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Holomorphic line bundles and divisors on a domain of a Stein manifold

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Dedicated to Professor Yoshihiro Mizuta on his sixtieth birthday

Abstract. Let D be an open set of a Stein manifold X of dimension n such that $H^k(D, \mathscr{O}) = 0$ for $2 \leq k \leq n-1$. We prove that D is Stein if and only if every topologically trivial holomorphic line bundle L on D is associated to some Cartier divisor $\mathfrak{d}$ on D.

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1. Introduction

For every holomorphic line bundle L on a reduced Stein space X there exists a global holomorphic section $\sigma \in \Gamma(X, \mathscr{O}(L))$ such that the zero set $\{\sigma = 0\}$ is nowhere dense in X. Therefore L is associated to the positive Cartier divisor $\operatorname{div}(\sigma)$ on X (see Gunning [9, pages 122–125]).

Conversely the author [1, Theorem 3] proved that an open set D of a Stein manifold X of dimension two is Stein if every holomorphic line bundle L on D is associated to some (not necessarily positive) Cartier divisor $\mathfrak{d}$ on D. Moreover Ballico [4, Theorem 1] proved that an open set D of a Stein manifold X of dimension more than two of the form $D = \{\varphi < c\}$, where $\varphi : X \to \mathbb{R}$ is a weakly 2-convex function of class $\mathscr{C}^2$ in the sense of Andreotti-Grauert, is Stein if every holomorphic line bundle L on D is associated to some Cartier divisor $\mathfrak{d}$ on D.

In this paper we prove that an open set D of a Stein manifold X of dimension n such that $H^k(D, \mathscr{O}) = 0$ for $2 \leq k \leq n-1$ is Stein if every topologically trivial holomorphic line bundle L on D is associated to some Cartier divisor $\mathfrak{d}$ on D (see Theorem 4.3). This generalizes both results above (see Corollaries 4.4 and 4.5).

The proof is by induction on $n = \dim X$ and the induction hypothesis is applied to the complex subspace $\left(Y, \left(\mathscr{O}_X / f\mathscr{O}_X\right)|_Y\right)$, where $f \in \mathscr{O}_X(X)$ and

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$Y := \{[f] = 0\}$. Therefore it is inevitable to consider complex spaces which are not necessarily reduced (see Theorem 4.1).

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2. Preliminaries

Throughout this paper complex spaces are always assumed to be second countable. We always denote by $\mathscr{O}$ without subscript the reduced complex structure sheaf of an arbitrary complex space. In other words we always set $\mathscr{O} := \mathscr{O}_X/\mathscr{N}_X$ for a complex space $(X, \mathscr{O}_X)$, where $\mathscr{N}_X$ is the nilradical of the complex structure sheaf $\mathscr{O}_X$.

Let $(X, \mathscr{O}_X)$ be a (not necessarily reduced) complex space and $(\mathrm{red}, \widetilde{\mathrm{red}}) : (X, \mathscr{O}) \to (X, \mathscr{O}_X)$ the reduction map. We denote by $[f]$ the valuation $x \mapsto f_x + \mathfrak{m}_x \in \mathscr{O}_{X,x}/\mathfrak{m}_x = \mathbb{C}$, $x \in U$, for every $f \in \mathscr{O}_X(U)$, where U is an open set of X. Then the assignment $f \mapsto [f]$ is identified with $\widetilde{\mathrm{red}} : \mathscr{O}_X(U) \to \mathscr{O}(U)$ (see Grauert-Remmert [8, page 87]).

Let X be a reduced complex space and $e : \mathscr{O} \to \mathscr{O}^*$ the homomorphism of sheaves on X defined by $e_x(f_x) := \exp\left(2\pi\sqrt{-1}\, f_x\right)$ for every $f_x \in \mathscr{O}_x$ and $x \in X$, where $\mathscr{O}^*$ denotes the multiplicative sheaf of invertible germs of holomorphic functions. Then e induces the homomorphism $e^* : H^1(X, \mathscr{O}) \to H^1(X, \mathscr{O}^*)$. As usual we identify the cohomology group $H^1(X, \mathscr{O}^*)$ with the set of holomorphic line bundles on X.

Let $\mathfrak{d}$ be a Cartier divisor on a reduced complex space X defined by the meromorphic Cousin-II distribution $\{(U_i, m_i)\}_{i\in I}$ on X (see Gunning [9, page 121]). We denote by $[\mathfrak{d}]$ the holomorphic line bundle on X defined by the cocycle $\left\{m_i/m_j\right\} \in Z^1(\{U_i\}_{i\in I}, \mathscr{O}^*)$ and we say that $[\mathfrak{d}]$ is the *holomorphic line bundle associated to* $\mathfrak{d}$. We say that $\mathfrak{d}$ is *positive* (or *effective*) if $\mathfrak{d}$ can be defined by a holomorphic Cousin-II distribution.

Let $(X, \mathscr{O}_X)$ be a (not necessarily reduced) complex space and D an open set of X. Then D is said to be *locally Stein* at a point $x \in \partial D$ if there exists a neighborhood U of x in X such that the open subspace $(D \cap U, \mathscr{O}_X|_{D\cap U})$ is Stein.

Throughout this paper we use the following notation:

$$\begin{aligned}
\Delta(r) &:= \{t \in \mathbb{C} \mid |t| < r\} \text{ for } r > 0, \quad \Delta := \Delta(1),\\
P(n, \varepsilon) &:= \Delta(1+\varepsilon)^n, \quad \text{and}\\
H(n, \varepsilon) &:= \Delta^n \cup \Big\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid 1-\varepsilon < |z_1| < 1+\varepsilon,\\
&\qquad\qquad |z_2| < 1+\varepsilon,\ |z_3| < 1+\varepsilon,\ \ldots,\ |z_n| < 1+\varepsilon\Big\}
\end{aligned}$$

for $n \geq 2$ and $0 < \varepsilon < 1$.

The pair $(P(n, \varepsilon), H(n, \varepsilon))$ is said to be a *Hartogs figure*. We have the following lemma which characterizes a Stein open set of $\mathbb{C}^n$.

Lemma 2.1 (Kajiwara-Kazama [10, Lemmas 1 and 2]). *Let D be an open set of $\mathbb{C}^n$. Then the following two conditions are equivalent.*

(1) *D is Stein.*
(2) *There do not exist a biholomorphic map $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, $\varepsilon \in (0, 1)$ and $b = (b_1, b_2, \dots, b_n) \in \mathbb{C}^n$ such that $\varphi(H(n, \varepsilon)) \subset D$, $|b_1| \leq 1-\varepsilon$, $\max_{2\leq\nu\leq n} |b_\nu| = 1$ and $\varphi(b) \in \partial D$.*[1]

3. Lemmas

In this section we denote by $\Phi_{(X, \mathscr{O}_X)}$ the composition of the induced homomorphisms

$$H^1(X, \mathscr{O}_X) \xrightarrow{\tilde{\mathrm{red}}^*} H^1(X, \mathscr{O}) \xrightarrow{e^*} H^1(X, \mathscr{O}^*)$$

for every complex space $(X, \mathscr{O}_X)$.

Lemma 3.1. *Let $(X, \mathscr{O}_X)$ be a Stein space of pure dimension 2 and D an open set of X. Let $(\theta, \tilde{\theta}) : (X, \mathscr{O}_X) \to \mathbb{C}^2$ be a holomorphic map, R and W open sets of X, $\varepsilon \in (0, 1)$ and $b = (b_1, b_2) \in \mathbb{C}^2$ such that $R \Subset W \subset X \setminus \mathrm{Sing}(X, \mathscr{O}_X)$, $\theta(W)$ is an open set of $\mathbb{C}^2$, the restriction $\theta|_W : W \to \theta(W)$ is biholomorphic, $\theta(R) = P(2, \varepsilon)$, $(\theta|_W)^{-1}(H(2, \varepsilon)) \subset D$, $|b_1| \leq 1 - \varepsilon$, $|b_2| = 1$ and $(\theta|_W)^{-1}(b) \in \partial D$. Then there exists a cohomology class $\alpha \in H^1(D, \mathscr{O}_X|_D)$ such that the holomorphic line bundle $\Phi_{(D, \mathscr{O}_X|_D)}(\alpha)|_{D\cap R}$ on $D \cap R$ is not associated to any Cartier divisor on $D \cap R$.*

Proof. Let $\theta_\nu := \tilde{\theta} z_\nu$ for $\nu = 1, 2$, where z_1 and z_2 are the coordinates of $\mathbb{C}^2$. Let $E_\nu := \{[\theta_\nu] \neq b_\nu\}$ for $\nu = 1, 2$. Since $(E_\nu, \mathscr{O}_X|_{E_\nu})$ is Stein and $1/([\theta_\nu] - b_\nu) \in \mathscr{O}(E_\nu)$, there exists $u_\nu \in \mathscr{O}_X(E_\nu)$ such that $[u_\nu] = 1/([\theta_\nu] - b_\nu)$ on E_ν for $\nu = 1, 2$. Let $T := \{|[\theta_1]| < 1 + \varepsilon\}$ and $T_1 := \{|[\theta_1]| < 1 + \varepsilon,\ |[\theta_2]| > 1 + \varepsilon/2\} \cup (T \setminus \bar{R})$. Then $(T, \mathscr{O}_X|_T)$ is Stein and $\{R, T_1\}$ is an open covering of T. Since $H^1(\{R, T_1\}, \mathscr{O}_X|_T) = 0$ and $R \cap T_1 \subset E_2$, there exist $v_0 \in \mathscr{O}_X(R) = \mathscr{O}(R)$ and $v_1 \in \mathscr{O}_X(T_1)$ such that $u_2 = v_1 - v_0$ on $R \cap T_1$. Let $F := (E_2 \cap R) \cup T_1$. Let $v \in \mathscr{O}_X(F)$ be defined by $v = v_0 + u_2$ on $E_2 \cap R$ and $v = v_1$ on T_1. Let $D_1 := D \cap E_1$ and $D_2 := D \cap \big((E_2 \cap T) \cup (T \setminus \bar{R})\big)$. Then $\{D_1, D_2\}$ is an open covering of D and $D_1 \cap D_2 \subset E_1 \cap F$. Let $\alpha \in H^1(\{D_1, D_2\}, \mathscr{O}_X|_D)$ be the cohomology class defined by $(u_1 v)|_{D_1\cap D_2} \in \mathscr{O}_X(D_1 \cap D_2) = Z^1(\{D_1, D_2\}, \mathscr{O}_X|_D)$. Then by

[1] An open set D of $\mathbb{C}^n$ satisfies condition (2) in Lemma 2.1 if and only if D is *p-convex* in the sense of Kajiwara-Kazama [10, page 2]. Note that the sentence "$\varphi(D)$ is a subset of Ω ..." should be "$\varphi(\mathring{D})$ is a subset of Ω ..." in the definition of a *boundary mapping* in Kajiwara-Kazama [10, page 2].

the argument in Abe [1, page 271] the holomorphic line bundle $\Phi_{(D,\,\mathscr{O}_X|_D)}(\alpha)|_{D\cap R}$ is not associated to any Cartier divisor on $D \cap R$. $\square$

A zero set $N(l)$ of a linear function $l(z_1, z_2, \dots, z_n) = \sum_{k=1}^{n} a_k z_k + b$ on $\mathbb{C}^n$, where $a_1, a_2, \dots, a_n, b \in \mathbb{C}$ and $(a_1, a_2, \dots, a_n) \neq (0, 0, \dots, 0)$, is said to be a *hyperplane* of $\mathbb{C}^n$.

Lemma 3.2. *Let D be an open set of $\mathbb{C}^n$ and H a hyperplane of $\mathbb{C}^n$. Let $Z := D \cap H$. Then for every Cartier divisor $\mathfrak{d}$ on D there exists a Cartier divisor $\mathfrak{d}'$ on D such that the support $\left|\mathfrak{d}'\right|$ of $\mathfrak{d}'$ is nowhere dense in Z and $[\mathfrak{d}]|_Z = [\mathfrak{d}'|_Z]$.*

Proof. As usual we identify a Cartier divisor on a complex manifold with a Weil divisor. Let $\mathfrak{d} = \sum_{\lambda\in\Lambda} \alpha_\lambda A_\lambda$, where A_λ is an irreducible analytic set of dimension $n-1$ and $\alpha_\lambda \in \mathbb{Z}$ for every $\lambda \in \Lambda$, be the expression of $\mathfrak{d}$ as a Weil divisor. Let Λ'' be the set of $\lambda \in \Lambda$ such that A_λ is a connected component of Z. Let $\Lambda' := \Lambda \setminus \Lambda''$, $\mathfrak{d}' := \sum_{\lambda\in\Lambda'} \alpha_\lambda A_\lambda$ and $\mathfrak{d}'' := \sum_{\lambda\in\Lambda''} \alpha_\lambda A_\lambda$. Then the support $\left|\mathfrak{d}'\right| = \bigcup_{\lambda\in\Lambda'} A_\lambda$ of $\mathfrak{d}'$ is nowhere dense in Z. Let $\left\{Z_\mu\right\}_{\mu\in M}$, where $M \subset \mathbb{N}$, be the set of connected components of Z. There exists a system $\left\{U_\mu\right\}_{\mu\in M}$ of mutually disjoint open sets of D such that U_μ is a neighborhood of Z_μ for every $\mu \in M$. Let $U_0 := D \setminus Z$. We choose a non-constant linear function l on $\mathbb{C}^n$ such that $\{l = 0\} = H$. If there exists $\lambda \in \Lambda''$ such that $Z_\mu = A_\lambda$, then let $\beta_\mu := \alpha_\lambda$. Otherwise let $\beta_\mu := 0$. Then $\mathfrak{d}''$ as a Cartier divisor is defined by the system $\{(U_0, 1)\} \cup \left\{(U_\mu, l^{\beta_\mu})\right\}_{\mu\in M}$. It follows that $[\mathfrak{d}'']$ is holomorphically trivial on $U := \bigcup_{\mu\in M} U_\mu$. Since U is a neighborhood of Z in D, the restriction $[\mathfrak{d}'']|_Z$ is also holomorphically trivial. Then we have that

$$[\mathfrak{d}]|_Z = [\mathfrak{d}' + \mathfrak{d}'']|_Z = \left([\mathfrak{d}'] \otimes [\mathfrak{d}'']\right)|_Z = [\mathfrak{d}']|_Z \otimes [\mathfrak{d}'']|_Z = [\mathfrak{d}'|_Z]. \qquad \square$$

A complex space $(X, \mathscr{O}_X)$ is said to be *Cohen-Macaulay* if the local $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$ is Cohen-Macaulay for every $x \in X$ (see Raimondo-Silva [12]).

Lemma 3.3. *Let $(X, \mathscr{O}_X)$ be a Cohen-Macaulay Stein space of pure dimension $n \geq 2$. Let D be an open set of X such that $H^k(D, \mathscr{O}_X|_D) = 0$ for $2 \leq k \leq n-1$. Let $(\theta, \tilde{\theta}) : (X, \mathscr{O}_X) \to \mathbb{C}^n$ be a holomorphic map, R and W open sets of X, $\varepsilon \in (0,1)$ and $b = (b_1, b_2, \dots, b_n) \in \mathbb{C}^n$ such that $R \Subset W \subset X \setminus \operatorname{Sing}(X, \mathscr{O}_X)$, $\theta(W)$ is an open set of $\mathbb{C}^n$, the restriction $\theta|_W : W \to \theta(W)$ is biholomorphic, $\theta(R) = P(n, \varepsilon)$, $(\theta|_W)^{-1}(H(n, \varepsilon)) \subset D$, $|b_1| \leq 1 - \varepsilon$, $\max_{2\leq\nu\leq n} |b_\nu| = 1$ and $(\theta|_W)^{-1}(b) \in \partial D$. Then there exists a cohomology class $\alpha \in H^1(D, \mathscr{O}_X|_D)$ such that the holomorphic line bundle $\Phi_{(D,\,\mathscr{O}_X|_D)}(\alpha)|_{D\cap R}$ on $D \cap R$ is not associated to any Cartier divisor on $D \cap R$.*

Proof. The proof proceed by induction on $n = \dim X$. By Lemma 3.1 the assertion is true if $n = 2$. We consider the case when $n \geq 3$. Let $\theta_\nu := \tilde{\theta} z_\nu$ for $\nu = 1, 2, \dots, n$, where $z_1, z_2, \dots, z_n$ are the coordinates of $\mathbb{C}^n$. We replace W by the connected component of W which contains $\bar{R}$. Let X_0 be the irreducible component of X which contains W. Since $(X, \mathscr{O}_X)$ is Stein, there exists $f \in \mathscr{O}_X(X)$ such that

$[f] = [\theta_n] - b_n$ on X_0 and $[f] \not\equiv 0$ on any irreducible component of X. Let $Y := \{[f] = 0\} = \operatorname{Supp}\left(\mathscr{O}_X / f\mathscr{O}_X\right)$ and $\mathscr{O}_Y := \left(\mathscr{O}_X / f\mathscr{O}_X\right)|_Y$. By the active lemma (see Grauert-Remmert [8, page 100]) we have that

$$\dim \mathscr{O}_{X,x}/f_x\mathscr{O}_{X,x} = \dim_x Y = n-1 = \dim \mathscr{O}_{X,x} - 1$$

for every $x \in Y$. Therefore f_x is not a zero divisor of $\mathscr{O}_{X,x}$ for every $x \in Y$ and $(Y, \mathscr{O}_Y)$ is a Cohen-Macaulay Stein space of pure dimension $n-1$ (see Grauert-Remmert [6, page 141] or Serre [15, page 85]). Let $m : \mathscr{O}_X \to \mathscr{O}_X$ be the homomorphism defined by $m_x(h_x) := f_x h_x$ for every $h_x \in \mathscr{O}_{X,x}$ and $x \in X$. Since the sequence

$$0 \to \mathscr{O}_X \xrightarrow{m} \mathscr{O}_X \xrightarrow{\tilde{\iota}} \mathscr{O}_X/f\mathscr{O}_X \to 0$$

is exact, we have the long exact sequence of cohomology groups

$$\cdots \to H^k(D, \mathscr{O}_X|_D) \to H^k(D \cap Y, \mathscr{O}_Y|_{D\cap Y}) \to H^{k+1}(D, \mathscr{O}_X|_D) \to \cdots.$$

Since by assumption $H^k(D, \mathscr{O}_X|_D) = 0$ for $2 \le k \le n-1$, we have that $H^k(D \cap Y, \mathscr{O}_X|_{D\cap Y}) = 0$ for $2 \le k \le n-2$ and that the homomorphism $\tilde{\iota}^* : H^1(D, \mathscr{O}_X|_D) \to H^1(D \cap Y, \mathscr{O}_Y|_{D\cap Y})$ is surjective. Let $(\theta', \tilde{\theta}') : (Y, \mathscr{O}_Y) \to \mathbb{C}^{n-1}$ be the holomorphic map such that $\tilde{\theta}' z_\nu = (\tilde{\iota}\theta_\nu)|_Y$ for $\nu = 1, 2, \ldots, n-1$ (see Grauert-Remmert [8, page 22]). Let $R' := R \cap Y$, $W' := W \cap Y$ and $b' := (b_1, b_2, \ldots, b_{n-1})$. Then $\theta(x) = (\theta'(x), b_n)$ for every $x \in W'$, $R' \Subset W' \subset Y \setminus \operatorname{Sing}(Y, \mathscr{O}_Y)$, $\theta'(W')$ is an open set of $\mathbb{C}^{n-1}$ and the restriction $\theta'|_{W'} : W' \to \theta'(W')$ is biholomorphic. We have that

$$\begin{aligned}
&\theta'(R') \times \{b_n\} = \theta(R') = P(n, \varepsilon) \cap \{z_n = b_n\} = P(n-1, \varepsilon) \times \{b_n\}, \\
&\left(\theta'|_{W'}\right)^{-1}(H(n-1, \varepsilon)) = (\theta|_W)^{-1}(H(n-1, \varepsilon) \times \{z_n = b_n\}) \\
&\qquad\qquad\qquad\qquad = (\theta|_W)^{-1}(H(n, \varepsilon)) \cap W' \subset D \cap Y, \quad \text{and} \\
&\left(\theta'|_{W'}\right)^{-1}(b') = (\theta|_W)^{-1}(b) \in \partial(D \cap Y),
\end{aligned}$$

where $\partial(D \cap Y)$ denotes the boundary of $D \cap Y$ in Y. By induction hypothesis there exists $\alpha' \in H^1(D \cap Y, \mathscr{O}_Y|_{D\cap Y})$ such that the holomorphic line bundle $\Phi_{(D\cap Y, \mathscr{O}_Y|_{D\cap Y})}(\alpha')|_{D\cap R'}$ on $D \cap R'$ is not associated to any Cartier divisor on $D \cap R'$. Since $\tilde{\iota}^*$ is surjective, there exists $\alpha \in H^1(D, \mathscr{O}_X|_D)$ such that $\tilde{\iota}^*(\alpha) = \alpha'$. Assume that there exists a Cartier divisor $\mathfrak{d}$ on $D \cap R$ such that $\Phi_{(D, \mathscr{O}_X|_D)}(\alpha)|_{D\cap R} = [\mathfrak{d}]$. By Lemma 3.2 there exists a Cartier divisor $\mathfrak{c}$ on $D \cap R$ such that the support $|\mathfrak{c}|$ is nowhere dense in $D \cap R'$ and $[\mathfrak{d}]|_{D\cap R'} = [\mathfrak{c}|_{D\cap R'}]$. Then we have that

$$\Phi_{(D\cap Y, \mathscr{O}_Y|_{D\cap Y})}(\alpha')|_{D\cap R'} = \Phi_{(D, \mathscr{O}_X|_D)}(\alpha)|_{D\cap R'} = [\mathfrak{d}]|_{D\cap R'} = [\mathfrak{c}|_{D\cap R'}]$$

and it is a contradiction. It follows that $\Phi_{(D, \mathscr{O}_X|_D)}(\alpha)|_{D\cap R}$ is not associated to any Cartier divisor on $D \cap R$. $\square$

4. Theorems

Theorem 4.1. *Let $(X, \mathscr{O}_X)$ be a (not necessarily reduced) Cohen-Macaulay Stein space of pure dimension n and D an open set of X. Assume that the following two conditions are satisfied.*

i) $H^k(D, \mathscr{O}_X|_D) = 0$ *for* $2 \leq k \leq n-1$.
ii) *For every holomorphic line bundle L on D which is an element of the image of the composition* Φ *of the homomorphisms*

$$H^1(D, \mathscr{O}_X|_D) \xrightarrow{\tilde{\mathrm{red}}^*} H^1(D, \mathscr{O}) \xrightarrow{e^*} H^1(D, \mathscr{O}^*)$$

there exists a Cartier divisor $\mathfrak{d}$ *on D such that* $L = [\mathfrak{d}]$.

Then D is locally Stein at every point $x \in \partial D \setminus \mathrm{Sing}\,(X, \mathscr{O}_X)$.

Proof. We may assume that $n \geq 2$. Assume that there exists a point $x_0 \in \partial D \setminus \mathrm{Sing}\,(X, \mathscr{O}_X)$ such that D is not locally Stein at x_0. Since X is Stein, there exist a holomorphic map $(f, \tilde{f}) : (X, \mathscr{O}_X) \to \mathbb{C}^n$ and an open set W of X such that $x_0 \in W \subset X \setminus \mathrm{Sing}\,(X, \mathscr{O}_X)$, $f(W)$ is an open set of $\mathbb{C}^n$ and $f|_W : W \to f(W)$ is biholomorphic (see Grauert-Remmert [7, page 151]). Take a Stein open set V of $\mathbb{C}^n$ such that $f(x_0) \in V \Subset f(W)$. Let $U := (f|_W)^{-1}(V)$. Then U is Stein, $x_0 \in U \Subset W$ and $f(U) = V$. Since D is not locally Stein at x_0, the open set $f(D \cap U)$ of $\mathbb{C}^n$ is not Stein. By Lemma 2.1 there exist a biholomorphic map $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, $\varepsilon \in (0, 1)$ and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$ such that $\varphi(H(n, \varepsilon)) \subset f(D \cap U)$, $|b_1| \leq 1 - \varepsilon$, $\max_{2 \leq \nu \leq n} |b_\nu| = 1$ and $\varphi(b) \in \partial\,(f(D \cap U))$. Let $(\theta, \tilde{\theta}) := \varphi^{-1} \circ (f, \tilde{f}) : (X, \mathscr{O}_X) \to \mathbb{C}^n$. We have that $\theta(W) = \varphi^{-1}(f(W))$ is an open set of $\mathbb{C}^n$ and $\theta|_W : W \to \theta(W)$ is biholomorphic. Let $P := P(n, \varepsilon)$ and $H := H(n, \varepsilon)$. Since V is Stein and $\varphi(H) \subset f(D \cap U) \subset V$, we have that $\varphi(P) \subset V \subset f(W)$. Let $R := (\theta|_W)^{-1}(P)$. Then we have that $\theta(R) = P$, $(\theta|_W)^{-1}(H) \subset U$ and $(\theta|_W)^{-1}(b) \in \partial D$. By Lemma 3.3 there exists a holomorphic line bundle $L \in \mathrm{im}\,\Phi$ such that $L|_{D \cap R}$ is not associated to any Cartier divisor on $D \cap R$. On the other hand by assumption there exists a Cartier divisor $\mathfrak{d}$ on D such that $L = [\mathfrak{d}]$ and therefore $L|_{D \cap R} = [\mathfrak{d}|_{D \cap R}]$, which is a contradiction. It follows that D is locally Stein at every point $x \in \partial D \setminus \mathrm{Sing}\,(X, \mathscr{O}_X)$. □

Remark 4.2. Condition i) in Theorem 4.1 can be replaced by the following weaker one:

i)′ The dimension of $H^k(D, \mathscr{O}_X|_D)$ is at most countably infinite for every integer k such that $2 \leq k \leq n-1$.

Proof. If condition i)′ is satisfied, then by Ballico [3, Proposizione 7], which generalizes Siu [16, Theorem A], we have that $\dim H^k(D, \mathscr{O}_X|_D) < +\infty$ for $2 \leq k \leq n-1$. We also have that $H^k(D, \mathscr{O}_X|_D) = 0$ for $k \geq n$ by Siu [17] and by Reiffen [13, page 277]. It follows that $H^k(D, \mathscr{O}_S|_D) = 0$ for $k \geq 2$ by Raimondo-Silva [12]. □

Every complex manifold is Cohen-Macaulay (see Grauert-Remmert [6, page 142]). The image of $H^1(D, \mathscr{O}) \to H^1(D, \mathscr{O}^*)$ coincides with the set of topologically trivial holomorphic line bundles on D. Therefore by Theorem 4.1 and by Docquier-Grauert [5] we obtain the following theorem.

Theorem 4.3. *Let X be a Stein manifold of dimension n and D an open set of X such that $H^k(D, \mathscr{O}) = 0$ for $2 \leq k \leq n-1$. Then the following four conditions are equivalent.*

(1) *D is Stein.*
(2) *For every holomorphic line bundle L on D there exists a positive Cartier divisor $\mathfrak{d}$ on D such that $L = [\mathfrak{d}]$.*
(3) *For every holomorphic line bundle L on D there exists a Cartier divisor $\mathfrak{d}$ on D such that $L = [\mathfrak{d}]$.*
(4) *For every topologically trivial holomorphic line bundle L on D there exists a Cartier divisor $\mathfrak{d}$ on D such that $L = [\mathfrak{d}]$.*

Corollary 4.4 (Abe [1, Theorem 3]). *Let X be a Stein manifold of dimension 2 and D an open set of X. Then the four conditions in Theorem 4.3 are equivalent.*

Let X be a complex manifold of dimension n and $\varphi : X \to \mathbb{R}$ a function of class $\mathscr{C}^2$. Then φ is said to be *weakly 2-convex* if for every $x \in X$ the Levi form of φ at x has at most one negative eigenvalue. By the theorem of Andreotti-Grauert [2] we have the following corollary.

Corollary 4.5 (Ballico [4, Theorem 1]). *Let X be a Stein manifold and $\varphi : X \to \mathbb{R}$ a weakly 2-convex function of class $\mathscr{C}^2$. Let $D := \{\varphi < c\}$, where $c \in \mathbb{R}$ is a constant. Then the four conditions in Theorem 4.3 are equivalent.*

We also have the following corollary (see Serre [14, page 65]).

Corollary 4.6 (Laufer [11, Theorem 4.1]). *Let X be a Stein manifold of dimension n and D an open set of X. Then the following two conditions are equivalent.*

(1) *D is Stein.*
(2) *$H^k(D, \mathscr{O}) = 0$ for $1 \leq k \leq n-1$.*

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