Theta - S_\Theta, \qquad a = 4\frac{Q+2}{Q-2},$$

where $\triangle_\Theta$ is the sub-laplacian associated to Θ, *i.e.*, $\triangle_\Theta u = \operatorname{tr}(\nabla^\Theta du)$–the horizontal trace of the Hessian of u, using the Biquard connection ∇^Θ of Θ, and S_Θ is the qc scalar curvature of Θ. Thus, letting $u=\Phi$ we come to $\mathcal{L}(1) = \Phi^{1-2^*}\tilde{\mathcal{L}}\Phi$, which shows $-S_\Theta = -4\frac{Q+2}{Q-2}K$. The latter is the same as that of $\tilde\eta$ since the two structures are isomorphic via the diffemorphism $\mathcal{C}$, or rather its extension, since we can consider $\mathcal{C}$ as a quaternionic contact conformal transformation between the whole sphere S^{4n+3} and the compactification $\hat\Sigma\cup\infty$ of the quaternionic Heisenberg group by adding the point at infinity, *cf.* [17, Section 5.2]. $\square$

We turn to the task of determining the first eigenvalue of the sub-laplacian on S^{4n+3}. In fact, we shall need only the fact that the restriction of every coordinate function is an eigenvalue. The proof of this fact can be seen directly without any reference to the Biquard connection. Alternatively, we can invoke [1] where spherical harmonics are studied on the homogeneous space (sphere) $\mathbb{S} = K/M$ which is 1-quasiconformal [3] to the group of Iwasawa type N via the Cayley transform, where $G = NAK$ is the Iwasawa decomposition of the rank one simple Lie group G and M is the centralizer of A in K. Since this will require setting a lot of notation unnecessary for the current goals, we prefer to use a result from [16].

Lemma 2.3. *If ζ is any of the (real) coordinate functions in $\mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}$, then*

$$\tilde{\triangle}\zeta = -\lambda_1\zeta, \quad \lambda_1 = \frac{\tilde{S}}{Q+2} = 2n \tag{2.5}$$

for the horizontal trace of the Hessian, where $\tilde{\triangle}$ is the sub-laplacian of the standard qc form $\tilde{\eta}$ of S^{4n+3}.

Proof. It is enough to furnish a proof for the sub-laplacian on the 3-Sasakain sphere since the two qc forms differ by a constant. We can see that every ζ of the considered type is an eigenfunction by using [16, Corollary 6.24]. It will be enough to see this for one coordinate function since the sub-laplacian on the sphere is rotation invariant. Thus, let us take $\zeta = t_1$. Notice that ζ is a quaternionic pluri-harmonic function [16, Definition 6.7] since it is the real part of the anti-regular function $t_1 + ix_1 - jy_1 - kz_1$. Thus, its restriction to the 3-Sasakain sphere is the real part of an anti-CRF function. Therefore we apply [16, Corollary 6.24] which gives $\mathrm{tr}(\nabla d\zeta) = 4\lambda n$ for the sub-laplacian of the 3-sasakain qc structure on the sphere. Next, we compute λ, which can be found in [16, Theorem 6.20]. Using that the sphere is 3-Sasakian it follows the Reeb vector fields are obtained from the outward pointing unit normal vector N as follows, $\xi_1 = iN$, $\xi_2 = jN$ and $\xi_3 = kN$, where for a point on the sphere we have $N(q) = q \in \mathbb{H}^{n+1}$. Therefore $\lambda = -t_1 = -\zeta$. To make this more apparent notice that only the first four coordinates of N matter. So, if we assume $n = 0$ (*i.e.* $N(q) = q \in \mathbb{H}^1$) we have $iN = -x + it + ky - jz$, $jN = -y + iz + jt - kx$ and $kN = -z - iy + jx + kt$, so we need to sum the *real dot product* of these vectors with i, j and k, respectively, which gives $-t$. Thus, for the sub-laplacian on the 3-Sasakian sphere we have

$$\mathrm{tr}(\nabla d\zeta) = -4n\zeta,$$

where ζ is the restriction any of the coordinate functions of $\mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}$. Since the qc contact form $\tilde{\Theta}$ is twice the 3-Sasakain qc contact form on the sphere it follows $\tilde{\triangle}$ is $1/2$ of the 3-Sasakain sub-laplacian. Thus

$$\tilde{\triangle} = -2n\zeta,$$

which shows $\lambda_1 = 2n = \frac{1}{2}(Q-6) = \tilde{S}/(Q+2)$. $\square$

We finish this section with a simple lemma which will be used to relate the various explicit constants. Its claim also follows from the conformal invariance of the Yamabe equation, but we prefer to give a proof, which is independent of the notion of qc scalar curvature. We recall, see [16, Chapter 8], that Vol_η will denote the volume form determined by the qc form η, thus $\mathrm{Vol}_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (\omega_1)^{2n}$. Also, for a qc form η we let $|\nabla^\eta F|^2 = \sum_{\alpha=1}^{4n} |dF(e_\alpha)|^2$ be the square of the length of the horizontal gradient of a function F taken with respect to an orthonormal basis of the horizontal space $H = Ker\ \eta$ and the metric determined by η.

Lemma 2.4. *Let* $F \in \overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})$, *cf.* (3.1), *be a positive function with* $\int_{\boldsymbol{G}} F^{2^*} \mathrm{Vol}_{\tilde\Theta} = 1$. *Then we have*

$$\int_{\boldsymbol{G}} a|\nabla^{\tilde\Theta} F|^2 \,\mathrm{Vol}_{\tilde\Theta} = \int_{S^{4n+3}} \left(a|\nabla^{\tilde\eta} g|^2 + \tilde S g^2 \right) \mathrm{Vol}_{\tilde\eta}, \qquad a = 4(2^* - 1), \tag{2.6}$$

and

$$\int_{\boldsymbol{G}} g^{2^*} \mathrm{Vol}_{\tilde\eta} = 1,$$

where

$$g = \mathcal{C}^*(F\Phi^{-1}), \tag{2.7}$$

and, as before, $\mathcal{C} : S^{4n+3} \to \Sigma$ *is the Cayley transform,* $\Theta = \Phi^{4/(Q-2)}\tilde\Theta$, *cf.* (2.3).

Remark 2.5. Notice that $\mathrm{Vol}_{\tilde\Theta} = 2^{-3}\,(2n)!\,dH$, where dH is the Lebesgue measure in $\mathbb{R}^{4n+3}$, which is a Haar measure on the group.

Proof. It will be convenient for the remaining of this proof to denote by small letters the pull-back by the Cayley transform of a function denoted with the corresponding capital letter. Thus, $f = \mathcal{C}^*F = F \circ \mathcal{C}$, $\phi = \mathcal{C}^*(\Phi)$ and $g = f\phi^{-1}$. By the conformality of the qc structures on the group and the sphere we have

$$\mathrm{Vol}_\Theta = \Phi^{2^*} \mathrm{Vol}_{\tilde\Theta} \tag{2.8}$$

By (2.8) we have $F^{2^*}\mathrm{Vol}_{\tilde\Theta} = f^{2^*}\phi^{-2^*}\mathrm{Vol}_{\tilde\eta}$, which motivates the definition (2.7) of the function g which is defined on the sphere and should be regarded as corresponding to the function F. Thus, we have for example $F = G\Phi$. By definition we have

$$\int_{\boldsymbol{G}} g^{2^*} \mathrm{Vol}_{\tilde\eta} = 1,$$

so our next task is to see that the Yamabe integral is preserved

$$\int_{\boldsymbol{G}} |\nabla^{\tilde\Theta} F|^2 \,\mathrm{Vol}_{\tilde\Theta} = \int_{S^{4n+3}} \left(|\nabla^{\tilde\eta} g|^2 + K g^2 \right) \mathrm{Vol}_{\tilde\eta}. \tag{2.9}$$

Here is where we shall exploit that a power of the conformal factor of the Cayley transform is a solution of the Yamabe equation. Let $\langle \nabla^\Theta \Phi, \nabla^\Theta G \rangle =$

$\sum_{a=1}^{4n}(e_a\Phi)\,(\mathbf{e}_a G)$ where $\{e_1,\dots,e_{4n}\}$ is an orthonormal basis of the horizontal space H. Using the divergence formula from [16, Section 8.1] we find

$$\begin{aligned}\int_{\boldsymbol{G}} |\tilde{\nabla}^{\Theta} F|^2 \,\mathrm{Vol}_{\tilde{\Theta}} &= \int_{\boldsymbol{G}} |\nabla^{\tilde{\Theta}}(G\Phi)|^2 \,\mathrm{Vol}_{\tilde{\Theta}}\\ &= \int_{\boldsymbol{G}} \Big(G^2|\nabla^{\tilde{\Theta}}\Phi|^2 + \Phi^2|\nabla^{\tilde{\Theta}}G|^2 + \Big\langle \Phi\nabla^{\tilde{\Theta}}\Phi,\, \nabla^{\tilde{\Theta}}G^2\Big\rangle\Big)\,\mathrm{Vol}_{\tilde{\Theta}}\\ &= \int_{\boldsymbol{G}} \Big(\Phi^2|\nabla^{\tilde{\Theta}}G|^2 - G^2\Phi\triangle_{\tilde{\Theta}}\Phi\Big)\mathrm{Vol}_{\tilde{\Theta}}.\end{aligned}$$

Now, the Yamabe equation (2.4) gives

$$\begin{aligned}\int_{\boldsymbol{G}} |\nabla^{\tilde{\Theta}} F|^2 \,\mathrm{Vol}_{\tilde{\Theta}} &= \int_{\boldsymbol{G}} \Big(\Phi^2|\nabla^{\tilde{\Theta}}G|^2 + KG^2\Phi^{2^*}\Big)\,\mathrm{Vol}_{\tilde{\Theta}}\\ &= \int_{S^{4n+3}} \Big(\phi^{2-2^*}(|\nabla^{\tilde{\Theta}}G|\circ\mathcal{C})^2 + Kg^2\Big)\,\mathrm{Vol}_{\tilde{\eta}}\\ &= \int_{S^{4n+3}} \Big(|\nabla^{\tilde{\eta}}g|^2 + Kg^2\Big)\,\mathrm{Vol}_{\tilde{\eta}},\end{aligned}$$

taking into account that $\mathcal{C}$ is a qc conformal map. Finally, a glance at (2.4) and (2.2) shows $\tilde{S}/K = 4(2^*-1) = (4(Q+2)/(Q-2)$ which allows to put (2.9) in the form (2.6). □

3. The best constant in the Folland-Stein inequality

In this section, following [11], we prove the main Theorem. It is important to observe that a suitable adaptation of the method of concentration of compactness due to P. L. Lions [23, 24] allows to prove that in any Carnot group the Yamabe constant and optimal constant in the Folland-Stein inequality is achieved in the space $\overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})$, see [27] and [28]. Here

$$\overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G}) = \overline{C_o^{\infty}(\boldsymbol{G})}^{\|\cdot\|_{\overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})}}.$$

The space $\overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})$ is endowed with the norm

$$\|u\|_{\overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G})} = \|\,|\nabla u|\,\|_{L^{2^*}(\boldsymbol{G})}, \tag{3.1}$$

where ∇u is the horizontal gradient of u and $|\nabla u|^2 = \sum_{a=1}^{4n}(e_a u)^2$ for an orthonormal basis $\{e_1,\dots,e_{4n}\}$ of horizontal left-invariant vector fields.

In this regard an elementary, yet crucial observation, is that if u is an entire solution to the Yamabe equation, then such are also the two functions

$$\tau_h u \overset{\text{def}}{=} u\circ\tau_h, \qquad h\in\boldsymbol{G}, \tag{3.2}$$

where $\tau_h : \boldsymbol{G} \to \boldsymbol{G}$ is the operator of left-translation $\tau_h(g) = hg$, and

$$u_\lambda \overset{\text{def}}{=} \lambda^{(Q-2)/2}\, u \circ \delta_\lambda, \qquad \lambda > 0. \tag{3.3}$$

The Heisenberg dilations are defined by

$$\delta_\lambda\left((q', \omega')\right) = \Big((\lambda q', \lambda^2 \omega')\Big), \qquad (q', \omega') \in \boldsymbol{G}$$

It is also well known, [27] and [28], that there are smooth positive minimizer of the Folland-Stein inequality on the quaternionic Heisenberg group $\boldsymbol{G}$. These facts will be used without further notice on regularity and existence.

We start with the "new" key, see [6, 11] and also [25] and [8], allowing the ultimate solution of the considered problem.

Lemma 3.1. *For every $v \in L^1(S^{4n+3})$ with $\int_{S^{4n+3}} v \, \mathrm{Vol}_{\tilde{\eta}} = 1$ there is a quaternionic contact conformal transformation ψ such that*

$$\int_{S^{4n+3}} \psi\, v \, \mathrm{Vol}_{\tilde{\eta}} = 0.$$

Proof. Let $P \in S^{4n+3}$ be any point of the quaternionic sphere and N be its antipodal point. Let us consider the local coordinate system near P defined by the Cayley transform $\mathcal{C}_N$ from N. It is known that $\mathcal{C}_N$ is a quaternionic contact conformal transformation between $S^{4n+3} \setminus N$ and the quaternionic Heisenberg group. Notice that in this coordinate system P is mapped to the identity of the group. For every r, $0 < r < 1$, let $\psi_{r,P}$ be the qc conformal transformation of the sphere, which in the fixed coordinate chart is given on the group by a dilation with center the identity by a factor δ_r. If we select a coordinate system in $\mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}$ so that $P = (1, 0)$ and $N = (-1, 0)$ and then apply the formulas for the Cayley transform from [16, Section 8.2] the formula for $(q^*, p^*) = \psi_{r,P}(q, p)$ becomes

$$\begin{aligned} q^* &= 2r\left(1 + r^2(1+p)^{-1}(1-p)\right)^{-1}(1+p)\,q \\ p^* &= \left(1 + r^2(1+p)^{-1}(1-p)\right)^{-1}\left(1 - r^2(1+p)^{-1}(1-p)\right), \textit{i.e}, \end{aligned}$$

We can define then the map $\Psi : B \to \bar{B}$, where B ($\bar{B}$) is the open (closed) unit ball in $\mathbb{R}^{4n+4}$, by the formula

$$\Psi(rP) = \int_{S^{4n+3}} \psi_{1-r,P}\, v \, \mathrm{Vol}_{\tilde{\eta}}.$$

Notice that Ψ can be continuously extended to $\bar{B}$ since for any point P on the sphere, where $r = 1$, we have $\psi_{1-r,P}(Q) \to P$ when $r \to 1$. In particular, $\Psi = id$ on S^{4n+3}. Since the sphere is not a homotopy retract of the closed ball it follows that there are r and $P \in S^{4n+3}$ such that $\Psi(rP) = 0$, *i.e.*, $\int_{S^{4n+3}} \psi_{1-r,P}\, v \, \mathrm{Vol}_{\tilde{\eta}} = 0$. Thus, $\psi = \psi_{1-r,P}$ has the required property. □

In the next step we prove that we can assume that the minimizer of the Folland-Stein inequality satisfies the zero center of mass condition. A number of well known invariance properties of the Yamabe functional will be exploited.

Lemma 3.2. *Let v be a smooth positive function on the sphere with $\int_{S^{4n+3}} v^{2^*}\mathrm{Vol}_{\tilde\eta} = 1$. There is a smooth positive function u such that $\int_{S^{4n+3}}\big(4\frac{Q+2}{Q-2}\,|\nabla u|^2 + \tilde S\, u^2\big)\mathrm{Vol}_{\tilde\eta} = \int_{S^{4n+3}}\Big(4\frac{Q+2}{Q-2}\,|\nabla v|^2 \;+\; \tilde S\, v^2\Big)\,\mathrm{Vol}_{\tilde\eta}$ and $\int_{S^{4n+3}} u^{2^*}\,\mathrm{Vol}_{\tilde\eta} = 1$. In addition,*

$$\int_{S^{4n+3}} P\, u^{2^*}(P)\,\mathrm{Vol}_{\tilde\eta} = 0, \qquad P \in \mathbb{R}^{4n+4} = \mathbb{H}^n \times \mathbb{H}. \tag{3.4}$$

In particular, the Yamabe constant

$$\begin{aligned}&\lambda(S^{4n+3}, [\tilde\eta])\\ &= \inf\left\{\int_{S^{4n+3}}\Big(4\frac{Q+2}{Q-2}\,|\nabla v|^2 \;+\; \tilde S\, v^2\Big)\mathrm{Vol}_{\tilde\eta} : \int_{S^{4n+3}} v^{2^*}\,\mathrm{Vol}_{\tilde\eta} = 1,\ v>0\right\}\end{aligned} \tag{3.5}$$

is achieved for a positive function u with a zero center of mass, i.e., for a function u satisfying (3.4).

Proof. By [16, Section 8.1], $\mathrm{Vol}_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge (\omega_1)^{2n}$ is a volume form on a qc manifold with contact form η. Thus if η is a qc structure on the sphere which is qc conformal to the standard qc structure $\tilde\eta$, $\eta = \phi^{4/(Q-2)}\tilde\eta$, then $\mathrm{Vol}_\eta = \phi^{2^*}\mathrm{Vol}_{\tilde\eta}$. This allows to cast equation (1.2) in the form

$$\phi^{-1} v\, \mathcal{L}(\phi^{-1} v)\ \mathrm{Vol}_\eta = v\tilde{\mathcal{L}}(v)\ \mathrm{Vol}_{\tilde\eta}.$$

Therefore, if we take a positive function v on the sphere $\int_{S^{4n+3}} v^{2^*}\,\mathrm{Vol}_{\tilde\eta} = 1$ and then consider the function

$$u = \phi^{-1}(v \circ \psi^{-1}), \tag{3.6}$$

where ψ is the qc conformal map of Lemma 3.1, $\eta \equiv (\psi^{-1})^*\tilde\eta$, and ϕ is the corresponding conformal factor of ψ, we can see that u achieves the claim of the lemma. $\square$

We shall call a function u on the sphere a *well centered* function when (3.4) holds true. In the next step, following [11], we show that a well centered minimizer has to be constant using the products of the coordinate functions with the optimizer.

Lemma 3.3. *If u is a well centered local minimum of the problem* (3.5)*, then $u \equiv$* const.

Proof. Let ζ be a smooth function on the sphere S^{4n+3}. After applying the divergence formula [16, Section 8] we obtain the formula

$$\begin{aligned}&\Upsilon(\zeta u)\\ &= \int_{S^{4n+3}} \zeta^2\left(4\frac{Q+2}{Q-2}\,|\tilde\nabla u|^2 + \tilde S\, u^2\right)\ \mathrm{Vol}_{\tilde\eta} - 4\frac{Q+2}{Q-2}\int_{S^{4n+3}} u^2\zeta\,\tilde\triangle\zeta\,\mathrm{Vol}_{\tilde\eta}.\end{aligned} \tag{3.7}$$

This suggests to take as a test function ζ an eigenfunction of the sub-laplacian $\tilde{\triangle}$ of the standard qc structure. In particular, we can let ζ be any of the coordinate functions in $\mathbb{H}^n \times \mathbb{H}$ in which case $\tilde{\triangle}\zeta = -\lambda_1\zeta$.

It will be useful to introduce the functional $N(v) = \left(\int_{S^{4n+3}} v^{2^*}\,\mathrm{Vol}_{\tilde\eta}\right)^{2/2^*}$ so that

$$\lambda(S^{4n+3},[\tilde\eta]) = \inf\{\mathcal{E}(v) : \; v \in D\,(S^{4n+3})\}, \qquad \mathcal{E}(v) \overset{\text{def}}{=} \Upsilon(v)/N(v). \tag{3.8}$$

Computing the second variation $\delta^2\mathcal{E}(u)v = \frac{d^2}{dt^2}\mathcal{E}(u+tv)_{|t=0}$ of $\mathcal{E}(u)$ we see that the local minimum condition $\delta^2\mathcal{E}(u)v \geq 0$ implies

$$\Upsilon(v) - (2^*-1)\Upsilon(u)\int_{S^{4n+3}} u^{2^*-2}v^2\,\mathrm{Vol}_{\tilde\eta} \;\geq 0$$

for any function v such that $\int_{S^{4n+3}} u^{2^*-1}v\,\mathrm{Vol}_{\tilde\eta} = 0$. Therefore, for ζ being any of the coordinate functions in $\mathbb{H}^n \times \mathbb{H}$ we have

$$\Upsilon(\zeta u) - (2^*-1)\Upsilon(u)\int_{S^{4n+3}} u^{2^*}\zeta^2\,\mathrm{Vol}_{\tilde\eta} \;\geq 0,$$

which after summation over all coordinate functions taking also into account (3.7) gives

$$\Upsilon(u) - (2^*-1)\Upsilon(u) + 4\lambda_1(2^*-1)\int_{S^{4n+3}} u^2\,\mathrm{Vol}_{\tilde\eta} \geq 0,$$

which implies, recall $2^*-1 = (Q+2)/(Q-2)$,

$$\begin{aligned} 0 \leq 4(2^*-1)\left(2^*-2\right)&\int_{S^{4n+3}} |\tilde\nabla u|^2\,\mathrm{Vol}_{\tilde\eta} \\ &\leq \left(4\lambda_1(2^*-1) - \left(2^*-2\right)\tilde S\right)\int_{S^{4n+3}} u^{2^*}\,\mathrm{Vol}_{\tilde\eta}. \end{aligned}$$

Thus, our task of showing that u is constant will be achieved once we see that

$$4\lambda_1(2^*-1) - \left(2^*-2\right)\tilde S \leq 0, \;\; \text{i.e,} \;\; \lambda_1 \leq \tilde S/(Q+2). \tag{3.9}$$

By Lemma 2.5 we have actually equality $\lambda_1 = \tilde S/(Q+2)$, which completes the proof. It is worth observing that inequality (3.9) can be written in the form

$$\lambda_1\,a \leq (2^*-2)\,\tilde S,$$

where a is the constant in front of the (sub-)laplacian in the conformal (sub-)laplacian, *i.e.*, $a = 4\frac{Q+2}{Q-2}$ in our case. □

At this point the proof of our main Theorem 1.1 is easily deduced as follows.

Proof of Theorem 1.1. Let F be a minimizer (local minimum) of the Yamabe functional $\mathcal{E}$ on $\boldsymbol{G}$ and g the corresponding function on the sphere defined in Lemma 2.4. By Lemma 3.2 and (3.6) the function $g_0 = \phi^{-1}(g \circ \psi^{-1})$ will be well centered and a minimizer (local minimum) of the Yamabe functional $\mathcal{E}$ on S^{4n+3}. The latter claim uses also the fact that the map $v \mapsto u$ of equation (3.6) is one-to-one and onto on the space of smooth positive functions on the sphere. Now, from Lemma 3.3 we conclude that $g_o = const$. Looking back at the corresponding functions on the group we see that

$$F_0 = \gamma \left[(1 + |q'|^2)^2 + |\omega'|^2 \right]^{-(Q-2)/4}$$

for some $\gamma = \text{const.} > 0$. Furthermore, the proof of Lemma 3.1 shows that F_0 is obtained from F by a translation (3.2) and dilation (3.3). Correspondingly, any positive minimizer (local maximum) of problem (3.11) is given up to dilation or translation by the function

$$F = \gamma \left[(1 + |q'|^2)^2 + |\omega'|^2 \right]^{-(Q-2)/4}, \qquad \gamma = \text{const.} > 0. \tag{3.10}$$

Of course, translations (3.2) and dilations (3.3) do not change the value of $\mathcal{E}$. Incidentally, this shows that any local minimum of the Yamabe functional $\mathcal{E}$ on the sphere or the group has to be a global one.

We turn to the determination of the best constant. Let us define the constants

$$\Lambda_{\tilde{\Theta}} \overset{\text{def}}{=} \inf \left\{ \int_{\boldsymbol{G}} |\nabla v|^2 \, \mathrm{Vol}_{\tilde{\Theta}} \ : \ v \in \overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G}), \ v \geq 0, \ \int_{\boldsymbol{G}} |v|^{2^*} \, \mathrm{Vol}_{\tilde{\Theta}} = 1 \right\}$$

and (3.11)

$$\Lambda \overset{\text{def}}{=} \inf \left\{ \int_{\boldsymbol{G}} |\nabla v|^2 \, dH \ : \ v \in \overset{o}{\mathcal{D}}{}^{1,2}(\boldsymbol{G}), \ v \geq 0, \ \int_{\boldsymbol{G}} |v|^{2^*} \, dH \ = \ 1 \right\}.$$

Clearly, $\Lambda_{\tilde{\Theta}} = S_{\tilde{\Theta}}^{-2}$, where $S_{\tilde{\Theta}}$ is the best constant in the L^2 Folland-Stein inequality

$$\left(\int_{\boldsymbol{G}} |u|^{2^*} \, \mathrm{Vol}_{\tilde{\Theta}} \right)^{1/2^*} \leq \ S_{\tilde{\Theta}} \left(\int_{\boldsymbol{G}} |\nabla^{\tilde{\Theta}} u|^2 \, \mathrm{Vol}_{\tilde{\Theta}} \right)^{1/2}, \tag{3.12}$$

while $\Lambda = S_2^{-2}$ is the best constant in the L^2 Folland-Stein inequality (1.3) (taken with respect to the Lebesgue measure!). By Remark 2.5 we have

$$\Lambda_{\tilde{\Theta}} = \left[2^{-3} (2n)! \right]^{1/(2n+3)} \Lambda.$$

Furthermore, by Lemma 3.3 and equations (2.6) and (2.7) with $g = const$, we have

$$\Lambda_{\tilde\Theta} = \frac{1}{S_2^2} = \frac{\int_{\boldsymbol{G}} |\nabla^{\tilde\Theta} F|^2 \,\mathrm{Vol}_{\tilde\Theta}}{\left[\int_{\boldsymbol{G}} |F|^{2^*} \,\mathrm{Vol}_{\tilde\Theta}\right]^{2/2^*}}$$
$$= \frac{\int_{S^{4n+3}} \left(|\nabla^{\tilde\eta} g|^2 + \frac{\tilde S}{a}\, g^2\right) \mathrm{Vol}_{\tilde\eta}}{\left[\int_{S^{4n+3}} |g|^{2^*} \,\mathrm{Vol}_{\tilde\eta}\right]^{2/2^*}} = 4n(n+1)\left[((2n)!)\,\omega_{4n+3}\right]^{1/(2n+3)}.$$

Here

$$\omega_{4n+3} = 2\pi^{2n+2}/\,\Gamma(2n+2) = 2\pi^{2n+2}/(2n+1)!$$

is the volume of the unit sphere $S^{4n+3} \subset \mathbb{R}^{4n+4}$ and we also took into account Remark 2.2 which shows that $\mathrm{Vol}_{\tilde\eta}$ gives $2^{2n+3}\,((2n)!)\,\omega_{4n+3}$ for the volume of S^{4n+3}. Thus,

$$S_{\tilde\Theta} = \left(4n(n+1)\left[((2n)!)\,\omega_{4n+3}\right]^{1/(2n+3)}\right)^{-1/2} = \frac{\left[((2n)!)\,\omega_{4n+3}\right]^{-1/(4n+6)}}{2\sqrt{n(n+1)}},$$

which completes the proof of part a).

b) The Yamabe constant of the sphere is calculated immediately by taking a constant function in (3.8)

$$\lambda(S^{4n+3}, [\tilde\eta]) = a\,\Lambda_{\tilde\Theta}, \qquad a = 4\frac{Q+2}{Q-2} = 4\frac{n+2}{n+1}. \tag{3.13}$$

This completes the proof of Theorem 1.1. □

Remark 3.4. In view of the above lemmas it follows that in the conformal class of the standard qc structure on the sphere (or the quaternionic Heisenberg group) there is an extremal qc contact form for problem (1.1) which is also qc-Einstein, see [16, Definition 4.1], and has partial symmetry, see [13, Definition 1.2], if viewed as a qc structure on the group. Thus, the above precise constants and extremals can also be taken directly from [16, Theorem 1.1 and 1.2] or the result of [13]. However, the functions (3.10) depend on one more arbitrary multiplicative parameter γ since in the current paper we are dealing with the functions realizing the infimum of (3.8) rather than with the qc Yamabe equation with a fixed qc scalar curvature.

References

[1] F. ASTENGO, M. COWLING and B. DI BLASIO, *The Cayley transform and uniformly bounded representations*, J. Funct. Anal. **213** (2004), 241–269.

[2] T. AUBIN, *Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. **32** (1979), 148–174.

[3] A. D. BANNER, *Some properties of boundaries of symmetric spaces of rank one*. Geom. Dedicata **88** (2001), 113–133.

[4] W. BECKNER, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. **138** (1993), 213–242.

[5] O. BIQUARD, "Métriques d'Einstein Asymptotiquement Symétriques", Astérisque, Vol. 265, 2000.

[6] T. P. BRANSON, L. FONTANA and C. MORPURGO, *Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere*, Ann. of Math., to appear, arXiv:0712.3905v3.

[7] L. CAFFARELLI, R. KOHN, and L. NIRENBERG, *First order interpolation inequality with weights*, Compositio Math. **53** (1984), 259–275.

[8] S. Y. CHANG and P. C. YANG, *Prescribing Gaussian curvature on S^2*, Acta Math. **159** (1987), 215–259.

[9] M. COWLING, A. H. DOOLEY, A. KORÁNYI and F. RICCI, *H-type groups and Iwasawa decompositions*, Adv. Math., **87** (1991), 1–41.

[10] G. FOLLAND and E. M. STEIN, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[11] R. L. FRANK and E. H. LIEB, *Sharp constants in several inequalities on the Heisenberg group*, Ann. of Math., to appear, arXiv:1009.1410.

[12] H. GLUCK, *Deformations of normal vector fields and the generalized Minkowski problem*. Bull. Amer. Math. Soc. **77** (1971), 1106–1110.

[13] N. GAROFALO and D. VASSILEV, *Symmetry properties of positive entire solutions of Yamabe type equations on groups of Heisenberg type*, Duke Math J. **106** (2001), 411–449.

[14] N. GAROFALO and D. VASSILEV, *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups*, Math Ann. **318** (2000), 453–516.

[15] N. GHOUSSOUB and C. YUAN, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. **352** (2000), 5703–5743.

[16] S. IVANOV, I. MINCHEV and D. VASSILEV, *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, math.DG/0611658.

[17] S. IVANOV, I. MINCHEV and D. VASSILEV, *Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem*, J. Eur. Math. Soc. (JEMS) **12** (2010), 1041–1067.

[18] S. IVANOV and D. VASSILEV, *Conformal quaternionic contact curvature and the local sphere theorem*, J. Math. Pures Appl. **93** (2010), 277-307.

[19] D. JERISON and J. LEE, *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. **1** (1988), 1–13.

[20] J. HERSCH, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*. C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A1645–A1648.

[21] A. KORÁNYI, *Kelvin transform and harmonic polynomials on the Heisenberg group*, Adv.Math. **56** (1985), 28–38.

[22] E. H. LIEB, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. **118** (1983), 349–374.

[23] P.-L. LIONS, *The concentration compactness principle in the calculus of variations. The limit case. Part 1*, Rev. Mat. Iberoamericana **1** (1985), 145–201.

[24] P.-L. LIONS, *The concentration compactness principle in the calculus of variations. The limit case. Part 2*, Rev. Mat. Iberoamericana **1** (1985), 45–121.

[25] E. ONOFRI, *On the positivity of the effective action in a theory of random surfaces*. Comm. Math. Phys. **86** (1982), 321–326.

[26] G. SZEGÖ, *Inequalities for certain eigenvalues of a membrane of given area* J. Rational Mech. Anal. **3** (1954), 343–356.

[27] D. VASSILEV, *Regularity near the characteristic boundary for sub-laplacian operators*, Pacific J. Math, **227** (2006), 361–397.

[28] D. VASSILEV, “Yamabe Type Equations on Carnot Groups”, Ph. D. thesis Purdue University, 2000.

[29] D. VASSILEV, *Lp estimates and asymptotic behavior for finite energy solutions of extremals to Hardy-Sobolev inequalities*, Trans. Amer. Math. Soc. **363** (2011), 37–62.

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