

Integral pinching results for manifolds with boundary

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Abstract. We prove that some Riemannian manifolds with boundary satisfying an explicit integral pinching condition are spherical space-forms. More precisely, we show that three-dimensional Riemannian manifolds with totally geodesic boundary, positive scalar curvature and an explicit integral pinching between the L^2 -norm of the scalar curvature and the L^2 -norm of the Ricci tensor are spherical space-forms with totally geodesic boundary. Moreover, we also prove that four-dimensional Riemannian manifolds with umbilic boundary, positive Yamabe invariant and an explicit integral pinching between the total integral of the (Q, T) -curvature and the L^2 -norm of the Weyl curvature are spherical space-forms with totally geodesic boundary. As a consequence, we show that a certain conformally invariant operator, which plays an important role in Conformal Geometry, is non-negative and has trivial kernel if the Yamabe invariant is positive and verifies a pinching condition together with the total integral of the (Q, T) -curvature. As an application of the latter spectral analysis, we show the existence of conformal metrics with constant Q -curvature, constant T -curvature, and zero mean curvature under the latter assumptions.

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1. Introduction

One of the most important questions about the relation between algebraic properties of the full curvature tensor and the topology of manifolds is under which conditions on its curvature tensor a Riemannian manifold is homeomorphic or diffeomorphic to a space of constant sectional curvature, namely a space form. A model example is the classical *sphere theorem* conjectured by Rauch [35], and which says that any closed, simply connected and $\frac{1}{4}$ -pinched Riemannian manifold is diffeomorphic to the standard sphere. The topological version was proved by Berger [5] and Klingenberg [27]. Just recently the original conjecture has been settled by Brendle and Schoen [6], using a result of Böhm and Wilking [7].

On the other hand, many sphere like theorems appeared in the literature in the last 30 years in connection to the celebrated Ricci flow. Just to mention some of

them which are related to our results, we start by recalling the pioneering work of R. Hamilton [22]. Using the Ricci flow, he proved the following theorem.

Theorem 1.1 (Hamilton). *If (M, g) is a closed three-dimensional Riemannian manifold with positive Ricci curvature, then M is diffeomorphic to a spherical space form, i.e. M admits a metric with constant positive sectional curvature.*

Later C. Margerin [30] proved an optimal curvature characterization of the smooth 4-sphere. We recall Margerin's theorem in a form where the optimality issue is not apparent, but enough for the link with our work. We define the weak pinching quantity

$$WP_g = \frac{|W_g|_g^2 + 2|E_g|_g^2}{R_g^2},$$

where W_g denoting the Weyl tensor, E_g the trace-free Ricci tensor and $|\cdot|_g$ the usual norm of a tensor with respect to the metric g . Here is the result.

Theorem 1.2 (Margerin). *Let (M, g) be a closed four-dimensional Riemannian manifold with positive scalar curvature. If the pinching condition $WP_g < \frac{1}{6}$ is satisfied, then M is diffeomorphic to a spherical space form. Moreover, we get that the manifold M is diffeomorphic to S^4 or $\mathbb{R}P^4$.*

Much later, Chang, Gursky and Yang [15] proved a remarkable improvement of Margerin's theorem with assumptions which are in integral form, and conformally invariant too.

Theorem 1.3 (Chang-Gursky-Yang). *Let (M, g) be a closed four-dimensional Riemannian manifold with positive Yamabe invariant. If the curvatures satisfy*

$$\int_M \left(|W_g|_g^2 + 2|E_g|_g^2 - \frac{1}{6}R_g^2 \right) dV_g < 0,$$

then M is diffeomorphic to a spherical space form. Moreover, we get that the manifold M is diffeomorphic to S^4 or $\mathbb{R}P^4$.

Notice that the integral pinching condition can be written in the following form (for the definition of Q_g , see below)

$$\int_M Q_g dV_g > \frac{1}{8} \int_M |W_g|_g^2 dV_g.$$

Recently, the first author and Z. Djadli [9] proved an integral pinching theorem in dimension three.

Theorem 1.4 (Catino-Djadli). *Let (M, g) be a closed three-dimensional Riemannian manifold with positive scalar curvature. If*

$$\int_M |\text{Ric}_g|_g^2 dV \leq \frac{3}{8} \int_M R_g^2 dV_g,$$

then M is diffeomorphic to a spherical space form.

A slightly weaker version of this result was also obtained by Y. Ge, C.-S. Lin and G. Wang [23].

On the other hand, the Ricci flow techniques have also been used to get sphere like theorems for manifolds with boundary. An example which is of interest to us is the following result of Shen [37].

Theorem 1.5 (Shen). *If (M, g) is a compact three-dimensional Riemannian manifold with totally geodesic boundary and positive Ricci curvature, then M admits a metric with constant positive sectional curvatures in the interior and totally geodesic boundary.*

Using the Ricci flow for manifolds with boundary defined by Shen [37], a very easy adaptation of the arguments of Margerin [30], immediately yields the following theorem:

Theorem 1.6. *Let (M, g) be a compact four-dimensional Riemannian manifold with totally geodesic boundary and positive scalar curvature. If the pinching condition $WP_g < \frac{1}{6}$ is satisfied, then M admits a metric with constant positive sectional curvatures in the interior and totally geodesic boundary.*

Our goal in this paper is to provide counterparts of the results of Chang-Gursky-Yang and Catino-Djadli for manifolds with boundary. The first result we will prove is the following:

Theorem 1.7. *Let (M, g) be a compact three-dimensional Riemannian manifold with totally geodesic boundary and positive scalar curvature. If*

$$\int_M |\text{Ric}_g|_g^2 dV_g \leq \frac{3}{8} \int_M R_g^2 dV_g,$$

then M admits a metric with constant positive sectional curvatures in the interior and totally geodesic boundary.

In order to state our second result on four manifolds with boundary, we need to recall some notions from Conformal Geometry. We start by recalling the Paneitz operator and its associated curvature invariant called Q -curvature. In 1983, Paneitz has discovered a conformally covariant differential operator on four dimensional compact smooth Riemannian manifolds with smooth boundary (M, g) (see [33]). To this operator, Branson [4] has associated a natural curvature invariant called Q -curvature. They are defined in terms of Ricci tensor Ric_g and scalar curvature R_g of the manifold (M, g) as follows

$$P_g^4 \varphi = \Delta_g^2 \varphi + \text{div}_g \left(\left(\frac{2}{3} R_g g - 2 \text{Ric}_g \right) d\varphi \right),$$

$$Q_g = -\frac{1}{12} \left(\Delta_g R_g - R_g^2 + 3 |\text{Ric}_g|^2 \right),$$

where φ is any smooth function on M , div_g is the divergence and d is the de Rham differential.

Similarly, Chang and Qing [13], have discovered a boundary operator P_g^3 defined on the boundary of compact four dimensional smooth Riemannian manifolds and a natural third-order curvature T_g associated to P_g^3 as follows

$$\begin{aligned}
 P_g^3 \varphi &= \frac{1}{2} \frac{\partial \Delta_g \varphi}{\partial n_g} + \Delta_{\hat{g}} \frac{\partial \varphi}{\partial n_g} - 2H_g \Delta_{\hat{g}} \varphi + (L_g)_{ab} (\nabla_{\hat{g}} \varphi)_a (\nabla_{\hat{g}} \varphi)_b + \nabla_{\hat{g}} H_g \cdot \nabla_{\hat{g}} \varphi \\
 &\quad + \left(F - \frac{R_g}{3} \right) \frac{\partial \varphi}{\partial n_g}. \\
 T_g &= -\frac{1}{12} \frac{\partial R_g}{\partial n_g} + \frac{1}{2} R_g H_g - \langle G_g, L_g \rangle + 3H_g^3 - \frac{1}{3} \text{Tr}(L^3) + \Delta_{\hat{g}} H_g,
 \end{aligned}$$

where φ is any smooth function on M , \hat{g} is the metric induced by g on ∂M , $L_g = (L_g)_{ab} = -\frac{1}{2} \frac{\partial g_{ab}}{\partial n_g}$ is the second fundamental form of ∂M , $H_g = \frac{1}{3} \text{tr}(L_g) = \frac{1}{3} g^{ab} L_{ab}$ ($g^{a,b}$ are the entries of the inverse g^{-1} of the metric g) is the mean curvature of ∂M , R_{bcd}^k is the Riemann curvature tensor $F = R_{nan}^a$, $R_{abcd} = g_{ak} R_{bcd}^k$ ($g_{a,k}$ are the entries of the metric g) and $\langle G_g, L_g \rangle = R_{anbn} (L_g)_{ab}$, $\frac{\partial}{\partial n_g}$ is the inward normal derivative with respect to g . We recall that (M, g) has umbilic boundary if $L_g = \lambda g$ for some constant λ . If $L_g = 0$ we say that the boundary is totally geodesic.

A remarkable property of the couple of operators (P_g^4, P_g^3) is that, as the couple Laplace-Beltrami operator and Neumann operator governs the transformation law of the Gauss curvature and the geodesic curvature on compact surfaces with boundary under conformal change of metric, (P_g^4, P_g^3) does the same for (Q_g, T_g) on compact four dimensional smooth Riemannian manifolds with boundary. In fact, after a conformal change of metric $g_u = e^{2u} g$ we have that

$$\begin{cases} P_{g_u}^4 = e^{-4u} P_g^4, \\ P_{g_u}^3 = e^{-3u} P_g^3; \end{cases} \quad \text{and} \quad \begin{cases} P_g^4 u + 2Q_g = 2Q_{g_u} e^{4u} & \text{in } M \\ P_g^3 u + T_g = T_{g_u} e^{3u} & \text{on } \partial M. \end{cases} \quad (1.1)$$

Another very important role played by the couple of curvatures (Q_g, T_g) in Conformal Geometry is that they arise in the well-known Gauss-Bonnet-Chern formula. More precisely

$$\int_M \left(Q_g + \frac{|W_g|^2}{8} \right) dV_g + \oint_{\partial M} (T + Z) dS_g = 4\pi^2 \chi(M) \quad (1.2)$$

where W_g and $Z dS_g$ (for the definition of Z see [13]) are pointwise conformally invariant. Moreover, it turns out that Z vanishes when the boundary is totally geodesic. Setting

$$\kappa_{P_g^4} = \int_M Q_g dV_g, \quad \kappa_{P_g^3} = \oint_{\partial M} T_g dS_g,$$

from (1.2), thanks to the fact that $W_g dV_g$ and ZdS_g are pointwise conformally invariant, we have that $\kappa_{P_g^4} + \kappa_{P_g^3}$ is conformally invariant, and will be denoted by

$$\kappa_{(P^4, P^3)} = \kappa_{P_g^4} + \kappa_{P_g^3}. \tag{1.3}$$

In addition to the conformally invariant quantity $\kappa_{(P^4, P^3)}$ of a compact four-dimensional Riemannian manifold with boundary, there exists also the Yamabe invariant of the conformal class $[g] = \{\tilde{g} = e^{2u}g, u \in C^\infty(M)\}$ defined by the

$$Y(M, \partial M, [g]) = \inf_{\tilde{g} \in [g], \text{vol}_{\tilde{g}}=1} \int_M R_{\tilde{g}} dV_{\tilde{g}} + \oint_{\partial M} H_{\tilde{g}} dS_{\tilde{g}}. \tag{1.4}$$

We recall that this invariant is defined for every compact Riemannian manifold with boundary of dimension greater or equal to 3.

Now we are ready to state our result on four manifolds with boundary.

Theorem 1.8. *Let (M, g) be a compact four-dimensional Riemannian manifold with umbilic boundary. If $Y(M, \partial M, [g]) > 0$ and if $\kappa_{(P^4, P^3)} > \frac{1}{8} \int_M |W_g|^2 dV_g$, then M admits a metric with constant positive sectional curvatures in the interior and totally geodesic boundary.*

The couple (P_g^4, P_g^3) gives rise to an operator defined on $H_{\frac{\partial}{\partial n}} = \left\{ u \in H^2(M) : \frac{\partial u}{\partial n_g} = 0 \right\}$ whose spectral property is very important for uniformization problems on four manifolds with boundary. The latter operator that we denote by $P_g^{4,3}$ is defined as follows

$$\begin{aligned} \left\langle P_g^{4,3} u, v \right\rangle_{L^2(M)} &= \int_M \left(\Delta_g u \Delta_g v + \frac{2}{3} R_g \nabla_g u \nabla_g v \right) dV_g \\ &\quad - 2 \int_M \text{Ric}_g(\nabla_g u, \nabla_g v) dV_g - 2 \oint_{\partial M} L_g(\nabla_{\hat{g}} u, \nabla_{\hat{g}} v) dS_g, \end{aligned}$$

for every $u, v \in H_{\frac{\partial}{\partial n}}$.

As a byproduct of our analysis, we obtain the following spectral property for $P_g^{4,3}$.

Theorem 1.9. *Let (M, g) be a compact four-dimensional Riemannian manifold with umbilic boundary. Assuming $Y(M, \partial M, [g]) > 0$ and $\kappa_{(P^4, P^3)} + \frac{1}{6} Y(M, \partial M, [g])^2 > 0$, then $P_g^{4,3}$ is non-negative and $\ker P_g^{4,3} \simeq \mathbb{R}$.*

A direct consequence of Theorem 1.9 is the existence of constant Q -curvature and constant T -curvature conformal metrics on four-manifolds which verify the assumptions of Theorem 1.9

Corollary 1.10. *Let (M, g) be a compact four-dimensional Riemannian manifold with umbilic boundary. Assuming $Y(M, \partial M, [g]) > 0$ and $\kappa_{(p^4, p^3)} + \frac{1}{6}Y(M, \partial M, [g])^2 > 0$, then M carries a metric conformal to g with constant Q -curvature, constant T -curvature and zero mean curvature.*

Proofs of Theorem 1.7 and Theorem 1.8 rely on the solution of some boundary value problems for fully nonlinear equations. Following [26] we will use the continuity method proving a priori estimates on the solutions to our equations. As a consequence of our work in dimension four, analysing the spectral property of a certain operator, we will show that this operator is non-negative and with trivial kernel (Theorem 1.9). As a byproduct, we will prove then existence of conformal metrics with constant Q -curvature, constant T -curvature and zero mean curvature under certain conformally invariant assumptions (Corollary 1.10).

The plan of the paper is the following: in Section 2 we will introduce some notations, set up the boundary value problem; in Section 3 and 4 we will prove Theorem 1.7 and Theorem 1.8 on three and four manifolds respectively; finally Section 5 will be devote to the proof of Theorem 1.9 and Corollary 1.10.

2. Preliminaries and notation

In this section, we give some notation and preliminaries like the notion of k -th symmetric elementary functions and some of their properties, the notion of σ_k -curvature of a Riemannian manifold, and some Moser-Trudinger type inequalities. For this end, let (M, g) be a compact, smooth, n -dimensional Riemannian manifold with boundary. We will denote by ν_g the inner normal vector field with respect to the metric g and by $\partial_\nu = \frac{\partial}{\partial n_g}$ the inward normal derivative. Moreover L_g and H_g will be the second fundamental form

$$L_{g,ab} = -\frac{1}{2} \frac{\partial g_{ab}}{\partial n_g},$$

and the mean curvature normalized, *i.e.*

$$H_g = \frac{1}{n-1} g^{ab} L_{g,ab}.$$

Given a section A of the bundle of symmetric 2-tensors, we can use the metric to raise an index and view A as a tensor of type $(1, 1)$, or equivalently as a section of $\text{End}(TM)$. This allows us to define $\sigma_k(g^{-1}A)$ the k -th elementary function of the eigenvalues of $g^{-1}A$. More precisely we define:

Definition 2.1. Let $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. We view the k -th elementary symmetric function as a function on \mathbb{R}^n :

$$\sigma_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

and we define

$$\Gamma_k^+ = \bigcap_{1 \leq j \leq k} \{\sigma_j(\lambda_1, \dots, \lambda_n) > 0\} \subset \mathbb{R}^n.$$

For a symmetric linear transformation $A : V \rightarrow V$, where V is an n -dimensional inner product space, the notation $A \in \Gamma_k^+$ will mean that the eigenvalues of A lie in the corresponding set. We note that this notation also makes sense for a symmetric 2-tensor on a Riemannian manifold. If $A \in \Gamma_k^+$, let $\sigma_k^{1/k}(A) = \{\sigma_k(A)\}^{1/k}$.

Definition 2.2. Let $A : V \rightarrow V$, where V is an n -dimensional inner product space. The $(k - 1)$ -th Newton transformation associated with A is

$$T_{(k-1)}(A) = \sum_{j=0}^{k-1} (-1)^{k-1-j} \sigma_j(A) A^{k-1-j}.$$

Also, for $t \in \mathbb{R}$ we define the linear transformation

$$L^t(A) = T_{(k-1)}(A) + \frac{1-t}{n-2} \sigma_1(T_{(k-1)}(A)) \cdot I.$$

We have the following list of properties (the proofs can be found in [8])

Lemma 2.3.

- (i) Γ_k^+ is an open convex cone with vertex at the origin, and we have the following sequence of inclusions

$$\Gamma_n^+ \subset \Gamma_{n-1}^+ \subset \dots \subset \Gamma_1^+.$$

- (ii) If $A \in \Gamma_k^+$, then $T_{k-1}(A)$ is positive definite. Hence for all $t \leq 1$, $L^t(A)$ is positive definite.
- (iii) We have the identities

$$T_{k-1}(A)^{ij} A_{ij} = k \sigma_k(A),$$

$$T_{k-1}(A)^{ll} = (n - k + 1) \sigma_{k-1}(A).$$

- (iv) If $A \in \Gamma_k^+$, then

$$\sigma_{k-1}(A) \geq \frac{k}{n-k+1} \binom{n}{k}^{\frac{1}{k}} \sigma_k(A)^{\frac{(k-1)}{k}}.$$

- (v) If A and B are symmetric linear transformations, $A, B \in \Gamma_k^+$, then $\forall \rho \in [0, 1]$, $\rho A + (1 - \rho)B \in \Gamma_k^+$, and

$$\sigma_k^{\frac{1}{k}}(\rho A + (1 - \rho)B) \geq \rho \sigma_k^{\frac{1}{k}}(A) + (1 - \rho) \sigma_k^{\frac{1}{k}}(B).$$

In particular this gives the concavity of the function $\sigma_k^{\frac{1}{k}}$ in the cone Γ_k^+ .

Next we give a lemma about the variation of the σ_k functional.

Lemma 2.4. *If $A : \mathbb{R} \rightarrow \text{Hom}(V, V)$, then*

$$\frac{d}{ds} \sigma_k(A)(s) = \sum_{i,j} T_{(k-1)}(A)_{ij}(s) \frac{d}{ds} (A)_{ij}(s),$$

i.e., the $(k - 1)$ -th Newton transformation is what arises when we differentiate σ_k .

We choose the tensor (here t is a real number)

$$A_g^t = \frac{1}{n-2} \left(\text{Ric}_g - \frac{t}{2(n-1)} R_g g \right),$$

where Ric_g and R_g denote the Ricci and the scalar curvature of g respectively. Note that for $t = 1$, A_g^1 is the classical Schouten tensor, namely $A_g^1 = A_g := \frac{1}{n-2}(\text{Ric}_g - \frac{1}{2(n-1)}R_g g)$, see [1]. Hence, with our notations, $\sigma_k(g^{-1}A_g^t)$ denotes the k -th elementary symmetric function of the eigenvalues of $g^{-1}A_g^t$.

Now, we give a lemma which shows that metrics g_1 , such that $A_{g_1}^t$ belong to the positive cone of order 2, verify also additional pointwise algebraic inequalities. More precisely, we have:

Lemma 2.5. *If for some metric g_1 on M we have $A_{g_1}^t \in \Gamma_2^+$, then*

$$\begin{aligned} -A_{g_1}^t + \sigma_1(g_1^{-1}A_{g_1}^t)g_1 &> 0, \\ A_{g_1}^t + \frac{n-2}{n}\sigma_1(g_1^{-1}A_{g_1}^t)g_1 &> 0. \end{aligned}$$

We will be concerned with the following equation for a conformal metric $\tilde{g} = e^{-2u}g$:

$$\begin{cases} \sigma_k^{1/k}(g^{-1}A_u^t) = fe^{2u} & \text{in } M, \\ \partial_\nu u = 0 & \text{on } \partial M. \end{cases} \tag{2.1}$$

Where f is a positive function on M . Let $\sigma_1(g^{-1}A_g^1)$ be the trace of A_g^1 with respect to the metric g . We have the following formula for the transformation of A_g^t under this conformal change of metric:

$$A_{\tilde{g}}^t = A_g^t + \nabla_g^2 u + \frac{1-t}{n-2}(\Delta_g u)g + du \otimes du - \frac{2-t}{2}|\nabla_g u|_g^2 g. \tag{2.2}$$

Since

$$A_g^t = A_g^1 + \frac{1-t}{n-2}\sigma_1(g^{-1}A_g^1)g,$$

this formula follows easily from the standard formula for the transformation of the Schouten tensor [36]:

$$A_{\tilde{g}}^1 = A_g^1 + \nabla_g^2 u + du \otimes du - \frac{1}{2} |\nabla_g u|_g^2 g. \tag{2.3}$$

Using this formula we may write (2.1) with respect to the background metric g

$$\sigma_k \left(g^{-1} \left(A_g^t + \nabla_g^2 u + \frac{1-t}{n-2} (\Delta_g u) g + du \otimes du - \frac{2-t}{2} |\nabla_g u|_g^2 g \right) \right)^{1/k} = f(x) e^{2u}.$$

Now, we discuss the ellipticity properties of equation (2.1).

Proposition 2.6 (Ellipticity property). *Let $u \in C^2(M)$ be a solution of equation (2.1) for some $t \leq 1$ and let $\tilde{g} = e^{-2u} g$. Assume that $A_{\tilde{g}}^t \in \Gamma_k^+$. Then the linearized operator at u , $\mathcal{L}^t : C^{2,\alpha}(M) \cap \{\partial_\nu u = 0 \text{ on } \partial M\} \rightarrow C^\alpha(M)$, is elliptic and invertible ($0 < \alpha < 1$).*

Proof. Define the operator

$$F_t[u, \nabla_g u, \nabla_g^2 u] = \sigma_k(g^{-1} A_{\tilde{g}}^t) - f(x)^k e^{2ku},$$

so that solutions of the equation (2.1) are exactly the zeroes of F_t . Define the function $u_s = u + s\varphi$, then the linearization at u of the operator F_t is defined by

$$\begin{aligned} \mathcal{L}^t(\varphi) &= \frac{d}{ds} F_t [u_s, \nabla_g u_s, \nabla_g^2 u_s] \Big|_{s=0} \\ &= \frac{d}{ds} \left(\sigma_k \left(g^{-1} A_{\tilde{g}}^t \right) \right) \Big|_{s=0} - \frac{d}{ds} \left(f(x)^k e^{2ku_s} \right) \Big|_{s=0}. \end{aligned}$$

From Lemma 2.4 we have

$$\frac{d}{ds} \left(\sigma_k \left(g^{-1} A_{\tilde{g}}^t \right) \right) \Big|_{s=0} = T_{k-1} \left(g^{-1} A_{\tilde{g}}^t \right)_{ij} \frac{d}{ds} \left(\left(A_{\tilde{g}}^t \right)_{ij} \right) \Big|_{s=0}.$$

We compute

$$\frac{d}{ds} \left(\left(A_{\tilde{g}}^t \right)_{ij} \right) \Big|_{s=0} = (\nabla_g^2 \varphi)_{ij} + \frac{1-t}{n-2} (\Delta_g \varphi) g_{ij} - (2-t) \nabla_g u \cdot \nabla_g \varphi g_{ij} + 2du \otimes d\varphi.$$

Easily we have also

$$\frac{d}{ds} \left(f(x)^k e^{2ku_s} \right) \Big|_{s=0} = 2kf(x)^k e^{2ku} \varphi.$$

Putting all together, we conclude

$$\mathcal{L}^t(\varphi) = T_{k-1} \left(g^{-1} A_{\tilde{g}}^t \right)_{ij} \left(\left(\nabla_g^2 \varphi \right)_{ij} + \frac{1-t}{n-2} (\Delta_g \varphi) g_{ij} \right) - 2kf(x)^k e^{2ku} \varphi + \dots$$

where the last terms denote additional ones which are linear in $\nabla_g \varphi$. The first term of the linearization is exactly the one defined in Definition 2.2, *i.e.*

$$L^t \left(A_{\tilde{g}}^t \right)_{ij} = T_{k-1} \left(A_{\tilde{g}}^t \right)_{ij} + \frac{1-t}{n-2} T_{k-1} \left(A_{\tilde{g}}^t \right)_{pp} \delta_{ij} .$$

So finally, we have

$$\mathcal{L}^t(\varphi) = L^t \left(A_{\tilde{g}}^t \right)_{ij} \left(\nabla_g^2 \varphi \right)_{ij} - 2kf(x)^k e^{2ku} \varphi + \dots$$

Since $A_{\tilde{g}}^t \in \Gamma_k^+$, by Lemma 2.3, we have that the tensor $L^t(A_{\tilde{g}}^t)$ is positive definite. So, the linearized operator at any solution u must be elliptic. Note also that, by the previous formula, the operator is of the form

$$\mathcal{L}^t(\varphi) = E(\varphi) - c(x)\varphi ,$$

where $E(\varphi)$ is a second order linear elliptic operator and $c(x)$ is a strictly positive function on M , since $c(x) = 2kf(x)^k e^{2ku}$ and $f(x) > 0$. This allows us to invert this operator between the Hölder spaces $C^{2,\alpha}(M) \cap \{\partial_\nu u = 0 \text{ on } \partial M\}$ and $C^\alpha(M)$ (see for instance [24]). □

Next, we recall some Moser-Trudinger type inequalities which will be used to prove Corollary 1.10.

Proposition 2.7. *Assume (M, g) is a compact four-dimensional Riemannian manifold with boundary such that $P_g^{4,3}$ is a non-negative operator with $\text{Ker} P_g^{4,3} \simeq \mathbb{R}$. Then we have that for all $\alpha < 16\pi^2$ there exists a constant $C = C(M, g, \alpha)$ such that*

$$\int_M e^{\frac{\alpha(u-\bar{u})^2}{\langle P_g^{4,3} u, u \rangle_{L^2(M)}}} dV_g \leq C,$$

for all $u \in H_{\frac{\partial}{\partial n}}$, and hence

$$\log \int_M e^{4(u-\bar{u})} \leq C + \frac{4}{\alpha} \langle P_g^{4,3} u, u \rangle_{L^2(M)} \quad \forall u \in H_{\frac{\partial}{\partial n}},$$

where $\bar{u} = \frac{1}{\text{Vol}_g(M)} \int_M u dV_g$, and $\text{Vol}_g(M) = \int_M dV_g$.

The latter proposition can be found in [31] together with its proof. The second inequality that we are going to state is a trace analogue of the previous one. Its proof can be found [32].

Proposition 2.8. *Assume $P_g^{4,3}$ is a non-negative operator with $\text{Ker} P_g^{4,3} \simeq \mathbb{R}$. Then we have that for all $\alpha < 12\pi^2$ there exists a constant $C = C(M, g, \alpha)$ such that*

$$\oint_{\partial M} e^{\frac{\alpha(u-\bar{u}_{\partial M})^2}{\langle P_g^{4,3} u, u \rangle_{L^2(M,g)}}} dS_g \leq C, \tag{2.4}$$

for all $u \in H_{\frac{\partial}{\partial n}}$, and hence

$$\log \oint_{\partial M} e^{3(u-\bar{u}_{\partial M})} dS_g \leq C + \frac{9}{4\alpha} \left\langle P_g^{4,3} u, u \right\rangle_{L^2(M,g)} \quad \forall u \in H_{\frac{\partial}{\partial n}}. \tag{2.5}$$

where $\bar{u}_{\partial M} = \frac{1}{\text{Vol}_g(\partial M)} \oint_{\partial M} u dS_g$, and $\text{Vol}_g(\partial M) = \oint_{\partial M} dS_g$.

Now, we give a lemma (whose proof can be found in [31]) which will be used together with the above Moser-Trudinger type inequalities in order to prove Corollary 1.10. It says that under the assumptions $\text{Ker} P_g^{4,3} \simeq \mathbb{R}$ and $P_g^{4,3}$ non-negative, the map

$$u \in H_{\frac{\partial}{\partial n}} \longrightarrow \|u\|_{P_g^{4,3}} = \left\langle P_g^{4,3} u, u \right\rangle_{L^2(M)}^{\frac{1}{2}}$$

induces an equivalent norm to the standard norm of $H^2(M)$ on $\{u \in H_{\frac{\partial}{\partial n}} \mid \bar{u} = 0\}$. More precisely we have the following:

Lemma 2.9. *Suppose $\text{Ker} P_g^{4,3} \simeq \mathbb{R}$ and $P_g^{4,3}$ non-negative then we have that $\|\cdot\|_{P_g^{4,3}}$ is an equivalent norm to $\|\cdot\|_{H^2}$ on $\{u \in H_{\frac{\partial}{\partial n}} \mid \bar{u} = 0\}$.*

Now we give a technical lemma which will be used to prove the above theorems.

Lemma 2.10. *Let (M, g) be a compact n -dimensional Riemannian manifold with totally geodesic boundary. Assuming $u \in C^2(M)$ with $\frac{\partial u}{\partial n_g} = 0$, then*

$$\frac{\partial |\nabla_g u|_g^2}{\partial n_g} = 0,$$

and

$$A_g(v, \nabla_g u) = 0.$$

Proof. First of all, using the fact that $\frac{\partial u}{\partial n_g} = 0$, we derive

$$|\nabla_g u|^2 = g^{ab} \partial_a u \partial_b u.$$

Thus we infer

$$\frac{\partial (|\nabla_g u|^2)}{\partial n_g} = \frac{\partial g^{ab}}{\partial n_g} \partial_a u \partial_b u + 2g^{ab} \frac{\partial (\partial_a u)}{\partial n_g} \partial_b u.$$

Next, using the fact that $L_g = 0$, one has $\frac{\partial g^{ab}}{\partial n_g} = 0$. Moreover from the trivial identity $\frac{\partial (\partial_a u)}{\partial n_g} = \partial_a \left(\frac{\partial u}{\partial n_g} \right)$, we infer

$$\frac{\partial (\partial_a u)}{\partial n_g} = 0.$$

Thus, we obtain

$$\frac{\partial (|\nabla_g u|^2)}{\partial n_g} = 0.$$

This prove the first point. For the second one, we have

$$A_g(v, \nabla_g u) = \frac{1}{n-2} \left(\text{Ric}_g(v, \nabla_g u) - \frac{1}{2(n-1)} R_g \frac{\partial u}{\partial n_g} \right).$$

Thus, we get

$$A_g(v, \nabla_g u) = \frac{1}{n-2} \text{Ric}_{v,a} \partial_a u.$$

Now using the Codazzi-Mainardi equation, we get

$$\text{Ric}_{v,a} = \nabla_b L_{g,ab} - \nabla_a H_g = 0.$$

So, we obtain.

$$A_g(v, \nabla_g u) = 0.$$

This completes the proof of the lemma. □

3. Three manifolds with boundary

In this section, we present the proof of Theorem 1.7. We will prove a more general theorem so that Theorem 1.7 will be a direct corollary. In fact, we have:

Theorem 3.1. *Let (M, g) be a compact three-dimensional Riemannian manifold with totally geodesic boundary and positive scalar curvature. There exists a positive constant $C = C(\text{diam}(M, g), \|\nabla^2 Rm\|)$ such that if*

$$\int_M \sigma_2(g^{-1} A_g^1) dV_g + C \left(\frac{7}{10} - t_0 \right) Y(M, [g])^2 > 0,$$

for some $t_0 \leq 2/3$, then there exists a conformal metric $\tilde{g} = e^{-2u} g$ with $R_{\tilde{g}} > 0$, $\sigma_2(g^{-1} A_{\tilde{g}}^{t_0}) > 0$ pointwise and totally geodesic boundary. Moreover, we have the inequalities

$$(3t_0 - 2)R_{\tilde{g}}\tilde{g} < 6\text{Ric}_{\tilde{g}} < 3(2 - t_0)R_{\tilde{g}}\tilde{g}. \tag{3.1}$$

Throughout the sequel, (M, g) will be a compact 3-dimensional Riemannian manifold with totally geodesic boundary and with positive scalar curvature. Since M is compact and $R_g > 0$, there exists $t_0 > \delta > -\infty$ such that A_g^δ is positive definite (i.e. $\text{Ric}_g - \frac{\delta}{4}R_g g > 0$ on M). Note that δ only depends on $\|Rm_g\|$.

For $t \in [\delta, t_0]$, consider the path of equations (in the sequel we use the notation $A^t_{u_t} := A^t_{g_t}$ for g_t given by $g_t = e^{-2u_t}g$)

$$\begin{cases} \sigma_2^{1/2}(g^{-1}A^t_{u_t}) = fe^{2u_t} & \text{in } M, \\ \partial_\nu u_t = 0 & \text{on } \partial M. \end{cases} \tag{3.2}$$

where $f = \sigma_2^{1/2}(g^{-1}A^\delta_g) > 0$. Note that $u \equiv 0$ is a solution for $t = \delta$.

We use the continuity method. Define

$$\mathcal{S} = \left\{ t \in [\delta, t_0] \mid \exists \text{ a solution } u_t \in C^{2,\alpha}(M) \text{ of (3.2) with } A^t_{u_t} \in \Gamma_2^+ \right\}.$$

Clearly, with our choice of f , $u \equiv 0$ is a solution for $t = \delta$. Since A^δ_g is positive definite, then $\delta \in \mathcal{S}$. Hence $\mathcal{S} \neq \emptyset$. Let $t \in \mathcal{S}$, and u_t be a solution. By Proposition 2.6, the linearized operator at u_t , $\mathcal{L}^t : C^{2,\alpha}(M) \cap \{\partial_\nu u = 0 \text{ on } \partial M\} \rightarrow C^\alpha(M)$, is invertible. The implicit function theorem tells us that \mathcal{S} is open. To prove that \mathcal{S} is close we need to establish a priori $C^{2,\alpha}$ estimates for solutions of the equation (3.2). To do this, we start by proving an upper bound estimate for solutions of (3.2).

Proposition 3.2 (Upper bound). *Let $u_t \in C^2(M)$ be a solution of (3.2) for some $t \in [\delta, t_0]$. If $g_t = e^{-2u_t}g \in \Gamma_2^+$, then $u_t \leq \bar{\delta}$, where $\bar{\delta}$ depends only on $\|Rm_g\|$.*

Proof. From Lemma 2.3 (iv), we have $\sqrt{3}\sigma_2^{1/2} \leq \sigma_1$, so for all $p \in M$

$$\sqrt{3}fe^{2u_t} \leq \sigma_1(g^{-1}A^t_{u_t}).$$

Let $p \in M$ be a maximum of u_t . Since the gradient terms vanish at p (this is true also if $p \in \partial M$, since $\partial_\nu u_t = 0$ on ∂M) we have $(\Delta u_t)(p) \leq 0$. Then, using (2.2), we have

$$\begin{aligned} \sqrt{3}f(p)e^{2u_t(p)} &\leq \sigma_1(g^{-1}A^t_{u_t})(p) \\ &= \sigma_1(g^{-1}A^t_g)(p) + (4 - 3t)(\Delta u_t)(p) \\ &\leq \sigma_1(g^{-1}A^t_g)(p) \\ &\leq \sigma_1(g^{-1}A^\delta_g)(p). \end{aligned}$$

Since M is compact, we have $u_t \leq \bar{\delta}$, for some $\bar{\delta}$ depending only on $\|Rm_g\|$. □

Next, we are going to show that solutions of (3.2) which verify upper-bound estimates enjoy also gradient ones:

Proposition 3.3 (Gradient estimate). *Let $u_t \in C^3(M)$ be a solution of (3.2) for some $\delta \leq t \leq t_0$. Assume that $u_t \leq \bar{\delta}$. Then $\|\nabla_g u\|_{g,\infty} < C_1$, where C_1 depends only on $\|\nabla Rm_g\|$ and $\bar{\delta}$.*

Proof. Let $H := |\nabla_g u|_g^2$. If the maximum of H is in the interior, then $\nabla_g H = 0$ and $\nabla_g^2 H$ is negative semi-definite. If the maximum of H is at the boundary, then by Lemma 2.10, $\frac{\partial H}{\partial n_g} = 0$. Thus, we also have that $\nabla_g H = 0$ and $\nabla_g^2 H$ is negative semi-definite. Interior gradient estimates for equation (3.2) were proved in [26, Proposition 4.1]. We remark that the same proof works for boundary gradient estimates. The reason is that, as we showed, at the maximal point once we have $\nabla_g H = 0$ and $\nabla_g^2 H$ is negative semi-definite, then the rest of computations in [26] is the same regardless of the point being in the interior or on the boundary. \square

As we proved before, there exist two constants $\bar{\delta}$ and C_1 depending only on $\|\nabla Rm_g\|$ such that all solutions of (3.2) for some $\delta \leq t \leq t_0$, satisfying $u_t \leq \bar{\delta}$ satisfy $\|\nabla_g u\|_\infty < C_1$. Consider now the following quantity:

$$I(M, \partial M, g) := \inf_{g'=e^{-2\varphi}g, |\nabla_g \varphi| \leq C_1, H_{g'}=0} \left(\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right).$$

We let, for $g' = e^{-2\varphi}g$

$$i(g') := \int_M R_{g'}^2 e^{-\varphi} dV_{g'}.$$

As one can easily check, if two metrics g_1 and g_2 are homothetic, then $i(g_1) = i(g_2)$. So, we have

$$I(M, \partial M, g) = \inf_{g'=e^{-2\varphi}g, \text{Vol}'_g(M)=1 \text{ and } |\nabla_g \varphi|_g \leq C_1, H_{g'}=0} \left(\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right).$$

Concerning $I(M, \partial M, g)$, we have the following lemma.

Lemma 3.4. *There exists a positive constant $C = C(\|\nabla Rm_g\|)$ such that*

$$I(M, \partial M, g) \geq C (Y(M, \partial M, [g]))^2.$$

Proof. As we have seen

$$I(M, \partial M, g) = \inf_{g'=e^{-2\varphi}g, \text{Vol}'_g(M)=1 \text{ and } |\nabla_g \varphi|_g \leq C_1, H_{g'}=0} \left(\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \right).$$

Take $\varphi \in C^\infty(M)$ such that, for $g' = e^{-2\varphi}g$, $\text{Vol}'_g(M) = 1$ and such that $|\nabla_g \varphi|_g \leq C_1$ where C_1 is given by Proposition 3.3. Since $\text{Vol}'_g(M) = 1$, if p is a point where φ attains its minimum we have

$$e^{-3\varphi(p)} \text{Vol}_g(M) \geq 1,$$

and then, there exists C_0 depending only on (M, g) such that $\varphi(p) \leq C_0$. Now, using the mean value theorem, it follows since $|\nabla_g \varphi|_g$ is controlled by a constant

depending only on (M, g) , that $\max \varphi \leq C'_0$ where C'_0 depends only on (M, g) . Using this, we clearly have that

$$\int_M R_{g'}^2 e^{-\varphi} dV_{g'} \geq e^{-C'_0} \int_M R_{g'}^2 dV_{g'}.$$

Using Hölder inequality and the definition of the Yamabe invariant, since $H_{g'} = 0$, we get (recall that $\text{Vol}'_g(M) = 1$)

$$\int_M R_{g'}^2 dV_{g'} \geq (Y(M, \partial M, [g]))^2,$$

and then $I(M, \partial M, g) \geq e^{-C'_0} (Y(M, \partial M, [g]))^2$. This ends the proof. □

We will prove a lower bound for a solution to the equation (3.2) following in [9, section 3]. Since we are dealing with manifolds with boundary we have to compute the conformal deformation of the integral of σ_2 in this context. Here is the formula.

Lemma 3.5. *For a conformal metric $\tilde{g} = e^{-2u}g$, we have the following integral transformation*

$$\begin{aligned} \int_M \sigma_2(\tilde{g}^{-1}A_{\tilde{g}}^1)e^{-4u} dV_g &= \int_M \sigma_2(g^{-1}A_g^1) dV_g + \frac{1}{8} \int_M R_g |\nabla_g u|_g^2 dV_g \\ &\quad - \frac{1}{4} \int_M |\nabla_g u|_g^4 dV_g + \frac{1}{2} \int_M \Delta_g u |\nabla_g u|_g^2 dV_g \\ &\quad - \frac{1}{2} \int_M A_g^1(\nabla_g u, \nabla_g u) dV_g \\ &\quad + \frac{1}{4} \oint_{\partial M} \partial_\nu u (R_g + 2\Delta_g u - 2|\nabla_g u|_g^2) dS_g \\ &\quad - \oint_{\partial M} A_g^1(\nu, \nabla_g u) dS_g - \frac{1}{4} \oint_{\partial M} \partial_\nu |\nabla_g u|_g^2 dS_g. \end{aligned}$$

In particular, if the boundary of M is totally geodesic and $\partial_\nu u = 0$, we get

$$\begin{aligned} \int_M \sigma_2(\tilde{g}^{-1}A_{\tilde{g}}^1)e^{-4u} dV_g &= \int_M \sigma_2(g^{-1}A_g^1) dV_g + \frac{1}{8} \int_M R_g |\nabla_g u|_g^2 dV_g \\ &\quad - \frac{1}{4} \int_M |\nabla_g u|_g^4 dV_g + \frac{1}{2} \int_M \Delta_g u |\nabla_g u|_g^2 dV_g \\ &\quad - \frac{1}{2} \int_M A_g^1(\nabla_g u, \nabla_g u) dV_g. \end{aligned}$$

Proof. For the computations, we will follow in [9, Section 3]. The final formula will be the same as in [9], but with some extra terms coming from the boundary.

Denote $\tilde{\sigma}_1 = \sigma_1(\tilde{g}^{-1}A_g^1)$, $\sigma_1 = \sigma_1(g^{-1}A_g^1)$, $\tilde{\sigma}_2 = \sigma_2(\tilde{g}^{-1}A_g^1)$, $\sigma_2 = \sigma_2(g^{-1}A_g^1)$. We have

$$2\tilde{\sigma}_2 = \tilde{\sigma}_1^2 - |A_g^1|_{\tilde{g}}^2.$$

By equation (2.3), we have

$$\tilde{\sigma}_1 e^{-2u} = \sigma_1 + \Delta_g u - \frac{1}{2}|\nabla_g u|_g^2,$$

so

$$\tilde{\sigma}_1^2 e^{-4u} = \sigma_1^2 + (\Delta_g u)^2 + \frac{1}{4}|\nabla_g u|_g^4 + 2\sigma_1 \Delta_g u - \Delta_g u |\nabla_g u|_g^2 - \sigma_1 |\nabla_g u|_g^2.$$

After an easy computation, we get

$$\begin{aligned} |A_g^1|_{\tilde{g}}^2 e^{-4u} &= |A_g^1|_g^2 + |\nabla_g^2 u|_g^2 + \frac{3}{4}|\nabla_g u|_g^4 - \sigma_1 |\nabla_g u|_g^2 - \Delta_g u |\nabla_g u|_g^2 \\ &\quad + 2(A_g^1)_{ij} \nabla_g^{2ij} u + 2(A_g^1)_{ij} \nabla_g^i u \nabla_g^j u + 2\nabla_{gij}^2 u \nabla_g^i u \nabla_g^j u. \end{aligned}$$

Putting all together, we obtain

$$\begin{aligned} 2\tilde{\sigma}_2 e^{-4u} &= 2\sigma_2 + (\Delta_g u)^2 - |\nabla_g^2 u|_g^2 - \frac{1}{2}|\nabla_g u|_g^4 + 2\sigma_1 \Delta_g u \\ &\quad - 2(A_g^1)_{ij} \nabla_g^{2ij} u - 2(A_g^1)_{ij} \nabla_g^i u \nabla_g^j u - 2\nabla_{gij}^2 u \nabla_g^i u \nabla_g^j u. \end{aligned}$$

Now, by simple computation, we have the following identities

$$\begin{aligned} -2 \int_M (A_g^1)_{ij} \nabla_g^{2ij} u \, dV_g &= -2 \int_M \sigma_1 \Delta_g u \, dV_g + 2 \oint_{\partial M} \partial_\nu u \, \sigma_1 \, dS_g \\ &\quad - 2 \oint_{\partial M} A_g^1(\nu, \nabla_g u) \, dS_g, \\ -2 \int_M \nabla_{gij}^2 u \nabla_g^i u \nabla_g^j u \, dV_g &= \int_M \Delta_g u |\nabla_g u|_g^2 \, dV_g - \oint_{\partial M} \partial_\nu u |\nabla_g u|_g^2 \, dS_g, \end{aligned}$$

where we integrated by parts and we used the Schur's lemma,

$$2\nabla_g^j(\text{Ric}_g)_{ij} = \nabla_i R_g,$$

for the first identity. Finally we get

$$\begin{aligned} 2 \int_M \tilde{\sigma}_2 e^{-4u} \, dV_g &= 2 \int_M \sigma_2 \, dV_g \\ &\quad + \int_M \left[(\Delta_g u)^2 - |\nabla_g^2 u|_g^2 - \frac{1}{2}|\nabla_g u|_g^4 + \Delta_g u |\nabla_g u|_g^2 \right. \\ &\quad \left. - 2A_g^1(\nabla_g u, \nabla_g u) \right] \, dV_g \\ &\quad + \oint_{\partial M} \partial_\nu u \left(\frac{1}{2}R_g - 2A_g^1(\nu, \nabla_g u) - |\nabla_g u|_g^2 \right) \, dS_g, \end{aligned}$$

Now, integrating the Bochner formula

$$\frac{1}{2} \Delta_g |\nabla_g u|_g^2 = |\nabla_g^2 u|_g^2 + Ric_g(\nabla_g u, \nabla_g u) dV_g + \nabla_i u, \nabla^i(\Delta_g u),$$

we get

$$\begin{aligned} \frac{1}{2} \oint_{\partial M} \partial_\nu |\nabla_g u|_g^2 dS_g &= \int_M \left[|\nabla_g^2 u|_g^2 - (\Delta_g u)^2 + Ric_g(\nabla_g u, \nabla_g u) \right] dV_g \\ &\quad + \oint_{\partial M} \partial_\nu u \Delta_g u dS_g. \end{aligned}$$

Using the definition of the Schouten tensor A_g^1 , we get the first point of the lemma.

Now, if the boundary is totally geodesic and $\partial_\nu u = 0$ on ∂M , then by Lemma 2.10 we have that all the boundary terms must vanish. Thus the second point of the lemma is proved. This completes the proof. \square

Since (M, g) has totally geodesic boundary, the boundary terms don't effect the conformal transformation of the integral of σ_2 . Hence, following in [9, Section 3] and using Lemma 3.4, we obtain the lower bound.

Proposition 3.6 (Lower Bound). *Assume that for some $t \in [\delta, t_0]$, $t_0 \leq 2/3$, the following estimate holds*

$$\int_M \sigma_2(g^{-1}A_g^1) dV_g + C \left(\frac{7}{10} - t \right) (Y(M, \partial M, [g]))^2 = \mu_t > 0, \tag{3.3}$$

for some C depending only on $\|\nabla Rm_g\|$. Then there exists $\underline{\delta}$ depending only on $\text{diam}_g(M)$ and $\|\nabla Rm_g\|$ such that if $u_t \in C^2(M)$ is a solution of (3.2) and if $A_{u_t}^t \in \Gamma_2^+$ then $u_t \geq \underline{\delta}$.

We have the following $C^{2,\alpha}$ estimate for solutions of the equation (3.2).

Proposition 3.7 ($C^{2,\alpha}$ estimate). *Let $u_t \in C^4(M)$ be a solution of (3.2) for some $\delta \leq t \leq t_0$, $t_0 \leq 2/3$, satisfying $\underline{\delta} < u_t < \bar{\delta}$, and $\|\nabla u_t\|_{g,\infty} < C_1$. Then, if $A_{u_t}^t \in \Gamma_2^+$, for $0 < \alpha < 1$, $\|u_t\|_{C^{2,\alpha}} \leq C_2$, where C_2 depends only on $\underline{\delta}, \bar{\delta}, C_1$ and $\|\nabla^2 Rm_g\|$.*

Proof. The interior C^2 estimate follows from the work of Chen [11] and the boundary C^2 estimate follows from [12, Theorem 6 (b)]. With the C^2 estimate at hand, we obtain high-order estimate (in particular $C^{2,\alpha}$ one) from the works of Evans [19], Krylov [28] and Lions-Trudinger [29]. \square

Since we proved $C^{2,\alpha}$ estimates for solutions of the equation (3.2), by the classical Ascoli-Arzela's Theorem, we have that \mathcal{S} is closed, therefore $\mathcal{S} = [\delta, t_0]$. In particular $t_0 \in \mathcal{S}$. Hence the metric

$\tilde{g} = e^{-2u_{t_0}} g$ then satisfies $\sigma_2\left(A_{\tilde{g}}^{t_0}\right) > 0$, $R_{\tilde{g}} > 0$ and $L_{\tilde{g}} = 0$. Furthermore, by Lemma 2.5 we have that the metric \tilde{g} satisfies

$$(3t_0 - 2)R_{\tilde{g}}\tilde{g} < 6\text{Ric}_{\tilde{g}} < 3(2 - t_0)R_{\tilde{g}}\tilde{g}. \tag{3.4}$$

Hence the proof of Theorem 3.1 is complete.

Now we are going to give the proof of Theorem 1.7.

Proof of Theorem 1.7. First of all from $R_g > 0$ and $L_g = 0$, we infer $Y(M, \partial M, [g]) > 0$. On the other hand, one can easily check that

$$\sigma_2\left(g^{-1}A_g\right) = \frac{3}{16}|R_g|^2 - \frac{1}{2}|\text{Ric}_g|^2.$$

Thus, we have $\int_M \sigma_2(g^{-1}A_g) \geq 0$ is equivalent to $\int_M |\text{Ric}_g|^2 dV_g \leq \frac{3}{8} \int_M |R_g|^2 dV_g$. Hence we can apply Theorem 3.1 with $t_0 = \frac{2}{3}$ and get the existence of a metric \tilde{g} conformal to g such that $\text{Ric}_{\tilde{g}} > 0$ and $L_{\tilde{g}} = 0$. Hence appealing to Theorem 1.5, we have the proof of Theorem 1.7 is complete. \square

4. Four manifolds with boundary

In this section, we give the proof of Theorem 1.8. As for the case of 3-manifolds, we are going to prove a more general theorem from which Theorem 1.8 becomes a direct application.

Theorem 4.1. *Let (M, g) be a compact four-dimensional Riemannian manifold with umbilic boundary and $0 \leq \alpha \leq 1$. If $Y(M, \partial M, [g]) > 0$, and*

$$\frac{1}{2}\kappa_{(P^4, P^3)} - \frac{\alpha}{16} \int_M |W_g|_g^2 dV_g + \frac{1}{24}(1 - t_0)(2 - t_0)Y(M, \partial M, [g])^2 > 0,$$

for some $t_0 \leq 1$, then there exists a conformal metric $\tilde{g} = e^{-2u} g$ whose curvature satisfies

$$R_{\tilde{g}} > 0, \quad \sigma_2(\tilde{g}^{-1}A_{\tilde{g}}^{t_0}) - \frac{\alpha}{16}|W_{\tilde{g}}|_{\tilde{g}}^2 > 0, \quad \text{and} \quad H_{\tilde{g}} = 0.$$

This implies the pointwise inequalities

$$(t_0 - 1)R_{\tilde{g}}\tilde{g} < 2\text{Ric}_{\tilde{g}} < (2 - t_0)R_{\tilde{g}}\tilde{g}.$$

Throughout the sequel, (M, g) will be a compact 4-dimensional Riemannian manifold with umbilic boundary and with positive Yamabe invariant $Y(M, \partial M, [g])$. Since all the hypothesis on the metric g are conformally invariant, then by a result of Escobar, see [18], we can choose in the conformal class the Yamabe metric, *i.e.* a metric with positive constant scalar curvature and zero mean curvature. Moreover, since umbilicity is also conformally invariant, we have that the boundary must be

totally geodesic. Hence, from now on, (M, g) will be a compact four-manifold with totally geodesic boundary, positive constant scalar curvature and satisfying the integral pinching condition.

On the other hand, since M is compact and $R_g > 0$, there exist $t_0 > \delta > -\infty$, $\delta < 0$ such that A_g^δ is positive definite (i.e. $\text{Ric} - \frac{\delta}{6}R > 0$ on M). Moreover we can choose δ so small such that

$$\sigma_2^{1/2} \left(g^{-1} A_g^\delta \right) - \frac{\sqrt{\alpha}}{4} |W_g|_g > 0.$$

Note that δ depends only on $\|Rm\|$.

Now we define a subclass of the positive cone of order 2 which will be useful in our arguments:

Definition 4.2. For a conformal metric $\tilde{g} = e^{-2u}g$, we define the set

$$\Lambda_{\tilde{g}}^+ = \left\{ t \in [\delta, t_0] \mid A_{\tilde{g}}^t \in \Gamma_2^+ \text{ and } \sigma_2^{1/2} \left(g^{-1} A_{\tilde{g}}^t \right) - \frac{\sqrt{\alpha}}{4} |W_g|_g > 0 \right\}.$$

In particular if $t \in \Lambda_{\tilde{g}}^+$ then $A_{\tilde{g}}^t \in \Gamma_2^+$.

We point out that $\delta \in \Lambda_{\tilde{g}}^+$.

For $t \in [\delta, t_0]$, consider the path of equations (in the sequel we use the notation $A_{u_t}^t := A_{g_t}^t$ for g_t given by $g_t = e^{-2u_t}g$)

$$\begin{cases} \sigma_2^{1/2} \left(g^{-1} A_{u_t}^t \right) - \frac{\sqrt{\alpha}}{4} |W_g|_g = f e^{2u_t} & \text{in } M, \\ \partial_\nu u = 0 & \text{on } \partial M. \end{cases} \tag{4.1}$$

where $f(x) = \sigma_2^{1/2} \left(g^{-1} A_g^\delta \right) - \frac{\sqrt{\alpha}}{4} |W_g|_g > 0$. Note that $u \equiv 0$ is a solution of (4.1) for $t = \delta$.

As for the tree-dimensional case, we use the continuity method. Define

$$\mathcal{S} = \left\{ t \in [\delta, t_0] \mid \exists \text{ a solution } u_t \in C^{2,\alpha}(M) \text{ of (4.1) with } t \in \Lambda_{u_t}^+ \right\}.$$

Clearly, with our choice of f , $u \equiv 0$ is a solution for $t = \delta$. Since $\delta \in \Lambda_{\tilde{g}}^+$, then $\delta \in \mathcal{S}$. Hence, we have $\mathcal{S} \neq \emptyset$. Let $t \in \mathcal{S}$, and u_t be a solution. By Proposition 2.6, the linearized operator at u_t , $\mathcal{L}^t : C^{2,\alpha}(M) \cap \{\partial_\nu u = 0 \text{ on } \partial M\} \rightarrow C^\alpha(M)$, is invertible (note that the additional term in the right hand side of the equation does not effect linearization). The implicit function theorem tells us that \mathcal{S} is open. To prove that \mathcal{S} is close we need to establish a priori $C^{2,\alpha}$ estimates for solutions of the equation (4.1). To do so, we start by establishing upper-bound estimate as for the case of 3-manifolds.

Proposition 4.3 (Upper bound). *Let $u_t \in C^2(M)$ be a solution of (4.1) for some $t \in [\delta, t_0]$, with $t \in \Lambda_{u_t}^+$. Then $u_t \leq \bar{\delta}$, where $\bar{\delta}$ depends only on $\|Rm_g\|$.*

Proof. From Lemma 2.3 (iv), we have $\frac{4}{\sqrt{6}}\sigma_2^{1/2} \leq \sigma_1$, so for all $p \in M$

$$\frac{4}{\sqrt{6}} \frac{\sqrt{\alpha}}{4} |W_g|_g + \frac{4}{\sqrt{6}} f e^{2u_t} \leq \sigma_1 \left(g^{-1} A_{u_t}^t \right).$$

Let $p \in M$ be the maximum of u_t , then (this is true also if $p \in \partial M$, since $\partial_\nu u = 0$ on ∂M) we have $(\Delta u_t)(p) \leq 0$. Then, using (2.2), we have

$$\begin{aligned} \frac{4}{\sqrt{6}} \frac{\sqrt{\alpha}}{4} (|W_g|_g)(p) + \frac{4}{\sqrt{6}} f(p) e^{2u_t(p)} &\leq \sigma_1(g^{-1} A_{u_t}^t)(p) \\ &= \sigma_1(g^{-1} A_g^t)(p) + (3 - 2t)(\Delta u_t)(p) \\ &\leq \sigma_1(g^{-1} A_g^t)(p) \\ &\leq \sigma_1(g^{-1} A_g^\delta)(p). \end{aligned}$$

This implies

$$\frac{4}{\sqrt{6}} f(p) e^{2u_t(p)} \leq \sigma_1(g^{-1} A_g^\delta)(p) - \frac{4}{\sqrt{6}} \frac{\sqrt{\alpha}}{4} (|W_g|_g)(p),$$

where the last term has positive sign. Since M is compact, this implies $u_t \leq \bar{\delta}$, for some $\bar{\delta}$ depending only on $\|Rm\|$. □

Following the previous section, once we have an upper bound of the solution, from Proposition 3.3, we get gradient estimates. Now we are going to establish the lower-bound estimates. To do that we need the following lemma.

Lemma 4.4. *If \hat{g} is a Riemannian metric on M conformal to g such that $L_{\hat{g}} = 0$, then*

$$\int_M \sigma_2(\hat{g}^{-1} A_{\hat{g}}) = \frac{1}{2} \kappa_{(P^4, P^3)}.$$

Proof. First of all, one can easily check that the following holds

$$Q_{\hat{g}} = 2\sigma_2(\hat{g}^{-1} A_{\hat{g}}) - \frac{1}{12} \Delta_{\hat{g}} R_{\hat{g}}.$$

Thus integrating this equation and using the divergence theorem, we get

$$\int_M Q_{\hat{g}} dV_{\hat{g}} = 2 \int_M \sigma_2(\hat{g}^{-1} A_{\hat{g}}) dV_{\hat{g}} + \frac{1}{12} \oint_{\partial M} \frac{\partial R_{\hat{g}}}{\partial n_{\hat{g}}} dS_{\hat{g}}.$$

On the other hand, since $L_{\hat{g}} = 0$, then

$$T_{\hat{g}} = -\frac{1}{12} \frac{\partial R_{\hat{g}}}{\partial n_{\hat{g}}}.$$

Thus we obtain

$$\int_M Q_{\hat{g}} dV_{\hat{g}} = 2 \int_M \sigma_2(\hat{g}^{-1} A_{\hat{g}}) dV_{\hat{g}} - \oint_{\partial M} T_{\hat{g}} dS_{\hat{g}}.$$

Hence, we get

$$\int_M \sigma_2(\hat{g}^{-1} A_{\hat{g}}) = \frac{1}{2} \kappa_{(P^4, P^3)}.$$

This completes the proof of the lemma. □

Proposition 4.5 (Lower bound). *Assume that for some $t \in [\delta, t_0]$ the following estimate holds*

$$\frac{1}{2} \kappa_{(P^4, P^3)} - \frac{\alpha}{16} \int_M |W_g|_g^2 dV_g + \frac{1}{24} (1-t)(2-t) Y(M, \partial M, [g])^2 = \mu_t > 0. \tag{4.2}$$

Then there exist $\underline{\delta}$ depending only on $\text{diam}(M, g)$ and $\|\nabla^2 Rm\|$ such that if $u_t \in C^2(M)$ is a solution of (4.1) and if $t \in \Lambda_{u_t}^+$ then $u_t \geq \underline{\delta}$.

Proof. Since $A_g^t = A_g^1 + \frac{1-t}{2} \sigma_1(A_g^1)g$, we easily have

$$\sigma_2(A_g^t) = \sigma_2(A_g^1) + \frac{3}{2} (1-t)(2-t) \sigma_1(A_g^1)^2.$$

Letting $\tilde{g} = e^{-2u_t}g$, since u_t is a solution of equation (4.1), we have

$$f^2 e^{4u_t} + \frac{\sqrt{\alpha}}{2} f |W_g|_g e^{2u_t} = \sigma_2(g^{-1} A_{u_t}^t) - \frac{\alpha}{16} |W_g|_g^2.$$

The left-hand side can be estimated by

$$f^2 e^{4u_t} + \frac{\sqrt{\alpha}}{2} f |W_g|_g e^{2u_t} \leq C' e^{2u_t},$$

where the positive constant C' depends only on $\|Rm\|$. So we get

$$\begin{aligned} C' e^{2u_t} &\geq \sigma_2(g^{-1} A_{u_t}^t) - \frac{\alpha}{16} |W_g|_g^2 \\ &= e^{-4u_t} \left(\sigma_2(\tilde{g}^{-1} A_{u_t}^1) + \frac{1}{24} (1-t)(2-t) R_{\tilde{g}}^2 \right) - \frac{\alpha}{16} |W_g|_g^2. \end{aligned}$$

Integrating this with respect to dV_g , we obtain

$$\begin{aligned} C' \int_M e^{2u_t} dV_g &\geq \int_M \sigma_2(\tilde{g}^{-1}A_{u_t}^1) dV_{\tilde{g}} - \frac{\alpha}{16} \int_M |W_g|_g^2 dV_g \\ &\quad + \frac{1}{24}(1-t)(2-t) \int_M R_{\tilde{g}}^2 dV_{\tilde{g}} \\ &= \frac{1}{2}\kappa_{(P^4, P^3)} - \frac{\alpha}{16} \int_M |W_g|_g^2 dV_g + \frac{1}{24}(1-t)(2-t) \int_M R_{\tilde{g}}^2 dV_{\tilde{g}} \\ &\geq \frac{1}{2}\kappa_{(P^4, P^3)} - \frac{\alpha}{16} \int_M |W_g|_g^2 dV_g \\ &\quad + \frac{1}{24}(1-t)(2-t)Y(M, \partial M, [g])^2 = \mu_t > 0, \end{aligned}$$

where we have used Lemma 4.4, and the fact that for any conformal metric $g' \in [g]$, if $H_{g'} = 0$, then

$$\int_M R_{g'}^2 dV_{g'} \geq Y(M, \partial M, [g])^2.$$

This gives

$$\max_M u_t \geq \log \mu_t - C(\text{diam}(M, g), \|Rm\|).$$

Since, as already remarked $\max_M |\nabla_g u_t|_g \leq C_1$ by the same arguments as the ones of Proposition 3.3, then we have the Harnack inequality

$$\max_M u_t \leq \min_M u_t + C(\text{diam}(M, g), \|\nabla^2 Rm\|),$$

by simply integrating along a geodesic connecting points at which u_t attains its maximum and minimum. Combining this two inequalities, we obtain

$$u_t \geq \min_M u_t \geq \log \mu_t - C =: \delta,$$

where C depends only on $\text{diam}(M, g)$ and $\|\nabla^2 Rm\|$. □

Once we have C^0 and C^1 estimates, using the same arguments as the ones of Proposition 3.7, we get $C^{2,\alpha}$ estimates. Thus we are ready to apply the continuity method as in the 3-dimensional case, and conclude the proof of Theorem 4.1.

Now we are ready to present the proof of Theorem 1.8.

Proof of Theorem 1.8. First of all, since $Y(M, \partial M, [g]) > 0$, and $\kappa_{(P^4, P^3)} > \frac{1}{8} \int_M |W_g|_g^2 dV_g$, then we can apply Theorem 4.1 with $t_0 = 1$ and $\alpha = 1$ and get the existence of a metric \tilde{g} conformal to g such that

$$\sigma_2(g^{-1}A_{\tilde{g}}) > \frac{1}{16}|W_g|_g^2, \quad \text{and} \quad L_{\tilde{g}} = 0.$$

This is equivalent to

$$\sigma_2 \left(\tilde{g}^{-1} A_{\tilde{g}} \right) > \frac{1}{16} |W_{\tilde{g}}|^2, \text{ and } L_{\tilde{g}} = 0.$$

On the other hand, one can check easily that the following holds

$$\sigma_2 \left(\tilde{g}^{-1} A_{\tilde{g}} \right) = \frac{1}{96} R_{\tilde{g}}^2 - \frac{1}{8} |E_{\tilde{g}}|^2.$$

Thus, we obtain

$$\frac{1}{6} R_{\tilde{g}}^2 - 2|E_{\tilde{g}}|^2 > |W_{\tilde{g}}|^2.$$

So rearranging the latter inequality, we get the Margerin’s weak pinching condition, namely

$$W P_{\tilde{g}} < \frac{1}{6}.$$

Hence, applying Theorem 1.6, we conclude the proof of Theorem 1.8. □

5. Principal eigenvalue of $P_g^{4,3}$ and applications

In this section, we provide the proof of Theorem 1.9 and Corollary 1.10. We start by giving a proposition which will be used for the proof of Theorem 1.9.

Proposition 5.1. *Let (M, g) be a compact four-dimensional Riemannian manifold with boundary such that $L_g = 0$. Assuming $Ric_g \leq R_g g$, then we have $P_g^{4,3}$ is a non-negative operator and $\ker P_g^{4,3} \simeq R$.*

Proof. First of all, since $L_g = 0$, then for every $u \in H_{\frac{\partial}{\partial n}}$, we have

$$\left\langle P_g^{4,3} u, u \right\rangle_{L^2(M)} = \int_M (\Delta_g u)^2 dV_g + \frac{2}{3} \int_M R_g |\nabla_g u|^2 dV_g - 2 \int_M Ric_g(\nabla_g u, \nabla_g u) dV_g.$$

Now we recall the Bochner identity

$$\frac{1}{2} \Delta_g (|\nabla_g u|^2) = |\nabla_g^2 u|^2 + Ric_g(\nabla_g u, \nabla_g u) + \langle \nabla_g u, \nabla_g(\Delta_g u) \rangle .$$

Integrating the latter formula, applying the divergence theorem and integrating by parts, we get

$$\begin{aligned} -\frac{1}{2} \oint_{\partial M} \frac{\partial (|\nabla_g u|^2)}{\partial n_g} dS_g &= \int_M |\nabla_g^2 u|^2 dV_g + \int_M Ric_g(\nabla_g u, \nabla_g u) - \int_M (\Delta_g u)^2 dV_g \\ &\quad - \oint_{\partial M} \frac{\partial u}{\partial n_g} \Delta_g u dS_g. \end{aligned}$$

Recalling that $u \in H_{\frac{\partial}{\partial n}}$, then

$$\oint_{\partial M} \frac{\partial u}{\partial n_g} \Delta_g u dS_g = 0.$$

Using Lemma 2.10, we obtain

$$\frac{\partial (|\nabla_g u|^2)}{\partial n_g} = 0.$$

Hence, we get

$$\int_M |\nabla_g^2 u|^2 dV_g + \int_M \text{Ric}_g(\nabla_g u, \nabla_g u) = \int_M (\Delta_g u)^2 dV_g.$$

Now, using the latter formula, we have

$$\begin{aligned} \langle P_g^{4,3} u, u \rangle_{L^2(M)} &= -\frac{1}{3} \int_M (\Delta_g u)^2 dV_g + \frac{4}{3} \int_M |\nabla_g^2 u|^2 dV_g + \frac{2}{3} \int_M R_g |\nabla_g u|^2 dV_g \\ &\quad - \frac{2}{3} \int_M \text{Ric}_g(\nabla_g u, \nabla_g u) dV_g. \end{aligned}$$

Next, setting

$$\bar{\nabla}_g^2 u = \nabla_g^2 u - \frac{1}{4} \Delta_g g;$$

we get

$$\langle P_g^{4,3} u, u \rangle_{L^2(M)} = \frac{4}{3} \int_M |\bar{\nabla}_g^2 u|^2 dV_g + \frac{2}{3} \int_M (R_g g - \text{Ric}_g)(\nabla_g u, \nabla_g u) dV_g.$$

So using the hypothesis $R_g g - \text{Ric}_g \geq 0$, we infer

$$\langle P_g^{4,3} u, u \rangle_{L^2(M)} \geq \frac{4}{3} \int_M |\bar{\nabla}_g^2 u|^2 dV_g.$$

Hence, we obtain $P_g^{4,3}$ is a non-negative operator. So to finish the proof of the proposition, it remains only to show that the kernel is constituted only by constants. In order to do that, we assume that there exists a non constant function $u \in H_{\frac{\partial}{\partial n}}$ such that $P_g^{4,3} u = 0$, and argue for a contradiction. From the fact that $u \in \ker P_g^{4,3}$, we infer that

$$\nabla_g^2 u - \frac{1}{4} \Delta_g g = 0.$$

Now calling the doubling of M by DM , and the reflected metric by \bar{g} , we have that \bar{g} is $C^{2,\alpha}$. Next we reflect u across ∂M and call the reflection by u_{DM} . Thus, we obtain an element in $H^2(DM)$ verifying

$$\nabla_{\bar{g}}^2 u_{DM} - \frac{1}{4} \Delta_{\bar{g}} \bar{g} = 0.$$

Thus using a result of Tashiro [38], we infer that (DM, \bar{g}) is conformally diffeomorphic to S^4 . Thus (M, g) is also conformally diffeomorphic to S^4_+ . So we derive the existence of a metric \tilde{g} conformal to g on M which is Einstein, of constant positive scalar curvature, and $L_{\tilde{g}} = 0$. Hence using the conformal invariance of $P_g^{4,3}$, we get

$$\frac{4}{3} \int_M |\bar{\nabla}_{\tilde{g}}^2 u|^2 dV_{\tilde{g}} + \frac{1}{2} R_{\tilde{g}} \int_M |\nabla_{\tilde{g}} u|^2 dV_{\tilde{g}} = 0.$$

Thus, we obtain u is constant and reach a contradiction. This completes the proof of the proposition. \square

Having this at hand, we are ready to give the proof of Theorem 1.9.

Proof of Theorem 1.9. Applying Theorem 4.1 with $t_0 = 0$, and $\alpha = 0$, we get the existence of a metric \tilde{g} conformal to g such that

$$\text{Ric}_{\tilde{g}} \leq R_{\tilde{g}} \tilde{g}, \quad \text{and} \quad L_{\tilde{g}} = 0.$$

Hence appealing to Proposition 5.1, we obtain that $P_{\tilde{g}}^{4,3}$ is non-negative and $\ker P_{\tilde{g}}^{4,3} \simeq \mathbb{R}$. Now recalling that the non-negativity of the operator $P_g^{4,3}$ and the triviality of its kernel are conformally invariant properties, we have that the proof of Theorem 1.9 is complete. \square

Next, we are going to present the proof of Corollary 1.10.

Proof of Corollary 1.10. Due to (1.1), the existence of constant Q -curvature, constant T -curvature ad zero mean curvature metrics conformal to the background one g is equivalent to solving the following (BVP)

$$\begin{cases} P_g^4 u + 2Q_g = 2\bar{Q}e^{4u} & \text{in } M; \\ P_g^3 u + T_g = \bar{T}e^{3u} & \text{on } \partial M; \\ \frac{\partial u}{\partial n_g} = 0 & \text{on } \partial M. \end{cases} \tag{5.1}$$

Where \bar{Q} and \bar{T} are constant real numbers. On the other hand it is easy to see that critical points of the functional

$$\begin{aligned} II(u) = & \left\langle P_g^{4,3} u, u \right\rangle_{L^2(M)} + 4 \int_M Q_g u dV_g + \oint_{\partial M} T_g u dS_g \\ & - \kappa_{P_g^4} \log \int_M e^{4u} dV_g - \frac{4}{3} \left(\kappa_{(P^4, P^3)} - \kappa_{P_g^4} \right) \log \oint_{\partial M} e^{3u} dS_g; \end{aligned}$$

are weak solution of (5.1), hence from standard elliptic regularity theory, they are smooth solutions. Thus to prove the corollary, we will prove the existence of critical

points. More precisely, under our assumption, we will prove the existence of a minimizer. To do so, we first point out that the functional II is invariant by translation by constant, and can also be written in the following form

$$\begin{aligned}
 II(u) = & \left\langle P_g^{4,3}u, u \right\rangle_{L^2(M)} + 4 \int_M Q_g(u - \bar{u})dV_g + \oint_{\partial M} T_g(u - \bar{u}_{\partial M})dS_g \\
 & - \kappa_{P_g^4} \log \int_M e^{4(u-\bar{u})}dV_g - \frac{4}{3} \left(\kappa_{(P^4, P^3)} - \kappa_{P_g^4} \right) \log \oint_{\partial M} e^{3(u-\bar{u}_{\partial M})}dS_g.
 \end{aligned}
 \tag{5.2}$$

Now exploiting this way of writing II , we have if $\kappa_{P_g^4} \leq 0$, then by Jensen’s inequality, we obtain

$$II(u) \geq \left\langle P_g^{4,3}u, u \right\rangle_{L^2(M)} + 4 \int_M Q_g(u - \bar{u})dV_g + \oint_{\partial M} T_g(u - \bar{u}_{\partial M})dS_g.$$

Hence, using Cauchy inequality, trace theorem, Sobolev embedding, Poincaré inequality, and Lemma 2.9, we get

$$II(u) \geq \gamma \|u - \bar{u}\|_{H^2} - C$$

for some $\gamma > 0$ and some large C . Next if $\kappa_{P_g^4} > 0$, we use Proposition 2.7 and Proposition 2.8 to obtain

$$\begin{aligned}
 II(u) \geq & \left\langle P_g^{4,3}u, u \right\rangle_{L^2(M)} + 4 \int_M Q_g(u - \bar{u})dV_g + \oint_{\partial M} T_g(u - \bar{u}_{\partial M})dS_g \\
 & + \left(-\frac{4}{\alpha_1} \kappa_{P_g^4} - \frac{3}{\alpha_2} \left(\kappa_{(P^4, P^3)} - \kappa_{P_g^4} \right) \right) \left\langle P_g^{4,3}u, u \right\rangle_{L^2(M)} - C_{\alpha_1, \alpha_2};
 \end{aligned}
 \tag{5.3}$$

for $\alpha_1 < 16\pi^2$ and $\alpha_2 < 12\pi^2$, and C_{α_1, α_2} a constant depending only on α_1, α_2 and (M, g) . To continue the proof we need the following rigidity result.

Lemma 5.2. *Let (M, g) be a compact four-dimensional Riemannian manifold with umbilic boundary. Assuming that $Y(M, \partial M, [g]) \geq 0$, we have $\kappa_{(P^4, P^3)} \leq 4\pi^2$ and equality holds if and only if (M, g) is conformally diffeomorphic to S_+^4 with its standard metric.*

Proof. Since $Y(M, \partial M, [g]) \geq 0$ and (M, g) has umbilic boundary, then by a result of Escobar [18], we can take the Yamabe metric \tilde{g} which has constant non-negative scalar curvature and such that $L_{\tilde{g}} = 0$. On the other hand, still by a result of Escobar [18], we have that

$$Y(M, \partial M, [g]) = R_{\tilde{g}} \text{Vol}_{\tilde{g}}(M)^{\frac{1}{2}} \leq Y(S_+^4, S^3, [g_S]) = 8\sqrt{3}\pi;$$

and equality holds if and only if (M, g) is conformally diffeomorphic to (S_+^4, g_S) . Now using Lemma 4.4, we have

$$\kappa_{(P^4, P^3)} = 2 \int_M \sigma_2(\tilde{g}^{-1}A_{\tilde{g}})dV_{\tilde{g}}.$$

On the other hand, we have also

$$\sigma_2(\tilde{g}^{-1}A_{\tilde{g}}) = \frac{1}{96}R_{\tilde{g}}^2 - \frac{1}{8}|E_{\tilde{g}}|^2.$$

Thus, we arrive to

$$\kappa_{(P^4, P^3)} = \frac{1}{4} \int_M \left(\frac{1}{12}R_{\tilde{g}}^2 - |E_{\tilde{g}}|^2 dV_{\tilde{g}} \right) \leq \frac{1}{48}R_{\tilde{g}}^2 \text{Vol}_{\tilde{g}}(M) \leq \frac{192}{48}\pi^2 = 4\pi^2$$

and equality holds if and only if (M, g) is conformally diffeomorphic to (S^4_+, g_S) . This completes the proof of the lemma. \square

Now coming back to our proof, we have that, since $Y(M, \partial M, [g]) > 0$, and (M, g) has an umbilic boundary, then by Lemma 5.2 $\kappa_{(P^4, P^3)} \leq 4\pi^2$ and equality holds if and only if (M, g) is conformally diffeomorphic to S^4_+ with its standard metric. Hence, we can assume that $\kappa_{(P^4, P^3)} < 4\pi^2$, otherwise there is nothing to do. Thus taking α_1 close to $16\pi^2$ and α_2 close to $12\pi^2$, and using Cauchy inequality, trace theorem, Sobolev embedding, Poincaré inequality, and Lemma 2.9, we get

$$II(u) \geq \gamma_0 \|u - \bar{u}\|_{H^2} - C_0;$$

for some $\gamma_0 > 0$ and some large C_0 . Hence in any case we obtain

$$II(u) \geq \gamma_1 \|u - \bar{u}\|_{H^2} - C_1 \tag{5.4}$$

for some $\gamma_1 > 0$ and some large C_1 . From this, and the fact that II is invariant by translation by constant, we have the existence of a minimizer u_n such that

$$\int_M e^{4u_n} dV_g = 1. \tag{5.5}$$

Thus by the coercivity property (5.4), we have

$$\|u_n - \bar{u}_n\|_{H^2} \leq C.$$

On the other hand, using Proposition 2.7, we infer

$$\log \int_M e^{4(u_n - \bar{u}_n)} dV_g \leq C. \tag{5.6}$$

So using (5.5), (5.6) and Jensen’s inequality we infer

$$|\bar{u}_n| \leq C.$$

Thus, we arrive to

$$\|u_n\|_{H^2} \leq C. \tag{5.7}$$

Hence up to a subsequence, we have

$$u_n \rightharpoonup u \quad \text{in } H^2.$$

Furthermore, we have $u \in H^2_{\frac{\partial}{\partial n}}$. On the other hand, it is easy to see that II is weakly lower semicontinuous on H^2 . Thus we have u is a minimizer of II . This completes the proof of Corollary 1.10. \square

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