

Elliptic operators with unbounded diffusion coefficients in L^p spaces

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Abstract. We prove that, for $N \geq 3$, $\alpha > 2$, $\frac{N}{N-2} < p < \infty$, the operator $Lu = m(x)(1 + |x|^\alpha)\Delta u$ generates an analytic semigroup in L^p which is contractive if and only if $p \geq \frac{N+\alpha-2}{N-2}$. Moreover, for $\alpha < \frac{N}{p}$, we provide an explicit description of the domain. Spectral properties of the operator L are also obtained.

Mathematics Subject Classification (2010): 47D07 (primary); 35B50, 35J25, 35J70 (secondary).

1. Introduction

In this paper we focus our attention on a class of elliptic operators with unbounded diffusion coefficients. We deal with operators of the form

$$Lu = m(x)(1 + |x|^\alpha)\Delta u, \quad (1.1)$$

for positive values of α , on $L^p = L^p(\mathbb{R}^N, dx)$ with respect to the Lebesgue measure. We assume that $m \in C_{\text{loc}}^\sigma(\mathbb{R}^N)$ for some $0 < \sigma \leq 1$ and that there exist constants $\kappa, \Sigma > 0$ such that $\kappa \leq m(x) \leq \Sigma$ for every $x \in \mathbb{R}^N$. We are interested both in parabolic problems associated with L

$$u_t - Lu = 0 \quad u(0) = f$$

and in the solvability of the elliptic equation

$$\lambda u - Lu = f$$

for $\lambda \in \mathbb{C}$. The case $\alpha \leq 2$ has been already investigated in literature and for this reason we shall assume $\alpha > 2$ throughout the paper, even when some arguments easily extend to lower values of α .

We refer to [2] where it is proved that, if $\alpha \leq 2$, then L generates an analytic semigroup in L^p , $1 \leq p \leq \infty$. Moreover, if $1 < p < \infty$, then

$$D(L_p) = \{u \in L^p : a^{1/2}\nabla u, aD^2u \in L^p\}$$

where $a(x) = (1 + |x|^\alpha)$. Actually the authors prove the same results for more general a satisfying $a > 0$ and $|\nabla a| \leq Ca^{1/2}$.

In the general case the situation appears more involved. The a-priori estimates

$$\| |x|^\alpha D^2u \|_p \leq C \| |x|^\alpha \Delta u \|_p, \quad \|(1 + |x|^\alpha)D^2u\|_p \leq C \|(1 + |x|^\alpha \Delta u)\|_p$$

hold in C_c^∞ for $\alpha < \frac{N}{p'}$ where p' is the conjugate exponent of p , see [6] and [11]. On the other hand, we are not aware of generation results for the operator L even for the above values of α .

We prove that, for $\frac{N}{N-2} < p < \infty$, the operator L generates an analytic semigroup in L^p , which is contractive if and only if $p \geq (N + \alpha - 2)/(N - 2)$. Moreover, according with the previous remarks, we provide an explicit description of the domain when $\alpha < \frac{N}{p'}$.

The paper is organized as follows. In Section 2 we recall known facts on L in spaces of continuous functions, important for our discussion. In Section 3 we prove some negative results in L^p . We consider the operator $\hat{L}_p = (L, \hat{D}_p)$ on any domain \hat{D}_p contained in the maximal domain

$$D_{p,\max}(L) = \{u \in W^{2,p} : (1 + |x|^\alpha)\Delta u \in L^p\}$$

and show that, if $m = 1$, $N = 1, 2$ and $1 \leq p \leq \infty$ or $N \geq 3$ and $p \leq N/(N - 2)$, then $\rho(\hat{L}_p) \cap [0, \infty[= \emptyset$.

In Section 4 we show that, if $N/(N - 2) < p < \infty$, the operator $\lambda - L$ is invertible on $D_{p,\max}(L)$ for every $\lambda \geq 0$, and moreover $(\lambda - L)^{-1} \leq (-L)^{-1}$ and $\|(\lambda - L)^{-1}\| \leq \|T\|$. The crucial point consists in proving (weighted) estimates for the integral operator

$$Tf(x) = \int_{\mathbb{R}^N} \frac{f(y) dy}{m(y)(1 + |y|^\alpha)|x - y|^{N-2}}.$$

Such estimates are then used in Section 5 to prove that, if $\alpha < N/p'$, then the maximal domain coincides with the weighted Sobolev space

$$D_p = \{u \in L^p(\mathbb{R}^N) : (1 + |x|^{\alpha-2})u, (1 + |x|^{\alpha-1})\nabla u, (1 + |x|^\alpha)D^2u \in L^p(\mathbb{R}^N)\}.$$

Explicit examples show that, when $\alpha \geq N/p'$, then D_p is a proper subset of $D_{p,\max}(L)$. If $\alpha < N$ a partial characterization of the domain in C_0 is obtained in Section 6.

In Section 7 we prove that the resolvent of $(L, D_{p,\max}(L))$ is compact in L^p and that the spectrum of L consists of eigenvalues independent of p . The introduction of a suitable weighted Hilbert spaces allows us to prove that the spectrum lies

in the negative real axis. In order to prove the solvability of the elliptic equation $\lambda u - Lu = f$ for complex λ and to show generation of an analytic semigroup we proceed in several steps. We use a Hardy-type inequality to prove that the operator $L_s = (s^\alpha + |x|^\alpha)\Delta, s > 0$, is dissipative if and only if $p \geq (N + \alpha - 2)/(N - 2)$. Then, under these restrictions on p , $(L, D_{p,\max}(L))$ generates a positive semigroup of contractions in L^p , which is analytic if $p > (N + \alpha - 2)/(N - 2)$. This is done in Section 8. In Section 9 we use an iteration procedure which, combined with perturbation and scaling arguments, shows that $(L, D_{p,\max}(L))$ generates an analytic semigroup in L^p for every $\frac{N}{N-2} < p < \infty$.

Notation 1.1. We use L^p for $L^p(\mathbb{R}^N, dx)$, where this latter is understood with respect to the Lebesgue measure. $C_b(\mathbb{R}^N)$ is the Banach space of all continuous and bounded functions in \mathbb{R}^N , endowed with the sup-norm, and $C_0(\mathbb{R}^N)$ its subspace consisting of all continuous functions vanishing at infinity. $C_c^\infty(\mathbb{R}^N)$ denotes the set of all C^∞ functions with compact support. $B(\rho)$ is the open ball with centre 0 and radius ρ . $C_{\text{loc}}^\sigma(\mathbb{R}^N)$ is the space of locally Hölder continuous functions in \mathbb{R}^N .

2. Solvability in spaces of continuous functions

Following [10], we recall the main results in spaces of continuous functions which will be useful throughout the paper.

Let A be a second order elliptic partial differential operator of the form

$$Au(x) = \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^N F_i(x)D_iu(x) \quad x \in \mathbb{R}^N$$

under the following hypotheses on the coefficients: $a_{ij} = a_{ji}$, a_{ij}, F_i are real-valued locally Hölder continuous functions of exponent $0 < \sigma \leq 1$ and the matrix (a_{ij}) satisfies the ellipticity condition

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2$$

for every $x, \xi \in \mathbb{R}^N$, with $\inf_K \lambda(x) > 0$ for every compact $K \subset \mathbb{R}^N$. The operator A is locally uniformly elliptic, that is uniformly elliptic on every compact subset of \mathbb{R}^N .

We endow A with its maximal domain in $C_b(\mathbb{R}^N)$ given by

$$D_{\max}(A) = \{u \in C_b(\mathbb{R}^N) \cap W_{\text{loc}}^{2,p}(\mathbb{R}^N) \text{ for all } p < \infty : Au \in C_b(\mathbb{R}^N)\}.$$

The main interest is in the existence of (spatial) bounded solutions of the parabolic problem

$$\begin{cases} u_t(t, x) = Au(t, x) & x \in \mathbb{R}^N, t > 0, \\ u(0, x) = f(x) & x \in \mathbb{R}^N \end{cases} \quad (2.1)$$

with initial datum $f \in C_b(\mathbb{R}^N)$. The unbounded interval $[0, \infty[$ can be changed to any bounded $[0, T]$ without affecting the results. Since the coefficients can be unbounded, the classical theory does not apply and existence and uniqueness for (2.1) are not clear. Quite surprisingly, existence is never a problem as stated in the following theorem.

Theorem 2.1. *There exists a positive semigroup $(T(t))_{t \geq 0}$ defined in $C_b(\mathbb{R}^N)$ such that, for any $f \in C_b(\mathbb{R}^N)$, $u(t, x) = T(t)f(x)$ belongs to the space $C_{\text{loc}}^{1+\frac{\sigma}{2}, 2+\sigma}((0, +\infty) \times \mathbb{R}^N)$, is a bounded solution of the following differential equation*

$$u_t(t, x) = \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(t, x) + \sum_{i=1}^N F_i(x)D_iu(t, x)$$

and satisfies

$$\lim_{t \rightarrow 0} u(t, x) = f(x)$$

pointwise.

When $f \in C_0(\mathbb{R}^N)$, then $u(t, \cdot) \rightarrow f$ uniformly as $t \rightarrow 0$. This, however, does not mean that $T(t)$ is strongly continuous on $C_0(\mathbb{R}^N)$ since this latter need not to be preserved by the semigroup.

The idea of the proof is to take an increasing sequence of balls filling the whole space and, in each of them, to find a solution of the parabolic problem associated with the operator. Then the sequence of solutions so obtained is proved to converge to a solution of the problem in \mathbb{R}^N . More precisely, let us fix a ball $B(\rho)$ in \mathbb{R}^N and consider the problem

$$\begin{cases} u_t(t, x) = Au(t, x) & x \in B(\rho), \quad t > 0, \\ u(t, x) = 0 & x \in \partial B(\rho), \quad t > 0 \\ u(0, x) = f(x) & x \in \mathbb{R}^N. \end{cases} \tag{2.2}$$

Since the operator A is uniformly elliptic and the coefficients are bounded in $B(\rho)$, there exists a unique solution u_ρ of problem (2.2). The next step consists in letting ρ to infinity in order to define the semigroup associated with A in \mathbb{R}^N . By using the parabolic maximum principle, it is possible to prove that the sequence u_ρ increases with ρ when $f \geq 0$ and is uniformly bounded by the sup-norm of f . In virtue of this monotonicity and since a general f can be written as $f = f^+ - f^-$, the limit

$$T(t)f(x) := \lim_{\rho \rightarrow \infty} u_\rho(t, x)$$

is well defined for $f \in C_b(\mathbb{R}^N)$ and one shows all relevant properties, using the interior Schauder estimates.

It is worth-mentioning that also the resolvent of A , namely $(\lambda - A)^{-1}$, is, for positive λ , the limit as $\rho \rightarrow \infty$ of the corresponding resolvents in the balls $B(\rho)$. The construction then shows that, for positive $f \in C_b(\mathbb{R}^N)$ and $\lambda > 0$, both the

semigroup $T(t)f$ (and the resolvent $(\lambda - A)^{-1}f$) select in a linear way the minimal solution among all bounded solutions of (2.1) (of $\lambda u - Au = f$). For this reason, from now on, the semigroup $T(t)$ will be called the minimal semigroup associated to A and will be denoted by $T_{\min}(t)$. Its generator (A, D) , where $D \subset D_{\max}(A)$, will be denoted by A_{\min} .

In contrast with the existence, the uniqueness is not guaranteed, in general, and relies on the existence of suitable Lyapunov functions. We do not deal here with such a topic and refer again to [10]. We only point out that uniqueness holds if and only if $D = D_{\max}(A)$, i.e. when A_{\min} coincides with $(A, D_{\max}(A))$.

Let us specialize to our operator L .

Proposition 2.2. *Let $L = m(x)(1 + |x|^\alpha)\Delta$.*

- (i) *If $\alpha \leq 2$, the semigroup preserves $C_0(\mathbb{R}^N)$ and neither the semigroup nor the resolvent are compact.*
- (ii) *If $\alpha > 2$ and $N = 1, 2$, the semigroup is generated by $(A, D_{\max}(A))$, $C_0(\mathbb{R}^N)$ and L^p are not preserved by the semigroup and the resolvent and both the semigroup and the resolvent are compact.*
- (iii) *If $\alpha > 2$, $N \geq 3$, then the semigroup is generated by $(A, D_{\max}(A)) \cap C_0(\mathbb{R}^N)$, the resolvent and the semigroup map $C_b(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$ and are compact.*

See ([10, Example 7.3]). In particular (ii) will imply that, if $\alpha > 2$ and $N = 1, 2$, problem (2.1) cannot be solved in L^p . Observe also that (iii) and the discussion above show that $(T_{\min}(t))_{t \geq 0}$ is strongly continuous on $C_0(\mathbb{R}^N)$.

3. Preliminary considerations in L^p

We consider the operator $\hat{L}_p = (L, \hat{D}_p)$ on any domain \hat{D}_p contained in the maximal domain in $L^p(\mathbb{R}^N)$ defined by

$$D_{p,\max}(L) = \{u \in L^p \cap W_{\text{loc}}^{2,p} : Lu \in L^p\}. \tag{3.1}$$

Note that $D_{p,\max}(L)$ is the analogous of $D_{\max}(L)$ for $p < \infty$. We are interested in solvability of elliptic and parabolic problems associated to L . We show that for certain values of p the equation

$$\lambda u - Lu = f$$

is not solvable in $L^p(\mathbb{R}^N)$ for positive λ .

In the following proposition we show that functions in $D_{p,\max}(L)$ are globally in $W^{2,p}$.

Proposition 3.1.

$$D_{p,\max}(L) = \{u \in W^{2,p} : (1 + |x|^\alpha)\Delta u \in L^p\}.$$

Proof. It is clear that the right hand side is included in the left one. Conversely, if $u \in D_{p,\max}(L)$, then $u, \Delta u \in L^p$ and we have to show that $u \in W^{2,p}$. Let $v \in W^{2,p}$ be such that $v - \Delta v = u - \Delta u$. Then $w = u - v \in L^p$ solves $w - \Delta w = 0$. Since w is a tempered distribution, by taking the Fourier transform it easily follows that $w = 0$, hence $u = v$. \square

The next lemma shows that the resolvent operator in L^p , if it exists, is a positive operator.

Lemma 3.2. *Suppose that $\lambda \in \rho(\hat{L}_p)$ for some $\lambda \geq 0$. Then for every $0 \leq f \in L^p$,*

$$(\lambda - \hat{L}_p)^{-1} f \geq 0.$$

Proof. By density we may assume that $0 \leq f \in C_c^\infty(\mathbb{R}^N)$. Suppose first that $\lambda > 0$. We set $u = (\lambda - \hat{L}_p)^{-1} f$. Suppose $\text{supp } f \subset B(R)$. Then u satisfies

$$\lambda u - Lu = f$$

in $B(R)$ and

$$\lambda u - Lu = 0$$

in $\mathbb{R}^N \setminus B(R)$. By local elliptic regularity, see [5, Theorem 9.19], $u \in C_{\text{loc}}^{2,\sigma}(\mathbb{R}^N)$. In $\mathbb{R}^N \setminus B(R)$, u satisfies

$$m(x)\Delta u = \frac{\lambda u}{1 + |x|^\alpha} \in L^p(\mathbb{R}^N).$$

By elliptic regularity again ([5, Theorem 9.19]), $u \in W^{2,p}(\mathbb{R}^N \setminus B(R))$. If $p > \frac{N}{2}$, we immediately deduce $u \in C_0(\mathbb{R}^N \setminus B(R))$. Otherwise $u \in L^{p_1}(\mathbb{R}^N \setminus B(R))$ where $\frac{1}{p_1} = s\frac{1}{p} - \frac{2}{N}$ (with the usual modification when $p = N/2$). As before it follows $\Delta u \in L^{p_1}(\mathbb{R}^N \setminus B(R))$ and $u \in W^{2,p_1}(\mathbb{R}^N \setminus B(R))$. By iterating this procedure until $p_i > \frac{N}{2}$ we deduce $u \in C_0(\mathbb{R}^N)$. Therefore u attains its minimum in a point $x_0 \in \mathbb{R}^N$. The equality

$$\lambda u(x_0) = m(x_0)(1 + |x_0|^\alpha)\Delta u(x_0) + f(x_0)$$

shows that $u(x_0) \geq 0$, since $\lambda > 0$, hence $u \geq 0$. If $\lambda = 0 \in \rho(\hat{L}_p)$, then $\lambda \in \rho(\hat{L}_p)$ for small positive values of λ and the thesis follows by approximation. \square

Lemma 3.3. *Suppose that $\lambda \in \rho(\hat{L}_p)$ for some $\lambda \geq 0$. Then for every $0 \leq f \in C_c(\mathbb{R}^N)$,*

$$(\lambda - \hat{L}_p)^{-1} f \geq (\lambda - L_{\min})^{-1} f.$$

Proof. Let $0 \leq f \in C_c^\infty(\mathbb{R}^N)$ and set $u = (\lambda - \hat{L}_p)^{-1} f$. Lemma 3.2 and its proof show that $0 \leq u \in D_{\max}(L)$. Since $(\lambda - L_{\min})^{-1} f$ is the minimal solution, we immediately have $u \geq (\lambda - L_{\min})^{-1} f$. \square

In the next two results we assume that $L = (1 + |x|^\alpha)\Delta$.

Proposition 3.4. *Assume that $m \equiv 1$, $N \geq 3$, $\alpha > 2$ and $p \leq \frac{N}{N-2}$. Then $\rho(\hat{L}_p) \cap [0, \infty[= \emptyset$.*

Proof. Let $\lambda > 0$ and $\chi_{B(0)} \leq f \leq \chi_{B(1)}$ be a smooth radial function. Denote by u the (minimal) solution of $\lambda u - Lu = f$ in $C_0(\mathbb{R}^N)$. Observe that from [10, Example 7.3] it follows that the above equation has a unique solution in $C_0(\mathbb{R}^N)$ (not in $C_b(\mathbb{R}^N)$). Hence, since the datum f is radial, the solution u is radial too and solves

$$\lambda u(\rho) - (1 + \rho^\alpha) \left(u''(\rho) + \frac{N-1}{\rho} u'(\rho) \right) = f(\rho).$$

For $\rho \geq 1$, u solves the homogeneous equation

$$\lambda u(\rho) - (1 + \rho^\alpha) \left(u''(\rho) + \frac{N-1}{\rho} u'(\rho) \right) = 0.$$

Let us write u as $u(\rho) = \eta(\rho)\rho^{2-N}$ for a suitable function η . Elementary computations show that η satisfies

$$\lambda \eta(\rho) - (1 + \rho^\alpha) \left(\eta''(\rho) + \frac{3-N}{\rho} \eta' \right) = 0 \tag{3.2}$$

for $\rho \geq 1$. First observe that, since $f \neq 0$ is nonnegative, the strong maximum principle, see [5, Theorem 3.5]), implies that u (and so η) is strictly positive. We use Feller’s theory to study the asymptotic behavior of the solutions of the previous equation (see [3, Section VI.4.c]). We introduce the Wronskian

$$W(\rho) = \exp \left\{ - \int_1^\rho \frac{3-N}{s} ds \right\} = \rho^{N-3}$$

and the functions

$$Q(\rho) = \frac{1}{(1 + \rho^\alpha)W(\rho)} \int_1^\rho W(s)ds = \frac{1}{N-2} \frac{1}{(1 + \rho^\alpha)\rho^{N-3}} (\rho^{N-2} - 1)$$

and

$$R(\rho) = W(\rho) \int_1^\rho \frac{1}{(1 + s^\alpha)W(s)} ds = \rho^{N-3} \int_1^\rho \frac{1}{(1 + s^\alpha)s^{N-3}} ds.$$

Since $\alpha > 2$ by assumption, we have $Q \in L^1(1, +\infty)$ and $R \notin L^1(1, \infty)$. This means that ∞ is an entrance endpoint. In this case there exists a positive decreasing solution η_1 of (3.2) satisfying $\lim_{\rho \rightarrow \infty} \eta_1(\rho) = 1$ and every solution of (3.2) independent of η_1 is unbounded at infinity. This shows that our solution u grows at infinity at least as ρ^{2-N} and therefore it does not belong to $L^p(\mathbb{R}^N)$. By Lemma 3.3 we deduce that $\lambda \notin \rho(\hat{L}_p)$. □

When $N = 1, 2$ and $\alpha > 2$, then (2.1) is never solvable in L^p .

Proposition 3.5. *Let $m \equiv 1$, $N = 1, 2$, $\alpha > 2$. Then $\rho(\hat{L}_p) \cap [0, \infty[= \emptyset$.*

This follows from Proposition 2.2 (ii), using Lemma 3.3. □

4. Solvability in L^p

In this section we investigate the solvability of the equation $\lambda u - Lu = f$ in L^p , for $\lambda \geq 0$. We start with $\lambda = 0$. Since the equation $-Lu = f$ is equivalent to $-\Delta u(x) = f(x)/m(x)(1 + |x|^\alpha)$, we can express u and its gradient through an integral operator involving the Newtonian potential. For $f \in L^p$ we set

$$Tf(x) = u(x) = C_N \int_{\mathbb{R}^N} \frac{f(y) dy}{m(y)(1 + |y|^\alpha)|x - y|^{N-2}} \tag{4.1}$$

and

$$Sf(x) = \nabla u(x) = C_N(N - 2) \int_{\mathbb{R}^N} \frac{f(y)(y - x) dy}{m(y)(1 + |y|^\alpha)|x - y|^N} \tag{4.2}$$

where $C_N = (N(N - 2)\omega_N)^{-1}$ and ω_N is the Lebesgue measure of the unit ball in \mathbb{R}^N .

We prove a preliminary result which will be useful to prove estimates for the norm of the operator T in L^p .

Lemma 4.1. *Let $2 < \beta < N$. Then*

$$\frac{1}{N(2 - N)\omega_N} \int_{\mathbb{R}^N} \frac{dy}{|x - y|^{N-2}|y|^\beta} = \frac{1}{(2 - \beta)(N - \beta)}|x|^{2-\beta}.$$

Proof. Set

$$u(x) = \frac{1}{N(2 - N)\omega_N} \int_{\mathbb{R}^N} \frac{dy}{|x - y|^{N-2}|y|^\beta}.$$

By writing x, y in spherical coordinates, $x = s\eta, y = r\omega$, with $\eta, \omega \in S_{N-1}, s, r \in [0, +\infty)$, the expression of u becomes

$$\begin{aligned} u(s\eta) &= \frac{1}{N(2 - N)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{r^{N-1} dr}{|s\eta - r\omega|^{N-2}|r|^\beta} \\ &= \frac{1}{N(2 - N)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{r^{N-1-\beta} dr}{s^{N-2} \left| \eta - \frac{r}{s}\omega \right|^{N-2}} \\ &= \frac{1}{N(2 - N)\omega_N} s^{2-\beta} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1-\beta} d\xi}{|\eta - \xi\omega|^{N-2}}. \end{aligned}$$

By the rotational invariance of the integral,

$$\begin{aligned} u(s\eta) &= \frac{1}{N(2 - N)\omega_N} s^{2-\beta} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1-\beta} d\xi}{|e_1 - \xi\omega|^{N-2}} \\ &= \frac{1}{N(2 - N)\omega_N} s^{2-\beta} \int_{\mathbb{R}^N} \frac{dy}{|e_1 - y|^{N-2}|y|^\beta}, \end{aligned}$$

where e_1 is the unitary vector in the canonical basis of S_{N-1} . Therefore $u(x) = C|x|^{2-\beta}$ with

$$C = \frac{1}{N(2-N)\omega_N} \int_{\mathbb{R}^N} \frac{dy}{|e_1 - y|^{N-2}|y|^\beta}.$$

To compute the constant C we note that u solves

$$\Delta u = \frac{1}{|x|^\beta}$$

or, in spherical coordinates,

$$u''(\rho) + \frac{N-1}{\rho}u'(\rho) = \frac{1}{\rho^\beta}.$$

Inserting $u(\rho) = C\rho^{2-\beta}$. we get $C = \frac{1}{(2-\beta)(N-\beta)}$. □

In the following lemma we investigate the boundedness of the operators T, S in weighted L^p -spaces. Even though we need here only the boundedness of T in the unweighted L^p -space, we prove the general result which will be of a central importance in the next sections.

Lemma 4.2. *Let $\alpha \geq 2$ and $N/(N-2) < p < \infty$. For every $0 \leq \beta, \gamma$ such that $\beta \leq \alpha - 2, \beta < \frac{N}{p'} - 2$ and $\gamma \leq \alpha - 1, \gamma < \frac{N}{p'} - 1$, there exists a positive constant C such that for any $f \in L^p$*

$$\begin{aligned} \| |\cdot|^\beta u \|_{L^p(\mathbb{R}^N)} &\leq C \| f \|_{L^p(\mathbb{R}^N)}; \\ \| |\cdot|^\gamma \nabla u \|_{L^p(\mathbb{R}^N)} &\leq C \| f \|_{L^p(\mathbb{R}^N)}, \end{aligned}$$

where u is defined in (4.1).

Proof. Since $m(x) \geq \kappa$, it is sufficient to prove the boundedness of T, S when $m \equiv 1$.

Set $x = s\eta, y = \rho\omega$ with $s, \rho \in [0, +\infty), \eta\omega \in S_{N-1}$, then

$$\begin{aligned} u(s\eta) &= \frac{1}{N(2-N)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{f(\rho\omega) \rho^{N-1} d\rho}{(1 + \rho^\alpha)|s\eta - \rho\omega|^{N-2}} \\ &= \frac{1}{N(2-N)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{s^2 f(s\xi\omega) \xi^{N-1} d\xi}{(1 + (s\xi)^\alpha)|\eta - \xi\omega|^{N-2}}. \end{aligned}$$

We compute the L^p norm of $|\cdot|^\beta u(\cdot)$. We start by integrating with respect to s the inequality above. We have, using Minkowski inequality for integrals,

$$\begin{aligned} & \left(\int_0^\infty |u(s\eta)|^p s^{\beta p + N - 1} ds \right)^{\frac{1}{p}} \\ & \leq \frac{1}{N(N-2)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-2}} \left(\int_0^\infty \frac{|f(s\xi\omega)|^p s^{N-1+2p+\beta p} ds}{(1+s^\alpha \xi^\alpha)^p} \right)^{\frac{1}{p}} \\ & = \frac{1}{N(N-2)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-2} \xi^{\frac{N}{p} + \beta + 2}} \left(\int_0^\infty \frac{|f(v\omega)|^p v^{N-1+2p+\beta p} dv}{(1+v^\alpha)^p} \right)^{\frac{1}{p}}. \end{aligned}$$

By recalling that $\beta \leq \alpha - 2$ and since

$$\frac{v^{2+\beta}}{1+v^\alpha} \leq \left(\frac{2+\beta}{\alpha-2+\beta} \right)^{\frac{2+\beta}{\alpha}} \frac{\alpha-2+\beta}{\alpha+2\beta},$$

we obtain

$$\begin{aligned} & \left(\int_0^\infty |u(s\eta)|^p s^{\beta p + N - 1} ds \right)^{\frac{1}{p}} \leq \left(\frac{2+\beta}{\alpha-2+\beta} \right)^{\frac{2+\beta}{\alpha}} \frac{\alpha-2+\beta}{\alpha+2\beta} \frac{1}{N(N-2)\omega_N} \\ & \times \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-2} \xi^{\frac{N}{p} + \beta + 2}} \left(\int_0^\infty |f(v\omega)|^p v^{N-1} dv \right)^{\frac{1}{p}}. \end{aligned}$$

Let us observe that, by Lemma 4.1 and the assumption $\beta < \frac{N}{p'} - 2$, we have

$$\begin{aligned} & \frac{1}{N(N-2)\omega_N} \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-2} \xi^{\frac{N}{p} + \beta + 2}} \\ & = \frac{p^2}{(N+\beta p)(Np - N - \beta p - 2p)}. \end{aligned} \tag{4.3}$$

By applying Jensen’s inequality with respect to probability measures

$$\frac{\xi^{N-1}}{c|\eta - \xi\omega|^{N-2} \xi^{\frac{N}{p} + \beta + 2}} d\xi d\omega,$$

where c is the right-hand side in (4.3), we obtain

$$\begin{aligned} & \int_0^\infty |u(s\eta)|^p s^{\beta p + N - 1} ds \leq \left(\frac{2+\beta}{\alpha-2+\beta} \right)^{\frac{(2+\beta)p}{\alpha}} \left(\frac{\alpha-2+\beta}{\alpha+2\beta} \right)^p \frac{c^{p-1}}{N(N-2)\omega_N} \\ & \times \int_{S_{N-1}} d\omega \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-2} \xi^{\frac{N}{p} + \beta + 2}} \int_0^\infty |f(v\omega)|^p v^{N-1} dv. \end{aligned}$$

By integrating with respect to η on S_{N-1} , we obtain

$$\int_{\mathbb{R}^N} |u(x)|^p |x|^{\beta p} ds \leq \left(\frac{2 + \beta}{\alpha - 2 + \beta}\right)^{\frac{(2+\beta)p}{\alpha}} \left(\frac{\alpha - 2 + \beta}{\alpha + 2\beta}\right)^p \frac{c^{p-1}}{N(N - 2)\omega_N} \\ \times \int_{S_{N-1}} d\omega \int_{S_{N-1}} d\eta \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-2} \xi^{\frac{N}{p} + \beta + 2}} \int_0^\infty |f(v\omega)|^p v^{N-1} dv.$$

A simple change of variables gives

$$\int_{S_{N-1}} d\eta \int_0^\infty \frac{\xi^{N-1} d\xi}{|\eta - \xi\omega|^{N-2} \xi^{\frac{N}{p} + \beta + 2}} = \int_{S_{N-1}} d\eta \int_0^\infty \frac{t^{N-1} dt}{|\eta t - \omega|^{N-2} t^{\frac{N}{p} - \beta}} \\ = \int_{\mathbb{R}^N} \frac{dy}{|y - \omega|^{N-2} |y|^{\frac{N}{p} - \beta}}.$$

By applying Lemma 4.1 again it follows that that

$$\int_{\mathbb{R}^N} |u(x)|^p |x|^{\beta p} dx \leq C^p \int_{\mathbb{R}^N} |f(x)|^p dx$$

with

$$C = \frac{p^2}{(N + \beta p)(Np - N - \beta p - 2p)} \left(\frac{2 + \beta}{\alpha - 2 + \beta}\right)^{\frac{2+\beta}{\alpha}} \frac{\alpha - 2 + \beta}{\alpha + 2\beta}. \tag{4.4}$$

The L^p norm of $|\cdot|^\gamma \nabla u(\cdot)$ is estimated in a similar way. □

By the previous lemma, the following estimate for the L^p -norm of the operator T immediately follows.

Corollary 4.3. *Let $\alpha \geq 2$ and $N/(N - 2) < p < \infty$. Then*

$$\|T\|_p \leq \kappa^{-1} \left(\frac{2}{\alpha - 2}\right)^{\frac{2}{\alpha}} \frac{\alpha - 2}{\alpha} \frac{p^2}{N(Np - N - 2p)}.$$

Proof. The estimate follows by setting $\beta = 0$ in (4.4) and estimating m from below with κ . □

Remark 4.4. The estimate with the constant C given by (4.4) is stable as $p \rightarrow \infty$ only if $\beta > 0$. On the other hand, the operator T is bounded also in L^∞ (and its norm will be computed later in Proposition 6.4). It is possible to prove that the operator T (with $\beta = 0$) is of weak-type $p - p$ with $p = N/(N - 2)$ and then interpolate between $N/(N - 2)$ and ∞ to obtain stable estimates for large p . The weak-type estimate is deduced as follows. Write Tf as the Riesz potential I_2

applied to the function $f(x)/(1 + |x|^\alpha)$ to get, using the classical estimate of the Riesz potentials through the Hardy-Littlewood maximal function M ,

$$|Tf(x)| \leq C \left(M \left(\frac{f(\cdot)}{1 + |\cdot|^\alpha} \right) (x) \right)^{1-2/N} \left\| \frac{f(\cdot)}{1 + |\cdot|^\alpha} \right\|_1^{2/N}$$

then Holder inequality to control the L^1 -norms in terms of the $L^{N/(N-2)}$ -norm of f and the weak $1 - 1$ estimate for M . Such a proof works only for $\beta = 0$ and gives constants depending on those of the Marzinkiewicz interpolation theorem and of the Hardy-Littlewood maximal function.

We can now prove the invertibility of L on $D_{p,\max}(L)$, defined in (3.1).

Proposition 4.5. *Let $\alpha > 2$ and $N/(N - 2) < p < \infty$. The operator L is closed and invertible on $D_{p,\max}(L)$ and the inverse of $-L$ is the operator T defined in (4.1).*

Proof. The closedness of L on $D_{p,\max}(L)$ follows from local elliptic regularity. If $u \in D_{p,\max}(L)$ satisfies $Lu = 0$, then $\Delta u = 0$ and then $u = 0$, since $u \in L^p$. This shows the injectivity of L . Finally, let $f \in L^p$ and $f_n \in C_c^\infty(\mathbb{R}^N)$ be such that $f_n \rightarrow f$ in L^p . Then $u_n = Tf_n \rightarrow u = Tf$ in L^p , since T is bounded (apply Lemma 4.2 with $\beta = 0$). By elementary potential theory

$$\Delta u_n(x) = \frac{f_n(x)}{m(x)(1 + |x|^\alpha)}$$

hence $u_n \in D_{p,\max}(L)$ and $Lu_n = f_n$. By the closedness of L , $u \in D_{p,\max}(L)$ and $Lu = f$. □

Theorem 4.6. *Let $\alpha > 2$, $N/(N - 2) < p < \infty$ and $\lambda \geq 0$. The operator $\lambda - L$ is invertible on $D_{p,\max}(L)$ and its inverse is a positive operator. Moreover, if $f \in L^p \cap C_0(\mathbb{R}^N)$, then $(\lambda - L)^{-1}f = (\lambda - L_{\min})^{-1}f$.*

Proof. Let ρ be the resolvent set of $(L, D_{p,\max}(L))$ and observe that the proposition above shows that $0 \in \rho$. Lemma 3.2 with $\hat{D} = D_{p,\max}(L)$ shows that if $0 \leq \lambda \in \rho$, then $(\lambda - L)^{-1} \geq 0$ and hence, by the resolvent equation, $(\lambda - L)^{-1} \leq (-L)^{-1} = T$ and therefore

$$\|(\lambda - L)^{-1}\| \leq \|T\| \tag{4.5}$$

where the norm above is the operator norm in L^p . Let $E = [0, \infty[\cap\rho$. Then E is non empty and open in $[0, \infty[$, since ρ is open, and closed since the operator norm of $(\lambda - L)^{-1}$ is bounded in E . Then $E = [0, \infty[$. To show the consistency of the resolvents we take $f \in C_c^\infty(\mathbb{R}^N)$ and let $u = (\lambda - L)^{-1}f$. As in Lemma 3.2 we see that $u \in C_0(\mathbb{R}^N) \cap C_{\text{loc}}^{2,\beta}(\mathbb{R}^N)$ for any $\beta < 1$. If $\text{supp } f \subset \mathbb{R}^N \setminus B(R)$, then the equation $Lu = \lambda u$ holds outside $B(R)$ and shows that u belongs to $D_{\max}(L) \cap C_0(\mathbb{R}^N)$, which is the domain of L_{\min} in $C_0(\mathbb{R}^N)$, see Proposition 2.2 (iii). Therefore $(\lambda - L_{\min})^{-1}f = u = (\lambda - L)^{-1}f$. By density, this equality extends to all functions $f \in L^p \cap C_0(\mathbb{R}^N)$. □

It is worth mentioning that the resolvents of L in L^p and L^q are consistent, provided that $p, q > N/(N - 2)$. This easily follows from above, together with a simple approximation argument, since both resolvents are consistent with $(\lambda - L_{\min})^{-1}$. Observe also that estimate (4.5) shows only that the resolvent is bounded on $[0, \infty[$ and is not sufficient to apply the Hille-Yosida theorem and prove results for parabolic problems.

5. Domain characterization

The main result of this section consists in showing that, for $N \geq 3, 1 < p < \infty, 2 < \alpha < N/p'$, the maximal domain $D_{p,\max}(L)$ defined in (3.1) coincides with the weighted Sobolev space D_p defined by

$$D_p = \{u \in W_{\text{loc}}^{2,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) : (1+|x|^{\alpha-2})u, (1+|x|^{\alpha-1})\nabla u, (1+|x|^\alpha)D^2u \in L^p(\mathbb{R}^N)\}$$

and endowed with its canonical norm

$$\begin{aligned} \|u\|_{D(L_p)} &= \|(1+|x|^{\alpha-2})u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^{\alpha-1})\nabla u\|_{L^p(\mathbb{R}^N)} + \|(1+|x|^\alpha)D^2u\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

As a tool we shall also use the space

$$\begin{aligned} \hat{D}_p &= \{u \in W_{\text{loc}}^{2,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) : (1+|x|^{\alpha-2})u, (1+|x|^{\alpha-1})\nabla u, (1+|x|^\alpha)\Delta u \in L^p(\mathbb{R}^N)\} \end{aligned}$$

endowed with its canonical norm, too.

Remark 5.1. Observe that the assumption $2 < \alpha < N/p'$ forces p to be strictly greater than $\frac{N}{N-2}$, according with Proposition 3.4. Observe also that the condition $\alpha < N/p'$ is equivalent to $p > N/(N - \alpha)$.

Lemma 5.2. *The space $C_c^\infty(\mathbb{R}^N)$ is dense both in D_p, \hat{D}_p with respect to their canonical norm.*

Proof. We give a proof for D_p , that for \hat{D}_p being identical. Let us first observe that a function $u \in W^{2,p}(\mathbb{R}^N)$ with compact support can be approximated by a sequence of C^∞ functions with compact support, in the $D(A_p)$ norm. Indeed, if ρ_n are standard mollifiers, $u_n = \rho_n * u \in C_c^\infty(\mathbb{R}^N)$, $\text{supp } u_n \subset \text{supp } u + B(1)$ for any $n \in \mathbb{N}$ and $u_n \rightarrow u$ in D_p since $(1+|x|^{\alpha-2}), (1+|x|^{\alpha-1}), (1+|x|^\alpha)$ are bounded (uniformly with respect to n) on $\text{supp } u + B(1)$. Next we show that any function u in D_p can be approximated, with respect to the norm of D_p , by a sequence of functions in $W^{2,p}(\mathbb{R}^N)$ each having a compact support. Let η be a smooth function such that $\eta = 1$ in $B(1), \eta = 0$ in $\mathbb{R}^N \setminus B(2), 0 \leq \eta \leq 1$ and set $\eta_n(x) = \eta(\frac{x}{n})$. If $u \in D_p$, then $u_n = \eta_n u$ are compactly supported functions in

$W^{2,p}(\mathbb{R}^N)$, $u_n \rightarrow u$ in $L^p(\mathbb{R}^N)$, $(1 + |x|^{\alpha-2})u_n \rightarrow (1 + |x|^{\alpha-2})u$ in $L^p(\mathbb{R}^N)$ by dominated convergence. Concerning the convergence of the derivatives we have

$$(1 + |x|^{\alpha-1})\nabla u_n = \frac{1}{n}(1 + |x|^{\alpha-1})\nabla\eta\left(\frac{x}{n}\right)u + (1 + |x|^{\alpha-1})\eta\left(\frac{x}{n}\right)\nabla u.$$

As before,

$$(1 + |x|^{\alpha-1})\eta\left(\frac{x}{n}\right)\nabla u \rightarrow (1 + |x|^{\alpha-1})\nabla u$$

in $L^p(\mathbb{R}^N)$. For the left term, since $\nabla\eta(x/n)$ can be different from zero only for $n \leq |x| \leq 2n$ we have

$$\frac{1}{n}(1 + |x|^{\alpha-1})\left|\nabla\eta\left(\frac{x}{n}\right)\right||u| \leq C(1 + |x|^{\alpha-2})|u|\chi_{\{n \leq |x| \leq 2n\}},$$

and the right hand side tends to 0 as $n \rightarrow \infty$. A similar argument shows the convergence of the second order derivatives (or the Laplacian) in the weighted L^p norm. \square

We now prove that L is closed on D_p and that $D_p = \hat{D}_p$.

Proposition 5.3. *Assume that $2 < \alpha < N/p'$. Then $D_p = \hat{D}_p$ and there exists a positive constant C such that for any $u \in D_p$*

$$\begin{aligned} &\|(1 + |x|^{\alpha-2})u\|_{L^p(\mathbb{R}^N)} + \|(1 + |x|^{\alpha-1})\nabla u\|_{L^p(\mathbb{R}^N)} + \|(1 + |x|^\alpha)D^2u\|_{L^p(\mathbb{R}^N)} \\ &\leq C\|Lu\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$ and set $f = -Lu$. Then

$$-\Delta u(x) = \frac{f(x)}{m(x)(1 + |x|^\alpha)}.$$

By elementary potential theory, u is given by (4.1). By setting $\beta = \alpha - 2$ and $\gamma = \alpha - 1$ in Lemma 4.2 and since $\alpha < \frac{N}{p'}$, we deduce that

$$\|(1 + |x|^{\alpha-2})u\|_{L^p(\mathbb{R}^N)} \leq C\|Lu\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{\hat{D}_p},$$

and

$$\|(1 + |x|^{\alpha-1})\nabla u\|_{L^p(\mathbb{R}^N)} \leq C\|Lu\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{\hat{D}_p}$$

for every $u \in C_c^\infty(\mathbb{R}^N)$. In order to prove the estimates of the second order derivatives, we apply the classical Calderón- Zygmund inequality to $(1 + |x|^\alpha)u$ and the

estimates of the lower order derivates obtained above. We deduce

$$\begin{aligned}
 & \|(1 + |x|^\alpha)D^2u\|_{L^p(\mathbb{R}^N)} \\
 & \leq C(\alpha)[\|D^2((1 + |x|^\alpha)u)\|_{L^p(\mathbb{R}^N)} \\
 & \quad + \|(1 + |x|^{\alpha-1})\nabla u\|_{L^p(\mathbb{R}^N)} + \|(1 + |x|^{\alpha-2})u\|_{L^p(\mathbb{R}^N)}] \\
 & \leq C(N, p, \alpha)[\|\Delta((1 + |x|^\alpha)u)\|_{L^p(\mathbb{R}^N)} + \|Lu\|_{L^p(\mathbb{R}^N)}] \\
 & \leq C(N, p, \alpha)[\|(1 + |x|^\alpha)\Delta u\|_{L^p(\mathbb{R}^N)} + \|Lu\|_{L^p(\mathbb{R}^N)}] \\
 & \leq C(N, p, \alpha)\|Lu\|_{L^p(\mathbb{R}^N)} \leq C(N, p, \alpha)\|u\|_{\hat{D}_p}.
 \end{aligned}$$

We have therefore proved that the graph norm of L , the D_p -norm and the \hat{D}_p -norm are all equivalent on $C_c^\infty(\mathbb{R}^N)$. Since, by Lemma 5.2, $C_c^\infty(\mathbb{R}^N)$ is a core both for D_p and \hat{D}_p , the proof is complete. \square

Finally we characterize the domain of L .

Theorem 5.4. *If $2 < \alpha < N/p'$, then D_p coincides with the maximal domain in L^p , that is*

$$D_p = \{u \in L^p \cap W_{\text{loc}}^{2,p} : Lu \in L^p\}.$$

Proof. The inclusion $D_p \subset D_{p,\max}(L)$ is obvious. Since the inverse of the operator $(L, D_{p,\max}(L))$ is the operator $-T$ defined in (4.1), see Proposition 4.5, it is sufficient to show that T maps L^p into D_p . Let $f \in L^p$ and $u = Tf$. We use Lemma 4.2 with $\beta = \alpha - 2$ and $\gamma = \alpha - 1$ and the equality $Lu = -f$ to deduce that $u \in \hat{D}_p$. Since $\hat{D}_p = D_p$, the thesis is achieved. \square

Next we show that if $m \equiv 1$ and $\alpha \geq N/p'$ then D_p is properly contained in $D_{p,\max}(L)$. This fact depends simply on the decay at infinity of the fundamental solution of the Laplace equation, together with the invertibility of L .

Proposition 5.5. *Let $m \equiv 1$, $N \geq 3$, $p > N/(N - 2)$, $\alpha \geq N/p'$. Then D_p is a proper subset of $D_{p,\max}(L)$.*

Proof. Observe that $\alpha > 2$. Let $\chi_{B(1)} \leq f \leq \chi_{B(2)}$ be a smooth radial function. Denote by u the unique solution in $D_{p,\max}(L)$ of $Lu(\rho) = (1 + \rho^\alpha)f(\rho)$, see Proposition 4.5. Since the datum f is radial, by uniqueness, the solution u is radial too, hence it solves

$$u''(\rho) + \frac{N-1}{\rho}u'(\rho) = f(\rho).$$

For $\rho \geq 2$, u solves the homogeneous equation

$$u''(\rho) + \frac{N-1}{\rho}u'(\rho) = 0,$$

hence it is given by $u = c\rho^{2-N}$ for some positive c (note that $c > 0$ by the strong maximum principle [5, Theorem 3.5]). Then

$$\begin{aligned} \int_{\mathbb{R}^N} |(1 + |x|^{\alpha-2})u|^p dx &\geq \alpha \int_2^\infty (1 + \rho^{\alpha-2})^p \rho^{p(2-N)} \rho^{N-1} d\rho \\ &\geq C \int_2^\infty \rho^{p\alpha - Np + N - 1} d\rho. \end{aligned}$$

The last integral converges if and only if $\alpha < \frac{N}{p'}$. In a similar way one can show that $(1 + |x|^{\alpha-1})\nabla u$ and $(1 + |x|^\alpha)D^2u$ are not in $L^p(\mathbb{R}^N)$. \square

A partial characterization of $D_{p,\max}(L)$ can be obtained from Lemma 4.2.

Proposition 5.6. *Let $N \geq 3$, $p > N/(N-2)$, $\alpha \geq N/p'$. If $0 \leq \beta < N/p' - 2$ and $0 \leq \gamma < N/p' - 1$, then $|x|^\beta u$ and $|x|^\gamma \nabla u$ belong to L^p , for every $u \in D_{p,\max}(L)$.*

Proof. This follows immediately from Lemma 4.2, since the operator $-T$ defined in (4.1) is the inverse of $(L, D_{p,\max}(L))$. \square

Observe that for $\beta = \gamma = 0$ the above result has been already proved in Proposition 3.1.

If $\alpha > N/p'$, $\alpha > 2$, then $C_c^\infty(\mathbb{R}^N)$ is not a core for L . This fact also gives $D_p \neq D_{p,\max}(L)$ in this case.

Proposition 5.7. *Let $\alpha > N/p'$. Then $L(C_c^\infty(\mathbb{R}^N))$ is not dense in $L^p(\mathbb{R}^N)$.*

Proof. It is sufficient to observe that $0 \neq \frac{1}{m(x)(1+|x|^\alpha)} \in L^{p'}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \frac{Lu}{m(x)(1 + |x|^\alpha)} dx = \int_{\mathbb{R}^N} \Delta u dx = 0$$

for every $u \in C_c^\infty(\mathbb{R}^N)$. \square

Proposition 5.8. *Let $\alpha = N/p'$. Then $L(C_c^\infty(\mathbb{R}^N))$ is dense in $L^p(\mathbb{R}^N)$.*

Proof. Let $g \in L^{p'}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} gLu dx = 0$$

for every $u \in C_c^\infty(\mathbb{R}^N)$. It follows that $\Delta(gm(1 + |x|^\alpha)) = 0$ in the distributional sense, hence in a classical sense. Set $h(x) = g(x)m(x)(1 + |x|^\alpha)$. Since h is an harmonic function, it satisfies

$$|\nabla h(0)| \leq \frac{C}{R^{N+1}} \int_{B(0,R)} |h| dx$$

for every $R > 0$. By assumption $g = \frac{h}{m(x)(1+|x|^\alpha)} \in L^{p'}(\mathbb{R}^N)$, therefore Hölder's inequality yields

$$\begin{aligned} |\nabla h(0)| &\leq \frac{C}{R^{N+1}} \int_{B(0,R)} \frac{|h|}{1+|x|^\alpha} (1+|x|^\alpha) dx \leq CR^{-N-1+\frac{N}{p}+\alpha} \\ &= CR^{\alpha-\frac{N}{p'}-1} = CR^{-1}. \end{aligned}$$

Letting R to infinity, we deduce that $|\nabla h(0)| = 0$. In a similar way one proves that $|\nabla h(x_0)| = 0$ for every $x_0 \in \mathbb{R}^N$. It means that $h = C$ for some constant C and $g = \frac{C}{m(x)(1+|x|^\alpha)} \in L^{p'}(\mathbb{R}^N)$. Since $\alpha = N/p'$, then $C = 0$ and the proof is complete. \square

It can be proved that the a-priori estimates of Proposition 5.3 for $p = 2$ still hold in $C_c^\infty(\mathbb{R}^N)$ if $\alpha \neq N/2$ and $m \equiv 1$. However, since $C_c^\infty(\mathbb{R}^N)$ is not a core for $(L, D_{2,\max}(L))$, for $\alpha > N/2$ they do not extend to the domain of L . Next we show that the a-priori estimates fail even in $C_c^\infty(\mathbb{R}^N)$ if $\alpha = N/p'$, which is a core by the Proposition above.

Proposition 5.9. *Let $m \equiv 1$, $p > N/(N - 2)$, $N \geq 3$, $\alpha = N/p'$. Then the estimates in Proposition 5.3 do not hold on $C_c^\infty(\mathbb{R}^N)$.*

Proof. Observe that $\alpha > 2$. Let $R \geq 2$, $\phi \in C_c^\infty(\mathbb{R}^N)$ be a radial function such that $\phi = 1$ in $B(R) \setminus B(2)$, $\phi = 0$ in $B(1) \cup (\mathbb{R}^N \setminus B(2R))$, $\|\phi'_R\|_\infty \leq \frac{C}{R}$, $\|\phi''_R\|_\infty \leq \frac{C}{R^2}$ and set $u(\rho) = \phi_R \rho^{2-N}$, $N > 2$. Then $u \in C_c^\infty(\mathbb{R}^N)$ (we omit to indicate the dependence of u on R) and

$$\Delta u = u'' + \frac{N-1}{\rho} u' = \phi''_R(\rho) \rho^{2-N} + (3-N)\phi'_R(\rho) \rho^{1-N}.$$

A straightforward computation shows that, for $\alpha = \frac{N}{p'}$,

$$\int_{\mathbb{R}^N} (|u|^p + |(1+|x|^\alpha)\Delta u|^p) dx \leq C$$

with C independent of R . On the other hand $u'(\rho) = \phi'_R(\rho) \rho^{2-N} + \phi_R(2-N)\rho^{1-N}$ and

$$\begin{aligned} &\int_0^\infty (1+\rho^{\alpha-1})^p \rho^{N-1} |u'(\rho)|^p d\rho \\ &= \int_1^{2R} (1+\rho^{\alpha-1})^p \rho^{N-1} \left| \phi'_R(\rho) \rho^{2-N} + \phi_R(2-N)\rho^{1-N} \right|^p d\rho. \end{aligned}$$

The last integral tends to ∞ as $R \rightarrow \infty$ since

$$\int_1^{2R} (1+\rho^{\alpha-1})^p \rho^{N-1} \left| \phi_R(2-N)\rho^{1-N} \right|^p d\rho \geq C \log R.$$

Therefore the L^p -norm of $(1 + |x|^{\alpha-1})\nabla u$ cannot be controlled by the L^p -norms of u and Lu on $C_c^\infty(\mathbb{R}^N)$. Similarly one shows that the L^p -norm of $(1 + |x|^\alpha)D^2u$ cannot be controlled by the L^p -norms of u and Lu . \square

6. The operator in $C_0(\mathbb{R}^N)$

Let

$$D(L) = D_{\max}(L) \cap C_0(\mathbb{R}^N)$$

be the generator of $(T_{\min}(t))_{t \geq 0}$ in C_0 , see Proposition 2.2 (iii) and note that we need the only restriction $N, \alpha > 2$. As in the L^p -case we give a description of the domain when $\alpha < N$ and a partial description when $\alpha \geq N$. We need the analogous of Lemma 4.2 for $p = \infty$

Lemma 6.1. *Let $\gamma, \beta > 0$ such that $\gamma < N$ and $\gamma + \beta > N$. Set*

$$J(x) = \int_{\mathbb{R}^N} \frac{dy}{|x - y|^\gamma (1 + |y|^\beta)}.$$

Then J is bounded in \mathbb{R}^N and has the following behaviour as $|x|$ goes to infinity

$$J(x) \simeq \begin{cases} c_1|x|^{N-(\gamma+\beta)} & \text{if } \beta < N \\ c_2|x|^{-\gamma} \log|x| & \text{if } \beta = N \\ c_3|x|^{-\gamma} & \text{if } \beta > N \end{cases}$$

for suitable positive constants c_1, c_2, c_3 .

Proof. Since $\frac{1}{|y|^\gamma}$ and $\frac{1}{1+|y|^\beta}$ are radial decreasing $J(x) \leq J(0) < \infty$. In order to prove the asymptotic behaviour, we write J in spherical coordinates. Set $x = s\eta$, $y = \rho\omega$ with $s, \rho \in [0, +\infty)$, $\eta \in S_{N-1}$, then

$$\begin{aligned} J(s\eta) &= \int_{S_{N-1}} d\omega \int_0^\infty \frac{\rho^{N-1} d\rho}{(1 + \rho^\alpha)|s\eta - \rho\omega|^{N-2}} \\ &= \int_{S_{N-1}} d\omega \int_0^\infty \frac{s^N \xi^{N-1} d\xi}{s^\gamma (1 + (s\xi)^\beta) |\eta - \xi\omega|^\gamma} = \int_{S_{N-1}} d\omega \int_0^\infty \frac{s^{N-\gamma} \xi^{N-1} d\xi}{(1 + s^\beta \xi^\beta) |e_1 - \xi\omega|^\gamma} \\ &= \int_{S_{N-1}} d\omega \int_0^{\frac{1}{2}} \frac{s^{N-\gamma} \xi^{N-1} d\xi}{(1 + s^\beta \xi^\beta) |e_1 - \xi\omega|^\gamma} + \int_{S_{N-1}} d\omega \int_{\frac{1}{2}}^\infty \frac{s^{N-\gamma} \xi^{N-1} d\xi}{(1 + s^\beta \xi^\beta) |e_1 - \xi\omega|^\gamma}. \end{aligned}$$

Set

$$J_1(s\eta) = \int_{S_{N-1}} d\omega \int_0^{\frac{1}{2}} \frac{s^{N-\gamma} \xi^{N-1} d\xi}{(1 + s^\beta \xi^\beta) |e_1 - \xi\omega|^\gamma}$$

and

$$J_2(s\eta) = \int_{S_{N-1}} d\omega \int_{\frac{1}{2}}^\infty \frac{s^{N-\gamma} \xi^{N-1} d\xi}{(1 + s^\beta \xi^\beta) |e_1 - \xi\omega|^\gamma}.$$

Concerning J_2 , we have

$$\lim_{s \rightarrow +\infty} s^{\gamma+\beta-N} J_2 = \int_{S_{N-1}} d\omega \int_{\frac{1}{2}}^{\infty} \frac{\xi^{N-1} d\xi}{\xi^\beta |e_1 - \xi\omega|^\gamma} = \int_{\mathbb{R}^N \setminus B(0, \frac{1}{2})} \frac{dy}{|y|^\beta |e_1 - y|^\gamma} = C > 0$$

for some positive constant C . Therefore

$$J_2(x) \simeq C|x|^{N-(\gamma+\beta)} \tag{6.1}$$

as $|x| \rightarrow \infty$. Let us estimate the remaining term. We have

$$J_1(s\eta) = \int_{S_{N-1}} d\omega \int_0^{\frac{1}{2}} \frac{s^{N-\gamma} \xi^{N-1} d\xi}{(1+s^\beta \xi^\beta) |e_1 - \xi\omega|^\gamma} = s^{-\gamma} \int_{S_{N-1}} d\omega \int_0^{\frac{s}{2}} \frac{t^{N-1} dt}{(1+t^\beta) |e_1 - \frac{t}{s}\omega|^\gamma}.$$

Since $\frac{1}{2} \leq |e_1 - \frac{t}{s}\omega| \leq \frac{3}{2}$,

$$c_1 J_1 \leq s^{-\gamma} \int_0^{\frac{s}{2}} \frac{t^{N-1} dt}{(1+t^\beta)} \leq c_2 J_1$$

for some positive c_1, c_2 . Evidently

$$s^{-\gamma} \int_0^{\frac{s}{2}} \frac{t^{N-1} dt}{(1+t^\beta)} \simeq \begin{cases} |s|^{N-(\gamma+\beta)} & \text{if } \beta < N, \\ |s|^{-\gamma} \log |s| & \text{if } \beta = N, \\ |s|^{-\gamma} & \text{if } \beta > N \end{cases}$$

as s goes to infinity. From (6.1) and the last estimate the asymptotic behaviour of J follows. □

The following two results are deduced from the lemma above as Theorem 5.4 and Proposition 5.6 are deduced from Lemma 4.2.

Theorem 6.2. *Let $2 < \alpha < N$. Then*

$$D(L) = \{u \in C_0 : (1 + |x|^{\alpha-2})u, (1 + |x|^{\alpha-1})\nabla u, (1 + |x|^\alpha)\Delta u \in C_0\}$$

Proposition 6.3. *Let $N \geq 3, \alpha \geq N$. If $0 \leq \beta < N - 2$ and $0 \leq \gamma < N - 1$, then for every $u \in D(L)$, $|x|^\beta u$ and $|x|^\gamma \nabla u$ belong to C_0 .*

Finally, in the case when $m \equiv 1$, we compute the operator norm in C_0 of the operator $T = (-L)^{-1}$ defined in (4.1)

Proposition 6.4. *If $N \geq 3$ and $\alpha > 2$ then*

$$\|T\|_\infty = \frac{\pi}{(N-2)\alpha \sin\left(\frac{2}{\alpha}\pi\right)}.$$

Proof. We have

$$\begin{aligned} \|T\| &= \frac{1}{N(N-2)\omega_N} \sup_{x \in \mathbb{R}^N} J(x) = \frac{1}{N(N-2)\omega_N} J(0) \\ &= \frac{1}{N(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{dy}{|y|^{N-2}(1+|y|^\alpha)}. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{N\omega_N} \int_{\mathbb{R}^N} \frac{dy}{|y|^{N-2}(1+|y|^\alpha)} &= \int_0^\infty \frac{s^{N-1}}{s^{N-2}(1+s^\alpha)} ds \\ &= \frac{1}{\alpha} \int_0^\infty \frac{t^{\frac{2}{\alpha}-1}}{1+t} dt = \frac{\pi}{\alpha \sin\left(\frac{2}{\alpha}\pi\right)} \end{aligned}$$

the proof is complete. □

7. Discreteness and location of the spectrum

Throughout this section, to unify the notation, when $p = \infty$, L^p stands for C_0 and $D_{p,\max}(L)$ for $D(L)$.

Proposition 7.1. *If $N/(N-2) < p < \infty$, $2 < \alpha \leq \infty$, then the resolvent of $(L, D_{p,\max}(L))$ is compact in L^p .*

Proof. Let us prove that $D_{p,\max}(L)$ is compactly embedded into L^p for $p < \infty$. Let \mathcal{U} be a bounded subset of $D_{p,\max}(L)$. Fixing $0 < \beta < \alpha - 2$, $N/p' - 2$ in Lemma 4.2 we obtain $\int_{\mathbb{R}^N} |(1+|x|^\beta)u|^p \leq M$ for some positive M and for every $u \in \mathcal{U}$. Then, given $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|>R} |u|^p < \varepsilon^p$$

for every $u \in \mathcal{U}$. Let \mathcal{U}' be the set of the restrictions of the functions in \mathcal{U} to $B(R)$. Since the embedding of $W^{2,p}(B(R))$ into $L^p(B(R))$ is compact, the set \mathcal{U}' which is bounded in $W^{2,p}(B(R))$ is totally bounded in $L^p(B(R))$. Therefore there exist $n \in \mathbb{N}$, $f_1, \dots, f_n \in L^p(B(R))$ such that

$$\mathcal{U}' \subseteq \bigcup_{i=1}^n \{f \in L^p(B(R)) : \|f - f_i\|_{L^p(B(R))} < \varepsilon\}.$$

Set $\bar{f}_i = f_i$ in $B(R)$ and $\bar{f}_i = 0$ in $\mathbb{R}^N \setminus B(R)$. Then $\bar{f}_i \in L^p(\mathbb{R}^N)$ and

$$\mathcal{U} \subseteq \bigcup_{i=1}^n \{f \in L^p(\mathbb{R}^N) : \|f - \bar{f}_i\|_{L^p(\mathbb{R}^N)} < 2\varepsilon\}.$$

It follows that \mathcal{U} is relatively compact in $L^p(\mathbb{R}^N)$. The compactness of the resolvent of $(L, D(L))$ in C_0 follows similarly from the results of the previous section or from ([10, Example 7.3]). \square

Clearly, the spectrum of L consists of eigenvalues. Let us show that it is independent of p .

Corollary 7.2. *If $N/(N - 2) < p \leq \infty$, $2 < \alpha < \infty$, then the spectrum of $(L, D_{p,\max}(L))$ is independent of p .*

Proof. Let ρ_p, ρ_q be the resolvent sets in L^p, L^q , respectively. Then $0 \in \rho_p \cap \rho_q$ and the inverse of L in L^p and in L^q is given by the operator $-T$ defined in (4.1), see Proposition 4.5. This shows the consistency of the resolvents at 0 and, since $\rho_p \cap \rho_q$ is connected, the consistency of the resolvents at any point of $\rho_p \cap \rho_q$, see [1, Proposition 2.2]. An application of [1, Proposition 2.6] concludes the proof. \square

In order to have more information on the spectrum of L , we introduce the Hilbert space L^2_μ , where $d\mu(x) = (m(x)(1 + |x|^\alpha))^{-1} dx$, endowed with its canonical inner product. Note that the measure μ is finite if and only if $\alpha > N$. We consider also the Sobolev space

$$H = \{u \in L^2_\mu : \nabla u \in L^2\}$$

endowed with the inner product

$$(u, v)_H = \int_{\mathbb{R}^N} (u\bar{v} d\mu + \nabla u \cdot \nabla \bar{v} dx)$$

and let \mathcal{V} be the closure of C_c^∞ in H , with respect to the norm of H . Observe that Sobolev inequality

$$\|u\|_{2^*}^2 \leq C_2^2 \|\nabla u\|_2^2 \tag{7.1}$$

holds in \mathcal{V} but not in H (consider for example the case where $\alpha > N$ and $u = 1$). Here $2^* = 2N/(N - 2)$ and C_2 is the best constant for which the equality above holds.

Lemma 7.3. *If $\alpha > 2$, the embedding of \mathcal{V} in L^2_μ is compact.*

Proof. The proof is very similar to that of Proposition 7.1 once one notes that on any ball $B(R)$ the measure μ is bounded above and below from zero. Therefore, it suffices to show that given \mathcal{U} a bounded subset of \mathcal{V} and $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|>R} |u|^2 d\mu < \varepsilon^2$$

for every $u \in \mathcal{U}$. This easily follows from (7.1) since

$$\int_{|x|>R} |u|^2 d\mu \leq \kappa^{-1} \left(\int_{|x|>R} |u|^{\frac{2N}{N-2}} dx \right)^{1-\frac{2}{N}} \left(\int_{|x|>R} \frac{1}{(1 + |x|^\alpha)^{\frac{N}{2}}} dx \right)^{\frac{2}{N}} . \square$$

Next we introduce the continuous and weakly coercive symmetric form

$$a(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v} \, dx \tag{7.2}$$

for $u, v \in \mathcal{V}$ and the self-adjoint operator \mathcal{L} defined by

$$D(\mathcal{L}) = \left\{ u \in L^2_\mu : \text{there exists } f \in L^2_\mu : a(u, v) = - \int_{\mathbb{R}^N} f \bar{v} \, d\mu \text{ for every } v \in \mathcal{V} \right\}$$

$$\mathcal{L}u = f.$$

Since $a(u, u) \geq 0$, the operator \mathcal{L} generates an analytic semigroup of contractions in $e^{t\mathcal{L}}$ in L^2_μ . An application of the Beurling-Deny criteria shows that the generated semigroup is positive and L^∞ -contractive. For our purposes we need to show that the resolvent of \mathcal{L} and of $(L, D_{p,\max}(L))$ are coherent. This is done in the following proposition.

Proposition 7.4.

$$D(\mathcal{L}) \subset \{u \in \mathcal{V} \cap W^{2,2}_{\text{loc}} : (1 + |x|^\alpha)\Delta u \in L^2_\mu\}$$

and $\mathcal{L}u = m(x)(1 + |x|^\alpha)u$ for $u \in D(\mathcal{L})$. If $\lambda > 0$ and $f \in L^p \cap L^2_\mu$, then

$$(\lambda - \mathcal{L})^{-1}f = (\lambda - L)^{-1}f.$$

Proof. The first part of the proposition easily follows from local elliptic regularity, testing with any $v \in C^\infty_c$ in (7.2). To show the coherence of the resolvents we consider $f \in C^\infty_c$, $\text{supp } f \subset B(R)$ and $u = (\lambda - L_{\min})^{-1}f$. Then $u \in D(L)$ solves

$$m(x)\Delta u = \frac{\lambda u}{1 + |x|^\alpha}$$

outside $B(R)$ and is a C^2 -function. Theorem 4.6 implies that $u \in D_{p,\max}(L)$ for every $p > N/(N - 2)$. If $N > 4$, then $u \in D_{2,\max}(L)$ hence $\nabla u \in L^2$, see Proposition 3.1, and clearly $u \in L^2_\mu$. This yields $u \in H$ but not yet $u \in \mathcal{V}$. To show that u can be approximated with a sequence of C^∞_c -functions, in the norm of H , we fix a smooth C^∞ function η such that $\eta \equiv 1$ in $B(1)$ and $\eta \equiv 0$ outside $B(2)$ and set $\eta_n(x) = \eta(x/n)$. Clearly $\eta_n u \rightarrow u$ in L^2_μ . Concerning the gradients we have $\nabla(\eta_n u) = \eta_n \nabla u + u \nabla \eta_n$. The term $\eta_n \nabla u$ converges to ∇u in L^2 , since $\nabla u \in L^2$ and we have to show that $u \nabla \eta_n \rightarrow 0$ in L^2 . Since $u \in L^{2^*}$ we can use Hölder’s inequality to deduce

$$\int_{\mathbb{R}^N} |u|^2 |\nabla \eta_n|^2 \, dx \leq \frac{C}{n^2} \int_{n \leq |x| \leq 2n} |u|^2 \, dx \leq C_1 \left(\int_{n \leq |x| \leq 2n} |u|^{2^*} \right)^{1 - \frac{2}{N}} \tag{7.3}$$

which tends to zero as $n \rightarrow \infty$. This shows u can be approximated with a sequence of $W^{1,2}$ compactly supported functions and to produce a sequence of smooth approximants it is now sufficient to use convolutions. Then $u \in \mathcal{V}$ and, by integration by parts,

$$a(u, v) = -(\lambda u - f, v)_{L^2_\mu},$$

that is $u \in D(\mathcal{L})$ and $\lambda u - \mathcal{L}u = f$. By density, this shows the coherence of the resolvents of L_{\min} and \mathcal{L} for $\lambda > 0$, hence of \mathcal{L} and $(L, D_{p,\max}(L))$, see Theorem 4.6. The cases $N = 3, 4$ require some variants, since $p = 2$ does not satisfy the inequality $p > N/(N - 2)$. To show that u belongs to L^2_μ we use the fact that $u \in L^p$ with $p = 2N/(N - 2)$ and therefore

$$\int_{\mathbb{R}^N} |u|^2 d\mu \leq \kappa^{-1} \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} dx \right)^{1-\frac{2}{N}} \left(\int_{\mathbb{R}^N} \frac{1}{(1 + |x|^\alpha)^{\frac{N}{2}}} dx \right)^{\frac{2}{N}}.$$

Next we show that ∇u belongs to L^2 . Since $u \in C^2$ and

$$\int_{\mathbb{R}^N} \nabla u \nabla v dx = \int_{\mathbb{R}^N} (\lambda u - f)v d\mu$$

for every $v \in C_c^\infty$, the same equality holds for every $v \in W^{1,2}$ having compact support. Taking $v_n = \eta_n u$ we get

$$\int_{\mathbb{R}^N} \eta_n |\nabla u|^2 dx = \int_{\mathbb{R}^N} (\lambda u - f)\eta_n u d\mu - \int_{\mathbb{R}^N} u \nabla u \cdot \nabla \eta_n dx.$$

Since

$$\int_{\mathbb{R}^N} u \nabla u \cdot \nabla \eta_n dx = -\frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \Delta \eta_n dx$$

we can proceed as in (7.3) to show that this term tends to zero and hence $\nabla u \in L^2$. From now on, the proof proceeds as in the case $N > 4$. □

We can now strengthen Corollary 7.2.

Proposition 7.5. *If $N/(N - 2) < p \leq \infty$, $2 < \alpha < \infty$, then the spectrum of $(L, D_{p,\max}(L))$ lies in $] - \infty, 0[$ and consists of a sequence λ_n of eigenvalues, which are simple poles of the resolvent and tend to $-\infty$. Each eigenspace is finite dimensional and independent of p .*

Proof. Since the resolvents of $(L, D_{p,\max}(L))$ and $(\mathcal{L}, (D(\mathcal{L})))$ are coherent and compact in L^p, L^2_μ , respectively all the assertions except the density of the eigenfunctions follow from [1, Proposition 2.2] (see also [9, Proposition 5.2] for more details). □

Observe that 0 is in the resolvent set of L , since it is injective. This is clear in L^p or C_0 because $\Delta u \in L^p$ implies $u = 0$. However, constant functions are in H if $\alpha > N$ and this explains why we work with \mathcal{V} (constant functions are never in \mathcal{V} , since \mathcal{V} embeds into L^{2^*}).

Next we show some methods to estimate the first eigenvalue λ_1 , when $m \equiv 1$.

Proposition 7.6. *Let $L = (1 + |x|^\alpha)\Delta$. The following estimates hold*

$$\lambda_1 \leq -\left(\frac{\alpha - 2}{2}\right)^{\frac{2}{\alpha}} \frac{\alpha}{\alpha - 2} \frac{(N - 2)^2}{4} \tag{7.4}$$

and

$$\lambda_1 \leq -(N - 2) \frac{\alpha \sin\left(\frac{2}{\alpha}\pi\right)}{\pi}. \tag{7.5}$$

Proof. By Corollary 4.3, we obtain

$$\|L^{-1}\| \leq \left(\frac{2}{\alpha - 2}\right)^{\frac{2}{\alpha}} \frac{\alpha - 2}{\alpha} \frac{p^2}{N(Np - N - 2p)}.$$

By classical spectral theory then

$$|\lambda_1| \geq \left(\frac{\alpha - 2}{2}\right)^{\frac{2}{\alpha}} \frac{\alpha}{\alpha - 2} \frac{N(Np - N - 2p)}{p^2}.$$

The function appearing on the right hand side attains its maximum for $p = \frac{2N}{N-2}$ where it reaches the value

$$\left(\frac{\alpha - 2}{2}\right)^{\frac{2}{\alpha}} \frac{\alpha}{\alpha - 2} \frac{(N - 2)^2}{4}.$$

Since the spectrum of L is independent of p we obtain (7.4). (7.5) is obtained in a similar way from Proposition 6.4. □

Observe that the coefficient

$$\left(\frac{\alpha - 2}{2}\right)^{\frac{2}{\alpha}} \frac{\alpha}{\alpha - 2}$$

is always greater than or equal to 1, and it is 1 for $\alpha = 2, \infty$. Then (7.4) improves the estimate $\lambda_1 \leq -(N - 2)^2/4$ which can be obtained using the classical Hardy inequality. On the other hand (7.5) is better than (7.4) for large α and small N , but worse for α close to 2 or large N .

Since \mathcal{L} is self-adjoint in L^2_μ , its eigenvalues can be computed through the Raleigh quotients and, in particular,

$$-\lambda_1 = \min \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : \int_{\mathbb{R}^N} |u|^2 d\mu = 1 \right\}.$$

Since

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^2 d\mu &\leq \left(\int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{1-2\frac{2}{N}} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|^\alpha)^{N/2}} dx \right)^{2/N} \\ &\leq C_2^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|^\alpha)^{N/2}} dx \right)^{2/N}, \end{aligned}$$

it follows that $-\lambda_1 \geq (C_2^2 L(\alpha))^{-1}$ where C_2 is given by [14] and

$$L(\alpha) = \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|^\alpha)^{N/2}} dx \right)^{2/N}.$$

Observe also that, when $\alpha \rightarrow \infty$, then (formally) λ_1 tends to the first eigenvalue of the Dirichlet Laplacian in the unit ball.

8. Dissipativity in L^p

In this section we show that $L = (s^\alpha + |x|^\alpha)\Delta$, $s > 0$, generates an analytic semigroup in L^p , for large p , which is contractive in a sector. Our proof works for more general operators like $a(x)\Delta$ whenever $a(x)$ satisfies the following pointwise estimates

$$|\nabla a(x)| \leq \alpha |x|^{\alpha-1} \quad |x|^\alpha \leq a(x). \tag{8.1}$$

Observe that the condition $\alpha \leq (p-1)(N-2)$ in the next Theorem can be rewritten as $p > (N + \alpha - 2)/(N - 2)$.

Theorem 8.1. *Let $N \geq 3$, $p > N/(N - 2)$, $2 < \alpha \leq (p - 1)(N - 2)$. If (8.1) holds, then $(L, D_{p,\max}(L))$ generates a positive semigroup of contractions in L^p . If $\alpha < (p - 1)(N - 2)$, the semigroup is also analytic.*

Proof. Observe that the positivity of the semigroup follows from Lemma 3.2, once its existence has been proved. Take $f \in L^p(\mathbb{R}^N)$, $\rho > 0$, $\lambda \in \mathbb{C}$ and consider the Dirichlet problem in $L^p(B(\rho))$

$$\begin{cases} \lambda u - Lu = f & \text{in } B(\rho), \\ u = 0 & \text{on } \partial B(\rho). \end{cases} \tag{8.2}$$

According to [5, Theorem 9.15], for $\lambda > 0$ there exists a unique solution u_ρ in $W^{2,p}(B(\rho)) \cap W_0^{1,p}(B(\rho))$. In order to show that the above problem is solvable

for complex values of λ and to obtain estimates independent of ρ , we show that $e^{\pm i\theta}L$ is dissipative in $B(\rho)$ for $0 \leq \theta \leq \theta_0$ and a suitable $0 < \theta_0 \leq \pi/2$. Set $u^* = \bar{u}_\rho |u_\rho|^{p-2}$. Multiply Lu_ρ by u^* and integrate over $B(\rho)$. The integration by parts is straightforward when $p \geq 2$. For $1 < p < 2$, $|u_\rho|^{p-2}$ becomes singular near the zeros of u_ρ . It is possible to prove the integration by parts is allowed also in this case (see [8]). Notice also that all boundary terms vanish since $u_\rho = 0$ at the boundary. So we get

$$\begin{aligned} \int_{B(\rho)} Lu_\rho u^* dx &= - \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx \\ &\quad - \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\operatorname{Im}(\bar{u}_\rho \nabla u_\rho)|^2 dx \\ &\quad - \int_{B(\rho)} \bar{u}_\rho |u_\rho|^{p-2} \nabla a(x) \nabla u_\rho dx - (p-2) \int_{B(\rho)} a(x) |u_\rho|^{p-4} \bar{u}_\rho \nabla u_\rho \operatorname{Re}(\bar{u}_\rho \nabla u_\rho) dx. \end{aligned}$$

By taking the real and imaginary part of the left and the right hand side, we have

$$\begin{aligned} \operatorname{Re} \left(\int_{B(\rho)} Lu_\rho u^* dx \right) &= -(p-1) \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx \\ &\quad - \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\operatorname{Im}(\bar{u}_\rho \nabla u_\rho)|^2 dx \\ &\quad - \int_{B(\rho)} |u_\rho|^{p-2} \nabla a(x) \operatorname{Re}(\bar{u}_\rho \nabla u_\rho) dx; \\ \operatorname{Im} \left(\int_{\mathbb{R}^N} Lu_\rho u^* dx \right) &= -(p-2) \int_{\mathbb{R}^N} a(x) |u_\rho|^{p-4} \operatorname{Im}(\bar{u}_\rho \nabla u_\rho) \operatorname{Re}(\bar{u}_\rho \nabla u_\rho) dx \\ &\quad - \int_{\mathbb{R}^N} |u_\rho|^{p-2} \nabla a(x) \operatorname{Im}(\bar{u}_\rho \nabla u_\rho) dx. \end{aligned}$$

By Hardy’s inequality as stated in Proposition 9.10,

$$\begin{aligned} \left| \int_{B(\rho)} |u_\rho|^{p-2} \nabla a(x) \operatorname{Re}(u_\rho \nabla u_\rho) dx \right| &\leq \alpha \int_{B(\rho)} |u_\rho|^{p-2} |x|^{\alpha-1} |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)| dx \\ &\leq \alpha \left(\int_{B(\rho)} |u_\rho|^{p-4} |x|^\alpha |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx \right)^{\frac{1}{2}} \left(\int_{B(\rho)} |u_\rho|^p |x|^{\alpha-2} dx \right)^{\frac{1}{2}} \\ &\leq \frac{\alpha p}{N + \alpha - 2} \int_{B(\rho)} |u_\rho|^{p-4} |x|^\alpha |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx \\ &\leq \frac{\alpha p}{N + \alpha - 2} \int_{B(\rho)} |u_\rho|^{p-4} a(x) |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx. \end{aligned}$$

It follows that

$$\begin{aligned}
 -\operatorname{Re} \left(\int_{B(\rho)} Lu_\rho u^* dx \right) &\geq \left(p-1 - \frac{\alpha p}{N+\alpha-2} \right) \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx \\
 &\quad + \int_{B(\rho)} a(x) |u_\rho|^{p-4} |\operatorname{Im}(\bar{u}_\rho \nabla u_\rho)|^2 dx
 \end{aligned}$$

and

$$\begin{aligned}
 &\left| \operatorname{Im} \left(\int_{B(\rho)} Lu_\rho u^* dx \right) \right| \\
 &\leq (p-2) \left(\int_{B(\rho)} |u_\rho|^{p-4} a(x) |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx \right)^{\frac{1}{2}} \\
 &\quad \times \left(\int_{B(\rho)} |u_\rho|^{p-4} a(x) |\operatorname{Im}(\bar{u}_\rho \nabla u_\rho)|^2 dx \right)^{\frac{1}{2}} \\
 &\quad + \alpha \int_{B(\rho)} |u_\rho|^{p-2} |x|^{\alpha-1} |\operatorname{Im}(\bar{u}_\rho \nabla u_\rho)| dx \\
 &\leq \left(p-2 + \frac{\alpha p}{N+\alpha-2} \right) \left(\int_{B(\rho)} |u_\rho|^{p-4} a(x) |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx \right)^{\frac{1}{2}} \\
 &\quad \times \left(\int_{B(\rho)} |u_\rho|^{p-4} a(x) |\operatorname{Im}(\bar{u}_\rho \nabla u_\rho)|^2 dx \right)^{\frac{1}{2}}.
 \end{aligned}$$

Setting

$$\begin{aligned}
 B^2 &= \int_{\mathbb{R}^N} |u_\rho|^{p-4} a(x) |\operatorname{Re}(\bar{u}_\rho \nabla u_\rho)|^2 dx, \\
 C^2 &= \int_{\mathbb{R}^N} |u_\rho|^{p-4} a(x) |\operatorname{Im}(\bar{u}_\rho \nabla u_\rho)|^2 dx,
 \end{aligned}$$

we proved that

$$-\operatorname{Re} \left(\int_{\mathbb{R}^N} Lu_\rho u^* dx \right) \geq \left(p-1 - \frac{\alpha p}{N+\alpha-2} \right) B^2 + C^2$$

and

$$\left| \operatorname{Im} \left(\int_{B(\rho)} Lu_\rho u^* dx \right) \right| \leq \left(p-2 + \frac{\alpha p}{N+\alpha-2} \right) BC.$$

Observe that, $p-1 - \frac{p\alpha}{N+\alpha-2}$ is positive for $\alpha < (N-2)(p-1)$. In this case it is possible to determine a positive constant l_α , independent of ρ , such that

$$\left(p-1 - \frac{p\alpha}{N+\alpha-2} \right) B^2 + C^2 \geq l_\alpha \left(p-2 + \frac{p\alpha}{N+\alpha-2} \right) BC$$

and, consequently,

$$\left| \operatorname{Im} \left(\int_{B(\rho)} Lu_\rho u^* dx \right) \right| \leq l_\alpha^{-1} \left\{ -\operatorname{Re} \left(\int_{B(\rho)} Lu_\rho u^* dx \right) \right\}.$$

If $\tan \theta_\alpha = l_\alpha$, then $e^{\pm i\theta}L$ is dissipative in $B(\rho)$ for $0 \leq \theta \leq \theta_\alpha$. The previous computations give also the dissipativity of L if $\alpha = (N-2)(p-1)$. Let us introduce the sector

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\operatorname{Arg} \lambda| < \pi/2 + \theta\}.$$

It follows from [12, Theorem I.3.9], that problem (8.2) has a unique solution for every $\lambda \in \Sigma_\theta$ and $0 \leq \theta < \theta_\alpha$ and that there exists a constant C_θ , independent of ρ , such that the solution u_ρ satisfies

$$\|u_\rho\|_{L^p(B(\rho))} \leq \frac{C_\theta}{|\lambda|} \|f\|_{L^p}. \tag{8.3}$$

In the case $\alpha = (N-2)(p-1)$ the solutions u_ρ exist for $\operatorname{Re} \lambda > 0$ and satisfy the estimate

$$\|u_\rho\|_{L^p(B(\rho))} \leq \frac{1}{\operatorname{Re} \lambda} \|f\|_{L^p}.$$

Next we use weak compactness arguments to produce a function $u \in D_{p,\max}(L)$ satisfying $\lambda u - Lu = f$. For definiteness, we consider the case $\alpha < (N-2)(p-1)$, the other one being simpler, and fix $\lambda \in \Sigma_\theta$, with $0 < \theta < \theta_\alpha$.

Let us fix a radius r and apply the interior L^p estimates ([5, Theorem 9.11]) together with (8.3) to the functions u_ρ with $\rho > r + 1$

$$\|u_\rho\|_{W^{2,p}(B(r))} \leq C_1 [\|\lambda u_\rho - Lu_\rho\|_{L^p(B(r+1))} + \|u_\rho\|_{L^p(B(r+1))}] \leq C_2 \|f\|_{L^p}.$$

By weak compactness and a diagonal argument, we can find a sequence $(\rho_n) \rightarrow \infty$ such that the functions (u_{ρ_n}) converge weakly in $W_{\text{loc}}^{2,p}$ to a function u . Clearly u satisfies $\lambda u - Lu = f$ and, by (8.3)

$$\|u\|_{L^p} \leq \frac{C_\theta}{|\lambda|} \|f\|_{L^p}. \tag{8.4}$$

In particular $u \in D_{p,\max}(L)$. To complete the proof we need only to show that $\lambda - L$ is injective on $D_{p,\max}(L)$ for $\lambda \in \Sigma_\theta$. Let

$$E = \{r > 0 : \Sigma_\theta \cap B(r) \subset \rho(L, D_{p,\max}(L))\}$$

and $R = \sup E$. Since 0 is in the resolvent set, by Proposition 4.5, R is positive. On the other hand the norm of the resolvent exists in $B(R) \cap \Sigma_\theta$ and is bounded by $C_\theta/|\lambda|$, by (8.4), hence cannot explode on the boundary of $B(R)$. This proves that $R = \infty$ and concludes the proof. \square

Finally we show that the condition on $\alpha \leq (p - 1)(N - 2)$ is necessary for the dissipativity.

Proposition 8.2. *Suppose that the operator $L = (s^\alpha + |x|^\alpha)\Delta$, $s > 0$, is dissipative. Then $\alpha \leq (p - 1)(N - 2)$.*

Proof. Suppose L dissipative. Then, for every $u \in D_{p,\max}(L)$, u real-valued,

$$\int_{\mathbb{R}^N} (s^\alpha + |x|^\alpha)u|u|^{p-2}\Delta u \, dx \leq 0.$$

If u above is in C_c^∞ , we may integrate by parts twice and, using the identity $\nabla|u|^p = pu|u|^{p-2}\nabla u$, we get

$$\int_{\mathbb{R}^N} |u|^p|x|^{\alpha-2} \, dx \leq \frac{p(p-1)}{\alpha(N+\alpha-2)} \int_{\mathbb{R}^N} (s^\alpha + |x|^\alpha)|u|^{p-2}|\nabla u|^2 \, dx. \tag{8.5}$$

By applying (8.5) to $u(\lambda \cdot)$, for $\lambda > 0$, we obtain

$$\int_{\mathbb{R}^N} |u(x)|^p|x|^{\alpha-2} \, dx \leq \frac{p(p-1)}{\alpha(N+\alpha-2)} \int_{\mathbb{R}^N} (s^\alpha\lambda^\alpha + |x|^\alpha)|u|^{p-2}|\nabla u(x)|^2 \, dx.$$

Letting $\lambda \rightarrow 0$ we get

$$\int_{\mathbb{R}^N} |u(x)|^p|x|^{\alpha-2} \, dx \leq \frac{p(p-1)}{\alpha(N+\alpha-2)} \int_{\mathbb{R}^N} |x|^\alpha|u|^{p-2}|\nabla u(x)|^2 \, dx$$

for every $u \in C_c^\infty(\mathbb{R}^N)$ and, by density, for every $u \in W^{1,p}(\mathbb{R}^N)$ with compact support. Since $\left(\frac{p}{N+\alpha-2}\right)^2$ is the best constant in Hardy’s inequality above (see Proposition 9.10), we obtain

$$\frac{p(p-1)}{\alpha(N+\alpha-2)} \geq \left(\frac{p}{N+\alpha-2}\right)^2,$$

which implies $\alpha \leq (p - 1)(N - 2)$. □

9. Analyticity in L^p

In this section we prove that $(L, D_{p,\max}(L))$ generates an analytic semigroup in L^p , for every $N/(N - 2) < p < \infty$. The results of the preceding Section show that this is the case if $p > (N + \alpha - 2)/(N - 2)$ and $L = L_s = (s^\alpha + |x|^\alpha)\Delta$; they are the starting point of an iteration procedure which proves the result, together with perturbation and scaling arguments.

In order to motivate the next lemmas we explain the main ideas behind the proof.

For $s > 0$ let $I_s : L^p \rightarrow L^p$ defined by $I_s u(x) = u(sx)$. Clearly I_s is invertible with inverse $I_{s^{-1}}$ and $\|I_s u\|_p = s^{-N/p} \|u\|_p$.

Since $L_1 = s^{2-\alpha} I_s L_s I_s^{-1}$ if $\lambda \in \mathbb{C}$, $\lambda = r\omega$ with ω in the resolvent set of L_s , then the equality

$$\lambda - L_1 = I_s r \left(\omega - \frac{s^{2-\alpha}}{r} \right) I_s^{-1}$$

yields the decay

$$\|\lambda - L_1\| \leq \frac{C}{|\lambda|} \|(\omega - L_s)^{-1}\|.$$

In order to make this argument rigorous, we need information on the resolvent sets of L_s and estimates on the norm of $(\omega - L_s)^{-1}$ which are independent of $0 < s \leq 1$. Formally we can define L_s also for $s = 0$. L_0 is scale invariant i.e. $L_0 = s^{2-\alpha} I_s L_0 I_s^{-1}$ but is singular at 0 and it would lead to extra difficulties due to the lack of regularity at the origin. For this reason we shall assume that $s > 0$. To treat the singular limit at 0 we introduce, for $0 < s \leq 1$, the operators $M_s = a_s \Delta$, where $a_s(x) = s^\alpha + |x|^\alpha$ for $|x| \leq 2$ and $a_s(x) = s^\alpha + 2^\alpha$, for $|x| \geq 2$. Observe that the operators M_s have bounded coefficients. Similarly, to investigate the behavior for large $|x|$, we set $N_s = b_s(x) \Delta$, where $b_s(x) = 1 + s^\alpha$ for $|x| \leq 1$ and $b_s(x) = s^\alpha + |x|^\alpha$ for $|x| \geq 1$.

For $0 < \theta < \pi$, $\rho > 0$, we denote by $\Sigma_{\theta, \rho}$ the closed set

$$\Sigma_{\theta, \rho} = \{\lambda \in \mathbb{C} : |\lambda| \geq \rho, |\text{Arg} \lambda| \leq \theta\}.$$

Let (L, D) be the operator $a\Delta$ with $D = D_{p, \max}(L)$ on L^p , where a is bounded below from 0. We say that (L, D) satisfies $P(\theta, \rho, C, \gamma)$, where $C, \rho > 0$, $\gamma \geq 0$ and $0 < \theta < \pi$ if $\Sigma_{\theta, \rho} \subset \rho(L)$ and for every $\lambda \in \Sigma_{\theta, \rho}$ the following estimates hold

$$\|(\lambda - L)^{-1}\| \leq \frac{C}{|\lambda|^\gamma}. \tag{9.1}$$

Finally, (L, D) satisfies $P(\theta, \rho, R, C, \gamma, \delta)$, $R > 0$, $\delta \in \mathbb{R}$ if it satisfies $P(\theta, \rho, C, \gamma)$, $a(x) \leq C$ for $x \in C(R)$ and moreover

$$\|(\lambda - L)^{-1}\|_{L^p \rightarrow W^{1,p}(C(R))} \leq \frac{C}{|\lambda|^\delta}, \tag{9.2}$$

Here $C(R) = B(2R) \setminus B(R)$ and the last norm is understood as the operator norm from L^p to $W^{1,p}(C(R))$.

Clearly, L generates an analytic semigroup if and only if $P(\theta, \rho, C, \gamma)$ holds for some $\theta > \pi/2$, $\gamma = 1$.

Lemma 9.1. *Let $L_1 = a_1 \Delta$, $L_2 = a_2 \Delta$ satisfy $P(\theta, \rho, R, C, \gamma, \delta)$ in L^p , with $\delta > 0$. Assume that $a_1(x) = a_2(x)$ for $R \leq |x| \leq 2R$ and let $L = a \Delta$ with $a(x) = a_1(x)$ if $|x| \leq R$ and $a(x) = a_2(x)$ if $|x| \geq R$. If there exists $\lambda_0 > 0$ such that $\lambda - L$ is injective for $\lambda > \lambda_0$, then L satisfies $P(\theta, \rho_1, C_1, \gamma)$ in L^p , where ρ_1, C_1 depend only on $p, \theta, \rho, R, C, \gamma$.*

Proof. Observe that the domains of L and L_2 coincide. Let $0 \leq \eta_1, \eta_2 \leq 1$ be positive C^∞ -functions supported in $B(2R)$ and $\mathbb{R}^N \setminus B(R)$, respectively, such that $\eta_1^2 + \eta_2^2 = 1$. For $\lambda \in \Sigma_{\theta, \rho}$ set $R_i(\lambda) = \eta_i(\lambda - L_i)^{-1}\eta_i$ for $i = 1, 2$. Observing that $L\eta_i = L_i\eta_i$, it follows that, if $f \in L^p(\mathbb{R}^N)$,

$$\begin{aligned} (\lambda - L)R_i(\lambda)f &= (\lambda - L)\eta_i(\lambda - L_i)^{-1}(\eta_i f) \\ &= \eta_i(\lambda - L_i)(\lambda - L_i)^{-1}(\eta_i f) + [\eta_i, L](\lambda - L_i)^{-1}(\eta_i f) \\ &= \eta_i^2 f + [\eta_i, L](\lambda - L_i)^{-1}(\eta_i f) \end{aligned}$$

where

$$[\eta_i, L_i]g = \eta_i a \Delta g - a \Delta(\eta_i g) = -2a \nabla \eta_i \nabla g - a(\Delta \eta_i)g$$

is a first order operator supported on $C(R)$. It follows that

$$(\lambda - L)R_i(\lambda)f = \eta_i^2 f + S_i(\lambda)f$$

where $S_i(\lambda)f = -2a \nabla \eta_i \nabla(\lambda - L_i)^{-1}(\eta_i f) - a(\Delta \eta_i)(\lambda - L_i)^{-1}(\eta_i f)$. By (9.2), it follows that

$$\|S_i(\lambda)f\|_p \leq \frac{c_1}{|\lambda|^\delta}$$

for $\lambda \in \Sigma_{\theta, \rho}$ and with c_1 depending only on C, R . Therefore

$$(\lambda - L)R(\lambda)f = f + S(\lambda)f$$

where

$$R(\lambda) = \sum_{i=1}^2 R_i(\lambda), \quad S(\lambda) = \sum_{i=1}^2 S_i(\lambda).$$

Choosing $|\lambda| > \rho_1$ large enough, we find

$$\|S(\lambda)\| \leq \frac{1}{2}$$

and we deduce that the operator $I + S(\lambda)$ is invertible. Setting $V(\lambda) = (I + S(\lambda))^{-1}$ we have

$$(\lambda - L)R(\lambda)V(\lambda)f = f$$

and hence the operator $R(\lambda)V(\lambda)$ is a right inverse of $\lambda - L$ and, by (9.1), satisfies

$$\|R(\lambda)V(\lambda)\| \leq \frac{2C}{|\lambda|^\gamma} \tag{9.3}$$

for $\lambda \in \Sigma_{\theta, \rho_1}$. Clearly $R(\lambda)V(\lambda)$ coincides with $(\lambda - L)^{-1}$ whenever this last is injective, in particular for $\lambda > \lambda_0$. The a-priori estimates (9.3) show that the norm of the resolvent cannot explode in the set Σ_{θ, ρ_1} hence this set is contained in $\rho(L)$ where the resolvent operator coincide with $R(\lambda)V(\lambda)$ and satisfies (9.3). \square

We specialize our results to the operators $M_s, N_s, 0 < s \leq 1$ defined before. We recall that $M_s = a_s \Delta$, where $a_s(x) = s^\alpha + |x|^\alpha$ for $|x| \leq 2$ and $a_s(x) = s^\alpha + 2^\alpha$, for $|x| \geq 2$ and $N_s = b_s(x) \Delta$, where $b_s(x) = 1 + s^\alpha$ for $|x| \leq 1$ and $b_s(x) = s^\alpha + |x|^\alpha$ for $|x| \geq 1$. Observe that $M_s = N_s$ for $1 \leq |x| \leq 2$. For later use we introduce also the operators $C_s = c_s \Delta$ where $c_s(x) = 1 + s^\alpha$ for $|x| \leq 1$, $c_s(x) = s^\alpha + |x|^\alpha$ for $1 \leq |x| \leq 2$ and $c_s(x) = 2^\alpha + s^\alpha$ for $|x| \geq 1$.

Lemma 9.2. *For every $1 < p < \infty$, there exist $\theta > \pi/2, \rho, C > 0$ such that for every $0 < s \leq 1$ M_s and C_s satisfy $P(\theta, \rho, 1, C, 1, 1/2)$.*

Proof. We give a proof for M_s , that for C_s being identical. Clearly there $a_s(x) \leq 1 + 2^\alpha$ in $C(1)$. The other properties follow from [2] after observing that the results proved in the paper remain true if the diffusion C^1 -coefficients a are replaced by Lipschitz coefficients. In fact a_s satisfies the estimate $|\nabla a_s(x)| \leq c a_s(x)^{1/2}$ a.e. in \mathbb{R}^N with c independent of $0 < s \leq 1$ and [2, Theorem 2.7] shows that there exist $\theta > \pi/2, C, \rho > 0$ (still independent of s) such that (9.1) holds. An application of [2, Lemma 2.4] (see also [2, Corollary 2.8]) shows that

$$\|a_s^{1/2} \nabla (\lambda - M_s)^{-1}\| \leq \frac{C}{|\lambda|^{1/2}}$$

for $\lambda \in \Sigma_{\theta, \rho}$ and C independent of s . Since $a_s(x) \geq 1$ in $C(1)$, (9.2) holds. □

We now turn to the operators N_s

Lemma 9.3. *Let $p > \frac{N+\alpha-2}{N-2}$. Then there exist $\theta > \pi/2, C > 0$ such that for every $0 < s \leq 1$ N_s satisfies $P(\theta, 0, 1, C, 1, 1/2)$.*

Proof. Clearly $b_s(x) \leq 1 + 2^\alpha$ in $C(1)$. Moreover $|\nabla b_s(x)| \leq \alpha |x|^{\alpha-1}$ and $|x|^\alpha \leq b_s(x)$ for every $x \in \mathbb{R}^N$. It follows from Theorem 8.1 that there exist $\theta > \pi/2, C > 0$, independent of $0 < s \leq 1$ such that (9.1) holds in the sector Σ_θ with $\gamma = 1$. Concerning the gradient estimate (9.2) we observe that there exist $c > 0$ such that for every $u \in W^{2,p}$ and $0 < s \leq 1$ $\|\Delta u\|_p \leq c \|N_s u\|_p$. If u solves $\lambda u - N_s u = f$, with $\lambda \in \Sigma_\theta$, then by the analyticity estimate proved before, $|\lambda| \|u\|_p + \|N_s u\|_p \leq C \|f\|_p$, hence $\|\Delta u\|_p \leq C \|f\|_p$ and the inequality $\|\nabla u\|_p^2 \leq c \|u\|_p \|\Delta u\|_p$ gives

$$\|\nabla u\|_p \leq \frac{C}{|\lambda|^{1/2}} \|f\|_p$$

with C independent of s . □

For smaller values of p , we start by showing the boundedness of the resolvent in a half-plane which includes the imaginary axis.

Lemma 9.4. *Let $p > \frac{N}{N-2}$. Then there exists $\lambda_0 < 0, C > 0$ independent of $0 < s \leq 1$, such that, for $\text{Re } \lambda > \lambda_0, \lambda \in \rho(N_s)$ and $\|(\lambda - N_s)^{-1}\| \leq C$.*

Proof. Since $b_s \geq 1$, Proposition 4.5 and Corollary 4.3 show that $0 \in \rho(N_s)$ and that there exists a positive constant c independent of s such that $\|N_s^{-1}\| \leq c$. It follows that the ball $B(c^{-1})$ is contained in the resolvent set of N_s and

$$\|(\lambda - N_s)^{-1}\| \leq \|(\lambda N_s^{-1} - I)^{-1} N_s^{-1}\| \leq \frac{c}{1 - |\lambda|c} \leq 2c$$

for $\lambda \in B((2c)^{-1})$.

We know from Corollary 7.2 that the spectra of N_s in L^p and C_0 coincide, as well their resolvent operators. Moreover, Proposition 7.5 shows that the spectrum is discrete and lies in the negative real axis, hence in $] -\infty, -(2c)^{-1}$, hence the spectral bound of N_s is less than $-(2c)^{-1}$. For $\operatorname{Re} \lambda > -(2c)^{-1}$ we can therefore write the resolvent in C_0 as the Laplace transform of the minimal semigroup $(T_s(t))_{t \geq 0}$, see [3, Lemma 1.9, Chapter VI].

Then for every $f \in C_0(\mathbb{R}^N)$, $\operatorname{Re} \lambda > -(2c)^{-1}$

$$\begin{aligned} |(\lambda - N_s)^{-1} f(x)| &= \left| \int_0^\infty e^{-\lambda t} T_s(t) f(x) dt \right| \leq \int_0^\infty e^{-(\operatorname{Re} \lambda)t} T_s(t) |f(x)| dt \\ &= (\operatorname{Re} \lambda - N_s)^{-1} |f|(x), \end{aligned}$$

since the minimal semigroup is positive. By integrating on \mathbb{R}^N , in virtue of the coherence of the resolvent and by a density argument it follows that

$$\|(\lambda - N_s)^{-1} f\|_p \leq \|(\operatorname{Re} \lambda - N_s)^{-1} f\|_p$$

for $f \in L^p(\mathbb{R}^N)$. On the other hand $(\lambda - N_s)^{-1}$ is bounded (in L^p) in the interval $[-(2c)^{-1}, (2c)^{-1}]$, uniformly with respect to s by the previous computations, and bounded in $[0, \infty]$ by the constant c , by Theorem 4.6 (see also equation (4.5)). \square

We start now the iteration procedure. Note that we can take as p_1 in the following Proposition any number greater than $(N + \alpha - 2)/(N - 2)$, by Lemma 9.3. Note also that gradient estimates (9.2) follow resolvent estimates since $W^{2,p}$ embeds into the domain of N_s

Lemma 9.5. *Let $N/(N - 2) < q < p_1 < \infty$ and let λ_0 be as in Proposition 9.4. Suppose that N_s satisfies $P(\theta, \rho, C, 1)$, relatively to p_1 , where $\theta > \pi/2$, $\rho, C > 0$ are independent of $0 < s \leq 1$. Then for every $0 \leq \tau \leq 1$ and*

$$\frac{1}{p} = \frac{1 - \tau}{p_1} + \frac{\tau}{q}$$

there are constants $\rho_1 = \rho \vee 1$, $C_1 > 0$ independent of s such that for every $\lambda \in \mathbb{C}$ with $|\lambda| > \rho_1$ and $\operatorname{Re} \lambda < \lambda_0$

$$\|(\lambda - N_s)^{-1}\|_p \leq C_1 |\lambda|^{\tau-1}, \quad \|\nabla(\lambda - N_s)^{-1}\|_p \leq C_1 |\lambda|^{\tau-\frac{1}{2}}.$$

Proof. The estimate for the resolvent follows using the Riesz Thorin theorem, interpolating between the resolvent estimates given by Lemmas 9.3, 9.4. The gradient estimates follows as in Lemma 9.3. In fact from $\lambda u - N_s u = f$ and the bound on u we deduce that $\|\Delta u\| \leq C(1 + |\lambda|^\tau)\|f\| \leq 2C|\lambda|^\tau\|f\|$ if $|\lambda| > \rho_1$. \square

The main step in the iteration procedure is contained in the next Lemma. Note that here τ is smaller than $1/2$ in order to produce decay in the estimate of gradients. We recall that $L_s = (s^\alpha + |x|^\alpha)\Delta$.

Lemma 9.6. *Let $N/(N - 2) < q < p_1 < \infty$ and suppose that N_s satisfies $P(\theta, \rho, C, 1)$ where $\theta > \pi/2$, $\rho, C > 0$ are independent of $0 < s \leq 1$. Then for every $0 \leq \tau < 1/2$ and*

$$\frac{1}{p} = \frac{1 - \tau}{p_1} + \frac{\tau}{q}$$

there are constants $\bar{\theta} > \pi/2$, $\bar{\rho}, \bar{C} > 0$ independent of $0 < s \leq 1$ such that L_s and N_s satisfy $P(\bar{\theta}, \bar{\rho}, \bar{C}, 1)$.

Proof. Let us first observe that L_s coincides with M_s in $B(2)$ and with N_s in $\mathbb{R}^N \setminus B(1)$. M_s satisfies $P(\theta_1, \rho_1, 1, C_1, 1, 1/2)$ in L^p for suitable $\theta_1 > \pi/2$, $\rho_1, C_1 > 0$ independent of $0 < s \leq 1$, by Lemma 9.2. By Lemma 9.5, N_s satisfies $P(\pi/2, \rho_2, 1, C_2, 1 - \tau, 1/2 - \tau)$ in L^p for suitable $\rho_2, C_2 > 0$ independent of $0 < s \leq 1$. Since $\tau < 1/2$, Lemma 9.1 then implies that L_s satisfies $P(\pi/2, \rho_3, 1, C_3, 1 - \tau, 1/2 - \tau)$ in L^p for suitable $\rho_3, C_3 > 0$ independent of $0 < s \leq 1$. The points $\pm i\rho_3$ are therefore in the resolvent set of L_s and the norms of $(\pm i\rho_3 - L_s)^{-1}$ are bounded uniformly in s . Since also the right half plane is in the resolvent set of L_s it follows that we can find $\bar{\theta} > \pi/2$ such that the arc $\Omega = \{\rho_3 e^{i\eta} : |\eta| \leq \bar{\theta}\}$ is contained in $\rho(L_s)$. Finally, $\|\omega - L_s\| \leq C_4$ for every $\omega \in \Omega$ and $0 < s \leq 1$, by $P(\pi/2, \rho_3, 1, C_3, 1 - \tau, 1/2 - \tau)$.

Finally, we use the scaling argument explained at the beginning of this section. Consider the dilations $I_s u(x) = u(sx)$ so that $L_1 = s^{2-\alpha} I_s L_s I_s^{-1}$ and fix $\omega \in \Omega$. Then, for every $\lambda = r\omega$, $r \geq 1$, we have

$$\lambda - L_1 = I_s r \left(\omega - \frac{s^{2-\alpha}}{r} L_s \right) I_s^{-1}.$$

By choosing $s = r^{\frac{1}{2-\alpha}} \leq 1$, we deduce

$$\|(\lambda - L_1)^{-1}\| \leq \frac{C_5}{r} \|(\omega - L_s)^{-1}\| \leq \frac{C_6}{|\lambda|}.$$

The identity $L_s = s^{\alpha-2} I_s^{-1} L_1 I_s$ then gives for $\lambda \in \Sigma_{\bar{\theta}, \rho_3}$

$$\lambda - L_s = s^{\alpha-2} I_s^{-1} \left(\frac{\lambda}{s^{\alpha-2}} - L_1 \right) I_s$$

hence

$$\|(\lambda - L_s)^{-1}\| \leq \frac{C_6}{|\lambda|}$$

and L_s satisfies $P(\bar{\theta}, \rho_3, C_6, 1)$. Property $P(\bar{\theta}, \bar{\rho}, \bar{C}, 1)$ for N_s (possibly for other parameters $\bar{\theta} > \pi/2, \bar{\rho}, \bar{C}$, still independent of s) follows from Lemmas 9.2 and 9.1 by gluing together the operators L_s and $C_s = c_s \Delta$ where $c_s(x) = 1 + s^\alpha$ for $|x| \leq 1, c_s(x) = s^\alpha + |x|^\alpha$ for $1 \leq |x| \leq 2$ and $c_s(x) = 2^\alpha + s^\alpha$ for $|x| \geq 1$. \square

Finally, we complete the iteration procedure.

Lemma 9.7. *Let $\frac{N}{N-2} < p < \infty, \alpha > 0$. Then, for every $0 < s \leq 1$, the operator $L_s = (s^\alpha + |x|^\alpha)$ generates an analytic semigroup.*

Proof. We may assume that $p \leq (N + \alpha - 2)/(N - 2)$ and fix $p_1 > (N + \alpha - 2)(N - 2)$ and q such that $N/(N - 2) < q < p$. Fixing $0 < \tau < 1/2$ and setting

$$\frac{1}{p_{n+1}} = \frac{(1 - \tau)}{p_n} + \frac{\tau}{q}$$

we apply repeatedly Lemma 9.6 obtaining sequences $\theta_n > \pi/2, \rho_n, C_n > 0$ such that for every $0 < s \leq 1$ L_s satisfies $P(\theta_n, \rho_n, C_n, 1)$. Since p_n converges to q we can find m such that $p_m < p$ and L_s is sectorial in L^{p_m} . Since L_s is also sectorial in L^{p_1} , by interpolation it is sectorial in L^p . \square

We can now prove the main result of this section.

Theorem 9.8. *Let $a(x)$ be a positive continuous function such that*

$$\lim_{|x| \rightarrow \infty} \frac{a(x)}{1 + |x|^\alpha} = l, \quad l > 0, \quad l \in \mathbb{R}.$$

If $p > N/(N - 2)$, then $(L, D_{p, \max}(L))$ generates an analytic semigroup.

Proof. For every $r > 0$, let $B_r = a_r \Delta$ where a_r is a continuous strictly positive function bounded from below such that $a_r(x) = a(x)$ for $|x| \geq r$ and $a_r \rightarrow l$ uniformly in \mathbb{R}^N , as $r \rightarrow \infty$. Obviously the operator $l(1 + |x|^\alpha)\Delta$ generates an analytic semigroup. Observe also that all the operators considered have the same domain. Given $\varepsilon > 0$ there exists R such that, for $r \geq R, |a_r(x) - l| < \varepsilon$ for every $x \in \mathbb{R}^N$ and hence

$$\|B_r - l(1 + |x|^\alpha)\Delta\| \leq \varepsilon \|l(1 + |x|^\alpha)\Delta\|$$

for $r \geq R$. By [3, Chapter III, Lemma 2.6], B_R generates an analytic semigroup in L^p . Evidently the operator L coincide with B_R for $|x| \geq R$ and with a uniformly elliptic operator \hat{L} for $|x| \leq 2R$ (consider any bounded and uniformly continuous extension of m_R outside $B(2R)$, bounded below from 0). Both operators B_R, \hat{L} satisfies $P(\theta, \rho, R, C, 1, 1/2)$ for suitable $\rho, C > 0$ and $\theta > \pi/2$. Estimate (9.2) holds for B_R with $\delta = 1/2$, by sectoriality, since $D(B_R)$ embeds continuously in $W^{2,p}$ and for \hat{L} since $D(\hat{L}) = W^{2,p}$. Lemma (9.1) shows that L has property $P(\theta, \rho, C, 1)$ and concludes the proof. \square

Finally, let us show that on $L^p \cap C_0(\mathbb{R}^N)$ the semigroup coincide with T_{\min} of Section 2. In particular, the semigroups are coherent in different L^p spaces.

Corollary 9.9. *Let $(T_p(t))$ be the semigroup generated by $(L, D_{p,\max}(L))$ in L^p , $p > N/(N - 2)$, and $(T_{\min}(t))$ be the minimal semigroup in $C_0(\mathbb{R}^N)$. Then for every $f \in C_0(\mathbb{R}^N) \cap L^p$, $T_p(t)f = T_{\min}(t)f$. Moreover, if $p, q > N/(N - 2)$ and $f \in L^p \cap L^q$, then $T_p(t)f = T_q(t)f$.*

Proof. Since $(\lambda - L)^{-1}f = (\lambda - L_{\min})^{-1}f$ for $f \in L^p \cap C_0(\mathbb{R}^N)$, see Theorem 4.6, the thesis follows by representing the semigroups as the limit of iterates of the corresponding resolvents, see [3, Chapter III, Corollary 5.5]. \square

Appendix

Here we prove a Hardy-type inequality used throughout the paper.

Proposition 9.10. *Let $u \in W^{1,p}(\mathbb{R}^N)$ with compact support, $1 < p < \infty$, $\gamma \geq 0$. Then*

$$\int_{\mathbb{R}^N} |u|^p |x|^\gamma dx \leq \left(\frac{p}{N + \gamma}\right)^2 \int_{\mathbb{R}^N} |u|^{p-4} |\operatorname{Re}(\bar{u}\nabla u)|^2 |x|^{\gamma+2} dx.$$

Moreover $\left(\frac{p}{N + \gamma}\right)^2$ is the best constant in the previous inequality.

Proof. Let first $u \in C_c^\infty(\mathbb{R}^N)$ and set $g(t) = u(tx)$. Then

$$\begin{aligned} |u(x)|^p &= |g(1)|^p = -p \int_1^\infty |g|^{p-2} \operatorname{Re}\left(\frac{\partial g}{\partial t}\right) dt \\ &= -p \int_1^\infty |u(tx)|^{p-2} \operatorname{Re}(\bar{u}\nabla u(tx))x dt. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x)|^p |x|^\gamma dx &\leq p \int_1^\infty dt \int_{\mathbb{R}^N} |u(tx)|^{p-2} |\operatorname{Re}(\bar{u}(tx)\nabla u(tx))||x|^{\gamma+1} dx \\ &= p \int_1^\infty dt \int_{S_{N-1}} d\omega \int_0^\infty |u(tr\omega)|^{p-2} |\operatorname{Re}(\bar{u}(tr\omega)\nabla u(tr\omega))|r^{\gamma+N} dr \\ &= p \int_1^\infty \frac{1}{t^{\gamma+N+1}} dt \int_{S_{N-1}} d\omega \int_0^\infty |u(s\omega)|^{p-2} |\operatorname{Re}(\bar{u}(s\omega)\nabla u(s\omega))|s^{\gamma+N} ds \\ &= \frac{p}{N + \gamma} \int_{\mathbb{R}^N} |u(x)|^{p-2} |\operatorname{Re}(\bar{u}(x)\nabla u(x))||x|^{\gamma+1} dx. \end{aligned}$$

By density this inequality holds for every $u \in W^{1,p}(\mathbb{R}^N)$ having compact support. At this point Hölder’s inequality yields

$$\int_{\mathbb{R}^N} |u(x)|^p |x|^\gamma dx \leq \frac{p}{N + \gamma} \left(\int_{\mathbb{R}^N} |u(x)|^p |x|^\gamma dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u(x)|^{p-4} |\operatorname{Re}(\bar{u}(x)\nabla u(x))| |x|^{\gamma+2} dx \right)^{\frac{1}{2}}.$$

Concerning the last assertion, we consider first the functions u_δ defined by $u_\delta(\rho) = \rho^{-\beta_1}$ for $\rho \leq 1$ and $u_\delta(\rho) = \rho^{-\beta_2}$ for $\rho \geq 1$ with

$$\beta_1 = \frac{N + \gamma - \delta}{p} \quad \beta_2 = \frac{N + \gamma + \delta}{p}$$

and then their truncated $u_{n,\delta} = \eta_n u_\delta$ with $\eta_n = \eta(\frac{x}{n})$, η being a smooth function supported in $B(2)$ and equal to 1 in $B(1)$. It is easy to check that

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^p |x|^\gamma dx &\geq \int_{|x| \leq n} |u|^p |x|^\gamma dx = \int_{|x| \leq 1} |u|^p |x|^\gamma dx + \int_{1 \leq |x| \leq n} |u|^p |x|^\gamma dx \\ &= \frac{1}{\beta_1^2} \int_{|x| \leq 1} |u|^{p-2} |\nabla u|^2 |x|^{\gamma+2} dx + \frac{1}{\beta_2^2} \int_{1 \leq |x| \leq n} |u|^{p-2} |\nabla u|^2 |x|^{\gamma+2} dx \\ &\geq \frac{1}{\beta_2^2} \int_{|x| \leq n} |u|^{p-2} |\nabla u|^2 |x|^{\gamma+2} dx = \frac{1}{\beta_2^2} \int_{|x| \leq n} |u_n|^{p-2} |\nabla u_n|^2 |x|^{\gamma+2} dx \\ &= \frac{1}{\beta_2^2} \left(\int_{\mathbb{R}^N} |u_n|^{p-2} |\nabla u_n|^2 |x|^{\gamma+2} dx - \int_{n \leq |x| \leq 2n} |u_n|^{p-2} |\nabla u_n|^2 |x|^{\gamma+2} dx \right) \\ &= \frac{1}{\beta_2^2} \left(\int_{\mathbb{R}^N} |u_n|^{p-2} |\nabla u_n|^2 |x|^{\gamma+2} dx - \frac{AC}{A\delta n^\delta} \right) \end{aligned}$$

where $A, C > 0$ do not depend on n, δ and satisfy $\int_{n \leq |x| \leq 2n} |u_n|^{p-2} |\nabla u_n|^2 |x|^{\gamma+2} dx \leq \frac{C}{\delta n^\delta}$ and $\int_{\mathbb{R}^N} |u_n|^{p-2} |\nabla u_n|^2 |x|^{\gamma+2} dx \geq A$. It follows

$$\int_{\mathbb{R}^N} |u_n|^p |x|^\gamma dx \geq \frac{1}{\beta_2^2} \left(1 - \frac{C}{A\delta n^\delta} \right) \int_{\mathbb{R}^N} |u_n|^{p-2} |\nabla u_n|^2 |x|^{\gamma+2} dx.$$

This proves that $\left(\frac{p}{N+\gamma}\right)^2$ is the best constant in the inequality. □

References

[1] W. ARENDT, *Gaussian estimates and interpolation of the spectrum in L^p* , Differential Integral Equations 7 (1994), 1153–1168.

- [2] S. FORNARO and L. LORENZI, *Generation results for elliptic operators with unbounded diffusion coefficients in L^p and C_b -spaces*, Discrete Contin. Dyn. Syst. **18** (2007), 747–772.
- [3] K. J. ENGEL and R. NAGEL, “One Parameter Semigroups for Linear Evolutions Equations”, Springer-Verlag, Berlin, 2000.
- [4] A. FRIEDMAN, “Partial Differential Equations of parabolic Type”, Prentice Hall, New Jersey, 1964.
- [5] D. GILBARG and N. TRUDINGER, “Elliptic Partial Differential Equations of Second Order”, Second edition, Springer, Berlin, 1983.
- [6] P. KRÉE, *Sur les multiplicateurs dans \mathcal{FL}^p avec poids*, Ann. Inst. Fourier (Grenoble) **16** (1966), 91–121.
- [7] A. LUNARDI, “Analytic Semigroups and Optimal Regularity in Parabolic Problems”, Progress in Nonlinear Differential Equations and Their Applications, Vol. 16, Birkhäuser, 1995.
- [8] G. METAFUNE and C. SPINA, *An integration by parts formula in Sobolev spaces*, Mediterr. J. Math. **5** (2008), 359–371.
- [9] G. METAFUNE and D. PALLARA, *Trace formulas for some singular differential operators and applications*, Math. Nachr. **211** (2000), 127–157.
- [10] G. METAFUNE, D. PALLARA and M. WACKER, *Feller Semigroups on \mathbb{R}^N* , Semigroup Forum **65** (2002), 159–205.
- [11] B. MUCKENHOUPT and R. WHEEDEN, *Weighted norm inequalities for singular and fractional integrals*, Trans. Amer. Math. Soc. **161** (1971), 249–258.
- [12] A. PAZY, “Semigroups of Linear Operators and Applications to Partial Differential Equations”, Applied mathematical sciences, Vol. 44, New York, Springer-Verlag, 1983.
- [13] E. M. STEIN and G. WEISS, “Introduction to Fourier Analysis on Euclidean Spaces”, Princeton University Press, 1971.
- [14] G. TALENTI, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

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