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L^p-Boundedness of Bergman projections in tube domains over homogeneous cones

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Abstract. In this paper, we generalize to all tube domains over homogeneous cones L^p-continuity properties of the Bergman projection.

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1. Introduction

Let D be a domain in $\mathbb{C}^n$ and dv the Lebesgue measure defined in $\mathbb{C}^n$. We denote by P the Bergman projection *i.e.*, the orthogonal projection of the Hilbert space $L^2(D, dv)$ onto its closed subspace $A^2(D, dv)$ consisting of holomorphic functions on D. It is well-known that, under weak assumptions, P is an integral operator defined on $L^2(D, dv)$ by

$$Pf(z) = \int_D B(z, w) f(w) dv(w),$$

where $B(\cdot, \cdot)$ is the Bergman kernel *i.e.*, the reproducing kernel of $A^2(D, dv)$. In this work we consider the Bergman projection in tube domains over homogeneous cones and we are interested in the values of $p \geq 1$ such that the Bergman projection P can be extended to a bounded operator on $L^p(D, dv)$.

The L^p-boundedness of Bergman projections on tube domains over cones has been studied by many authors. In [1], D. Békollé and A. Bonami considered the tube domain over the forward light cone; they obtained some sufficient conditions using Schur's Lemma. They proved that this condition is necessary and sufficient for the positive Bergman operator, that is, the Bergman operator with kernel $|B(\cdot, \cdot)|$. Jointly with M. M. Peloso and F. Ricci they improved this result in [5]. To take care of cancellations, they introduced the mixed-normed spaces $L^{p,q}$. The consideration of the case $p = 2$ of these spaces has been used in [2] by D. Békollé,

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A. Bonami and G. Garrigós to generalize this improvement to the case of general symmetric cones via a Fourier transform in the x variables. Indeed, they have found values of p for which the Bergman projection is bounded while the positive Bergman operator is unbounded. Together with the first author of this paper, they presented all these results with more details in the Lecture Notes [3] of the Workshop "PDE, Classical Analysis and Applications" held in Yaoundé in December 2001. Moreover, D. Debertol in [10] obtained the generalization of the sufficient conditions above for general weighted measures. We follow the same direction in this paper.

On the other hand, D. Békollé and A. Temgoua in [7] generalized results in [1] to the case of Siegel domains of type II, not necessarily symmetric; again they applied Schur's Lemma to the positive Bergman operator.

As it is proved in [3] or [5], it is important to mention that all these sufficient conditions are far from being necessary when the rank of the cone is greater than 1. However, in [4], an improvement has been obtained in the case of the forward light cone. This is pursued in [14], where G. Garrigós and A. Seeger improved previous work of T. Wolff on the cone multiplier.

The aim of our work is the generalization of all the theory to tube domains over convex homogeneous cones. More precisely, we shall consider general weighted measures, which coincide in the case of symmetric cones with those obtained by D. Debertol [10]. A particular case of this work has been done in [6] by D. Békollé and the first author, who considered the tube domain over the Vinberg cone. This is the simplest example of a non self-dual cone. In this case and in the case of rank 2, the sufficient conditions obtained for the positive Bergman operator are also necessary. We do not know whether this is the case for any arbitrary open convex homogeneous cone.

In this paper, we prove all the results mentioned above in the case of tube domains over homogeneous cones. The main difficulty of this work is to develop for all homogeneous cones, the necessary tools that are well known in the case of symmetric cones and of the Vinberg cone. Once this is done for any arbitrary homogeneous cone, one can easily proceed as in the previous cases. We deeply rely on the Vinberg's description of homogeneous cones presented in [20].

This paper is divided into 8 sections. In Section 2, we give some geometric properties of homogeneous cones which are necessary to state our results. Section 3 is devoted to the statement of the results. In Section 4, we recall some useful results about homogeneous cones, such as the Whitney decomposition and the gamma function. Section 5 deals with Bergman spaces. In Sections 6 and 7, we give the proofs of results announced in the third section. The last section is devoted to some comments about necessary conditions.

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2. Algebraic structure of homogeneous cones

Let V be a n-dimensional real vector space and Ω be an open convex cone in V *i.e.*, for $x,\ y \in \Omega$, and $\lambda,\ \mu > 0$, we have $\lambda x \in \Omega$ and $\lambda x + \mu y \in \Omega$. We assume that Ω does not contain straight lines and that it is *homogeneous*, that is, the group $G(\Omega)$ of all transformations of $GL(V)$ which leave invariant Ω acts transitively on Ω. In [20], Vinberg described convex homogeneous cones as the cones of Hermitian positive matrices in a T-algebra. We recall the definition of a T-algebra.

Definition 2.1. A matrix algebra of rank r is a real algebra[1] $\mathcal{U}$ bigraded by subspaces $\mathcal{U}_{ij},\ i, j = 1, \ldots, r$ *i.e.*, $\mathcal{U} = \bigoplus_{i,j} \mathcal{U}_{ij}$, such that

$$\mathcal{U}_{ij}\mathcal{U}_{jk} \subset \mathcal{U}_{ik}$$

and for $j \neq l$,

$$\mathcal{U}_{ij}\mathcal{U}_{lk} = 0.$$

As was recalled in [8], if we represent each $a \in \mathcal{U}$ by the generalized matrix $(a_{ij})_{i,j=1}^{r}$, where a_{ij} denotes the projection of a onto $\mathcal{U}_{ij}$, then the representation of ab is given by the matrix product $(a_{ij})(b_{ij})$.

Definition 2.2. An involution of a matrix algebra $\mathcal{U}$ is a linear mapping $\star : x \mapsto x^\star$ of $\mathcal{U}$ onto itself that satisfies the following conditions:

(i) $x^{\star\star} = x$;
(ii) $(xy)^\star = y^\star x^\star$;
(iii) $\mathcal{U}_{ij}^\star = \mathcal{U}_{ji}$ for all $x,\ y \in \mathcal{U}$.

In its matrix representation, an involution corresponds to taking the transpose, *i.e.*, $(a^\star)_{ij} = a_{ji}^\star$. A consequence of the existence of an involution is that

$$n_{ij} = n_{ji}, \tag{2.1}$$

where

$$n_{ij} = \dim \mathcal{U}_{ij}.$$

Let $\mathcal{U}$ be an algebra with an involution $\star$. As in [19], we define the subspace of "Hermitian matrices" in $\mathcal{U}$,

$$\mathcal{X} = \{x \in \mathcal{U} : x^\star = x\},$$

and

$$\mathcal{T} = \bigoplus_{1 \le i \le j \le r} \mathcal{U}_{ij},$$

the subalgebra of $\mathcal{U}$ consisting of upper triangular matrices.

[1] Associativity of the multiplication is not assumed.

We shall always assume that $\mathcal{U}_{ii} = \mathbb{R}c_i$ where $c_i^2 = c_i$. Let ρ denote the unique isomorphism of $\mathcal{U}_{ii}$ onto the algebra of real numbers $\mathbb{R}$. For a matrix $x \in \mathcal{U}$,

$$x = \sum_{i=1}^{r} x_{ii} + \sum_{i \neq j} x_{ij},$$

we define its trace by

$$\operatorname{tr} x = \sum_{i=1}^{r} \rho(x_{ii}). \tag{2.2}$$

Definition 2.3. A matrix algebra $\mathcal{U}$ with an involution $x \mapsto x^\star$ is called a T-algebra if the following conditions are satisfied:

(i) $\mathcal{U}_{ii} = \mathbb{R}c_i$ for $i = 1, \ldots, r$;
(ii) for $x_{ij} \in \mathcal{U}_{ij}$, $c_i x_{ij} = x_{ij} c_j = x_{ij}$;
(iii) for all $x, y \in \mathcal{U}$, $\operatorname{tr}(xy) = \operatorname{tr}(yx)$;
(iv) for all $x, y, z \in \mathcal{U}$, $\operatorname{tr}[x(yz)] = \operatorname{tr}[(xy)z]$;
(v) if $x \neq 0$, then $\operatorname{tr}(xx^\star) > 0$;
(vi) for all $t, u, v \in \mathcal{T}$, $t(uv) = (tu)v$;
(vii) for all $t, u \in \mathcal{T}$, $t(uu^\star) = (tu)u^\star$.

Remark 2.4. From (v) in the definition above the formula

$$(x|y) = \operatorname{tr}(xy^\star)$$

defines a scalar product in $\mathcal{U}$. Therefore a matrix algebra with an involution is Euclidean. Under this inner product, $\mathcal{U}_{ij}$ is orthogonal to $\mathcal{U}_{kl}$ unless $(i, j) = (k, l)$.

We denote by $\mathbf{e}$ the unit element of the matrix T-algebra $\mathcal{U}$, *i.e.*,

$$\mathbf{e} = \sum_{j=1}^{r} c_j.$$

Let

$$H = \{t \in \mathcal{T} : \rho(t_{ii}) > 0, \ i = 1, \ldots, r\}$$

be the subgroup of upper triangular matrices whose diagonal elements are positive and let

$$\Omega(\mathcal{X}) = \{ss^\star : s \in H\} \subset \mathcal{X}.$$

Note that the product in H is associative by (vi). The transformations

$$\pi(w) : uu^\star \mapsto (wu)(u^\star w^\star) \quad (w, u \in H) \tag{2.3}$$

of $\Omega(\mathcal{X})$ correspond to the left translations of $\Omega(\mathcal{X})$ ([20, page 383]). Note that from properties (vi) and (vii) of Definition 2.3, for $v, w \in H$,

$$\pi(v)\pi(w) = \pi(vw). \tag{2.4}$$

We have the following important result, due to Vinberg, which relates homogeneous cones to T-algebras.

Proposition 2.5 ([20, Proposition 1, page 384]). *For every T-algebra $\mathcal{U}$, the set $\Omega(\mathcal{X})$ is a convex homogeneous cone in which, by* (2.3)*, the group H acts linearly and transitively. Moreover, all convex homogeneous cones can be described as $\Omega(\mathcal{X})$ for some T-algebra.*

Therefore, in the sequel, we shall consider the open convex homogeneous cone Ω defined by

$$\Omega = \{ss^\star : s \in H\},$$

where the n-dimensional vector space V containing Ω is

$$V = \{x \in \mathcal{U} : x^\star = x\}.$$

Since the mapping $H \ni s \mapsto ss^\star \in \Omega$ is one-to-one, the group H acts simply transitively on Ω by (2.3). Hence, by homogeneity, one can write $\Omega = H \cdot \mathbf{e}$, where we use the notation

$$\pi(t)\mathbf{e} = t \cdot \mathbf{e}$$

for all $t \in H$. As it is mentioned in [19], we have the factorization

$$H = NA$$

where

$$N = \{t \in H : \ \forall i,\ \rho(t_{ii}) = 1\}, \quad A = \{t \in H : \ \forall i < j,\ t_{ij} = 0\}.$$

As in [13, page 14 and page 20], we introduce the following notation, related to the vector space V containing the homogeneous cone Ω. We recall that n_{ij} is the dimension of $\mathcal{U}_{ij}$; we define

$$n_i = \sum_{j=1}^{i-1} n_{ji}$$

and

$$m_i = \sum_{j=i+1}^{r} n_{ij};$$

then

$$\dim V = n = r + \sum_{i=1}^{r} m_i = r + \sum_{i=1}^{r} n_i. \tag{2.5}$$

2.1. The equation of the cone Ω

This is exactly what is done in [20, page 385]. In the T-algebra $\mathcal{U}$ of rank r, we consider the subspaces

$$\mathcal{U}^k = \bigoplus_{1\leq i,j\leq k} \mathcal{U}_{ij},$$

and with every element $x \in V$ we associate a sequence of matrices to $x^{(k)} \in \mathcal{U}^k$, as follows

$$\begin{aligned} x^{(r)} &= x \\ x^{(k-1)} &= \sum_{i,j=1}^{k-1} \left[\rho(x_{kk}^{(k)})x_{ij}^{(k)} - x_{ik}^{(k)}x_{kj}^{(k)}\right], \end{aligned}$$

where we consider that the matrix $x^{(k-1)}$ is formed from the second order "minors" of $x^{(k)}$. We put

$$p_k(x) = \rho(x_{kk}^{(k)}), \quad k = 1, \dots, r\,.$$

We notice that $p_k(x)$ is a homogeneous polynomial of degree 2^{r-k}. In [15] the polynomials p_k are called the *determinant-type polynomials* associated to the cone Ω and p_1 is the *composite determinant*. Since the computation of the composite determinant is hard to carry, H. Ishi in [15, Proposition 1.4] gave recurrence relations between determinant-type polynomials p_k. For $k = 1, \dots, r$ and $x \in \Omega$, we put

$$Q_k(x) = \frac{p_k(x)}{\prod_{j=k+1}^{r} p_j(x)};$$

the functions Q_k are homogeneous of degree 1. These functions are denoted by χ_k in [13].

Lemma 2.6 ([20, Proposition 2, Chapter III]). *The cone Ω is determined by the inequalities*

$$p_k(x) > 0, \qquad k = 1, \dots, r.$$

Also

$$\Omega = \{x \in V : Q_k(x) > 0, \ \ k = 1, \dots, r\}.$$

2.2. The adjoint cone

We consider the matrix algebra with involution $\mathcal{U}'$ which differs from $\mathcal{U}$ only in its grading, and we put

$$\mathcal{U}'_{ij} = \mathcal{U}_{r+1-i,\,r+1-j} \quad (i, j = 1, \dots, r).$$

It is proved in [20, Chapter 3, Section 6] that $\mathcal{U}'$ is also a T-algebra and $V' = V$ where V' is the subspace of $\mathcal{U}'$ consisting of Hermitian matrices. The dual cone of the convex homogeneous cone Ω is the set

$$\Omega^* = \{\xi \in V' : (x|\xi) > 0,\ \forall x \in \overline{\Omega} \setminus \{0\}\}.$$

The cone Ω^* is also convex and homogeneous and the group

$$H' = H^\star = \{t^\star,\ t \in H\}$$

acts simply transitively in Ω^*. See [20, Chapter 1, Proposition 9]. Therefore,

$$\Omega^* = \{t^\star t, t \in H\}.$$

As previously, we write

$$(\mathcal{U}')^k = \bigoplus_{1 \le i,j \le k} (\mathcal{U}')_{ij}$$

and to every element $\xi \in V'$ we associate the determinant-type polynomials denoted $p_k^*(\xi)$ of degree 2^{r-k} and the functions

$$Q_k^*(\xi) = \frac{p_k^*(\xi)}{\prod_{j=k+1}^r p_j^*(\xi)}.$$

Thus

$$\Omega^* = \{\xi \in V' : Q_k^*(\xi) > 0,\ \ k = 1, \dots, r\}.$$

In the sequel, we will use the following notations: for all $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r$, $x \in \Omega$ and $\xi \in \Omega^*$,

$$Q^\alpha(x) = \prod_{j=1}^r Q_j^{\alpha_j}(x) \ \text{ and } \ (Q^*)^\alpha(\xi) = \prod_{j=1}^r (Q_j^*)^{\alpha_j}(\xi).$$

We put $\tau = (\tau_1, \tau_2, \cdots, \tau_r) \in \mathbb{R}^r$ with

$$\tau_i = 1 + \frac{1}{2}(m_i + n_i).$$

For $x \in \Omega$, we have $x = t \cdot \mathbf{e}$ where $t \in H$. Then from [20, Chapter 3, Section 3] we have

$$Q_j(x) = t_{jj}^2 \qquad j = 1, \dots, r. \tag{2.6}$$

Let $y \in \Omega$. We have, for $j = 1, \dots, r$

$$Q_j(\pi(t)y) = Q_j(x) Q_j(y), \tag{2.7}$$

since, for $y = s \cdot \mathbf{e}$, by (2.4) and (2.6) we can write

$$Q_j(\pi(t)y) = Q_j(\pi(ts)\mathbf{e}) = t_{jj}^2 s_{jj}^2 = Q_j(x)Q_j(y).$$

Therefore, for any $s \in H$,

$$Q_j(\pi(s)x) = Q_j(s \cdot \mathbf{e})Q_j(x) \qquad j = 1, \dots, r, \tag{2.8}$$

and

$$Q^\tau(\pi(s)x) = \det \pi(s) Q^\tau(x), \tag{2.9}$$

since

$$\det \pi(s) = Q^\tau(s \cdot \mathbf{e}). \tag{2.10}$$

See [20, page 388].

The above properties are also valid if we replace Q_j by Q_j^* and $x \in \Omega$ by $\xi \in \Omega^*$.

Definition 2.7. Let C be an open cone. We say that C is *self-dual* if $C = C^*$. A homogeneous cone that is self-dual is said to be a *symmetric* cone.

In the following examples, that have been treated in [15], we compute the determinant-type polynomials.

Example 2.8. *The cone of positive-definite symmetric matrices*. This is a symmetric cone. We describe the above concepts for the cone $\Omega = \mathrm{Sym}_+(r, \mathbb{R})$, contained in the vector space $V = \mathrm{Sym}(r, \mathbb{R})$. The matrix algebra of rank r is the usual algebra $\mathcal{U} = V = V'$ and the involution the transpose map $V \ni X \mapsto {}^tX$. The unit element of V is the usual identity matrix $\mathbf{e} = I$ and $c_j = D_j$ are diagonal matrices whose entries are 0 except for the jth which is equal to 1. Obviously, $n_{ij} = 1$, for all $i, j \in \{1, \dots, r\}$.

In this example, the group $\mathcal{T}$ consists in the upper triangular matrices in $GL(r, \mathbb{R})$ and the factorization $y = t \cdot I$ is precisely the Gauss decomposition of a positive symmetric matrix. See [11, Chapter VI, Section 3]. The subgroup N consists of all triangular matrices in $GL(r, \mathbb{R})$ with 1s on the diagonal, while A is given by the diagonal matrices $P(a) = \mathrm{diag}\{a_1, \dots, a_r\}$.

Finally, for each matrix $X = (x_{ij})_{1 \leq i, j \leq r} \in V$, we define the matrix $\xi = (x_{r+1-i, r+1-j})_{1 \leq i, j \leq r} \in V$. Then

$$Q_j(X) = \frac{\Delta_{r+1-j}(\xi)}{\Delta_{r-j}(\xi)}$$

and

$$Q_j^*(X) = \frac{\Delta_{r+1-j}(X)}{\Delta_{r-j}(X)}$$

where Q_j and Q_j^* are the functions defined above and Δ_j are the usual principal minors from linear algebra, that is, the determinant of the $j \times j$ symmetric submatrices obtained by restriction to the first j coordinates. Observe that

$$\det X = \prod_{j=1}^{r} Q_j(X) = \prod_{j=1}^{r} Q_j^*(X).$$

Henceforth

$$\mathrm{Sym}_+(r, \mathbb{R}) = \{X \in \mathrm{Sym}(r, \mathbb{R}) : \Delta_j(X) > 0, \quad j = 1, \dots, r\}.$$

Example 2.9. *Vinberg's Cone*. This is the simplest example of a convex homogeneous non selfadjoint cone of rank 3, given by Vinberg in [20, page 397]. Consider the Euclidean vector space $\mathcal{U} = V$ of 3×3 matrices with real entries given by

$$x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix};$$

clearly, $n_{23} = 0, \quad n_{12} = n_{13} = 1$ so that $m_1 = 2, m_2 = m_3 = 0$ and $n_1 = 0, n_2 = n_3 = 1$. As in the case of real symmetric matrices above, the unit element of V is the usual identity matrix $\mathbf{e} = I$ and $c_j = D_j$ are diagonal matrices whose entries are 0, except for the jth entry which is equal to 1. We have

$$H = \left\{ s = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix} : \ s_{jj} > 0, \ j = 1, 2, 3 \right\}.$$

For every $x \in V$,

$$p_3(x) = x_{33}, \ \ p_2(x) = x_{33}x_{22} \text{ and } p_1(x) = x_{33}x_{22}(x_{33}x_{11} - x_{13}^2) - x_{33}^2 x_{12}^2,$$

so that

$$Q_3(x) = x_{33}, \ \ Q_2(x) = x_{22} \text{ and } Q_1(x) = x_{11} - \frac{x_{12}^2}{x_{22}} - \frac{x_{13}^2}{x_{33}};$$

the convex homogeneous cone is then equal to the set

$$\{x \in V : Q_j(x) > 0, \quad j = 1, 2, 3\} = \{x \in V : x \text{ is positive definite}\}.$$

Note that since $n_{23} = 0$, the space V' is identified with the set

$$\left\{ \xi = (\xi_{(2)}, \xi_{(3)}) : \xi_{(k)} = \begin{pmatrix} \xi_{11} & \xi_{1k} \\ \xi_{1k} & \xi_{kk} \end{pmatrix}, k = 2, \ 3 \right\},$$

and the cone Ω^* is the set of elements $\xi = (\xi_{(2)}, \xi_{(3)})$ such that both components are positive definite; the group H' is the set

$$H' = \left\{ t = (t_{(2)}, t_{(3)}) : t_{(k)} = \begin{pmatrix} t_{11} & 0 \\ t_{1k} & t_{kk} \end{pmatrix};\ k = 2, 3;\ t_{jj} > 0,\ \ j = 1, 2, 3 \right\},$$

the unit element is $\mathbf{e} = (\mathbf{e}_{(1)}, \mathbf{e}_{(2)})$ with $\mathbf{e}_{(1)} = \mathbf{e}_{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, for $\xi \in V'$, we have

$$p_3^*(\xi) = \xi_{11},\ \ p_2^*(\xi) = \xi_{11}\xi_{22} - \xi_{12}^2 \text{ and } p_1^*(\xi) = (\xi_{11}\xi_{22} - \xi_{12}^2)(\xi_{11}\xi_{33} - \xi_{12}^2)$$

so that

$$Q_3^*(\xi) = \xi_{11},\ \ Q_2^*(\xi) = \xi_{22} - \frac{\xi_{12}^2}{\xi_{11}} \text{ and } Q_1^*(\xi) = \xi_{33} - \frac{\xi_{13}^2}{\xi_{11}};$$

observe that $\det \xi \neq \prod_{j=1}^3 Q_j^*(\xi)$, hence ξ is not positive definite.

Remark 2.10. From [13, page 19], a necessary and sufficient condition for a cone to be self-conjugate or self-dual is that all n_{ij} are equal when $i \neq j$. Let d denotes this common dimension for the spaces $\mathcal{U}_{ij}$. Then for every symmetric cone of rank r, we have $m_i = (r - i)d$ and $n_i = (i - 1)d$ so that from (2.5) above, we obtain

$$(r-1)\frac{d}{2} = \frac{n}{r} - 1.$$

In particular, $d = 1$ for the cone of positive-definite symmetric matrices.

3. Statement of the results

Let $T_\Omega = V + i\Omega$ be the tube domain over the open convex homogeneous cone Ω. For each $w \in T_\Omega$,

$$Q^{-2\tau}(\Im m\, w)dv(w)$$

is the invariant measure with respect to the group of automorphisms of T_Ω. Let $\nu = (\nu_1, \nu_2, \cdots, \nu_r) \in \mathbb{R}^r$. We denote by $L^p_\nu(T_\Omega)$, $1 \le p \le \infty$, the Lebesgue space $L^p(T_\Omega, Q^{\nu-\tau}(\Im m\, w)dv(w))$.

The *weighted Bergman space* $A^p_\nu(T_\Omega)$ is the closed subspace of $L^p_\nu(T_\Omega)$ consisting of holomorphic functions. In order to have a non-trivial subspace, we take $\nu = (\nu_1, \nu_2, \cdots, \nu_r) \in \mathbb{R}^r$ such that $\nu_i > \frac{m_i + n_i}{2}$, $i = 1, \ldots, r$. [2]

The orthogonal projection of the Hilbert space $L^2_\nu(T_\Omega)$ on its closed subspace $A^2_\nu(T_\Omega)$ is the *weighted Bergman projection* P_ν. We recall that P_ν is defined by the integral

$$P_\nu f(z) = \int_{T_\Omega} B_\nu(z, w) f(w) Q^{\nu-\tau}(\Im m\, w)dv(w),$$

[2] If there is $k \in \{1, \ldots, r\}$ such that $\nu_k \le \frac{m_k}{2}$, then $A^p_\nu(T_\Omega) = \{0\}$.

where

$$B_\nu(z,\ w) = d_\nu Q^{\nu-\tau}\left(\frac{z-\bar{w}}{i}\right)$$

is the *weighted Bergman kernel i.e.*, the reproducing kernel of $A^2_\nu(T_\Omega)$.

In this paper, we discuss boundedness of P_ν on $L^p_\nu(T_\Omega)$ for values of p different from 2. Let us consider the positive Bergman operator P^+_ν defined on $L^2_\nu(T_\Omega)$ by

$$P^+_\nu f(z) = \int_{T_\Omega} |B_\nu(z, w)| f(w) Q^{\nu-\tau}(\Im m\, w) dv(w).$$

Theorem 3.1. *The operator P^+_ν is bounded on $L^p_\nu(T_\Omega)$ when*

$$1 + \max_{1\le i\le r} \frac{\frac{n_i}{2}}{\nu_i - \frac{m_i}{2}} < p < 1 + \min_{1\le i\le r} \frac{\nu_i - \frac{m_i}{2}}{\frac{n_i}{2}}.$$

Hence P_ν is bounded for this range of p.

Recall that this theorem has been proved by D. Békollé and A. Temgoua in [7]. We give a new proof of this theorem within the framework of T-algebra construction of convex homogeneous cones. This sufficient condition is also necessary for some open convex homogeneous cones, for example when the rank is 2 and for the case of Vinberg cone and its dual. (See [6, Theorem 1.1].) Moreover, for general symmetric cones, if we assume that $\nu = (\nu, \dots, \nu) \in \mathbb{R}^r$, then this sufficient condition is also necessary. (See [3, Theorem 4.10].)

Moreover, for the case of tube domains over symmetric cones and the tube domain over the Vinberg cone, the authors of [3] and [6] respectively established that there are values of p for which P_ν is bounded, but P^+_ν is unbounded. We extend this result to the tube domain over open convex homogeneous cones. We have the following theorem, which is the main result of this paper.

Theorem 3.2.

i) *When the Bergman projector is bounded from $L^p_\nu(T_\Omega)$ to $A^p_\nu(T_\Omega)$, we have*

$$1 + \max_{1\le i\le r} \frac{\frac{n_i}{2}}{\nu_i + 1 + \frac{m_i}{2} + \frac{n_i}{2}} < p < 1 + \min_{1\le i\le r} \frac{\nu_i + 1 + \frac{m_i}{2} + \frac{n_i}{2}}{\frac{n_i}{2}}.$$

ii) *The Bergman projector P_ν extends to a bounded operator on $L^p_\nu(T_\Omega)$ for*

$$1 + \max_{1\le i\le r} \frac{\frac{n_i}{2}}{\nu_i - \frac{m_i}{2} + \frac{n_i}{2}} < p < 1 + \min_{1\le i\le r} \frac{\nu_i - \frac{m_i}{2} + \frac{n_i}{2}}{\frac{n_i}{2}}.$$

The necessary condition is not hard to prove. We describe the main ideas in the proof of the sufficient condition. As in [3] and [6], we must take advantage of the oscillations of the Bergman kernel. Hence, we are induced to use the Fourier transform in the x variables and consequently to focus on L^2-norms in these variables. For this reason, we introduce mixed norms spaces. For $1 \leq p, q \leq \infty$, let $L_\nu^{p,q}(T_\Omega) = L^q(\Omega, Q^{\nu-\tau}(y)dy; L^p(V, dx))$ be the space of functions f on T_Ω such that

$$\|f\|_{L_\nu^{p,q}(T_\Omega)} := \left(\int_\Omega \left(\int_V |f(x+iy)|^p dx\right)^{\frac{q}{p}} Q^{\nu-\tau}(y)dy\right)^{\frac{1}{q}}$$

is finite (with obvious modification if $p = \infty$.) As before, we call $A_\nu^{p,q}(T_\Omega)$ the closed subspace of $L_\nu^{p,q}(T_\Omega)$ consisting of holomorphic functions.

For $p = 2$, we prove that P_ν is bounded on $L_\nu^{2,q}(T_\Omega)$ when

$$2\left(1 + \max_{1\leq i\leq r} \frac{\frac{n_i}{2}}{\nu_i - \frac{m_i}{2}}\right) < q < 2\left(1 + \min_{1\leq i\leq r} \frac{\nu_i - \frac{m_i}{2}}{\frac{n_i}{2}}\right).$$

Then Theorem 3.2 follows by interpolation with Theorem 3.1. Note that in the case of symmetric cones, which Debertol considered, the necessary condition of the $L_\nu^{2,q}(T_\Omega)$-boundedness of the weighted Bergman projector P_ν has been left open. We still have the same difficulty here. Nevertheless, we observe that, for the case of rank 2 and the Vinberg cone [6], the sufficient condition above is also necessary.

4. Some useful results in a convex homogeneous cone

In this section, we recall some important facts about homogeneous cones such as the Riemannian structure that yields an isometry between the cone and its dual and the Whitney decomposition of the cone. Most of these results have been established in [3] and [6].

4.1. The Riemannian structure Ω and its dual

We denote by φ (respectively φ_*) the *characteristic function* of the cone Ω (respectively Ω^*); then for $x \in \Omega$ and $\xi \in \Omega^*$,

$$\varphi(x) = \int_{\Omega^*} e^{-(x|\xi)} d\xi \quad \text{and} \quad \varphi_*(\xi) = \int_\Omega e^{-(\xi|x)} dx.$$

Recall that the gradient of a differentiable function f at the point $x \in \mathbb{R}^n$ is defined by

$$(\nabla f(x)|u) = D_u f(x) = \frac{d}{dt} f(x+tu)\bigg|_{t=0}$$

for all $u \in \mathbb{R}^n$.

For $x \in \Omega$ we define $x' \in \Omega^*$ by

$$x' = -\nabla \log \varphi(x).$$

Similarly, for $\xi \in \Omega^*$ we define

$$\xi' = -\nabla \log \varphi_*(\xi).$$

Note that for each $x \in \Omega$ and $\xi \in \Omega^*$, we have $x'' = x$ and $\xi'' = \xi$. (See [20, Chapter 1, Section 4].)

For $x = t \cdot \mathbf{e}$,

$$\begin{aligned}(x'|\pi(t)y) = (-\nabla \log \varphi(x)|\pi(t)y) &= -\frac{d}{du} \log \varphi(x + u\pi(t)y)\Big|_{u=0} \\ &= -\frac{d}{du} \log \varphi(\pi(t)(\mathbf{e} + uy))\Big|_{u=0} \\ &= -\frac{d}{du} \log(\varphi \circ \pi(t))(\mathbf{e} + uy)\Big|_{u=0} \\ &= -\frac{d}{du} \log \varphi(\mathbf{e} + uy)\Big|_{u=0} \\ &= (\mathbf{e}'|y)\end{aligned}$$

so that

$$x' = t^{\star -1} \cdot \mathbf{e}'.$$

Moreover, for all $t \in H$, we have

$$Q_j(t \cdot \mathbf{e}) Q_j^*(t^{\star -1} \cdot \mathbf{e}) = 1,$$

where $j = 1, \ldots, r$. Let $\mathbf{e}_0$ be the unique fixed point of the map $\sigma : x \mapsto x'$, (*cf.* [11, Proposition I.3.5]). Since Ω is a homogeneous cone, every $x \in \Omega$ can be written as $x = \pi(t)\mathbf{e}_0$; therefore, $x' = \pi(t^{\star -1})\mathbf{e}_0$ and by (2.8) we have

$$Q_j(x) Q_j^*(x') = Q_j(\mathbf{e}_0) Q_j^*(\mathbf{e}_0) = \text{constant} \tag{4.1}$$

for $j = 1, \ldots, r$.

Since the function $\log \varphi$ is strictly convex (*cf.* [11, Proposition I.3.3]), the symmetric bilinear form on $\mathbb{R}^n$

$$G_x(u, v) = D_u D_v \log \varphi(x) \quad \text{(respectively} \quad G_{\xi'}(u, v) = D_u D_v \log \varphi_*(\xi))$$

where $u,\ v \in \mathbb{R}^n$ defines on Ω (respectively Ω^*) a structure of Riemannian manifold. The corresponding Riemannian distances are given by

$$d(x, y) = \inf_{\gamma} \left\{ \int_0^1 \sqrt{G_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt \right\}$$

and

$$d_*(\xi, \eta) = \inf_{\gamma^*} \left\{ \int_0^1 \sqrt{G^*_{\gamma^*(t)}(\dot{\gamma^*(t)}, \dot{\gamma^*(t)})}dt \right\},$$

where the infimum is taken on the smooth paths $\gamma : [0, 1] \to \Omega$ (respectively $\gamma^* : [0, 1] \to \Omega^*$) such that $\gamma(0) = x,\ \gamma(1) = y$ (respectively $\gamma^*(0) = \xi,\ \gamma^*(1) = \eta$).

The Riemannian distances d and d_* are invariant under the action of $G(\Omega)$ and $G(\Omega^*)$ respectively, *i.e.*,

$$\forall x,\ y \in \Omega,\ \forall g \in G(\Omega),\ d(gx, gy) = d(x, y)$$

and

$$\forall \xi,\ \eta \in \Omega^*,\ \forall g \in G(\Omega),\ d_*(g^*\xi, g^*\eta) = d_*(\xi, \eta).$$

(See [11, pages 15-16].) We have the following:

Theorem 4.1. *The map $\sigma : x \mapsto x'$ between the Riemannian manifolds Ω and Ω^* is an isometry; that is*

$$d_*(x',\ y') = d(x, y).$$

4.2. The invariant measure on Ω

Since we also have the identification $\Omega^* \equiv H' \cdot \mathbf{e}$, we deduce from (2.9) that the measure

$$dm(x) = Q^{-\tau}(x)dx \quad \text{(respectively } dm_*(\xi) = (Q^*)^{-\tau}(\xi)d\xi\text{)}$$

is H-invariant on Ω (respectively H'-invariant on Ω^*).

Lemma 4.2. *Given $\lambda > 0$, there is a constant $C = C(\lambda) > 0$ such that:*

i) *if $d(y,\ t) \leq \lambda$ then $\frac{1}{C} \leq \frac{Q_j(y)}{Q_j(t)} \leq C$ for all $j = 1, \dots, r$ and $x,\ y \in \Omega$;*

ii) *if $d_*(\xi,\ \eta) \leq \lambda$ then $\frac{1}{C} \leq \frac{Q^*_j(\xi)}{Q^*_j(\eta)} \leq C$ for all $j = 1, \dots, r$ and $\xi,\ \eta \in \Omega^*$.*

Let $\lambda > 0,\ \ y \in \Omega$ (respectively $\xi \in \Omega^*$) and d (respectively d_*) the $G(\Omega)$-invariant (respectively $G(\Omega^*)$-invariant) distance defined in Ω (respectively Ω^*). We denote by

$$B_\lambda(y) = \{x \in \Omega : d(y,\ x) < \lambda\}$$

and

$$B^*_\lambda(\xi) = \{\eta \in \Omega^* : d_*(\eta,\ \xi) < \lambda\}$$

the d-ball (respectively d_*-ball) centered at the point y (respectively ξ) with radius λ.

Lemma 4.3. *Let $0 < \lambda < 1$. Then*

$$m(B_\lambda(y)) \sim \lambda^n \quad \text{and} \quad m_*(B^*_\lambda(\xi)) \sim \lambda^n.$$

Proof. By the $G(\Omega)$-invariance of the distance, we have, for all $t \in H$, $B_\lambda(\mathbf{e}) = t \cdot B_\lambda(\mathbf{e})$, so that $m(B_\lambda(y)) = m(B_\lambda(\mathbf{e}))$ for all $y \in \Omega$. We have

$$m(B_\lambda(\mathbf{e})) = \int_{B_\lambda(\mathbf{e})} dm(y) = \int_{B_\lambda(\mathbf{e})} Q(y)^{-\tau} dy \sim \int_{B_\lambda(\mathbf{e})} dy.$$

It is well known that the distance d is equivalent to the Euclidean distance on compact subsets of V (*cf.* [17]); hence there are two positive constants c_1 and c_2 such that

$$\{y \in V : |y - \mathbf{e}| \le c_1\lambda\} \subset B_\lambda(\mathbf{e}) \subset \{y \in V : |y - \mathbf{e}| \le c_2\lambda\}$$

and the result follows. □

4.3. The Whitney decomposition of the cone Ω

We give now the Whitney decomposition of the cone Ω, which is obtained, for instance, as in Lemma 3.5 of [6].

Lemma 4.4. *Given $0 < \lambda < 1$, there exists a sequence $\{y_j\}_j$ of points of Ω such that the following three properties hold:*

i) *the balls $B_{\frac{\lambda}{2}}(y_j)$ are pairwise disjoint;*
ii) *the balls $B_\lambda(y_j)$ form a covering of Ω;*
iii) *there is an integer $N = N(\Omega)$ such that every $y \in \Omega$ belongs to at most N balls $B_\lambda(y_j)$.*

Remark 4.5. This lemma is also true for the dual cone Ω^*.

Definition 4.6. Sequences $\{y_j\}_j$ (respectively $\{\xi_j\}_j$) of points of Ω (respectively Ω^*) that satisfy properties of Lemma 4.4 are called λ-*lattices* of Ω (respectively Ω^*.).

The family $\{B_\lambda(y_j)\}_j$ (respectively $\{B^*_\lambda(\xi_j)\}_j$) is called the *Whitney decomposition* of the cone Ω (respectively Ω^*).

Proposition 4.7. *The sequence $\{y_j\}_j$ is a λ-lattice of Ω if and only if $\{y'_j\}_j$ is a λ-lattice in Ω^*. The sequence $\{y'_j\}_j$ is called the dual lattice of the λ-lattice $\{y_j\}_j$.*

Lemma 4.8. *Let $(y_0, \xi_0) \in \Omega \times \Omega^*$; then*

$$|B_\lambda(y_0)| = C_\lambda Q^\tau(y_0) \quad \text{and} \quad |B^*_\lambda(\xi_0)| = C_\lambda (Q^*)^\tau(\xi_0). \tag{4.2}$$

Proof. We know that $y_0 = t \cdot \mathbf{e}$ with $t \in H$; if we use the change of variables $y = \pi(t)x$, $dy = Q^\tau(y_0)dx$ and since the distance d is $G(\Omega)$-invariant, $d(y, y_0) = d(\pi(t)x, \pi(t)\mathbf{e}) = d(x, \mathbf{e})$. Hence,

$$|B_\lambda(y_0)| = Q^\tau(y_0) \int_{B_\lambda(\mathbf{e})} dx = C_\lambda Q^\tau(y_0).$$

The same argument holds for $B^*_\lambda(\xi_0)$. □

Proposition 4.9. *Let* $y \in \Omega$ (*respectively* $\xi \in \Omega^*$). *There is a constant* $\gamma = \gamma(\Omega, \Omega^*) \geq 1$ *such that*

$$\frac{1}{\gamma} < \frac{(y|\xi)}{(y|\xi_0)} < \gamma \quad \left(\text{respectively} \quad \frac{1}{\gamma} < \frac{(y|\xi)}{(y_0|\xi)} < \gamma\right)$$

whenever $\xi \in B^*_\lambda(\xi_0)$ (*respectively* $y \in B_\lambda(y_0)$).

Corollary 4.10. *Let* $(y_0, \xi_0) \in \Omega \times \Omega^*$. *There is a constant* $\gamma > 0$ *such that*

$$\frac{n}{\gamma} \leq (y|\xi) \leq n\gamma$$

for all $(y, \xi) \in B_\lambda(y_0) \times B^*_\lambda(\xi_0)$.

Lemma 4.11. *There is a constant* $c > 0$ *such that for all* $t \in H$,

$$\|\pi(t)x\| \leq c(t \cdot \mathbf{e}|\mathbf{e})\|x\|,$$

where $x \in \Omega$.

Proof. Let us first remark that, since the function $(x, y) \in \mathcal{U} \times \mathcal{U} \mapsto \|xy\|$ is continuous, there is $C > 0$ such that, for all $x, y \in \mathcal{U}$,

$$\|xy\| \leq C\|x\|\|y\|. \tag{4.3}$$

Let $x \in \Omega$. Then $x = s \cdot \mathbf{e}$ with $s \in H$. Then, by (2.4) and (4.3),

$$\|\pi(t)x\| = \|\pi(ts)\mathbf{e}\| = \|(ts)(ts)^\star\| \leq C^3\|t\|^2\|s\|^2 = C^3(t \cdot \mathbf{e}|\mathbf{e})(s \cdot \mathbf{e}|\mathbf{e}).$$

Applying the Cauchy-Schwarz inequality, we obtain

$$(s \cdot \mathbf{e}|\mathbf{e}) = (x|\mathbf{e}) \leq \sqrt{r}\|x\|;$$

hence

$$\|\pi(t)x\| \leq C^3\sqrt{r}(t \cdot \mathbf{e}|\mathbf{e})\|x\|.$$ □

4.4. The gamma function of a homogeneous cone

The following lemmas are given in order to define the holomorphic determination of the logarithm of $Q_j^{\alpha_j}$ and hence define the gamma function of Ω and Ω^*. We consider once more the T-subalgebras

$$\mathcal{U}^k = \bigoplus_{1 \leq i,\, j \leq k} \mathcal{U}_{ij},$$

with the units

$$\mathbf{e}_k = c_1 + \cdots + c_k,$$

and we denote by $\Omega^{(k)}$ the associated open convex homogeneous cone.

Lemma 4.12. *For* $y \in \bigoplus_{i=1}^{r-1} \mathcal{U}_{ir}$, *we have*

$$\mathbf{e}_{r-1} + yy^\star \in \Omega^{(r-1)} \tag{4.4}$$

and

$$yy^\star \in \overline{\Omega^{(r-1)}}. \tag{4.5}$$

Proof. Let $\delta \in \mathbb{R}^+$. Then $\mathbf{e}_{r-1} + y + \delta c_r \in H$ and we have

$$(\mathbf{e}_{r-1} + y + \delta c_r) \cdot \mathbf{e} = \mathbf{e}_{r-1} + yy^\star + \delta y + \delta y^\star + \delta^2 c_r \in \Omega.$$

Let $\xi \in (\Omega^{(r-1)})^*$. Then $\xi = t^\star t$, with $t \in H^{(r-1)}$; moreover, $\xi + c^r = (t + c_r)^\star (t + c_r)$, so that $\xi + c^r \in \Omega^*$. It follows that

$$(\xi + c^r | \mathbf{e}_{r-1} + yy^\star + \delta y + \delta y^\star + \delta^2 c_r) = (\xi | \mathbf{e}_{r-1} + yy^\star) + \delta^2 > 0.$$

Taking $\delta \to 0$, we see that

$$\mathbf{e}_{r-1} + yy^\star \in \overline{\Omega^{(r-1)}}.$$

Replacing y by $\varepsilon^{-1} y$, we can write

$$\varepsilon^2 \mathbf{e}_{r-1} + yy^\star \in \overline{\Omega^{(r-1)}}.$$

Thus, as ε tends to 0 we get (4.5). Finally, since $\mathbf{e}_{r-1} \in \Omega^{(r-1)}$, we get (4.4). □

Lemma 4.13. *For* $x \in \Omega$ *and* $y \in \overline{\Omega}$, *we have*

$$Q_j(x + y) \geq Q_j(x)$$

for all $j = 1, \dots, r$.

Proof. We prove this by induction on the rank r. Since by (2.8), we have $Q_j(\pi(t)y) = Q_j(\pi(t)\mathbf{e}) Q_j(y)$ for $t \in H$, we may take $x = \mathbf{e}$. Then $Q_r(\mathbf{e} + y) = 1 + \rho(y_{rr}) \geq 1 = Q_r(\mathbf{e})$. Now, assume that for any $1 \leq k \leq r - 1$, we have $Q_j^{(k)}(\mathbf{e}_k + v^{(k)}) \geq 1 = Q_j^{(k)}(\mathbf{e}_k)$ where $j = 1, \dots, k$ and $v^{(k)} \in \overline{\Omega^{(k)}}$. Let $u = \mathbf{e} + y$ and

$$w = \frac{1}{1 + \rho(y_{rr})} u^{(r-1)}.$$

Then $w = \mathbf{e}_{r-1} + \frac{1}{1+\rho(y_{rr})} v^{(r-1)}$ where $v = y + c_r$. Since $v \in \overline{\Omega}$, we have $v^{(r-1)} \in \overline{\Omega^{(r-1)}}$ and so, by the induction hypothesis,

$$Q_j^{(r-1)}(w) \geq 1$$

for $j = 1, \dots, r - 1$. The functions $Q_j^{(r-1)}$ are homogeneous of degree 1, so that

$$Q_j^{(r-1)}(w) = \frac{1}{1 + \rho(y_{rr})} Q_j^{(r-1)}(u^{(r-1)}) = Q_j(u).$$

□

Lemma 4.14. *For $x \in V$ and $y \in \Omega$, we have*

$$\Im m\, Q_j(x+iy) \geq Q_j(y) \tag{4.6}$$

for $j = 1, \ldots, r$.

Proof. The proof of (4.6) is inductive over the rank r. By (2.8),

$$Q_j(\pi(t)(x+iy)) = Q_j(\pi(t)\mathbf{e})Q_j(x+iy),$$

so that we can assume $y = \mathbf{e}$. Let $z = x + i\mathbf{e}$. We have

$$\Im m\, Q_r(z) = \Im m\, p_r(z) = 1.$$

Now, assume that for any $1 \leq k \leq r-1$ we have $\Im m\, Q_j^{(k)}(x + iv^{(k)}) \geq Q_j^{(k)}(v^{(k)})$ where $j = 1, \ldots, k$ and $v^{(k)} \in \Omega^{(k)}$. Put $w = \frac{1}{p_r(z)} z^{(r-1)}$. Then

$$w_{jj} = x_{jj} - \frac{\rho(x_{rr})x_{jr}x_{rj}}{1+\rho(x_{rr})^2} + i\left(c_j + \frac{x_{jr}x_{rj}}{1+\rho(x_{rr})^2}\right)$$

and

$$w_{jk} = x_{jk} - \frac{\rho(x_{rr})x_{jr}x_{rk}}{1+\rho(x_{rr})^2} + i\left(\frac{x_{jr}x_{rk}}{1+\rho(x_{rr})^2}\right).$$

Observe that $\Im m\, w = \mathbf{e}_{r-1} + \frac{1}{1+\rho(x_{rr})^2} yy^\star$ where $y \in \bigoplus_{i=1}^{r-1} \mathcal{U}_{ir}$. Hence, by Lemma 4.12, $\Im m\, w \in \Omega^{(r-1)}$. Thus, by the induction hypothesis and Lemma 4.13, we have

$$\Im m\, Q_j^{(r-1)}(w) \geq Q_j^{(r-1)}(\Im m\, w) \geq 1.$$

Since $Q_j^{(r-1)}$ are homogeneous of degree 1, we have

$$Q_j^{(r-1)}(w) = \frac{1}{p_r(z)} Q_j^{(r-1)}(z^{(r-1)}) = Q_j(z). \qquad \square$$

Notation : For $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$, we shall denote

$$Q_j^{\nu_j}(z/i) \qquad (z \in T_\Omega)$$

the determination of the ν_j-th power that corresponds to the holomorphic determination of the logarithm of $Q_j^{\nu_j}(z/i)$ which is real and positive on $i\Omega$.

Likewise, for the dual cone we use the notation

$$(Q_j^*)^{\nu_j}(z/i) \qquad (z \in T_{\Omega^*}).$$

We now define the gamma functions.

Proposition 4.15 ([13, Theorem 2.1]). *Let* $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$.

i) *The integral*

$$\Gamma_\Omega(\alpha) = \int_\Omega e^{-(\mathbf{e}|x)} Q^{\alpha-\tau}(x)dx$$

converges if and only if $\alpha_i > \frac{m_i}{2}$. *In this case,*

$$\Gamma_\Omega(\alpha) = \pi^{\frac{n-r}{2}} \prod_{i=1}^r \Gamma\left(\alpha_i - \frac{m_i}{2}\right)$$

where Γ *is the usual gamma function.*

ii) *The integral*

$$\Gamma_{\Omega^*}(\alpha) = \int_{\Omega^*} e^{-(\mathbf{e}|\xi)} (Q^*)^{\alpha-\tau}(\xi)d\xi$$

converges if and only if $\alpha_i > \frac{n_i}{2}$. *In this case,*

$$\Gamma_{\Omega^*}(\alpha) = \pi^{\frac{n-r}{2}} \prod_{i=1}^r \Gamma\left(\alpha_i - \frac{n_i}{2}\right)$$

where Γ *is the usual gamma function.*

Corollary 4.16. *Let* $\nu = (\nu_1, \nu_2, \dots, \nu_r) \in \mathbb{R}^r$. *The integral*

$$\int_\Omega e^{-(\xi|y)} Q^{\nu-\tau}(y)dy \qquad \left(\textit{respectively} \quad \int_{\Omega^*} e^{-(y|\xi)} (Q^*)^{\nu-\tau}(\xi)d\xi\right)$$

is finite for all $\xi \in \Omega^*$ (*respectively* $y \in \Omega$) *if and only if*

$$\nu_j > \frac{m_j}{2}, \quad j = 1, \dots, r \quad \left(\textit{respectively} \quad \nu_j > \frac{n_j}{2}, \quad j = 1, \dots, r\right).$$

For these values of ν *and for all* $\zeta = \eta + i\xi \in T_{\Omega^*}$ (*respectively* $z = x + iy \in T_\Omega$),

$$\int_\Omega e^{i(\zeta|y)} Q^{\nu-\tau}(y)dy = \Gamma_\Omega(\nu)(Q^*)^{-\nu}\left(\frac{\zeta}{i}\right) \tag{4.7}$$

$$\left(\textit{respectively} \int_{\Omega^*} e^{i(z|\xi)} (Q^*)^{\nu-\tau}(\xi)d\xi = \Gamma_{\Omega^*}(\nu) Q^{-\nu}\left(\frac{z}{i}\right)\right). \tag{4.8}$$

Proof. To prove (4.7) and (4.8), by homogeneity, it suffices to compute the integrals

$$\int_\Omega e^{-(\mathbf{e}|y)} Q^{\nu-\tau}(y)dy \quad \text{and} \quad \int_{\Omega^*} e^{-(\mathbf{e}|\xi)} (Q^*)^{\nu-\tau}(\xi)d\xi$$

respectively. The result then follows by Proposition 4.15. □

Remark 4.17. From the previous corollary, we deduce the characteristic function of the cone Ω and of its dual Ω^*.

$$\varphi(x) = \varphi(\mathbf{e})Q^{-\tau}(x) = \Gamma_{\Omega^*}(\tau)Q^{-\tau}(x) \tag{4.9}$$

and

$$\varphi_*(\xi) = \varphi_*(\mathbf{e})(Q^*)^{-\tau}(\xi) = \Gamma_{\Omega}(\tau)(Q^*)^{-\tau}(\xi) \tag{4.10}$$

for all $x \in \Omega$ and $\xi \in \Omega^*$.

Remark 4.18. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r$ be such that $\alpha_j > \frac{m_j+n_j}{2}$, $j = 1, \dots, r$; for all $(x, y, t) \in V \times \Omega \times \Omega$,

$$\left| Q^{-\alpha}\left(\frac{x+iy}{i}\right)\right| \le Q^{-\alpha}(y) \tag{4.11}$$

$$Q^{-\alpha}(y+t) < Q^{-\alpha}(y). \tag{4.12}$$

Inequality (4.11) is a direct application of Lemma 4.14 and inequality (4.12) follows from Corollary 4.16.

Lemma 4.19. *Let $\mu = (\mu_1, \mu_2, \dots, \mu_r) \in \mathbb{R}^r$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathbb{R}^r$.*

i) *For all $y \in \Omega$, the integral*

$$J_{\mu\lambda}(y) = \int_\Omega Q^\mu(y+v)Q^{\lambda-\tau}(v)dv$$

is finite if and only if

$$\lambda_j > \frac{m_j}{2}, \quad \mu_j + \lambda_j < -\frac{n_j}{2}, \qquad j = 1, \dots, r.$$

In this case,

$$J_{\mu\lambda}(y) = M_{\lambda\mu}Q^{\mu+\lambda}(y)$$

where

$$M_{\lambda\mu} = \frac{\Gamma_\Omega(\lambda)\Gamma_{\Omega^*}(-\mu-\lambda)}{\Gamma_{\Omega^*}(-\mu)}.$$

ii) *For all $\xi \in \Omega^*$, the integral*

$$K_{\mu\lambda}(y) = \int_{\Omega^*} (Q^*)^\mu(\xi+\eta)(Q^*)^{\lambda-\tau}(\eta)d\eta$$

is finite if and only if

$$\lambda_j > \frac{n_j}{2}, \quad \mu_j + \lambda_j < -\frac{m_j}{2}, \qquad j = 1, \dots, r.$$

In this case,

$$K_{\mu\lambda}(\xi) = \frac{\Gamma_{\Omega^*}(\lambda)\Gamma_\Omega(-\mu-\lambda)}{\Gamma_\Omega(-\mu)}\,(Q^*)^{\mu+\lambda}(\xi).$$

Proof. One can observe that the convergence of the integral $J_{\lambda\mu}(y)$ is established for $y = \mathbf{e}$; the rest follows from the identification $\Omega \equiv H \cdot \mathbf{e}$ and the fact that $Q(\pi(t)x) = Q(y)Q(x)$ where $y = t \cdot \mathbf{e}$, with $t \in H$.

By (4.8), we write

$$Q^{\mu}(\mathbf{e}+v) = \frac{1}{\Gamma_{\Omega^*}(-\mu)} \int_{\Omega^*} e^{-(\mathbf{e}+v|\xi)} (Q^*)^{-\mu-\tau}(\xi) d\xi,$$

if and only if $\mu_j < -\frac{n_j}{2}$, $j = 1, \ldots, r$. According to Fubini's Theorem, (4.7) and (4.8), we obtain that

$$\begin{aligned} J_{\mu\lambda}(\mathbf{e}) &= \frac{1}{\Gamma_{\Omega^*}(-\mu)} \int_{\Omega^*} e^{-(\mathbf{e}|\xi)} (Q^*)^{-\mu-\tau}(\xi) \left(\int_{\Omega} e^{-(v|\xi)} Q^{\lambda-\tau}(v) dv \right) d\xi \\ &= \frac{\Gamma_{\Omega}(\lambda)}{\Gamma_{\Omega^*}(-\mu)} \int_{\Omega^*} e^{-(\mathbf{e}|\xi)} (Q^*)^{-\mu-\lambda-\tau}(\xi) d\xi < +\infty \end{aligned}$$

if and only if

$$\lambda_j > \frac{m_j}{2}, \quad \mu_j + \lambda_j < -\frac{n_j}{2}, \qquad j = 1, \ldots, r.$$

This prove i). The proof of ii) is the same. □

Lemma 4.20. *Let* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r$.

i) *The integral*

$$J_{\alpha}(y) = \int_V \left| Q^{-\alpha}\left(\frac{x+iy}{i}\right) \right| dx \qquad (y \in \Omega) \tag{4.13}$$

converges if and only if $\alpha_j > 1 + n_j + \frac{m_j}{2}$, $j = 1, \ldots, r$. *In this case,*

$$J_{\alpha}(y) = c_{\alpha} Q^{-\alpha+\tau}(y),$$

where

$$c_{\alpha} = \frac{(2\pi)^n 2^{-|\alpha|+|\tau|} \Gamma_{\Omega^*}(\alpha-\tau)}{[\Gamma_{\Omega^*}(\alpha/2)]^2}.$$

ii) *The function*

$$F(z) = Q^{-\alpha}\left(\frac{z+it}{i}\right) \qquad (z \in T_{\Omega}),$$

with $t \in \Omega$, *belongs to* $A^{p,q}_{\nu}(T_{\Omega})$ *if and only if*

$$\alpha_j > \max\left\{ \frac{1+n_j+\frac{m_j}{2}}{p}, \frac{\nu_j+\frac{n_j}{2}}{q} + \frac{\tau_j}{p} \right\}, \quad j = 1, \ldots, r.$$

Proof. For fixed $y \in \Omega$, interpret (4.13) as the L^2-norm in dx of $Q^{\frac{-\alpha}{2}}\left(\frac{x+iy}{i}\right)$. By (4.8) and Plancherel's formula, the integral in (4.13) is finite if and only if the integral

$$\int_{\Omega^*} e^{-2(y|\xi)}(Q^*)^{\alpha-2\tau}(\xi)d\xi$$

is finite. This proves (i). The rest follows by Lemma 4.19. □

Lemma 4.21. *Let* $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ *and* $0 < \lambda < \frac{1}{4}$. *There is a constant* C_α *such that for all* $y \in \Omega$, $\|y\| < \lambda$,

$$\int_{\{x\in V:\,\|x\|<1\}} \left|Q^{-\alpha}\left(\frac{x+iy}{i}\right)\right| dx \geq C_\alpha Q^{-\alpha+\tau}(y).$$

Proof. We set $x = \pi(t)u$ where $y = \pi(t)\mathbf{e} = t \cdot \mathbf{e}$ the fact that $dx = Q^\tau(y)du$. Then

$$\begin{aligned}\int_{\{x\in V:\|x\|<1\}} \left|Q^{-\alpha}\left(\frac{x+iy}{i}\right)\right| dx &= Q^{-\alpha+\tau}(y)\int_{\{u\in V:\|\pi(t)u\|<1\}} \left|Q^{-\alpha}\left(\frac{u+i\mathbf{e}}{i}\right)\right| du \\ &\geq Q^{-\alpha+\tau}(y)\int_{\{u\in\Omega:\ \|\pi(t)u\|<1\}} \left|Q^{-\alpha}\left(\frac{u+i\mathbf{e}}{i}\right)\right| du \\ &\geq C_\alpha Q^{-\alpha+\tau}(y),\end{aligned}$$

with

$$C_\alpha = \int_{\{u\in\Omega:\ \|u\|<4/c\sqrt{r}\}} \left|Q^{-\alpha}\left(\frac{u+i\mathbf{e}}{i}\right)\right| du.$$

In fact, by our assumption, $\|y\| < \frac{1}{4}$; so Lemma 4.11 states that $\|\pi(t)u\| < c\sqrt{r}/4$. It follows that set $\{u \in \Omega : \ \|\pi(t)u\| < 1\}$ contains the set $\{u \in \Omega : \ \|u\| < 4/c\sqrt{r}\}$. □

5. The Bergman spaces

Here we recall some basic facts about Bergman spaces. Once we have the preliminary results above, the proof of all these results are basically the same as those obtained in the papers [5] and [6]. The reader can look at these papers to have more details of proofs omitted here. For $\nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j+n_j}{2}$, $j = 1, \ldots, r$; we shall denote $L^2_{(\nu)}(\Omega^*) = L^2(\Omega^*,\ (Q^*)^\nu(\xi)d\xi)$ and by

$$\mathcal{L}g(z) = (2\pi)^{-\frac{n}{2}} \int_{\Omega^*} e^{i(z|\xi)} g(\xi)d\xi$$

the Laplace transform of a locally integrable function g. We have this Paley-Wiener type theorem, whose proof is analogous to [11, Proposition IX.3.3].

Theorem 5.1. *Let $\nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathbb{R}^r$ with $\nu_j > \frac{m_j+n_j}{2}$, $j = 1, \ldots, r$. A function F belongs to $A^2_\nu(T_\Omega)$ if and only if $F = \mathcal{L}g$, with $g \in L^2_{(-\nu)}(\Omega^*)$. Moreover*

$$\|F\|^2_{A^2_\nu(T_\Omega)} = e_\nu \|g\|^2_{L^2_{(-\nu)}(\Omega^*)} \tag{5.1}$$

where

$$e_\nu = 2^{-|\nu|}\Gamma_\Omega(\nu).$$

We denote by $\langle , \rangle_\nu$ the Hermitian form induced by the $A^2_\nu(T_\Omega)$-norm. Since the Bergman kernel is a reproducing kernel of $A^2_\nu(T_\Omega)$, it follows, by polarization of (5.1), that for $F \in A^2_\nu(T_\Omega)$,

$$F(w) = \langle F, B_\nu(\cdot,\ w)\rangle_\nu = e_\nu \langle g,\ g_w\rangle_{L^2_{(-\nu)}(\Omega^*)} = \int_{\Omega^*} g(\xi) e_\nu \overline{g_w(\xi)} (Q^*)^{-\nu}(\xi) d\xi.$$

Since $F = \mathcal{L}g$, one has

$$g_w(\xi) = (2\pi)^{-\frac{n}{2}} e_\nu^{-1} e^{-i(\bar{w}|\xi)} (Q^*)^\nu(\xi).$$

Hence, by (4.8),

$$B_\nu(z, w) = (2\pi)^{-\frac{n}{2}} \mathcal{L}g_w(z) = d_\nu Q^{-\nu-\tau}\left(\frac{z - \bar{w}}{i}\right),$$

with

$$d_\nu = \frac{(2\pi)^n 2^{-|\nu|}\Gamma_{\Omega^*}(\nu + \tau)}{\Gamma_\Omega(\nu)}.$$

The operator

$$P_\nu f(z) = \int_{T_\Omega)} B_\nu(z, w) f(w) Q^{\nu-\tau}(\Im m\ w) dv(w)$$

is the identity of $A^2_\nu(T_\Omega)$; it provides the orthogonal projection of $L^2_\nu(T_\Omega)$ onto $A^2_\nu(T_\Omega)$ *i.e.*, it is the Bergman projection.

Lemma 5.2. *Let $F \in A^{p,q}_\nu(T_\Omega)$. The following assertions hold:*

i) *There is a constant $C = C(p, q, \nu) > 0$ such that, for all $z = x + iy \in T_\Omega$,*

$$|F(x + iy)| \le C Q^{-\frac{\nu}{q} - \frac{\tau}{2p}}(y) \|F\|_{A^{p,q}_\nu(T_\Omega)}. \tag{5.2}$$

ii) *There is a constant $C = C(p, q, \nu) > 0$ such that, for all $y \in \Omega$,*

$$\|F(\cdot + iy)\|_p \le C Q^{-\frac{\nu}{q}}(y) \|F\|_{A^{p,q}_\nu(T_\Omega)}. \tag{5.3}$$

iii) *There is a constant* $C = C(p, q, \nu) > 0$ *such that, for all* $y \in \Omega$ *and all* $s > p$,

$$\|F(\cdot + iy)\|_s \leq C Q^{-\frac{\nu}{q} - \frac{\tau}{2}\left(\frac{1}{p} - \frac{1}{s}\right)}(y)\|F\|_{A_\nu^{p,q}(T_\Omega)}. \tag{5.4}$$

Corollary 5.3. *The Bergman space* $A_\nu^{p,q}(T_\Omega)$ *is a Banach space.*

Proof. Taking $s = \infty$ in Lemma 5.2, we see that convergence in $A_\nu^{p,q}(T_\Omega)$ implies convergence over compact subsets of T_Ω. So $A_\nu^{p,q}(T_\Omega)$ is a closed subspace of $L_\nu^{p,q}(T_\Omega)$. This one is known to be a Banach space. □

Corollary 5.4. *Let* $\nu = (\nu_1, \nu_2, \dots, \nu_r) \in \mathbb{R}^r$ *be such that* $\nu_j > \frac{m_j + n_j}{2}$, $j = 1, \dots, r$, *and* $F \in A_\nu^{p,q}(T_\Omega)$;

i) *for every* $t \in \Omega$, *the function* $F_t(z) = F(z + it)$ *belongs to the Hardy space* $\mathsf{H}^s(T_\Omega)$ *for* $s \geq p$;

ii) *for* $y, t \in \Omega$,

$$\|F(\cdot + i(y + t))\|_s \leq \|F(\cdot + iy)\|_s;$$

iii) *for* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r$ *such that* $\alpha_j > \frac{m_j + n_j}{2}$, $j = 1, \dots, r$ *and* $\varepsilon > 0$, *let*

$$F_{\varepsilon,\alpha}(z) = F(z + i\varepsilon\underline{\mathbf{e}})Q^{-\alpha}\left(\frac{\varepsilon z + i\underline{\mathbf{e}}}{i}\right).$$

Then $F_{\varepsilon,\alpha} \in A_\nu^{p,q}(T_\Omega)$ *and we have*

$$\lim_{\varepsilon \to 0} \|F - F_{\varepsilon,\alpha}\|_{A_\nu^{p,q}(T_\Omega)} = 0.$$

Corollary 5.5. *Let* $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{R}^r$ *and* $\mu = (\mu_1, \dots, \mu_r) \in \mathbb{R}^r$ *such that* $\nu_j > \frac{m_j + n_j}{2}$, $j = 1, \dots, r$ *and* $\mu_j > \frac{m_j + n_j}{2}$, $j = 1, \dots, r$. *The subspace* $A_\nu^{p,q}(T_\Omega) \cap A_\mu^{s,r}(T_\Omega)$ *of the Bergman spaces* $A_\nu^{p,q}(T_\Omega)$ *and* $A_\mu^{s,r}(T_\Omega)$ *is dense in each of them.*

6. Proof of Theorem 3.1

In order to prove Theorem 3.1, we will state that the $L_\nu^{p,q}(T_\Omega)$-boundedness of the operator P_ν^+ is related to the $L^q(\Omega, Q^{\nu-\tau}(y)dy)$-boundedness of a positive integral operator on the cone Ω.

Consider the positive integral operator S defined on Ω by

$$Sg(y) = \int_\Omega Q^{-\nu}(y + v)g(v)Q^{\nu-\tau}(v)dv. \tag{6.1}$$

It is easy to verify that S is a self-adjoint operator. We put

$$q_\nu = 1 + \min_{1 \leq i \leq r} \frac{\nu_i - \frac{m_i}{2}}{\frac{n_i}{2}}.$$

Theorem 6.1. *Let $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{R}^r$ be such that $\nu_j > \frac{m_j+n_j}{2}$, $j = 1, \dots, r$. The operator S is bounded on $L^q(\Omega, Q^{\nu-\tau}(v)dv)$ when $q'_\nu < q < q_\nu$.*

Proof. We will use Schur's Lemma (see [12]). The kernel of the operator S relative to the measure $Q^{\nu-\tau}(v)dv$ is given by

$$N(y, v) = Q^{-\nu}(y + v)$$

and it is positive. By Schur's Lemma, it is sufficient to find a positive and measurable function φ defined on Ω such that

$$\int_\Omega N(y, v)\varphi(v)^{q'} Q^{\nu-\tau}(v)dv \le C\varphi(y)^{q'} \tag{6.2}$$

and

$$\int_\Omega N(y, v)\varphi(y)^{q} Q^{\nu-\tau}(y)dy \le C\varphi(v)^{q}. \tag{6.3}$$

We take as test functions $\varphi(v) = Q^\gamma(v)$ where $\gamma = (\gamma_1, \dots, \gamma_r) \in \mathbb{R}^r$ has to be determined. An application of Lemma 4.19 gives that (6.2) holds whenever

$$\frac{-\nu_j + \frac{m_j}{2}}{q'} < \gamma_j < \frac{-\frac{n_j}{2}}{q'}, \quad j = 1, \dots, r$$

and (6.3) holds when

$$\frac{-\nu_j + \frac{m_j}{2}}{q} < \gamma_j < \frac{-\frac{n_j}{2}}{q}, \quad j = 1, \dots, r.$$

The inequalities (6.2) and (6.3) are simultaneously satisfied if each γ_j, $j = 1, \dots, r$, satisfies the condition

$$\gamma_j \in \left]\frac{-\nu_j + \frac{m_j}{2}}{q'}, \frac{-\frac{n_j}{2}}{q'}\right[\bigcap \left]\frac{-\nu_j + \frac{m_j}{2}}{q}, \frac{-\frac{n_j}{2}}{q}\right[. \tag{6.4}$$

The intersection in (6.4) is not empty if $\frac{-\nu_j+\frac{m_j}{2}}{q'} < \frac{-\frac{n_j}{2}}{q}$ and $\frac{-\nu_j+\frac{m_j}{2}}{q} < \frac{-\frac{n_j}{2}}{q'}$; that is if, for any $j = 1, \dots, r$,

$$\frac{\nu_j + \frac{-m_j+n_j}{2}}{\nu_j - \frac{m_j}{2}} < q < \frac{\nu_j + \frac{-m_j+n_j}{2}}{\frac{n_j}{2}}$$

i.e.,

$$1 + \frac{\frac{n_j}{2}}{\nu_j - \frac{m_j}{2}} < q < 1 + \frac{\nu_j - \frac{m_j}{2}}{\frac{n_j}{2}}. \qquad \square$$

Theorem 3.1 is obtained by taking $p = q$ in the following theorem, whose proof is analogous to [6, Theorem 6.2].

Theorem 6.2. *Let $1 \le p \le +\infty$ and $1 \le q < +\infty$. The operator P_ν^+ is bounded on $L_\nu^{p,q}(T_\Omega)$ when $q'_\nu < q < q_\nu$. Moreover, the weighted Bergman projector P_ν is bounded from $L_\nu^{p,q}(T_\Omega)$ to $A_\nu^{p,q}(T_\Omega)$ when $q'_\nu < q < q_\nu$.*

7. L^p-estimates of the Bergman projector P_ν

In this section we prove Theorem 3.2. We shall start by the necessity. Assume that P_ν is bounded from $L^{p,q}_\nu(T_\Omega)$ to $L^{p,q}_\nu(T_\Omega)$, since P_ν is self-adjoint, then P_ν is also bounded from $L^{p',q'}_\nu(T_\Omega)$ to $L^{p',q'}_\nu(T_\Omega)$. Consider the test function $f(z) = Q^{-\nu+\tau}(\Im m\, z)\chi_{D(i\mathbf{e},\delta)}(z)$ where $\chi_{D(i\mathbf{e},\delta)}$ is the characteristic function of the Euclidean ball centered at $i\mathbf{e}$ with radius δ. Then $f \in \bigcap L^{p,q}_\nu(T_\Omega)$ and, by the mean value formula,

$$P_\nu f(z) = C_\delta Q^{-\nu-\tau}\left(\frac{z+i\mathbf{e}}{i}\right).$$

Therefore, by ii) of Lemma 4.20, $P_\nu f \in A^{p,q}_\nu(T_\Omega)$ if and only if for every $j = 1,\dots,r$,

$$\nu_j + \tau_j > \max\left\{\frac{1}{p}\left(1 + n_j + \frac{m_j}{2}\right), \frac{1}{q}\left(\nu_j + \frac{n_j}{2}\right) + \frac{\tau_j}{p}\right\}.$$

It follows that, for $p = q$, if the weighted Bergman projection is bounded, then, for every $j = 1,\dots,r$,

$$\nu_j + \tau_j > \frac{1}{p}\left(\nu_j + \frac{n_j}{2} + \tau_j\right).$$

Since P_ν is self-adjoint, we also have that, for every $j = 1,\dots,r$,

$$\nu_j + \tau_j > \frac{1}{p'}\left(\nu_j + \frac{n_j}{2} + \tau_j\right).$$

This proves part i) of Theorem 3.2.

For the sufficiency, we shall find values of p for which the Bergman projector P_ν is bounded whenever the operator P^+_ν is not bounded. We will use the Paley-Wiener Theorem (Theorem 5.1) to prove that the Laplace transform is an isomorphism between $A^{2,q}_\nu(T_\Omega)$ and the space $b^q_\nu(\Omega^*)$. We then conclude by interpolation. The results here are the analogues of those in [3] and [6]. We will only give statements of the proofs that emphasize differences.

In the sequel, we consider the following disjoint covering of the cone Ω^*,

$$E^*_1 = B^*_1,\quad E^*_j = B^*_j \setminus \bigcup_{k=1}^{j} B^*_k,\quad j = 2,\dots,$$

where $B^*_j = B^*_\lambda(y'_j)$ and $\{y'_j\}_j$ is the dual lattice of the λ-lattice $\{y_j\}_j$. We have $\Omega^* = \bigcup_j E^*_j$ and

$$|E^*_j| \sim |B^*_j| \sim (Q^*)^\tau(y'_j).$$

Definition 7.1. Let $q \geq 1$, $0 < \lambda < 1$ and $\{\xi_j\}$ a λ-lattice in Ω'. We denote by $b_\nu^q(\Omega^*)$ the space of all measurable functions g which are locally square integrable and satisfy the estimate

$$\|g\|_{b_\nu^q(\Omega^*)} := \left(\sum_j (Q^*)^{-\nu}(\xi_j)\left(\int_{E_j^*}|g(\xi)|^2 d\xi\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} < +\infty.$$

We say that a sequence $\{\lambda_j\}_j$ belongs to l_ν^q if it satisfies

$$\sum_j |\lambda_j|^q (Q^*)^{-\nu}(\xi_j) < +\infty.$$

Lemma 7.2. *The space $b_\nu^q(\Omega^*)$ is a Banach space.*

Proof. Just remark that $b_\nu^q(\Omega^*) = l_\nu^q(L^2(E_j^*))$. □

Remark 7.3. Let $\{a_j\}_j$ be a positive sequence. Then

$$\left(\sum_j a_j\right)^\delta \leq \sum_j a_j^\delta \qquad \text{if} \qquad 0 < \delta \leq 1 \tag{7.1}$$

and

$$\sum_j a_j^\delta \leq \left(\sum_j a_j\right)^\delta \qquad \text{if} \qquad \delta \geq 1. \tag{7.2}$$

7.1. The boundedness of the Bergman projector P_ν on $L_\nu^{2,q}(T_\Omega)$

We shall show that the Laplace transform $\mathcal{L}$ is isomorphically bounded from $b_\nu^q(\Omega^*)$ onto $A_\nu^{2,q}(T_\Omega)$.

The following proposition proves the statement for $q = 2$.

Proposition 7.4. *There is a constant $C = C(\nu) > 1$ such that for all $F \in A_\nu^{2,2}(T_\Omega)$,*

$$\frac{1}{C}\sum_j (Q^*)^{-\nu}(\xi_j)\int_{E_j^*}|g(\xi)|^2 d\xi \leq \|F\|^2_{A_\nu^{2,2}} \leq C\sum_j (Q^*)^{-\nu}(\xi_j)\int_{E_j^*}|g(\xi)|^2 d\xi;$$

where $F = \mathcal{L}g$ with $g \in L^2_{(-\nu)}(\Omega^*)$.

Lemma 7.5. *Let $q \geq 1$. There is a constant $C = C(\nu, \tau, q) > 0$ such that for all $g \in b_\nu^q(\Omega^*)$ and all $y \in \Omega$,*

$$\int_{\Omega^*}|g(\xi)|e^{-(y|\xi)}d\xi \leq C\|g\|_{b_\nu^q(\Omega^*)}Q^{-\frac{\nu}{q}-\frac{\tau}{2}}(y).$$

In particular, g is locally integrable on Ω^*.

Theorem 7.6. *Let $q \geq 1$. For all $F \in A_\nu^{2,q}(T_\Omega)$ there is a unique function $g \in b_\nu^q(\Omega^*)$ such that $F = \mathcal{L}g$ and*

$$\|g\|_{b_\nu^q(\Omega^*)} \leq C\|F\|_{A_\nu^{2,q}(T_\Omega)}.$$

Proof. By density (see Corollary 5.5), take $F \in A_\nu^{2,q}(T_\Omega) \cap A_\nu^{2,2}(T_\Omega)$. By the Paley-Wiener Theorem (Theorem 5.1), there exists a function $g \in L^2_{(-\nu)}(\Omega^*)$ such that

$$F(x+iy) = \mathcal{L}g(x+iy) = (2\pi)^{-\frac{n}{2}} \int_{\Omega^*} g(\xi) e^{i(x+iy|\xi)} d\xi.$$

Let $\{y_j\}_j$ be a λ-lattice of Ω and let $\{y'_j\}_j$ be the dual lattice of the λ-lattice $\{y_j\}_j$. We saw that the map $x \mapsto x'$ is an isometry from Ω on Ω^* (*cf.* Theorem 4.1). Thus, for $y \in B_j = B_\lambda(y_j)$, one has $y' \in B_j^* = B_\lambda^*(y'_j)$; moreover, by Corollary 4.10, there is a constant γ such that $\frac{1}{\gamma} \leq (y|\xi) \leq \gamma$ whenever $y \in B_j$ and $\xi \in B_j^*$. Then, for $y \in B_j$, according to Corollary 4.10, we have

$$\int_{E_j^*} |g(\xi)|^2 d\xi \leq c_\gamma \int_{\Omega^*} |g(\xi)|^2 e^{-2(y|\xi)} d\xi = C' \int_V |F(x+iy)|^2 dx,$$

by Plancherel's formula. It follows from (4.2) that

$$\left(\int_{E_j^*} |g(\xi)|^2 d\xi \right)^{\frac{q}{2}} \leq c'_q Q^{-\tau}(y_j) \int_{B_j} \left(\int_V |F(x+iy)|^2 dx \right)^{\frac{q}{2}} dy.$$

If we denote by $\{\xi_j\}_j$ the dual λ-lattice of $\{y_j\}_j$, then by (4.1) and i) of Lemma 4.2,

$$\begin{aligned}
(Q^*)^{-\nu}(\xi_j) \left(\int_{E_j^*} |g(\xi)|^2 d\xi \right)^{\frac{q}{2}} &\leq c_q (Q^*)^{-\nu}(\xi_j) Q^{-\tau}(y_j) \int_{B_j} \left(\int_V |F(x+iy)|^2 dx \right)^{\frac{q}{2}} dy \\
&\leq c_{\nu q} Q^{\nu-\tau}(y_j) \int_{B_j} \left(\int_V |F(x+iy)|^2 dx \right)^{\frac{q}{2}} dy \\
&\leq c'_{\nu q} \int_{B_j} \left(\int_V |F(x+iy)|^2 dx \right)^{\frac{q}{2}} Q^{\nu-\tau}(y) dy;
\end{aligned}$$

hence

$$\|g\|_{b_\nu^q(\Omega^*)} \leq C_{\nu,q} \|F\|_{A_\nu^{2,q}(T_\Omega)}. \tag{7.3}$$

This finishes the proof. □

We prove now the converse of the previous theorem.

Theorem 7.7. *Assume* $1 \le q < 2q_\nu$. *Given* $g \in b_\nu^q(\Omega^*)$, *then* $\mathcal{L}g \in A_\nu^{2,q}(T_\Omega)$ *and*

$$\|\mathcal{L}g\|_{A_\nu^{2,q}(T_\Omega)} \le C\|g\|_{b_\nu^q(\Omega^*)}.$$

Proof. Write $F(x+iy) = F_y(x) = \mathcal{L}g(x+iy)$. For every $y \in \Omega$, the function $x \mapsto F_y(x)$ is the inverse Fourier transform of the function $\xi \mapsto \psi_y(\xi) = g(\xi)e^{-(y|\xi)}$. By Plancherel's formula,

$$\|F\|^q_{A_\nu^{2,q}(T_\Omega)} = \int_\Omega \left(\int_{\Omega^*} |g(\xi)|^2 e^{-2(y|\xi)} d\xi\right)^{\frac{q}{2}} Q^{\nu-\tau}(y)dy.$$

By (ii) of Lemma 4.4 and Proposition 4.9, we deduce that

$$\|F\|^q_{A_\nu^{2,q}(T_\Omega)} \le \int_\Omega \left(\sum_j e^{-2\gamma(y|\xi_j)} \int_{E_j^*} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} Q^{\nu-\tau}(y)dy. \tag{7.4}$$

First assume that $1 \le q \le 2$. Since $\frac{q}{2} \le 1$, we deduce from inequality (7.1) and Corollary 4.16 that

$$\begin{aligned}\|F\|^q_{A_\nu^{2,q}(T_\Omega)} &\le \int_\Omega \sum_j e^{-q\gamma(y|\xi_j)} \left(\int_{E_j^*} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} Q^{\nu-\tau}(y)dy \\ &\le \sum_j \left(\int_{E_j^*} |g(\xi)|^2 d\xi\right)^{\frac{q}{2}} \int_\Omega e^{-q\gamma(y|\xi_j)} Q^{\nu-\tau}(y)dy \le C_{\nu q\gamma}\|g\|^q_{b_\nu^q(\Omega^*)}.\end{aligned}$$

Assume next that $2 \le q < 2q_\nu$. Let $\rho = \frac{q}{2}$ and $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$. By Hölder's inequality,

$$\begin{aligned}\sum_j e^{-2\gamma(y|\xi_j)} \left(\int_{E_j^*} |g(\xi)|^2 d\xi\right) &\le \left(\sum_j e^{-2\gamma(y|\xi_j)} \left(\int_{E_j^*} |g(\xi)|^2 d\xi\right)^\rho (Q^*)^{-\alpha\rho}(\xi_j)\right)^{\frac{1}{\rho}} \\ &\quad \times \left(\sum_j e^{-2\gamma(y|\xi_j)} (Q^*)^{\alpha\rho'}(\xi_j)\right)^{\frac{1}{\rho'}}.\end{aligned}$$

It follows from (7.4) that

$$\begin{aligned}\|F\|^q_{A_\nu^{2,q}(T_\Omega)} \le \int_\Omega \Bigg[&\left(\sum_j e^{-2\gamma(y|\xi_j)} \left(\int_{E_j^*} |g(\xi)|^2 d\xi\right)^\rho (Q^*)^{-\alpha\rho}(\xi_j)\right) \\ &\times \left(\sum_j e^{-2\gamma(y|\xi_j)} (Q^*)^{\alpha\rho'}(\xi_j)\right)^{\frac{\rho}{\rho'}}\Bigg] Q^{\nu-\tau}(y)dy.\end{aligned} \tag{7.5}$$

By (4.2), ii) of Lemma 4.2, Proposition 4.9 and iii) of Lemma 4.4, we have

$$\sum_j e^{-2\gamma(y|\xi_j)}(Q^*)^{\alpha\rho'}(\xi_j) \le c\sum_j e^{-2\gamma(y|\xi_j)}(Q^*)^{\alpha\rho'-\tau}(\xi_j)\int_{E_j^*} d\xi$$

$$\le CN\int_{\Omega^*} e^{-2(y|\xi)}(Q^*)^{\alpha\rho'-\tau}(\xi)d\xi.$$

We deduce from (4.8) that

$$\sum_j e^{-2\gamma(y|\xi_j)}(Q^*)^{\alpha\rho'}(\xi_j) \le C_{\alpha\rho}Q^{-\alpha\rho'}(y),$$

whenever $\alpha_j\rho' > \frac{n_j}{2},\ j = 1,\dots,r$.

So for $\alpha_j\rho' > \frac{n_j}{2},\ j = 1,\dots,r$, from inequality (7.5) we obtain:

$$\|F\|^q_{A_\nu^{2,q}(T_\Omega)} \le C_{\alpha\rho}\int_\Omega\left(\sum_j e^{-2\gamma(y|\xi_j)}\left(\int_{E_j^*}|g(\xi)|^2d\xi\right)^\rho(Q^*)^{-\alpha\rho}(\xi_j)\right)Q^{-\alpha\rho+\nu-\tau}(y)dy$$

$$\le C_{\alpha\rho}\sum_j\left(\int_{E_j^*}|g(\xi)|^2d\xi\right)^\rho(Q^*)^{-\alpha\rho}(\xi_j)\int_\Omega e^{-2\gamma(y|\xi_j)}Q^{-\alpha\rho+\nu-\tau}(y)dy.$$

Moreover, if $-\alpha_j\rho + \nu_j > \frac{m_j}{2},\ j = 1,\dots,r$, by (4.7), we have

$$\int_\Omega e^{-2\gamma(y|\xi_j)}Q^{-\alpha\rho+\nu-\tau}(y)dy = c_{\alpha\nu\rho}(Q^*)^{\alpha\rho-\nu}(\gamma\xi_j);$$

it follows that

$$\|F\|^q_{A_\nu^{2,q}(T_\Omega)} \le C_{\alpha\nu\rho}\|g\|^q_{b_\nu^q(\Omega^*)}.$$

Therefore, the conclusion follows if we choose $\alpha_1,\dots,\alpha_r$ such that

$$\alpha_j\rho' > \frac{n_j}{2},\ -\alpha_j\rho+\nu_j > \frac{m_j}{2},\ j = 1,\dots,r.$$

Each parameter $\alpha_j,\ j = 1,\dots,r$ must lie in $\left]\frac{n_j}{2\rho'}, \frac{\nu_j-\frac{m_j}{2}}{\rho}\right[$ which is a non-empty interval. □

We have proved that the Laplace transform $\mathcal{L}$ maps $b_\nu^q(\Omega^*)$ isomorphically onto $A_\nu^{2,q}(T_\Omega)$ whenever $1 \le q < 2q_\nu$. Let us now consider the operator $R = \mathcal{L}^{-1}P_\nu$. We will now show that the operator R is bounded from $L_\nu^{2,q}(T_\Omega)$ to $b_\nu^q(\Omega^*)$.

Let $\phi \in L^2_\nu(T_\Omega)$; by Paley-Wiener Theorem, $F \in A^2_\nu(T_\Omega)$ if and only if $F = \mathcal{L}g$ with $g \in L^2_{(-\nu)}(\Omega^*)$. The self-adjointness of P_ν implies

$$\langle P_\nu\phi, F\rangle_{A^2_\nu(T_\Omega)} = \langle \phi, F\rangle_{L^2_\nu(T_\Omega)} = \langle \phi, \mathcal{L}g\rangle_{L^2_\nu(T_\Omega)}.$$

Now, by Plancherel's formula and Fubini's Theorem

$$\begin{aligned}\langle\phi,\mathcal{L}g\rangle_{L^2_\nu(T_\Omega)} &= \int_\Omega \left(\int_V \phi_y(x)\overline{\mathcal{F}^{-1}(g(\xi)e^{-(y|\xi)})(x)}dx\right) Q^{\nu-\tau}(y)dy \\ &= \int_\Omega \left(\int_{\Omega^*} \mathcal{F}(\phi_y)(\xi)\overline{g(\xi)}e^{-(y|\xi)}d\xi\right) Q^{\nu-\tau}(y)dy \qquad (7.6) \\ &= \int_{\Omega^*}\left((Q^*)^\nu(\xi)\int_\Omega \mathcal{F}(\phi_y)(\xi)e^{-(y|\xi)}Q^{\nu-\tau}(y)dy\right)\overline{g(\xi)}(Q^*)^{-\nu}(\xi)d\xi,\end{aligned}$$

where $\mathcal{F}$ is the Fourier transform. Therefore, for $g \in L^2_{(-\nu)}(\Omega^*)$, equality (7.6) and the polarization of isometry (5.1) in the Paley-Wiener Theorem imply that

$$\begin{aligned}\langle\phi, \mathcal{L}g\rangle_{L^2_\nu(T_\Omega)} &= \langle P_\nu\phi, F\rangle_{A^2_\nu(T_\Omega)} \\ &= e_\nu\langle \mathcal{L}^{-1}P_\nu, g\rangle_{L^2_{(-\nu)}(\Omega^*)} \qquad (7.7) \\ &= e_\nu\langle R\phi, g\rangle_{L^2_{(-\nu)}(\Omega^*)}.\end{aligned}$$

Comparing (7.6) and (7.7) then gives

$$R\phi(\xi) = e_\nu^{-1}(Q^*)^\nu(\xi)\left(\int_\Omega \mathcal{F}\phi_y(\xi)e^{-(y|\xi)}Q^{\nu-\tau}(y)dy\right).$$

We shall need the following lemma:

Lemma 7.8. *If $q \geq 2$, then R extends into a bounded operator from $L^{2,q}_\nu(T_\Omega)$ to $b^q_\nu(\Omega^*)$ i.e.,*

$$\|R\phi\|_{b^q_\nu(\Omega^*)} \leq C\|\phi\|_{L^{2,q}_\nu(T_\Omega)}.$$

Let

$$Q_\nu = 2q_\nu = 2 + 2\min_{1\leq j\leq r}\frac{\nu_j - \frac{m_j}{2}}{\frac{n_j}{2}};$$

we can prove now the following result:

Corollary 7.9. *The Bergman projector P_ν extends to a bounded operator from $L^{2,q}_\nu(T_\Omega)$ to $A^{2,q}_\nu(T_\Omega)$ when $Q'_\nu < q < Q_\nu$.*

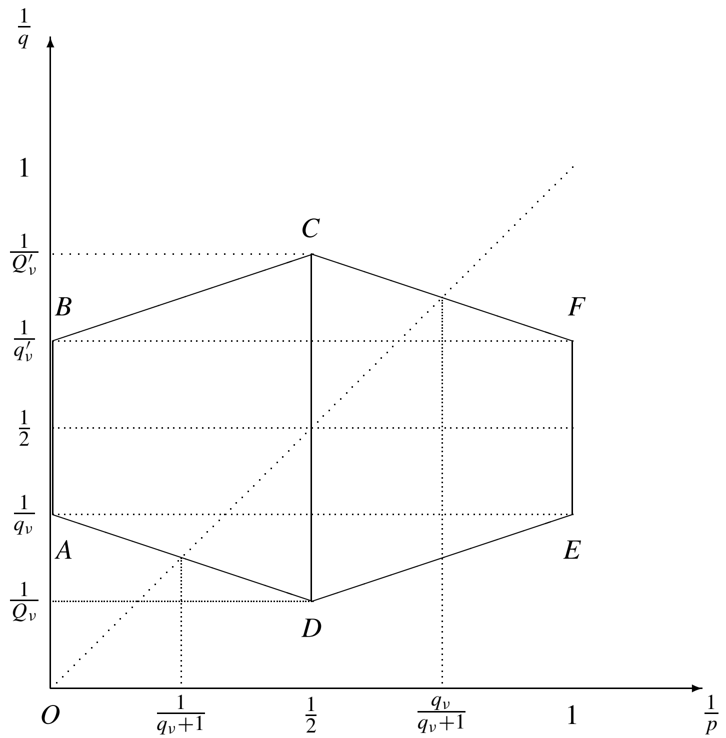
Proof. Assume that $2 \le q < Q_\nu$. By Lemma 7.8, the operator R is bounded from $L^{2,q}_\nu(T_\Omega)$ to $b^q_\nu(\Omega^*)$ and, according to the Theorem 7.7, the Laplace transform $\mathcal{L}$ is bounded from $b^q_\nu(\Omega^*)$ to $A^{2,q}_\nu(T_\Omega)$. We conclude that the Bergman projector $P_\nu = \mathcal{L} \circ R$ is bounded from $L^{2,q}_\nu(T_\Omega)$ to $A^{2,q}_\nu(T_\Omega)$. We obtain the other part by the self-adjointness of P_ν. □

7.2. Proof of Theorem 3.2

Theorem 7.10. *The Bergman projector P_ν extends to a bounded operator from $L^{p,q}_\nu(T_\Omega)$ to $A^{p,q}_\nu(T_\Omega)$ if*

$$\begin{cases} 0 \le \dfrac{1}{p} \le \dfrac{1}{2} \\ \dfrac{1}{q_\nu p'} < \dfrac{1}{q} < 1 - \frac{1}{q_\nu p'} \end{cases} \qquad \text{or} \qquad \begin{cases} \dfrac{1}{2} \le \dfrac{1}{p} \le 1 \\ \dfrac{1}{q_\nu p} < \dfrac{1}{q} < 1 - \dfrac{1}{q_\nu p}. \end{cases}$$

Proof. For a fixed $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ that satisfies $\nu_j > \frac{m_j+n_j}{2}$, $j = 1, \ldots, r$ let us consider the following picture



By interpolation, P_ν is bounded on $L^{p,q}_\nu(T_\Omega)$ for $\left(\frac{1}{p}, \frac{1}{q}\right)$ in the interior of the hexagon of vertices

$$A\left(0, \frac{1}{q_\nu}\right), \quad D\left(\frac{1}{2}, \frac{1}{Q_\nu}\right), \quad E\left(1, \frac{1}{q_\nu}\right)$$

and their symmetric points with respect to $\left(\frac{1}{2}, \frac{1}{2}\right)$. □

Theorem 3.2 is the particular case $p = q$ of Theorem 7.10. It is important to say that, for the dual cone Ω^*, we obtain

$$q_\nu = 1 + \min_{1\le j\le r} \frac{\nu_j - \frac{n_j}{2}}{\frac{m_j}{2}}.$$

Note that, for symmetric cones, this value is the same as the one we obtained in this paper. However, this is not the case for homogeneous nonself-dual cone as shown by Vinberg's cone (see [6]).

8. Final remarks

The techniques that have been exposed in this paper reveal some shortcomings, since we were expecting necessary and sufficient conditions in Theorem 3.1, Theorem 6.1 and Corollary 7.9. Unfortunately, with the method exposed here, the converse of Theorem 3.1 can be stated as follows: when the positive Bergman operator P^+_ν is bounded in $L^p_\nu(T_\Omega, dv)$, then

$$1 + \max_{1\le j\le r} \frac{n_j/2}{\nu_j} < q < 1 + \min_{1\le j\le r} \frac{\nu_j}{n_j/2}. \tag{8.1}$$

Moreover, the converse of Corollary 7.9 can be stated: when the Bergman projector P_ν is bounded from $L^{2,q}_\nu(T_\Omega)$ to $A^{2,q}_\nu(T_\Omega)$, then

$$2\left(1 + \max_{1\le j\le r} \frac{n_j/2}{\nu_j}\right) < q < 2\left(1 + \min_{1\le j\le r} \frac{\nu_j}{n_j/2}\right). \tag{8.2}$$

Thus, we realise that conditions (8.1) and (8.2) are somehow linked. As a matter of fact, if Ω is an open convex homogeneous cone such that $m_{j_0} = 0$ where $j_0 \in \{1, \ldots, r\}$ satisfies

$$1 + \min_{1\le j\le r} \frac{\nu_j - \frac{m_j}{2}}{\frac{n_j}{2}} = 1 + \frac{\nu_{j_0} - \frac{m_{j_0}}{2}}{\frac{n_{j_0}}{2}} = q_\nu,$$

then conditions (8.1) and (8.2) are necessary and sufficient in Theorem 3.1 (or Theorem 6.1) and Corollary 7.9 respectively. This is exactly what has happened in [3], [5] and [6]. Moreover, it is also the case for general symmetric cones, using general weighted measures, when $\nu_j \geq \nu_r - \frac{r-j}{r-1}\left(\nu_r - (r-1)\frac{d}{2}\right)$ for $j = 1, \ldots, r-1$.

On the other hand, whether the necessary condition of Theorem 3.2 coincides with the sufficient condition is still an open problem even for the symmetric cones of rank 2.

Finally, even for the positive Bergman operator P_ν^+, for the values of p satisfying $1 + \max_{1\leq j\leq r} \frac{n_j/2}{\nu_j} < p < 1 + \max_{1\leq j\leq r} \frac{n_j/2}{\nu_j - \frac{m_j}{2}}$ or $1 + \min_{1\leq j\leq r} \frac{\nu_j - \frac{m_j}{2}}{n_j/2} < p < 1 + \min_{1\leq j\leq r} \frac{\nu_j}{n_j/2}$ (which are not always empty intervals), we do not know whether there is L^p_ν-boundedness.

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