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Varieties with $q(X) = \dim(X)$ and $P_2(X) = 2$

ZHI JIANG

Abstract. We give a complete description of all smooth projective complex varieties with $q(X) = \dim(X)$ and $P_2(X) = 2$.

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It has always been a goal of algebraic geometers to classify algebraic varieties, and a good part of their work in the twentieth century was devoted to the classification of algebraic surfaces. Of course, one can obtain a complete description only in particular cases, such as when the numerical invariants of the surface are small.

In the past thirty years, the development of new techniques made it possible to obtain results in higher dimensions as well. The numerical (birational) invariants of a smooth projective complex variety X that are commonly used are its *irregularity* $q(X) := h^1(X, \mathcal{O}_X)$ and its *plurigenera* $P_m(X) := h^0(X, \omega_X^m)$.

One can quote for example a beautiful result of Kawamata ([13]), who proved that X is birational to an Abelian variety if and only if $q(X) = \dim(X)$ and the Kodaira dimension $\kappa(X)$ is 0 (this means $\max_{m>0} P_m(X) = 1$). This result has been improved on by many authors, and the optimal version can be found in [2] and [12]: *X is birational to an Abelian variety if and only if $q(X) = \dim(X)$, and $P_2(X) = 1$ or $0 < P_m(X) \leq m - 2$ for some $m \geq 3$.*

When $q(X) = \dim(X)$, but the numerical invariants of X are a little bit higher than these bounds, one can still obtain a complete birational description of X. Hacon and Pardini treated the case $P_3(X) = 2$ and proved that X is birational to a smooth double cover of its Albanese variety $\mathrm{Alb}(X)$, with explicit and very specific branch locus ([11]). Hacon then gave an equally precise description in the case $P_3(X) = 3$ (X is birational to a smooth bidouble cover of $\mathrm{Alb}(X)$) in [9], and Chen and Hacon dealt with the case $P_3(X) = 4$, where the description obtained is still complete but more complicated ([4]).

In this article, following the strategy of Chen and Hacon in [4], but building on the results of [12], we give a complete birational description of X in the case $P_2(X) = 2$ (Theorem 3.3).

Theorem. *Let X be a smooth projective variety with $q(X) = \dim(X)$ and $P_2(X) = 2$. Then $\kappa(X) = 1$ and X is birational to a quotient $(K \times C)/G$, where K is an Abelian*

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variety, C is a curve, G is a finite group which acts diagonally and freely on $K \times C$, and $C \to C/G$ is branched at 2 points.

A main ingredient in the above classifications is to bound the possible Kodaira dimensions of X. In this direction, we have the following result (Theorem 2.4).

Theorem. *Let X be a smooth projective variety with $q(X) = \dim(X)$ and $0 < P_m(X) \leq 2m - 2$, for some $m \geq 4$. Then $\kappa(X) \leq 1$.*

Throughout this article, we work over the field of complex numbers.

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1. Preliminaries

In this section we recall some definitions and prove preliminary results. Let X be a smooth projective variety.

1.1. Albanese variety

There is an Abelian variety $\mathrm{Alb}(X)$, called the *Albanese variety* of X, together with a morphism $a_X : X \to \mathrm{Alb}(X)$ called the *Albanese morphism* of X, which has a universal property for morphisms from X to Abelian varieties ([16], Section 4.4). We say that X has *maximal Albanese dimension* if $\dim(a_X(X)) = \dim(X)$. We recall a criterion for the surjectivity of the Albanese morphism ([12, Theorem 2.9]).

Theorem 1.1. *Let X be a smooth projective variety. If*

$$0 < P_m(X) \leq 2m - 2,$$

for some $m \geq 2$, the Albanese morphism $a_X : X \to \mathrm{Alb}(X)$ is surjective.

1.2. Cohomological loci

Let X be a smooth projective variety and let F be a coherent sheaf on X. The cohomological support loci of F are defined as

$$V_i(F) = \{P \in \mathrm{Pic}^0(X) \mid H^i(X, F \otimes P) \neq 0\}.$$

1.3. Iitaka fibration

We denote by $X \dashrightarrow I(X)$ the *Iitaka fibration* of X, where $I(X)$ has dimension $\kappa(X)$ ([16, Definition 2.1.36]). Given a surjective morphism $f : X \to Y$, we say that *the Iitaka model of X dominates Y* if there exist an integer $N > 0$ and an ample divisor H on Y such that $NK_X \succeq f^*H$.

The following proposition is [12, Lemma 2.2].

Proposition 1.2. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties and assume that the Iitaka model of X dominates Y. Fix a torsion element $Q \in \mathrm{Pic}^0(X)$ and an integer $m \geq 2$. Then $h^0(X, \omega_X^m \otimes Q \otimes f^*P)$ is constant for all $P \in \mathrm{Pic}^0(Y)$.*

We deduce a corollary in the case where Y is a curve, which we will use several times.

Corollary 1.3. *Let $f : X \to C$ be a surjective morphism between a smooth projective variety X and a smooth projective curve C of genus ≥ 1 and assume that the Iitaka model of X dominates C. If for some torsion element $Q \in \mathrm{Pic}^0(X)$ and some integer $m \geq 2$, we have $h^0(X, \omega_X^m \otimes Q) \neq 0$, then $f_*(\omega_X^m \otimes Q)$ is an ample vector bundle on C.*

Proof. Since C is a smooth curve, the torsion-free sheaf $f_*(\omega_X^m \otimes Q)$ is locally free. Since Q is torsion, there exists an étale cover $\pi : X' \to X$ such that $\omega_X^m \otimes Q$ is a direct summand of $\pi_*\omega_{X'}^m$. By [19, Corollary 3.6], the vector bundle $(f \circ \pi)_*\omega_{X'/C}^m$ is nef, hence so is $f_*(\omega_{X/C}^m \otimes Q)$.

If $g(C) \geq 2$, ω_C is ample, hence so is $f_*(\omega_X^m \otimes Q)$.

If $g(C) = 1$, we claim the following standard fact for which we could not find a reference:

♣ for any nef vector bundle F on an elliptic curve C, the cohomological locus $V_1(F)$ is finite.

We prove the claim by induction on the rank of F. The rank-1 case is trivial. Let $r > 0$ be an integer. Assuming ♣ proved for all nef vector bundles of rank $\leq r$, we will prove ♣ for any nef vector bundle F of rank $r + 1$.

We consider the Harder-Narasimhan filtration ([16, Proposition 6.4.7])

$$0 = F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset F_0 = F,$$

where F_i are subbundles of F with the properties that F_i/F_{i+1} is a semistable bundle for each i and

$$\mu(F_{n-1}/F_n) > \cdots > \mu(F_1/F_2) > \mu(F_0/F_1).$$

Since $F = F_0$ is nef, so is F_0/F_1, hence $\mu(F_0/F_1) \geq 0$. So F_i/F_{i+1} is a semistable vector bundle with positive slope, for each $i \geq 1$. Hence, for each $i \geq 1$, F_i/F_{i+1} is an ample vector bundle (see Main Claim in the proof of [16, Theorem 6.4.15]). Thus F_1 is also ample, and $V_1(F_1)$ is empty. We just need to prove that $V_1(F_0/F_1)$ is finite.

If $\mu(F_0/F_1) > 0$, we again have that F_0/F_1 is ample and $V_1(F)$ is empty; so we are done. If $\mu(F_0/F_1) = 0$, take $P \in V_1(F_0/F_1)$. Then $h^1(C, F_0/F_1 \otimes P) \neq 0$. Hence, by Serre duality, there exists a non-trivial homomorphism of bundles

$$\pi_P : F_0/F_1 \to P^\vee.$$

Since F_0/F_1 is semistable and $\mu(F_0/F_1) = 0$, π_P is surjective. We have an exact sequence of vector bundles:

$$0 \to G \to F_0/F_1 \to P^\vee \to 0.$$

The rank of G is $\leq r$. Since F_0/F_1 is semistable and $\mu(G) = \mu(F_0/F_1) = 0$, G is also semistable. Hence G is a nef vector bundle (by the Main Claim quoted above) of rank $\leq r$ and, by induction, $V_1(G)$ is finite. We conclude that $V_1(F) \subset V_1(G) \cup \{P\}$ is finite. We have finished the proof of the Claim.

Let the line bundle Q and the integer m be as in the assumptions. By Proposition 1.2, for $m \geq 2$, $h^0(X, \omega_X^m \otimes Q \otimes f^*P) = h^0(C, f_*(\omega_X^m \otimes Q) \otimes P)$ is constant for all $P \in \mathrm{Pic}^0(C)$, hence $h^1(C, f_*(\omega_X^m \otimes Q) \otimes P)$ is also constant for all $P \in \mathrm{Pic}^0(C)$. By the claim ♣, $h^1(C, f_*(\omega_X^m \otimes Q) \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(C)$. Hence $f_*(\omega_X^m \otimes Q)$ is an I.T. vector bundle of index 0, hence is ample ([5, Corollary 3.2]). □

1.4. Iitaka fibration of a variety of maximal Albanese dimension

We assume in this section that X has maximal Albanese dimension and we consider a model $f : X \to Y$ of the Iitaka fibration of X, where Y is a smooth projective variety. We have the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ \downarrow f & & \downarrow f_* \\ Y & \xrightarrow{a_Y} & \mathrm{Alb}(Y). \end{array} \tag{1.1}$$

By [11, Proposition 2.1], a_Y is generically finite, f_* is an algebraic fiber space, $\mathrm{Ker}(f_*)$ is an Abelian variety denoted by K, and a general fiber of f is birational to an Abelian variety $\widetilde{K}$ isogenous to K. Let G be the kernel of the group morphism

$$\mathrm{Pic}^0(X) = \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(K) \to \mathrm{Pic}^0(\widetilde{K}).$$

Then $f^*\mathrm{Pic}^0(Y)$ is contained in G, and $\overline{G} = G/f^*\mathrm{Pic}^0(Y)$ is a finite group consisting of elements $\chi_1, \ldots, \chi_r$. Let $P_{\chi_1}, \ldots, P_{\chi_r} \in G$ be torsion line bundles representing lifts of the elements of $\overline{G}$, so that

$$G = \bigsqcup_{i=1}^{r} (P_{\chi_i} + f^*\mathrm{Pic}^0(Y)).$$

There is an easy observation:

Lemma 1.4. *Under the above assumptions and notation, let moreover $P \in \mathrm{Pic}^0(X)$. If $H^0(X, \omega_X^m \otimes P) \neq 0$ for some $m > 0$, we have $P \in G$.*

Proof. If $P \notin G$, since a general fiber F of f is birational to the Abelian variety $\widetilde{K}$ and $P|_{\widetilde{K}}$ is non-trivial, any section of $\omega_X^m \otimes P$ vanishes on F. Hence $H^0(X, \omega_X^m \otimes P) = 0$, which is a contradiction. □

Chen and Hacon made several useful observations about the cohomological locus $V_0(\omega_X)$ ([3, Lemma 2.2]) which we summarize in the following proposition.

Proposition 1.5 (Chen-Hacon). *Under the above assumptions and notation, we have the following.*

(1) *$V_0(\omega_X) \subset G$.*
(2) *Denote by $V_0(\omega_X, \chi_i)$ the union of irreducible components of $V_0(\omega_X)$ contained in $P_{\chi_i} + f^* \mathrm{Pic}^0(Y)$. Then for each i, $V_0(\omega_X, \chi_i)$ is not empty.*
(3) *If $P_{\chi_i} \notin f^* \mathrm{Pic}^0(Y)$, the dimension of $V_0(\omega_X, \chi_i)$ is positive.*

Since every component of $V_0(\omega_X)$ is a translate by a torsion point of an Abelian subvariety of $\mathrm{Pic}^0(X)$ ([7, 8, 18]), we can write by item (1):

$$V_0(\omega_X) = \bigcup_{1 \leq i \leq r} \bigcup_s (P_{\chi_{i,s}} + T_{\chi_{i,s}}) \subset G,$$

where $P_{\chi_{i,s}} \in P_{\chi_i} + f^* \mathrm{Pic}^0(Y)$ is a torsion point and $T_{\chi_{i,s}}$ is an Abelian subvariety of $f^* \mathrm{Pic}^0(Y)$.

Definition 1.6. We call $T_{\chi_{i,s}}$ a *maximal component* of $V_0(\omega_X)$ if $T_{\chi_{i,s}}$ is maximal for the inclusion among all $T_{\chi_{j,t}}$.

By [3, Theorem 2.3], note that necessarily, if $\kappa(X) > 0$ and $T_{\chi_{i,s}}$ is a maximal component of $V_0(\omega_X)$, we have $\dim(T_{\chi_{i,s}}) \geq 1$.

We conclude this section with a technical result on the structure of the locus $V_0(\omega_X)$ when $\kappa(X) > 0$.

Proposition 1.7. *Let X be a smooth projective variety with maximal Albanese dimension, such that $\kappa(X) > 0$. Let $T_{\chi_{i,s}}$ be a maximal component of $V_0(\omega_X)$. Then, for any (j, t) such that $\dim(T_{\chi_{j,t}}) \geq 1$, we have $\dim(T_{\chi_{i,s}} \cap T_{\chi_{j,t}}) \geq 1$.*

Proof. Let $\widehat{T}_{\chi_{i,s}}$ and $\widehat{T}_{\chi_{j,t}}$ be the dual of $T_{\chi_{i,s}}$ and $T_{\chi_{j,t}}$ respectively. Let π_1 and π_2 be the natural morphisms of Abelian varieties $\mathrm{Alb}(X) \to \widehat{T}_{\chi_{i,s}}$ and $\mathrm{Alb}(X) \to \widehat{T}_{\chi_{j,t}}$. Take an étale cover $t : \widetilde{X} \to X$ which is induced by an étale cover of $\mathrm{Alb}(X)$ such that $t^* P_{\chi_{i,s}}$ and $t^* P_{\chi_{j,t}}$ are trivial. Let f_1 and f_2 be the compositions of morphisms

$\pi_1 \circ a_X \circ t$ and $\pi_2 \circ a_X \circ t$, respectively. We then take the Stein factorizations of f_1 and f_2:

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow g_1 & \searrow f_1 & \\ X_1 & \xrightarrow{h_1} & \widehat{T}_{\chi_{i,s}} \end{array} \qquad \begin{array}{ccc} \widetilde{X} & & \\ \downarrow g_2 & \searrow f_2 & \\ X_2 & \xrightarrow{h_2} & \widehat{T}_{\chi_{j,t}}. \end{array}$$

After modifications, we can assume that X_1 and X_2 are smooth. We claim the following:

- $h^0(X_1, \omega_{X_1} \otimes h_1^* P) > 0$ for all $P \in T_{\chi_{i,s}}$, and similarly $h^0(X_2, \omega_{X_2} \otimes h_2^* Q) > 0$ for all $Q \in T_{\chi_{j,t}}$.

The argument to prove the claim is due to Chen and Debarre. Let c be the codimension of $T_{\chi_{i,s}}$ in $\mathrm{Pic}^0(X)$. By the proof of [6, Theorem 3], $P_{\chi_{i,s}} + T_{\chi_{i,s}}$ is an irreducible component of $V_c(\omega_X)$. Hence

$$h^c(\widetilde{X}, \omega_{\widetilde{X}} \otimes t^* P) \geq h^c(X, \omega_X \otimes P \otimes P_{\chi_{i,s}}) > 0$$

for any $P \in T_{\chi_{i,s}}$.

Again by the proof of [6, Theorem 3], the dimension of a general fiber of g_1 is also c. Since g_1 is an algebraic fiber space, we have $R^c g_{1*}\omega_{\widetilde{X}} = \omega_{X_1}$ ([14, Proposition 7.6]), and

$$R^c f_{1*}\omega_{\widetilde{X}} = h_{1*}(R^c g_{1*}\omega_{\widetilde{X}}) = h_{1*}\omega_{X_1}$$

([15, Theorem 3.4]). Moreover, the sheaves $R^k f_{1*}\omega_{\widetilde{X}}$, satisfy the generic vanishing theorem ([10, Corollary 4.2]), hence $V_j(R^k f_{1*}\omega_{\widetilde{X}}) \neq T_{\chi_{i,s}}$ for any $j > 0$. Pick $P \in T_{\chi_{i,s}} - \bigcup_{j>0,k} V_j(R^k f_{1*}\omega_{\widetilde{X}})$, so that

$$H^j(\widehat{T}_{\chi_{i,s}}, R^k f_{1*}\omega_{\widetilde{X}} \otimes P) = 0$$

for all $j > 0$ and all k. By the Leray spectral sequence, we have

$$0 \neq h^c(\widetilde{X}, \omega_{\widetilde{X}} \otimes f_1^* P) = h^0(\widehat{T}_{\chi_{i,s}}, R^c f_{1*}\omega_{\widetilde{X}} \otimes P) = h^0(\widehat{T}_{\chi_{i,s}}, h_{1*}\omega_{X_1} \otimes P).$$

Hence we conclude the claim by semicontinuity.

If $\dim(T_{\chi_{i,s}} \cap T_{\chi_{j,t}}) = 0$, the morphism

$$\mathrm{Alb}(X) \xrightarrow{(\pi_1,\pi_2)} \widehat{T}_{\chi_{i,s}} \times \widehat{T}_{\chi_{j,t}}$$

is surjective. Now we consider the following diagram

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{t} & X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ \downarrow g_1 & & & \searrow & \downarrow \pi_1 \\ X_1 & & \xrightarrow{h_1} & & \widehat{T}_{\chi_{i,s}}. \end{array}$$

From the proof of [6, Theorem 3], we know that the fibers of g_1 fill up the fibers of π_1. Hence we have a surjective morphism $\widetilde{X} \xrightarrow{(g_1,g_2)} X_1 \times X_2$. Since $a_X \circ t : \widetilde{X} \to \mathrm{Alb}(X)$ and $(h_1, h_2) : X_1 \times X_2 \to \widehat{T}_{\chi_{i,s}} \times \widehat{T}_{\chi_{j,t}}$ are generically finite and surjective, by the proof of [2, Lemma 3.1], $K_{\widetilde{X}/X_1 \times X_2}$ is effective. Therefore, it follows from the claim that

$$h^0(\widetilde{X}, \omega_{\widetilde{X}} \otimes t^*P \otimes t^*Q) > 0 \tag{1.2}$$

for all $P \in T_{\chi_{i,s}}$ and $Q \in T_{\chi_{j,t}}$. Since $t : \widetilde{X} \to X$ is birationally equivalent to an étale cover of X induced by an étale cover of $\mathrm{Alb}(X)$, $t_*\mathcal{O}_{\widetilde{X}} = \bigoplus_\alpha P_\alpha$, where P_α is a torsion line bundle on X for every α. Let

$$T = T_{\chi_{i,s}} + T_{\chi_{j,t}}$$

be the Abelian variety generated by $T_{\chi_{i,s}}$ and $T_{\chi_{j,t}}$. Then (1.2) implies that there exists an α such that

$$P_\alpha + T \subset V_0(\omega_X).$$

Since $\dim(T_{\chi_{j,t}}) \geq 1$ and $\dim(T_{\chi_{i,s}} \cap T_{\chi_{j,t}}) = 0$, we obtain $T_{\chi_{i,s}} \subsetneq T$, contradicting the assumption that $T_{\chi_{i,s}}$ is a maximal component of $V_0(\omega_X)$. This finishes the proof of the proposition. ◻

2. Varieties with $q(X) = \dim(X)$ and $0 < P_m(X) \leq 2m - 2$

In this section, we prove that the Iitaka model of a smooth projective variety X with $q(X) = \dim(X)$ and $0 < P_m(X) \leq 2m - 2$ for some $m \geq 2$ is birational to an Abelian variety. We begin with a useful easy lemma ([11, Lemma 2.14]).

Lemma 2.1. *Let X be a smooth projective variety, let L and M be line bundles on X, and let $T \subset \mathrm{Pic}^0(X)$ be an irreducible subvariety of dimension t. If for some positive integers a and b and all $P \in T$, we have $h^0(X, L \otimes P) \geq a$ and $h^0(X, M \otimes P^{-1}) \geq b$, then $h^0(X, L \otimes M) \geq a + b + t - 1$.*

Our next result is a consequence of the proof of [4, Proposition 3.6 and Proposition 3.7], although not explicitly stated there.

Proposition 2.2. *Let X be a smooth projective variety with $q(X) = \dim(X)$ and $0 < P_m(X) \leq 2m - 2$ for some $m \geq 2$. Let $f : X \to Y$ be an algebraic fiber space onto a smooth projective variety Y, which is birationally equivalent to the Iitaka fibration of X. Then Y is birational to an Abelian variety.*

Proof. Since we have $0 < P_m(X) \leq 2m - 2$ for some $m \geq 2$, a_X is surjective by Theorem 1.1. Since $q(X) = \dim(X)$, we saw in Section 1.4 that a_X and a_Y are both surjective and generically finite. We then use diagram (1.1) and the notation of Section 1.4.

If $\kappa(X) = 1$, then Y is an elliptic curve, because a_Y is surjective.

If $\kappa(X) \geq 2$, we use the same argument as in the proof of [4, Proposition 3.6]. We claim that

(†) $V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y) = \{\mathcal{O}_X\}$.

Let δ be the maximal dimension of a component of $V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y)$.

If $\delta = 0$, $V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y) = \{\mathcal{O}_X\}$ by [3, Proposition 1.3.3].

If $\delta \geq 2$, by Lemma 2.1, there exists $P_0 \in f^* \operatorname{Pic}^0(Y)$ such that

$$h^0(X, \omega_X^2 \otimes P_0) \geq 1 + 1 + 2 - 1 = 3.$$

By Proposition 1.2, $h^0(X, \omega_X^2 \otimes P) = h^0(X, \omega_X^2 \otimes P_0) \geq 3$ for any $P \in f^* \operatorname{Pic}^0(Y)$. We iterate this process and get $P_m(X) \geq 2m - 1$, which is a contradiction.

If $\delta = 1$, there is a 1-dimensional component T of $V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y)$. Let $E = \operatorname{Pic}^0(T)$ and let $g : X \to E$ be the induced surjective morphism. By [4, Corollary 2.11 and Lemma 2.13], for some torsion element $P \in T$, there exist a line bundle L of degree 1 on E and an inclusion $g^*L \hookrightarrow \omega_X \otimes P$, and $P|_F = \mathcal{O}_F$, where F is a general fiber of g. Since $\kappa(X) \geq 2$, we have $\kappa(F) \geq 1$. Again by [2, Theorem 3.2], $\operatorname{rank}(g_*(\omega_X^2 \otimes P^2)) = P_2(F) \geq 2$. Consider the exact sequence of sheaves on E:

$$0 \to L^2 \to g_*(\omega_X^2 \otimes P^2) \to \mathcal{Q} \to 0,$$

where $\operatorname{rank}(\mathcal{Q}) \geq 1$. Since $g : X \to E$ is dominated by $f : X \to Y$, the Iitaka model of X (*i.e.* Y) dominates E, hence $g_*(\omega_X^2 \otimes P^2)$ is ample by Corollary 1.3, thus so is $\mathcal{Q}$ and $h^0(X, \mathcal{Q}) \geq 1$. Hence $h^0(X, \omega_X^2 \otimes P^2) \geq 3$.

For any $k \geq 3$, we apply Lemma 2.3 (to be proved below) to get

$$h^0(X, \omega_X^k \otimes P^k) \geq h^0(X, \omega_X^{k-1} \otimes P^{k-1}) + 2.$$

By induction, we have $h^0(X, \omega_X^m \otimes P^m) \geq 2m - 1$ for all $m \geq 2$. Since $P \in T \subset f^* \operatorname{Pic}^0(Y)$, we have, by Proposition 1.2, $P_m(X) = h^0(X, \omega_X^m \otimes P^m) \geq 2m - 1$, which is a contradiction. We have proved claim (†).

Since X and Y are of maximal Albanese dimension, $K_{X/Y}$ is effective (see the proof of [2, Lemma 3.1]). This implies

$$f^* V_0(\omega_Y) \subset V_0(\omega_X) \cap f^* \operatorname{Pic}^0(Y) = \{\mathcal{O}_X\},$$

and hence $\kappa(Y) = 0$ by [3, Theorem 1]. By Kawamata's Theorem ([13]), a_Y is birational. □

We still need to prove the following result used in the proof of the proposition. It is an analogue of [4, Corollary 3.2].

Lemma 2.3. *Let X be a smooth projective variety of maximal Albanese dimension with $\kappa(X) \geq 2$. Suppose that there exist a surjective morphism $g : X \to C$ onto an* *elliptic curve C and an ample line bundle L on C with an inclusion $g^*L \hookrightarrow \omega_X \otimes P_2$ for some torsion line bundle $P_2 \in \mathrm{Pic}^0(X)$. Then we have*

$$h^0(X, \omega_X^m \otimes P_1 \otimes P_2) \geq h^0(X, \omega_X^{m-1} \otimes P_1) + 2,$$

for all torsion line bundles $P_1 \in V_0(\omega_X)$ and all $m \geq 3$.

Proof. From the inclusion, we obtain $H^0(X, \omega_X \otimes P_2) \neq 0$, and by items (1) and (2) in Proposition 1.5, we conclude that $P_2 \in V_0(\omega_X, \chi_i)$ for some i and we get an exact sequence of sheaves on C:

$$0 \to g_*(\omega_X^{m-1} \otimes P_1) \otimes L \hookrightarrow g_*(\omega_X^m \otimes P_1 \otimes P_2) \to \mathcal{Q} \to 0. \qquad (2.1)$$

By item (2) in Proposition 1.5, we have $h^0(X, \omega_X \otimes P_2^\vee \otimes P) \neq 0$ for some $P \in f^* \mathrm{Pic}^0(Y)$ such that $P_2^\vee \otimes P \in V_0(\omega_X, -\chi_i)$. Hence we have an inclusion $g^*L \hookrightarrow \omega_X^2 \otimes P$. Moreover, since P_2 is a torsion line bundle and each irreducible component of $V_0(\omega_X)$ is a subtorus of $\mathrm{Pic}^0(X)$ translated by a torsion point, we may assume that $P \in f^* \mathrm{Pic}^0(Y)$ is also a torsion line bundle. Therefore the Iitaka model of X dominates C. Thus, by Corollary 1.3, both $g_*(\omega_X^{m-1} \otimes P_1)$ and $g_*(\omega_X^m \otimes P_1 \otimes P_2)$ are ample, and so is $\mathcal{Q}$. By Serre duality, for any ample vector bundle V on C, we have $H^1(C, V) = 0$. Hence, Riemann-Roch gives

$$h^0(X, \omega_X^{m-1} \otimes P_1) = h^0(C, g_*(\omega_X^{m-1} \otimes P_1)) = \deg(g_*(\omega_X^{m-1} \otimes P_1)),$$

and

$$h^0(\omega_X^m \otimes P_1 \otimes P_2) = h^0(C, g_*(\omega_X^{m-1} \otimes P_1) \otimes L) + h^0(C, \mathcal{Q}).$$

Let F be a connected component of a general fiber of g. Since $\kappa(X) \geq 2$, we have $\kappa(F) \geq 1$ by the easy addition formula ([17, Corollary 1.7]). Hence we have $P_2(F) \geq 2$ by [2, Theorem 3.2] (see also Remark 3.2). Since $P_1 \in V_0(\omega_X)$, we have $h^0(X, \omega_X^m \otimes P_1 \otimes P_2) \geq h^0(X, \omega_X^{m-1} \otimes P_2) > 0$. Hence we have $h^0(F, \omega_F^m \otimes P_1 \otimes P_2) \neq 0$. Then, by Lemma 1.4 and Proposition 1.5, there exists $P' \in \mathrm{Pic}^0(F)$ which is pulled back by the Iitaka fibration of F such that $(P_1 \otimes P_2)|_F \otimes P' \in V_0(\omega_F)$. On the other hand, since $P_1 \otimes P_2$ is torsion, we have $h^0(F, \omega_F^m \otimes P_1 \otimes P_2) = h^0(F, \omega_F^m \otimes P_1 \otimes P_2 \otimes P')$ by Proposition 1.2. Therefore, we conclude

$$h^0(F, \omega_F^m \otimes P_1 \otimes P_2) = h^0(F, \omega_F^m \otimes P_1 \otimes P_2 \otimes P') \geq P_{m-1}(F) \geq 2,$$

where the last inequality holds since $m \geq 3$. Hence,

$$\mathrm{rank}(g_*(\omega_X^m \otimes P_1 \otimes P_2)) = h^0(F, \omega_F^m \otimes P_1 \otimes P_2) \geq 2.$$

Since $P_1 \in V_0(\omega_X)$ by assumption, we have $\mathrm{rank}(g_*(\omega_X^{m-1} \otimes P_1)) \geq 1$.

If $\operatorname{rank}(g_*(\omega_X^{m-1} \otimes P_1)) \geq 2$, we have

$$\begin{aligned} h^0(C, g_*(\omega_X^m \otimes P_1 \otimes P_2)) &\geq h^0(C, g_*(\omega_X^{m-1} \otimes P_1) \otimes L) \\ &\geq \deg(g_*(\omega_X^{m-1} \otimes P_1)) + \operatorname{rank}(g_*(\omega_X^{m-1} \otimes P_1)) \\ &\geq h^0(X, \omega_X^{m-1} \otimes P_1) + 2. \end{aligned}$$

If $\operatorname{rank}(g_*(\omega_X^{m-1} \otimes P_1)) = 1$, $\mathcal{Q}$ has rank ≥ 1. Since $\mathcal{Q}$ is ample, $h^0(C, \mathcal{Q}) \geq 1$. We also have

$$\begin{aligned} h^0(X, \omega_X^m \otimes P_1 \otimes P_2) &= h^0(C, g_*(\omega_X^{m-1} \otimes P_1) \otimes L) + h^0(C, \mathcal{Q}) \\ &\geq h^0(X, \omega_X^{m-1} \otimes P_1) + \operatorname{rank}(\omega_X^{m-1} \otimes P_1) + 1 \\ &= h^0(X, \omega_X^{m-1} \otimes P_1) + 2. \end{aligned}$$

Hence the lemma is proved. □

Under the hypotheses of Proposition 2.2, it turns out that when $m \geq 4$, we can bound the Kodaira dimension of X by 1 (the case $m = 2$ is the object of the next section; when $m = 3$, the bound $\kappa(X) \leq 2$ was obtained in [4] and there are examples when there is equality).

Theorem 2.4. *Let X be a smooth projective variety with $q(X) = \dim(X)$ and $0 < P_m(X) \leq 2m - 2$ for some $m \geq 4$. Then $\kappa(X) \leq 1$.*

Proof. By Theorem 1.1, the Albanese morphism $a_X : X \to \operatorname{Alb}(X)$ is surjective and hence generically finite. We then use diagram (1.1). By Proposition 2.2, we may assume that Y, the image of the Iitaka fibration of X, is an Abelian variety.

We assume $\kappa(X) \geq 2$ and under this assumption we will deduce a contradiction.

Let $T_{\chi_{1,s}}$ be a maximal component of $V_0(\omega_X)$ in the sense of Definition 1.6. If $\dim(T_{\chi_{1,s}}) = 1$, by Proposition 1.7, we conclude that $T_{\chi_{i,t}} \subset T_{\chi_{1,s}}$ for any (i, t) such that $\dim(T_{\chi_{i,t}}) > 0$. By [3, Theorem 2.3], $\operatorname{Pic}^0(Y) = T_{\chi_{1,s}}$. Then $\dim(Y) = \dim(\operatorname{Pic}^0(Y)) = 1$, which contradicts our assumption that $\kappa(X) \geq 2$. Hence we get $\dim(T_{\chi_{1,s}}) \geq 2$.

We then iterate Lemma 2.1 to get

$$h^0(X, \omega_X^{m-i} \otimes P_{\chi_{1,s}}^{m-i}) \geq (m - i - 1) \dim(T_{\chi_{1,s}}) + 1.$$

By Proposition 1.2, we have

$$h^0(X, \omega_X^{m-i} \otimes P_{\chi_{1,s}}^{m-i} \otimes f^* P) \geq (m - i - 1) \dim(T_{\chi_{1,s}}) + 1, \tag{2.2}$$

for all $0 \leq i \leq m-2$ and all $P \in \operatorname{Pic}^0(Y)$. According to item (2) in Proposition 1.5, $V_0(\omega_X, -(m-1)\chi_1)$ is not empty, namely there exists $P_0 \in \operatorname{Pic}^0(Y)$ such that

$h^0(X, \omega_X \otimes P_{\chi_{1,s}}^{-(m-1)} \otimes P_0) > 0$. Thus $h^0(X, \omega_X^m \otimes P_0) \geq h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-1})$. Again by Proposition 1.2, we have

$$P_m(X) = h^0(X, \omega_X^m \otimes P_0) \geq h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-1}).$$

We also have

$$2m - 2 \geq P_m(X) \geq h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-1}) \geq (m-2)\dim(T_{\chi_{1,s}}) + 1,$$

where the last inequality holds by taking $i = 1$ in (2.2). Hence we deduce that $\dim(T_{\chi_{1,s}}) = 2$.

Claim 1: $(m-1)\chi_1 = 0$.

If $(m-1)\chi_1 \neq 0$, by item (3) in Proposition 1.5, there exists a torsion point $P_{-(m-1)\chi_{1,t}} \in \mathrm{Pic}^0(X)$ such that $P_{-(m-1)\chi_{1,t}} + T_{-(m-1)\chi_{1,t}} \subset V_0(\omega_X)$ with $\dim(T_{-(m-1)\chi_{1,t}}) \geq 1$.

If $\dim(T_{-(m-1)\chi_{1,t}}) \geq 2$, by (2.2) (let $i = 1$) and Lemma 2.1, we get $P_m(X) \geq 2m - 3 + 1 + 2 - 1 = 2m - 1$, which is a contradiction.

Hence $\dim(T_{-(m-1)\chi_{1,t}}) = 1$. Let $C = \widehat{T}_{-(m-1)\chi_{1,t}}$ and let $\pi : \mathrm{Alb}(X) \to C$ be the dual of the inclusion $T_{-(m-1)\chi_{1,t}} \hookrightarrow \mathrm{Pic}^0(X)$. Then we set $f = \pi \circ a_X$ as in the following commutative diagram:

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle a_X} & \searrow^{f} & \\ \mathrm{Alb}(X) & \xrightarrow{\pi} & C. \end{array}$$

Since we assume $\kappa(X) \geq 2$ and $\dim(T_{-(m-1)\chi_{1,t}}) = 1$, we have $V_0(\omega_X) \neq \mathrm{Pic}^0(X)$, therefore $\chi(\omega_X) = 0$. By [4, Lemma 2.10 and Corollary 2.11], there exists an ample line bundle L on C such that $f^*L \hookrightarrow \omega_X \otimes P_{-(m-1)\chi_{1,t}}$. We then apply Lemma 2.3 to conclude that

$$\begin{aligned} P_m(X) &= h^0(X, \omega_X^m \otimes P_{\chi_{1,s}}^{m-1} \otimes P_{-(m-1)\chi_{1,t}}) \\ &\geq h^0(X, \omega_X^{(m-1)} \otimes P_{\chi_{1,s}}^{m-1}) + 2 \\ &\geq 2m - 1, \end{aligned}$$

where the last inequality holds by (2.2). This is a contradiction. We have proved Claim 1.

Let $\overline{G}$ be defined as in the beginning of Section 1.4.

Claim 2: $\overline{G} \simeq \mathbb{Z}/2$, namely $\overline{G}$ contains only one nonzero element χ_1. In particular, by Claim 1, m is an odd number.

Assuming the claim is not true, there exists $0 \neq \chi_2 \in \overline{G}$ such that $(m-2)\chi_1 + \chi_2 \neq 0$. According to item (3) in Proposition 1.5, there exists $P_{\chi_{2,t}} + T_{\chi_{2,t}} \subset$

$V_0(\omega_X, \chi_2)$ with $\dim(T_{\chi_{2,t}}) \geq 1$. Then as in the proof of Claim 1, by Lemma 2.1 and Lemma 2.3, we conclude

$$h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-2} \otimes P_{\chi_{2,t}}) \geq h^0(X, \omega_X^{m-2} \otimes P_{\chi_{1,s}}^{m-2}) + 2 \geq 2m - 3,$$

where the last inequality holds because of (2.2).

Since $(m-2)\chi_1 + \chi_2 \neq 0$, we may repeat the above process to get

$$P_m(X) \geq h^0(X, \omega_X^{m-1} \otimes P_{\chi_{1,s}}^{m-2} \otimes P_{\chi_{2,t}}) + 2 \geq 2m - 1,$$

which is a contradiction. Hence we have proved Claim 2.

As $m \geq 4$ is odd, $m - 2 \geq 3$ and $(m-3)\chi_1 = 0$. By (2.2) (with $i = 3$), $P_{m-3}(X) \geq 2m - 7$. Since $\kappa(X) \geq 2$, by Proposition 1.2 and Lemma 2.1, we have

$$\begin{aligned} 2m - 2 \geq P_m(X) &\geq P_{m-3}(X) + P_3(X) + \kappa(X) - 1 \\ &\geq 2m - 6 + P_3(X). \end{aligned}$$

Hence $P_3(X) \leq 4$. According to Chen and Hacon's classification of these varieties (see [4, Theorems 1.1 and 1.2]) and Claim 2, the only possibility is that X is a double cover of its Albanese variety and $\kappa(X) = 2$, as described in [4, Example 2]. Namely, there exists an algebraic fiber space

$$q : \mathrm{Alb}(X) \to S$$

from an Abelian variety of dimension ≥ 3 to an Abelian surface, and $a_X : X \to \mathrm{Alb}(X)$ is birational to a double cover of $\mathrm{Alb}(X)$ such that $a_{X*}\mathcal{O}_X = \mathcal{O}_{\mathrm{Alb}(X)} \oplus (q^*L \otimes P)^\vee$, where L is an ample divisor of S with $h^0(S, L) = 1$ and $P \in \mathrm{Pic}^0(A) - \mathrm{Pic}^0(S)$ and $2P \in \mathrm{Pic}^0(S)$. However, for such a variety, we have the inclusion of sheaves $a_X^*(q^*L \otimes P) \hookrightarrow \omega_X$ (see the proof of [4, Claim 4.6]). Thus, as $m \geq 4$ is odd,

$$\begin{aligned} P_m(X) &= h^0(X, \omega_X^m) \\ &\geq h^0(\mathrm{Alb}(X), q^*L^m \otimes P^m \otimes a_{X*}\mathcal{O}_X) \\ &= h^0(\mathrm{Alb}(X), q^*L^{m-1} \otimes P^{m-1}) \\ &= (m-1)^2 > 2m - 2, \end{aligned}$$

which is a contradiction. This concludes the proof of Theorem 2.4. □

3. Varieties with $q(X) = \dim(X)$ and $P_2(X) = 2$

In this section, we describe explicitly all smooth projective varieties X with $q(X) = \dim(X)$ and $P_2(X) = 2$. We first show that the Iitaka model of X is an elliptic curve. In particular, $\kappa(X) = 1$.

Proposition 3.1. *Let X be a smooth projective variety with $q(X) = \dim(X)$ and $P_2(X) = 2$. Assume that $f : X \to Y$ is a birational model of the Iitaka fibration of X and Y is a smooth projective variety. Then Y is an elliptic curve.*

Proof. We use diagram (1.1). By Theorem 1.1, a_X is surjective and hence generically finite. By Proposition 2.2, we may assume that $a_Y : Y \to \mathrm{Alb}(Y)$ is an isomorphism.

If $\dim(Y) = 1$, Y is an elliptic curve and we are done. We now assume that $\dim(Y) \geq 2$ and deduce a contradiction.

Let $T_{\chi_{i,s}}$ be a maximal component of $V_0(\omega_X)$. By the claim (†) in the proof of Proposition 2.2, we know that $P_{\chi_i} \notin f^* \mathrm{Pic}^0(Y)$. By item (3) of Proposition 1.5, there exist a torsion line bundle $P_{-\chi_{i,t}} \in P_{\chi_i}^{\vee} + f^* \mathrm{Pic}^0(Y)$ and a positive-dimensional Abelian subvariety $T_{-\chi_{i,t}} \subset f^* \mathrm{Pic}^0(Y)$ such that $P_{-\chi_{i,t}} + T_{-\chi_{i,t}}$ is a connected component of $V_0(\omega_X)$. Let T be the neutral component of $T_{\chi_{i,s}} \cap T_{-\chi_{i,t}}$. By Proposition 1.7, $\dim(T) \geq 1$.

If $\dim(T) \geq 2$, then by Lemma 2.1,

$$h^0(X, \omega_X^2 \otimes P \otimes Q) \geq 1 + 1 + 2 - 1 = 3,$$

for all $P \in P_{\chi_{i,s}} + T$ and all $Q \in P_{-\chi_{i,t}} + T$. Since $P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}} \in f^* \mathrm{Pic}^0(Y)$, we obtain, by Proposition 1.2, $P_2(X) \geq 3$, which is a contradiction.

Hence T is an elliptic curve and we denote by $\widehat{T}$ its dual. There exists a projection $\pi : Y \to \widehat{T}$. We then consider the commutative diagram:

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & \overset{\bar{f}}{\searrow} & \\ Y & \xrightarrow{\pi} & \widehat{T} \end{array}$$

and define

$$\begin{aligned} F_1 &= \bar{f}_*(\omega_X \otimes P_{\chi_{i,s}}), \\ F_2 &= \bar{f}_*(\omega_X \otimes P_{-\chi_{i,t}}), \\ F_3 &= \bar{f}_*(\omega_X^2 \otimes P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}}). \end{aligned}$$

These are vector bundles on the elliptic curve $\widehat{T}$ and by [19, Corollary 3.6] and Corollary 1.3, F_1 and F_2 are nef and F_3 is ample.

Since $P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}} \in f^* \mathrm{Pic}^0(Y)$ and $f : X \to Y$ is a model of the Iitaka fibration of X, we have

$$2 = P_2(X) = h^0(X, \omega_X^2 \otimes P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}}) = h^0(\widehat{T}, F_3).$$

There exists a natural morphism

$$F_1 \otimes F_2 \xrightarrow{\upsilon} F_3,$$

corresponding to the multiplication of sections. Let R_1, R_2, and R_3 be the respective ranks of F_1, F_2, and F_3. I claim:

♠ $R_3 > \min\{R_1, R_2\}$.

Indeed if $R_1 \geq 2$ and $R_2 \geq 2$, then by Lemma 2.1, $R_3 \geq R_1 + R_2 - 1$. If either R_1 or R_2 is 1, we just need to prove $R_3 \geq 2$. Let $f|_{X_t} : X_t \to Y_t$ be the restriction of f to a general fiber of $\bar{f}$. Since $f : X \to Y$ is a model of the Iitaka fibration of X, fixing an ample divisor H on Y, there exists an integer $N > 0$ such that $NK_X - H$ is effective. Hence $(NK_X - H)|_{X_t} \succeq 0$, therefore the Iitaka model of X_t dominates Y_t. Indeed, $f|_{X_t} : X_t \to Y_t$ is a birational model of the Iitaka fibration of X_t since a general fiber of $f|_{X_t}$ is isomorphic to a general fiber of f which is birational to an Abelian variety. As we have assumed $\dim(Y) \geq 2$, we have $\dim(Y_t) \geq 1$. Thus X_t is of maximal Albanese dimension and $\kappa(X_t) \geq 1$, hence $P_2(X_t) \geq 2$ ([2, Theorem 3.2]). Since $(P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}})|_{X_t}$ is pulled back from Y_t, we have

$$R_3 = h^0(X_t, (\omega_X^2 \otimes P_{\chi_{i,s}} \otimes P_{-\chi_{i,t}})|_{X_t}) = P_2(X_t) \geq 2,$$

where the second equality holds again because of Proposition 1.2. This proves the claim ♠.

Consider the Harder-Narasimhan filtration of F_1 (respectively F_2) and let G_1 (respectively G_2) be the unique maximal subbundle of F_1 (respectively F_2) of largest slope. By definition, G_1 and G_2 are semistable. Let r_1 and r_2 be their respective ranks. I claim:

♣ $r_1 > 0$ and $r_2 > 0$ and therefore G_1 and G_2 are ample.

If $\deg(G_1) = 0$, we conclude from the definition of G_1 that $0 = \mu(G_1) \geq \mu(F_1)$. Since F_1 is nef, $\deg(F) = \mu(F) = 0$, hence $V_0(F_1) = V_1(F_1)$. By the generic vanishing theorem (see for example [10]), $V_1(F_1)$ is finite. However, since $T \subset T_{\chi_{i,s}}$, we have $h^0(F_1 \otimes P) > 0$ for all $P \in T$, which is a contradiction. So $r_1 > 0$ and similarly, $r_2 > 0$. Since G_1 and G_2 are semistable, they are ample (see the Main Claim in the [16, proof of Theorem 6.4.15]). This proves the claim ♣.

Set $G_3 = \upsilon(G_1 \otimes G_2)$ and let r_3 be its rank. Again by Lemma 2.1, we have

$$r_3 \geq r_1 + r_2 - 1 \geq \max\{r_1, r_2\}.$$

Since G_1 and G_2 are semistable and ample, so is $G_1 \otimes G_2$ ([16, Corollary 6.4.14]). Therefore the slopes satisfy

$$\mu(G_3) \geq \mu(G_1 \otimes G_2) = \mu(G_1) + \mu(G_2),$$

and G_3 is also ample.

We then apply Riemann-Roch,

$$\begin{aligned} h^0(\widehat{T}, G_3) &\geq r_3(\mu(G_1) + \mu(G_2)) \\ &\geq \deg(G_1) + \deg(G_2) \\ &\geq 2, \end{aligned}$$

where the second inequality holds because $r_3 \geq \max\{r_1, r_2\}$ and the third inequality holds because $\deg(G_1) > 0$ and $\deg(G_2) > 0$.

Since G_3 is a subbundle of F_3 and $h^0(\widehat{T}, F_3) = 2$, we have $h^0(\widehat{T}, G_3) = 2$, hence all the inequalities above should be equalities. In particular, $r_3 = r_1 = r_2$. Hence by the claim ♠, $r_3 \leq \min\{R_1, R_2\} < R_3$. Therefore, F_3/G_3 is a sheaf of rank ≥ 1. Since F_3 is ample, so is F_3/G_3, hence $h^0(\widehat{T}, F_3/G_3) \geq 1$. Since G_3 is ample, $h^1(\widehat{T}, G_3) = 0$. Hence,

$$h^0(\widehat{T}, F_3) = h^0(\widehat{T}, G_3) + h^0(\widehat{T}, F_3/G_3) \geq 3,$$

which is a contradiction. Thus $\dim(Y) = 1$. This finishes the proof of Proposition 3.1. □

Remark 3.2. It is easy to see that combining Proposition 1.7 and the proof of Proposition 3.1, one can give another proof of Chen and Hacon's characterization of Abelian varieties ([2]): *a smooth projective variety X with maximal Albanese dimension and $P_2(X) = 1$ is birational to an Abelian variety.*

The following theorem is the main result of this article. It describes explicitly all smooth projective varieties X with $q(X) = \dim(X)$ and $P_2(X) = 2$.

Theorem 3.3. *Let X be a smooth projective variety with $P_2(X) = 2$ and $q(X) = \dim(X)$. Then $\kappa(X) = 1$ and X is birational to a quotient $(K \times C)/G$, where K is an Abelian variety and C is a smooth projective curve, G is a finite group that acts diagonally and freely on $K \times C$, and $C \to C/G$ is branched at 2 points.*

Proof. Since we know by Proposition 3.1 that Y is an elliptic curve, the proof is parallel to the [4, proof of Theorem 5.1]. By [13], Theorem 13, there exists a curve C of genus $g \geq 2$, an Abelian variety $\widetilde{K}$, and a finite Abelian group G, which acts faithfully on C and $\widetilde{K}$, such that X is birational to $(\widetilde{K} \times C)/G$, where G acts diagonally and freely on $\widetilde{K} \times C$.

We then consider the induced morphism $\phi : C \to C/G = Y$. Following [1], Section VI.12, we have

$$2 = P_2(X) = \dim(H^0(C, \omega_C^2)^G) = h^0\left(Y, \mathcal{O}_Y\left(\sum_{P \in Y} \left\lfloor 2\left(1 - \frac{1}{e_P}\right)\right\rfloor P\right)\right),$$

where P is a branch point of ϕ, and e_P is the ramification index of a ramification point lying over P.

Since $\lfloor 2(1 - \frac{1}{e_P})\rfloor = 1$ for any branch point P, we have only two branch points. □

Example 3.4. Let C be a bi-elliptic curve of genus 2, let $\phi : C \to E$ be the morphism such that ϕ is branched at two points, and let τ be the induced involution. Take an Abelian variety K and set $G = \mathbb{Z}_2$. Let G act on C by τ and on K by translation by a point of order 2. Set $X = (K \times C)/G$, where G acts diagonally. Then $P_2(X) = h^0(C, \omega_C^2)^\tau = 2$ (this construction actually gives all varieties with $q(X) = \dim(X)$ and $P_3(X) = 2$; see [11]).

Remark 3.5. The family of varieties with $q(X) = \dim(X)$ and $P_2(X) = 2$ is not bounded (see [4, Example 1]).

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Max-Planck-Institut für Mathematik
Vivatsgasse, 7
53111 Bonn, Germany
jiang@dna.ens.fr