

One-dimensional symmetry of periodic minimizers for a mean field equation

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Abstract. We consider on a two-dimensional flat torus T defined by a rectangular periodic cell the following equation

$$\Delta u + \rho \left(\frac{e^u}{\int_T e^u} - \frac{1}{|T|} \right) = 0, \quad \int_T u = 0.$$

It is well-known that the associated energy functional admits a minimizer for each $\rho \leq 8\pi$. The present paper shows that these minimizers depend actually only on one variable. As a consequence, setting $\lambda_1(T)$ to be the first eigenvalue of the Laplacian on the torus, the minimizers are identically zero whenever $\rho \leq \min\{8\pi, \lambda_1(T)|T|\}$. Our results hold more generally for solutions that are Steiner symmetric, up to a translation.

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1. Introduction

This paper is concerned with the following nonlinear equation on a two-dimensional flat torus T :

$$\Delta u + \rho \left(\frac{e^u}{\int_T e^u} - \frac{1}{|T|} \right) = 0, \quad u \in H^1(T), \quad (1.1)$$

where ρ is a real parameter and $H^1(T)$ denotes the classical Sobolev space. Since the above equation is invariant by adding a constant to a solution, we define

$$\mathring{H}(T) := \left\{ u \in H^1(T) : \int_T u = 0 \right\},$$

and consider the equivalent problem

$$\Delta u + \rho \left(\frac{e^u}{\int_T e^u} - \frac{1}{|T|} \right) = 0, \quad u \in \mathring{H}(T). \quad (1.2)$$

As a consequence of the Moser-Trudinger inequality as established by Fontana on compact manifolds [11], Problem (1.2) admits a variational formulation and is the Euler-Lagrange equation of the functional:

$$J^\rho : \overset{\circ}{H}(T) \rightarrow \mathbb{R}, \quad u \mapsto \frac{1}{2} \int_T |\nabla u|^2 - \rho \log \left(\frac{1}{|T|} \int_T e^u \right). \quad (1.3)$$

When $\rho \leq 0$, it is easy to verify that zero is the only solution of (1.2) since in this range of the parameter the functional is strictly convex. But for $\rho > 0$, the study of existence and uniqueness of solutions becomes much more difficult. Using the Moser-Trudinger inequality [11], one easily checks the existence of a minimizer in the range $\rho < 8\pi$. At the critical value $\rho = 8\pi$ the work of Ding *et al.* [10] and Nolasco-Tarantello [19] prove that such a minimizer persists. More generally, it has been proved in [6] that the solutions of (1.2) are uniformly bounded for $\rho \leq 8\pi$.

But these existence results could give the solution $u \equiv 0$ which trivially satisfies (1.2). Actually Struwe-Tarantello proved in [24] that $u \equiv 0$ is the unique solution of (1.2) when the parameter ρ is close to zero. The arguments of [24] hold in any torus but do not give the explicit range of the parameter where zero is the unique solution. For torus defined by rectangular periodic cell $(-a, a) \times (-b, b)$ with $\frac{b}{a} \leq \frac{1}{2}$, it has been shown in [3] that $u \equiv 0$ is the unique solution of Problem (1.2) if and only if $\rho \leq \lambda_1(T)|T|$ where $\lambda_1(T)$ denotes the first non-zero eigenvalue of the Laplacian on T . For other types of torus a uniqueness result is contained in [17]. Typically when the periodic cell is a square, it is proved in [17] that Problem (1.2) admits only the trivial solution when $\rho \leq 8\pi$. This result is optimal since above 8π existence of non-trivial solutions is known by [24] and [23]. Both [3] and [17] give optimal results for a large class of torus, but do not cover all cases. At the light of [3] and [17] we actually expect that $u \equiv 0$ is the unique solution of (1.2) whenever $\rho \leq \min\{8\pi, \lambda_1(T)|T|\}$.

A natural question related to the uniqueness is whether the solutions of (1.2) are invariant under some translations. The present work addresses this question when the torus is defined by a rectangular periodic cell. Henceforth we will make the following assumption:

(H) T is a flat two-dimensional torus with periodic cell $\Omega = (-a, a) \times (-b, b)$.

We shall say that $u : T \rightarrow \mathbb{R}$ is one-dimensional if

$$\frac{\partial u}{\partial x_1} \equiv 0 \text{ in } \Omega \quad \text{or} \quad \frac{\partial u}{\partial x_2} \equiv 0 \text{ in } \Omega.$$

In [3], it has been established that every solution depends only on one variable when $\rho \leq \rho^*(T)$, where $\rho^*(T)$ is an explicit positive constant depending on the maximum conformal radius of the rectangle Ω . In particular such $\rho^*(T)$ is strictly greater than 4π . From the work of [3], it is expected that actually $\rho^*(T) = 8\pi$.

The main purpose of this paper is to show that any *global minimizers* of the functional (1.3) must be one-dimensional whenever $\rho \leq 8\pi$. As a consequence of

our result, we derive that $u \equiv 0$ is the unique minimizer of (1.3) for a suitable range of the parameter. Indeed, by setting $\lambda_1(T)$ to be the first eigenvalue of the torus, Ricciardi and Tarantello [23] proved that $\rho > \lambda_1(T)|T|$ is a necessary and sufficient condition for the existence of at least one nonzero one-dimensional solution for (1.2). Our main result reads more precisely as follows:

Theorem 1.1. *Assume (H) holds and $\rho \leq 8\pi$. Then*

- (a) *any global minimizer of the functional J^ρ is one-dimensional;*
- (b) *for each $\rho \leq \min\{8\pi, \lambda_1(T)|T|\}$, $u \equiv 0$ is the unique global minimizer of the functional J^ρ .*

By setting $\rho_0 := \min\{8\pi, \lambda_1(T)|T|\}$ we derive in particular:

$$\frac{1}{|T|} \int_T e^u \leq e^{\frac{1}{2\rho_0} \int_T |\nabla u|^2}, \quad \forall u \in \mathring{H}(T), \tag{1.4}$$

with equality if and only if $u \equiv 0$. Inequality (1.4) is sharp since an appropriate choice of test functions shows that any minimizer of J^ρ is nonzero whenever $\lambda_1(T)|T| < \rho \leq 8\pi$. An optimal inequality like (1.4) has been previously derived on the two-dimensional canonical sphere by Onofri [20], who proved that in such a case $u \equiv 0$ is the unique minimizer of the functional (1.3). The same conclusion was obtained via another method by Hong [13]. Solutions which are non-zero do exist at 8π when the manifold is the sphere. But below this critical value, the works of Chanillo-Kiessling [5] and Lin [15] have shown that $u \equiv 0$ is actually the unique solution of Problem (1.2) when the domain is the sphere. Our Theorem 1.1 is the analogue of Onofri’s result on rectangular two-dimensional torus.

In order to prove Theorem 1.1 we will first derive that, up to a translation, any global minimizer u of the functional J^ρ satisfies in the periodic cell Ω the following symmetry and monotonicity properties

$$\left\{ \begin{array}{ll} u(x_1, x_2) = u(-x_1, x_2) = u(x_1, -x_2), & \forall (x_1, x_2) \in \Omega, \\ \frac{\partial u}{\partial x_1}(x) \leq 0, & \forall x \in (0, a) \times (-b, b), \\ \frac{\partial u}{\partial x_2}(x) \leq 0, & \forall x \in (-a, a) \times (0, b), \end{array} \right. \tag{1.5}$$

namely u is *Steiner symmetric* in the rectangular periodic cell Ω . Actually we will prove that (1.5) holds for any *semi-stable* critical point of J^ρ , *i.e.* function u satisfying

$$DJ_{(u)}^\rho = 0 \quad \text{and} \quad D^2J_{(u)}^\rho \geq 0. \tag{1.6}$$

We will then investigate the one-dimensional property of any critical point u of the functional J^ρ satisfying (1.5). With this aim, we consider for $t \in [0, 1]$ the family of periodic functions

$$\varphi_t := t \frac{\partial u}{\partial x_1} + (1 - t) \frac{\partial u}{\partial x_2}.$$

A careful analysis will show that for some $t_0 \in [0, 1]$, the nodal line of the periodic function φ_{t_0} encloses a simply connected domain in \mathbb{R}^2 . This is a crucial step in order to apply the so-called Bol's isoperimetric inequality. When $\rho \leq 8\pi$, this inequality provides precious information on the invertibility of the linearized problem and will imply

$$t_0 \frac{\partial u}{\partial x_1} + (1 - t_0) \frac{\partial u}{\partial x_2} \equiv 0, \quad \text{when } \rho \leq 8\pi.$$

The fact that u is one-dimensional will follow immediately. Hence our main Theorem 1.1 will be a consequence of the following stronger statement:

Theorem 1.2. *Assume (H) holds. Then,*

- (a) *up to a translation, any semi-stable critical point of J^ρ is Steiner symmetric (i.e. satisfies (1.5));*
- (b) *any Steiner symmetric solution of (1.2) is one-dimensional when $\rho \leq 8\pi$;*
- (c) *for each $\rho \leq \min\{8\pi, \lambda_1(T)|T|\}$, $u \equiv 0$ is the unique Steiner symmetric solution of (1.2).*

The article is organized as follows. In Section 2, we study the symmetry of global minimizers and more generally of semi-stable critical point of the functional (1.3). We show that they are always Steiner symmetric in the periodic cell of the torus, up to a translation. In order to derive a stronger statement in the range of parameter $\rho \leq 8\pi$, we discuss in Section 3 Bol's isoperimetric inequality, and a Faber-Krahn inequality that is tightly related to it. These ingredients are crucially used in Section 4 to prove that any Steiner symmetric solution of Problem (1.1) is one-dimensional whenever $\rho \leq 8\pi$. This will provide a complete proof of Theorem 1.1.

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2. Steiner symmetry

The following proposition is a particular case of a result due to Kawohl (see [14, page 82]). It establishes that the global minimizers of the functional (1.3) are Steiner symmetric in the periodic cell, up to a translation.

Proposition 2.1 (Kawohl, [14]). *Let (H) be satisfied and u be a global minimizer of J^ρ with its maximum located at the origin. Then,*

$$\begin{cases} u(x_1, x_2) = u(-x_1, x_2) = u(x_1, -x_2) & \forall (x_1, x_2) \in \Omega, \\ \frac{\partial u}{\partial x_i}(x) \leq 0 & \forall x \in (0, a) \times (0, b), i = 1, 2. \end{cases} \tag{2.1}$$

Proof. The conclusion of this proposition holds for very general functional. It relies on the Steiner symmetrization that has already been introduced by Pólya and Szegő in [22] (see also [14]). Let us sketch quickly the arguments in our case. Consider a function u defined in the rectangular periodic cell Ω . For each $x_1 \in [-a, a]$, let us set

$$\Omega_c := \{x \in \Omega : u(x) \geq c\} \quad \text{and} \quad \Omega_c(x_1) := \Omega_c \cap (\{x_1\} \times \mathbb{R}).$$

The Steiner symmetrization Ω_c^* of the level set Ω_c with respect to the line $\{x_2 = 0\}$ is defined by

$$\Omega_c^* := \bigcup_{x_1 \in [-a, a]} \left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 \leq |x_2| \leq \frac{\mathcal{L}^1(\Omega_c(x_1))}{2} \right\},$$

where \mathcal{L}^1 denotes the Lebesgue-measure in \mathbb{R} . The *Steiner symmetrization* u^* of u with respect to the line $\{x_2 = 0\}$ is then defined by

$$u^*(x_1, x_2) := \sup\{c \in \mathbb{R} : (x_1, x_2) \in \Omega_c^*\} \quad \text{for } (x_1, x_2) \in \Omega.$$

By construction we have

$$\begin{aligned} u^*(x_1, x_2) &= u^*(-x_1, x_2), \quad \forall (x_1, x_2) \in \Omega, \\ \frac{\partial u^*}{\partial x_1}(x) &\leq 0, \quad \forall x \in (0, a) \times (0, b). \end{aligned}$$

The two additional main properties of this symmetrization are

$$\int_{\Omega} |\nabla u|^2 \geq \int_{\Omega} |\nabla u^*|^2 \quad \text{and} \quad \int_{\Omega} e^u = \int_{\Omega} e^{u^*},$$

furthermore the Dirichlet integral decreases strictly unless u and u^* coincide up to a translation (see [14]). Hence $J^\rho(u) > J^\rho(u^*)$ unless, up to a translation, $u = u^*$.

Similarly, we can define the Steiner symmetrization of a function with respect to the line $\{x_1 = 0\}$. By doing successively a Steiner symmetrization of u with respect to $\{x_2 = 0\}$ and then $\{x_1 = 0\}$, we construct a function u^{**} which fulfills (2.1) and is such that $J(u) > J(u^{**})$ unless $u = u^{**}$, modulo a translation. \square

Remark 2.2. A function satisfying (2.1) is said to be “Steiner symmetric”. Let us emphasize that for any Steiner symmetric solution of (1.2) the following alternative holds for $i = 1, 2$:

$$(a) \text{ either } \frac{\partial u}{\partial x_i} \equiv 0 \text{ in } \Omega, \quad (b) \text{ or } \frac{\partial u}{\partial x_i}(x) < 0 \quad \forall x \in (0, a) \times (0, b). \quad (2.2)$$

This follows easily by applying Hopf’s lemma to the equation

$$\Delta \left(\frac{\partial u}{\partial x_i} \right) + \rho \frac{e^u}{\int_{\Omega} e^u} \frac{\partial u}{\partial x_i} = 0.$$

In our setting, Proposition 2.1 can also be proved by using a variant of the “moving planes method” and admits the following extension:

Proposition 2.3. *Assume (H) holds and let u be a semi-stable critical point of J^ρ (i.e. satisfying (1.6)) with its maximum located at the origin. Then*

$$u(x_1, x_2) = u(-x_1, x_2) = u(x_1, -x_2), \quad \forall (x_1, x_2) \in \Omega. \quad (2.3)$$

Furthermore, for each $i = 1, 2$, the alternative (2.2) holds.

Proof. The proof of (2.3) follows the arguments of [8, Lemma 2.2]. We only prove that $u(x_1, x_2) = u(-x_1, x_2)$ because the same arguments yield the other identity. By assumption the second variation of J^ρ at u is a nonnegative-definite bilinear form, namely

$$\int_T |\nabla \xi|^2 - \rho \left\{ \int_T \frac{e^u}{\int_T e^u} \xi^2 - \left(\int_T \frac{e^u}{\int_T e^u} \xi \right)^2 \right\} \geq 0, \quad \forall \xi \in \mathring{H}(T). \quad (2.4)$$

Since the left hand-side of (2.4) is invariant by replacing ξ with $\xi + c$ for any $c \in \mathbb{R}$, we actually have

$$\int_T |\nabla \xi|^2 - \rho \left\{ \int_T \frac{e^u}{\int_T e^u} \xi^2 - \left(\int_T \frac{e^u}{\int_T e^u} \xi \right)^2 \right\} \geq 0, \quad \forall \xi \in H^1(T). \quad (2.5)$$

Consider on the rectangular periodic cell Ω , the functions u^* and w defined as:

$$u^*(x_1, x_2) := u(-x_1, x_2), \quad w := u - u^*.$$

The function w satisfies the linear problem

$$\Delta w + c(x)w = 0, \quad \text{with } c(x) := \frac{\rho}{\int_{\Omega} e^u} \frac{e^u - e^{u^*}}{u - u^*}. \quad (2.6)$$

By defining $\Omega_+ := \{x \in \Omega : x_1 > 0\}$, we claim that exactly one of the following alternative holds:

$$(i) \ w > 0 \text{ in } \Omega_+, \quad (ii) \ w < 0 \text{ in } \Omega_+, \quad (iii) \ w \equiv 0 \text{ in } \Omega_+. \quad (2.7)$$

To prove this claim, assume by contradiction that w changes sign in Ω_+ . Therefore, since $w(x_1, x_2) = -w(-x_1, x_2)$, the following two sets have positive measure:

$$D_+ := \{x \in \Omega_+ : w(x) > 0\} \quad D_- := \{x \in \Omega \setminus \Omega_+ : w(x) > 0\}.$$

We define a new function in Ω defined by

$$\phi(x) = \begin{cases} w(x) & \text{if } x \in D_+, \\ -tw(x) & \text{if } x \in D_-, \\ 0 & \text{otherwise,} \end{cases}$$

where t is a positive constant such that

$$\int_{\Omega} e^u \phi(x) dx = 0. \quad (2.8)$$

Note that this choice of t is made possible because D_+, D_- are assumed to have positive measure. From the convexity of the exponential function, we note that the following inequality holds in the open set $D_+ \cup D_-$:

$$c(x) = \frac{\rho}{\int_{\Omega} e^u} \frac{e^u - e^{u^*}}{u - u^*}(x) < \frac{\rho}{\int_{\Omega} e^u} e^{u(x)}, \quad \forall x \in D_+ \cup D_-. \quad (2.9)$$

By using (2.9), we deduce that

$$0 = \Delta w + c(x)w < \Delta w + \rho \frac{e^u}{\int_{\Omega} e^u} w, \quad \text{in } D_+ \cup D_-. \quad (2.10)$$

By (2.10), the fact that $w > 0$ in $D_+ \cup D_-$ and the definition of ϕ imply

$$\int_{\Omega} |\nabla \phi|^2 - \rho \int_{\Omega} \frac{e^u}{\int_{\Omega} e^u} \phi^2 < 0. \quad (2.11)$$

Recalling the choice done in (2.8), we see that (2.11) yields a contradiction to (2.5). Therefore w cannot change sign in Ω_+ , namely

$$w(x) \geq 0 \ \forall x \in \Omega_+ \quad \text{or} \quad w(x) \leq 0 \ \forall x \in \Omega_+. \quad (2.12)$$

By applying the strong maximum principle, the alternative (2.7) follows.

Let us see now why cases (i) and (ii) in (2.7) can be excluded. Since both functions u, u^* coincide when $x_1 = 0$ and achieve their maximum at the origin, we also have

$$w(0, x_2) = 0 \quad \forall x_2 \in (-b, b), \quad \frac{\partial w}{\partial x_1}(0, 0) = 0. \tag{2.13}$$

By applying to Equation (2.6) Hopf’s lemma at the point $(0, 0)$ (in the domain Ω_+) together with properties (2.12) and (2.13), we deduce that $w \equiv 0$. This proves that $u(x_1, x_2) = u(-x_1, x_2)$ for all $(x_1, x_2) \in \Omega$. Arguing in the same way, we can also prove that $u(x_1, x_2) = u(x_1, -x_2)$, and so (2.3) holds.

The proof of the alternative (2.2) is inspired by some arguments found in [12, Section 3]. We give the arguments for $\frac{\partial u}{\partial x_1}$, since the same proof holds for $\frac{\partial u}{\partial x_2}$. Assume first by contradiction that $\frac{\partial u}{\partial x_1}$ changes sign in Ω_+ . In this case both the following sets have positive measure:

$$E_+ := \left\{ x \in \Omega_+ : \frac{\partial u}{\partial x_1}(x) > 0 \right\} \quad E_- := \left\{ x \in \Omega_+ : \frac{\partial u}{\partial x_1} \leq 0 \right\}. \tag{2.14}$$

Let us introduce in Ω the function

$$\varphi(x) = \begin{cases} \frac{\partial u}{\partial x_1}(x) & \text{if } x \in E_+, \\ -t \frac{\partial u}{\partial x_1}(x) & \text{if } x \in E_-, \\ 0 & \text{otherwise,} \end{cases} \tag{2.15}$$

where t is a positive constant chosen such that

$$\int_{\Omega} e^u \varphi(x) dx = 0. \tag{2.16}$$

Due to the symmetry property (2.3) satisfied by u , we have $\frac{\partial u}{\partial x_1}(0, x_2) = 0$ and therefore the function φ is a periodic $H^1_{loc}(\mathbb{R}^2)$ -function. From the fact that

$$\Delta \left(\frac{\partial u}{\partial x_1} \right) + \rho \frac{e^u}{\int_{\Omega} e^u} \frac{\partial u}{\partial x_1} = 0 \quad \text{in } \Omega, \tag{2.17}$$

we easily deduce that

$$\int_{\Omega} |\nabla \varphi|^2 - \rho \int_{\Omega} \frac{e^u}{\int_{\Omega} e^u} \varphi^2 = 0. \tag{2.18}$$

Hence (2.18) together with (2.16) show that the function φ realizes the equality in (2.5). So φ satisfies the equation

$$\Delta \varphi + \rho \frac{e^u}{\int_{\Omega} e^u} \left(\varphi - \int_{\Omega} \frac{e^u}{\int_{\Omega} e^u} \varphi \right) = 0 \quad \text{in } \Omega,$$

and because of (2.16), we actually get

$$\Delta\varphi + \rho \frac{e^u}{\int_{\Omega} e^u} \varphi = 0 \quad \text{in } \Omega.$$

Since by construction φ vanishes when $x_1 \leq 0$, the unique continuation principle shows that $\varphi \equiv 0$, and therefore $\frac{\partial u}{\partial x_1} \equiv 0$. This is in contradiction with the fact that E_+, E_- have positive measure. Therefore, by also taking into account that u achieves its maximum point at 0, we deduce that $\frac{\partial u}{\partial x_1} \leq 0$ in Ω_+ . By applying the strong maximum principle to (2.17), the alternative (2.2) follows. \square

We conclude this section with some observations on the critical points of solutions of Problem (1.2) which are Steiner symmetric in the periodic cell.

Remark 2.4. Consider any solution u of Problem (1.2) which satisfies the additional properties:

- (a) the function u is Steiner symmetric in Ω , *i.e.* it enjoys properties (2.1),
- (b) $\frac{\partial u}{\partial x_i} \neq 0$ for $i = 1, 2$.

Then the set of critical points of the periodic solution $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ is exactly given by $\mathcal{C} := \{(ma, nb) : m, n \in \mathbb{Z}\}$. Since $\frac{\partial u}{\partial x_1}(0, x_2) = 0$ and $\frac{\partial u}{\partial x_2}(x_1, 0) = 0$, we also derive

$$\frac{\partial^2 u}{\partial x_1 \partial x_2}(c) = 0, \quad \forall c \in \mathcal{C}. \tag{2.19}$$

Since for $i = 1, 2$ we have

$$\Delta \left(\frac{\partial u}{\partial x_i} \right) + \rho \frac{e^u}{\int_{\Omega} e^u} \frac{\partial u}{\partial x_i} = 0, \quad \frac{\partial u}{\partial x_i} \leq 0 \text{ in } (0, a) \times (0, b),$$

an application of Hopf’s lemma yields for $i = 1, 2$

$$\frac{\partial^2 u}{\partial x_i^2}(c) \neq 0, \quad \forall c \in \mathcal{C}. \tag{2.20}$$

Therefore (2.19) and (2.20) show that each critical point of u (Steiner symmetric in Ω) is non-degenerate. By Proposition 2.3, this remark applies in particular to any global or local minimizers of the functional J^ρ .

3. Bol’s Isoperimetric Inequality

Let us start by recalling the following isoperimetric inequality which goes back to Bol [2].

Proposition 3.1 (Bol’s inequality). *Let Ω be a simply-connected domain of \mathbb{R}^2 and $v \in C^2(\Omega)$ satisfying*

$$-\Delta v \leq e^v \quad \text{and} \quad \int_{\Omega} e^v \leq 8\pi. \tag{3.1}$$

Then, for any $\omega \subset\subset \Omega$ of class C^1 the following inequality holds:

$$\left(\int_{\partial\omega} e^{v/2} \right)^2 \geq \frac{1}{2} \left(\int_{\omega} e^v \right) \left(8\pi - \int_{\omega} e^v \right). \tag{3.2}$$

Above result with $\omega \subset\subset \Omega$ simply-connected and v analytic can be found in Bandle [1]. In [25], the proof given by Suzuki assumes only the function v to be of class C^2 but ω to be simply connected. In Chang *et al.* [4, Lemma 4.2], it was noted that the assumption on ω to be simply connected is not necessary. But let us emphasize that the assumption on the domain Ω to be *simply connected* is crucial. Indeed, consider for each $t > 0$ the radial harmonic function

$$v(x) = -2 \log(t|x|), \quad x \neq 0.$$

On the annulus $A := \{1 < |x| < R\}$, we have

$$\int_{\partial A} e^{v/2} = \frac{4\pi}{t} \quad \text{and} \quad \int_A e^v = \frac{2\pi}{t^2} \log R,$$

and in particular

$$\frac{\left(\int_{\partial A} e^{v/2} \right)^2}{\left(\int_A e^v \right) \left(8\pi - \int_A e^v \right)} = 8\pi (\log R)^{-1} \left(8\pi - \frac{2\pi}{t^2} \log R \right)^{-1}. \tag{3.3}$$

Hence by choosing $t^2 = \log R$, we see that the assumptions (3.1) are satisfied. But, letting $R \rightarrow \infty$, the ratio (3.3) tends to zero, and therefore Bol’s inequality (3.2) cannot hold.

In order to handle in Problem (1.2) the case $\rho = 8\pi$, we shall need the Bol’s isoperimetric inequality in the following form:

Proposition 3.2. *Let Ω be a simply-connected domain of \mathbb{R}^2 and $v \in C^2(\overline{\Omega})$ satisfying*

$$-\Delta v < e^v \text{ in } \overline{\Omega} \quad \text{and} \quad \int_{\Omega} e^v \leq 8\pi. \tag{3.4}$$

Then, for any $\omega \subset\subset \Omega$ of class C^1 the following strict inequality holds:

$$\left(\int_{\partial\omega} e^{v/2} \right)^2 > \frac{1}{2} \left(\int_{\omega} e^v \right) \left(8\pi - \int_{\omega} e^v \right). \tag{3.5}$$

Proof. By (3.4) we can find $\varepsilon \in (0, 1)$ such that $-\Delta v(x) \leq (1 - \varepsilon)e^{v(x)}$ for all $x \in \Omega$. Therefore the function $\tilde{v} := v + \log(1 - \varepsilon)$ satisfies assumptions (3.1). Hence, by applying Bol's inequality (3.2) to the function \tilde{v} we get:

$$\left(\int_{\partial\omega} e^{v/2}\right)^2 \geq \frac{1}{2} \left(\int_{\omega} e^v\right) \left(8\pi - (1 - \varepsilon) \int_{\omega} e^v\right),$$

and so the conclusion (3.5) follows. □

Bol's inequality will be used to derive the following.

Proposition 3.3. *Let Ω be a simply-connected domain of \mathbb{R}^2 and $v \in C^2(\overline{\Omega})$ such that $-\Delta v < e^v$ in $\overline{\Omega}$. Assume there exists $\varphi \in C^1(\overline{\Omega})$ such that*

$$\Delta\varphi + e^v\varphi = 0 \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega, \quad \varphi \not\equiv 0. \tag{3.6}$$

Then $\int_{\Omega} e^v > 4\pi$.

Proof. Assume $\int_{\Omega} e^v \leq 4\pi$, we shall prove that any $\varphi \in C^1(\overline{\Omega})$ with $\varphi \not\equiv 0$ satisfies the following type of Faber-Krahn inequality:

$$\frac{\int_{\Omega} |\nabla\varphi|^2}{\int_{\Omega} e^v\varphi^2} > 1.$$

Note that since φ is continuous up to the boundary the following property on the upper level sets holds:

$$\Omega_t := \{|\varphi| > t\} \subset\subset \Omega, \quad \forall t \geq 0.$$

The following arguments are borrowed from [1] and [25]. We give it for the sake of completeness. Set $U(x) = -2\log(1 + \frac{|x|^2}{8})$ which satisfies $\Delta U + e^U = 0$ in \mathbb{R}^2 . Note that this function realizes the equality in (3.2) when ω is a ball centered at the origin. We shall make a rearrangement of the function $|\varphi|$ with respect to the measure e^U and e^v . To this end, define first the balls Ω^* and Ω_t^* centered at the origin as follows:

$$\int_{\Omega^*} e^{U(x)} dx = \int_{\Omega} e^{v(x)}, \quad \int_{\Omega_t^*} e^{U(x)} dx = \int_{\Omega_t} e^{v(x)}.$$

The balls Ω_t^* can be seen geometrically as geodesic balls on the two-dimensional sphere having same measure as the set $\{|\varphi| > t\}$ endowed with the measure $e^v dx$. Define the symmetrization $\varphi^* : \Omega^* \rightarrow \mathbb{R}$ of the function $|\varphi|$ as follows:

$$\varphi^*(x) = \sup\{t \in \mathbb{R} : x \in \Omega_t^*\}.$$

We define in this way an equimeasurable rearrangement with respect to the measures $e^U dx$ and $e^v dx$, *i.e.*

$$\int_{\{\varphi^* > t\}} e^U = \int_{\Omega_t} e^v, \quad \forall t > 0. \tag{3.7}$$

In particular, we have (Cavalieri’s principle):

$$\int_{\Omega^*} e^U |\varphi^*|^2 = \int_{\Omega} e^v |\varphi|^2. \tag{3.8}$$

Let us prove that the Dirichlet integral decreases by making such an arrangement. By applying coarea formula, Schwarz inequality, Bol’s inequality (3.4) and also (3.7), we get:

$$\begin{aligned} -\frac{d}{dt} \int_{\Omega_t} |\nabla|\varphi||^2 &= \int_{\{|\varphi|=t\}} |\nabla\varphi| \\ &\geq \left(\int_{\{|\varphi|=t\}} e^{v/2} \right)^2 \left(\int_{\{|\varphi|=t\}} \frac{e^v}{|\nabla\varphi|} \right)^{-1} \\ &= \left(\int_{\{|\varphi|=t\}} e^{v/2} \right)^2 \left(-\frac{d}{dt} \int_{\Omega_t} e^v \right)^{-1} \\ &> \frac{1}{2} \left(\int_{\Omega_t} e^v \right) \left(8\pi - \int_{\Omega_t} e^v \right) \left(-\frac{d}{dt} \int_{\Omega_t} e^v \right)^{-1} \\ &= \frac{1}{2} \left(\int_{\Omega_t^*} e^v \right) \left(8\pi - \int_{\Omega_t^*} e^v \right) \left(-\frac{d}{dt} \int_{\Omega_t^*} e^v \right)^{-1}, \end{aligned} \tag{3.9}$$

for almost every $t \geq 0$. Furthermore, since e^U realizes on each ball ω the equality in (3.2), we check easily that:

$$-\frac{d}{dt} \int_{\Omega_t^*} |\nabla\varphi^*|^2 = \frac{1}{2} \left(\int_{\Omega_t^*} e^v \right) \left(8\pi - \int_{\Omega_t^*} e^v \right) \left(-\frac{d}{dt} \int_{\Omega_t^*} e^v \right)^{-1}. \tag{3.10}$$

Hence, (3.9) and (3.10) yield:

$$-\frac{d}{dt} \int_{\Omega_t} |\nabla\varphi|^2 > -\frac{d}{dt} \int_{\Omega_t^*} |\nabla\varphi^*|^2 \quad \text{a.e. } t \geq 0. \tag{3.11}$$

By integrating (3.11) with respect to t , we obtain

$$\int_{\Omega} |\nabla\varphi|^2 > \int_{\Omega^*} |\nabla\varphi^*|^2. \tag{3.12}$$

Hence, from (3.8) and (3.12), we deduce that

$$\frac{\int_{\Omega} |\nabla \varphi|^2}{\int_{\Omega} e^v |\varphi|^2} > \frac{\int_{\Omega^*} |\nabla \varphi^*|^2}{\int_{\Omega^*} e^U |\varphi^*|^2} \geq \lambda_1(e^U, \Omega^*), \tag{3.13}$$

where we set whenever $B \subset \subset \mathbb{R}^2$:

$$\lambda_1(e^U, B) = \inf \left\{ \frac{\int_B |\nabla \xi|^2}{\int_B e^U \xi^2} : \xi \in H_0^1(B), \xi \not\equiv 0 \right\}. \tag{3.14}$$

Under the assumption that $\int_{\Omega} e^v \leq 4\pi$, we claim that $\lambda_1(e^U, \Omega^*) \geq 1$. Indeed, on the one hand a straightforward computation shows that $\psi = \frac{8-r^2}{8+r^2}$ solves

$$-\Delta \psi = e^U \psi, \quad \psi > 0 \text{ in } B_{\sqrt{8}}, \quad \psi \in H_0^1(B_{\sqrt{8}}),$$

where $B_{\sqrt{8}}$ denotes the ball $B(0, \sqrt{8})$. Therefore, we get $\lambda_1(e^U, B_{\sqrt{8}}) = 1$. On the other hand, from (3.7), we know that $\int_{\Omega} e^U \leq 4\pi$. By explicit calculation, we get $\Omega^* \subseteq B_{\sqrt{8}}$. Hence, we deduce

$$\lambda_1(e^U, \Omega^*) \geq \lambda_1(e^U, B_{\sqrt{8}}) = 1. \tag{3.15}$$

Inequalities (3.13) and (3.15) allow to conclude the proof of the theorem. □

4. One-dimensional symmetry

To study the one-dimensional properties of a solution u of Problem (1.2), we consider for each $t \in [0, 1]$, the periodic function $\varphi_t : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$\varphi_t := t \frac{\partial u}{\partial x_1} + (1-t) \frac{\partial u}{\partial x_2}. \tag{4.1}$$

In [21], Payne was considering a similar family in order to prove that the first eigenfunction of the Dirichlet Laplacian in a strictly convex domain has at most one critical point. But in our case, the purpose and the way how we are going to exploit the family (4.1) differs notably from [21].

By differentiating with respect to each variable the equation (1.2) satisfied by u , we note that

$$\Delta \varphi_t + \rho \frac{e^u}{\int_T e^u} \varphi_t = 0, \quad \varphi_t \in \mathring{H}(T).$$

Therefore, by setting $v := u + \log(\frac{\rho}{\int_T e^u})$, the function φ_t is a solution of the linear problem

$$\Delta \varphi_t + e^v \varphi_t = 0, \quad \varphi_t \in \mathring{H}(T), \tag{4.2}$$

i.e., φ_t is an eigenfunction of the linear operator $\Delta + e^v$. Let us set

$$\mathcal{Z}_t = \left\{ x \in \mathbb{R}^2 : \varphi_t(x) = 0 \right\}.$$

Let us restrict our attention only to solutions u which are Steiner symmetric in the periodic cell Ω , i.e. satisfying

$$\begin{cases} u(x_1, x_2) = u(-x_1, x_2) = u(x_1, -x_2) & \forall (x_1, x_2) \in \Omega, \\ \frac{\partial u}{\partial x_i}(x) \leq 0 & \forall x \in (0, a) \times (0, b), i = 1, 2. \end{cases} \quad (4.3)$$

The following proposition will be the crucial point to derive our one-dimensional symmetry results.

Proposition 4.1. *Let (H) be satisfied and (ρ, u) be a solution of Problem (1.2) with u Steiner symmetric. Then, by considering the family of functions φ_t defined by (4.1), the following alternative holds:*

- (a) either $\frac{\partial u}{\partial x_i} \equiv 0$ for some $i = 1, 2$;
- (b) or else, for some $t_0 \in [0, 1]$ we can find a bounded simply connected domain $D_{t_0} \subset \mathbb{R}^2$ such that

$$\partial D_{t_0} \subset \mathcal{Z}_{t_0} \quad \text{and} \quad \int_{D_{t_0}} e^u \leq \frac{1}{2} \int_{\Omega} e^u. \quad (4.4)$$

Proof. Assume $\frac{\partial u}{\partial x_1} \not\equiv 0$ and $\frac{\partial u}{\partial x_2} \not\equiv 0$. We need to show the existence of a simply connected domain satisfying alternative (b). The main idea rests on the observation that the nodal lines of φ_0 and φ_1 are respectively parallel to the x_1 -axis and x_2 -axis. So as t varies the nodal lines of φ_t must come into contact and create a simply connected region in \mathbb{R}^2 . This idea can be made rigorous by arguing as follows.

Consider the points $P_1 = (0, 0)$, $P_2 = (a, 0)$ and introduce the set

$$\mathcal{T} := \{t \in [0, 1] : P_1, P_2 \text{ are in the same connected component of } \mathcal{Z}_t\},$$

Note first that

$$0 \in \mathcal{T} \quad \text{and} \quad 1 \notin \mathcal{T}. \quad (4.5)$$

Indeed for $t = 0$ we have: $\varphi_0(x_1, 0) = \frac{\partial u}{\partial x_2}(x_1, 0) = 0$ for all $x_1 \in \mathbb{R}$, and therefore P_1, P_2 belong to the same connected component of \mathcal{Z}_0 (see Figure 4.1).

When $t = 1$, we have $\varphi_1 \equiv \frac{\partial u}{\partial x_1}$. Since $\frac{\partial u}{\partial x_1}(x) < 0$ for $x_1 \in (0, a)$ (see Remark 2.2) we see that $\mathcal{Z}_1 = \bigcup_{m \in \mathbb{Z}} (\{ma\} \times \mathbb{R})$. Hence P_1, P_2 belong to different connected components of \mathcal{Z}_1 (see Figure 4.2).

Consider the C^1 function $F(t, x) := t \frac{\partial \varphi_t}{\partial x_1} + (1 - t) \frac{\partial \varphi_t}{\partial x_2}$ defined in $[0, 1] \times T$. By applying to this family the results of the Appendix A, we deduce that the set \mathcal{T} is closed in $[0, 1]$ (see Lemma A.1) and also open in $[0, 1]$ if (see Lemma A.2)

$$\nabla \varphi_t(x) \neq 0, \quad \forall x \in \mathcal{Z}_t, \forall t \in [0, 1]. \quad (4.6)$$

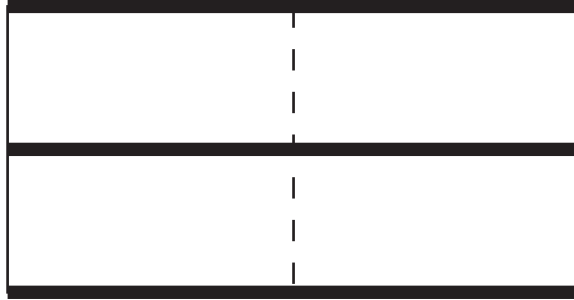


Figure 4.1. Nodal lines of φ_0 in Ω .

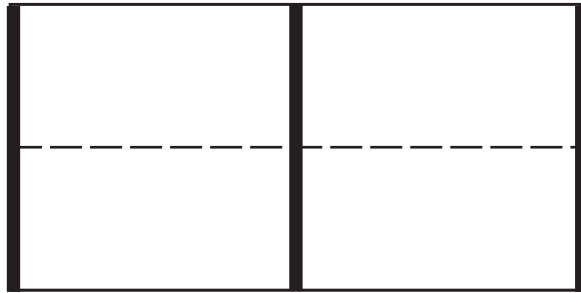


Figure 4.2. Nodal lines of φ_1 in Ω .

As a consequence, if (4.6) holds then \mathcal{T} must coincide with $[0, 1]$. This would be a contradiction because $1 \notin \mathcal{T}$ (see (4.5)). Hence

for some $t_0 \in [0, 1]$, φ_{t_0} has at least one critical point in \mathcal{Z}_{t_0} .

In order to study further the structure of \mathcal{Z}_{t_0} , note that the solution u is assumed to be Steiner symmetric in the periodic cell Ω . As a consequence

$$\varphi_{t_0} \text{ is odd} \quad \text{and} \quad \mathcal{Z}_{t_0} \cap \Omega \subseteq \{0\} \cup R \cup (-R),$$

where $R := (0, a) \times (-b, 0)$. Therefore we may restrict the study of \mathcal{Z}_{t_0} to the rectangular domain R . We distinguish two cases.

Case I: There exists a connected set $\Gamma \subseteq \mathcal{Z}_{t_0} \cap \overline{R}$ homeomorphic to a circle S^1 .

In such a case, by the Jordan's and Schoenflies' Theorem (see Thm. 10.2 and Thm. 17.1 in [18]), the bounded component D_{t_0} of $\mathbb{R}^2 \setminus \Gamma$ is contained in \overline{R} and is simply connected. Since u is axis-symmetric in Ω we also have

$$\int_{D_{t_0}} e^u \leq \int_R e^u = \frac{1}{4} \int_{\Omega} e^u,$$

and therefore (4.4) holds.

Case II: None of the connected sets in $\mathcal{Z}_{t_0} \cap \overline{R}$ is homeomorphic to S^1 .

Since φ_{t_0} is an eigenfunction of the linear problem (4.2), the results of [9] show that φ_{t_0} has only finite number of critical points in $\mathcal{Z}_{t_0} \cap \overline{R}$: c_1, \dots, c_n . Hence

$$(\mathcal{Z}_{t_0} \cap \overline{R}) \setminus \{c_1, \dots, c_n\} = \gamma_1 \cup \dots \cup \gamma_m,$$

where each $\overline{\gamma}_i$ is a C^∞ one-dimensional connected manifold. It is well-known that $\overline{\gamma}_i$ is either homeomorphic to a closed interval of \mathbb{R} (a “simple arc”) or to a circle S^1 (a “simple closed curve”). The second possibility has been considered in Case I above, and so we only consider the situation where each γ_i is a simple arc. In this case, each end-point of $\overline{\gamma}_i$ is

(i) either on ∂R and therefore coincides with one of the points

$$P_1 = (0, 0), \quad P_2 = (a, 0), \quad P_3 = (0, -b), \quad P_4 = (a, -b), \quad (4.7)$$

(ii) or coincides with one of the critical points c_1, \dots, c_n .

Consider then a critical point $c \in \overline{R}$ of φ_{t_0} on \mathcal{Z}_{t_0} . It follows easily from Remark 2.4 that actually $c \in R$. By applying Hopf’s lemma to equation (4.2), we deduce that the nodal sets $\{\varphi_{t_0} > 0\}$ and $\{\varphi_{t_0} < 0\}$ cannot satisfy the interior ball boundary condition at $c \in \mathcal{Z}_{t_0}$. Therefore, for some ball $B(c, \epsilon) \subset R$, the set $(\mathcal{Z}_{t_0} \setminus \{c\}) \cap B(c, \epsilon)$ has at least 4 connected components $\gamma_1, \dots, \gamma_4$. Consider now in $(\mathcal{Z}_{t_0} \setminus \{c\}) \cap \overline{R}$ the connected components $\Gamma_1, \dots, \Gamma_4$ containing respectively $\gamma_1, \dots, \gamma_4$. If $\Gamma_i = \Gamma_j$ for some $i \neq j$, then $\Gamma_i \cup \{c\}$ is a simple closed curve, the situation considered in Case I. So $(\mathcal{Z}_{t_0} \setminus \{c\}) \cap \overline{R}$ has at least 4 different connected components Γ_i , each of which being a finite union of Jordan arcs. Hence $\Gamma_i \cap \partial R \neq \emptyset$ and therefore we may numerate each of this component in such a way that $\Gamma_i \cap \partial R = \{P_i\}$ (with P_i defined in (4.7)).

Consider then the sets: $-\overline{\Gamma}_1, -\overline{\Gamma}_2, \overline{\Gamma}_3 - 2a, \overline{\Gamma}_4 - 2a$ (subsets of \mathcal{Z}_{t_0}). Since φ_t is odd and periodic, we derive that

$$\tilde{\Gamma} := (-\overline{\Gamma}_1) \cup (-\overline{\Gamma}_2) \cup (\overline{\Gamma}_3 - 2a) \cup (\overline{\Gamma}_4 - 2a),$$

is a simple closed curve in \mathbb{R}^2 (see Figure 4.3) contained in \mathcal{Z}_{t_0} .

By applying Jordan-Schoenflies Theorem [18], we deduce that the bounded connected component D_{t_0} of $\mathbb{R}^2 \setminus \tilde{\Gamma}$ is simply connected and $\partial D_{t_0} = \tilde{\Gamma}$. Moreover let R_i be the subsets of R defined by:

$$\begin{aligned} \partial R_1 &:= \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \{(x, 0) \in \mathbb{R}^2 : x \in [0, a]\}, \\ \partial R_2 &:= \overline{\Gamma}_2 \cup \overline{\Gamma}_3 \cup \{(a, y) \in \mathbb{R}^2 : y \in [0, -b]\}, \\ \partial R_3 &:= \overline{\Gamma}_3 \cup \overline{\Gamma}_4 \cup \{(x, -b) \in \mathbb{R}^2 : x \in [0, a]\}, \\ \partial R_4 &:= \overline{\Gamma}_4 \cup \overline{\Gamma}_1 \cup \{(a, y) \in \mathbb{R}^2 : y \in [0, -b]\}, \end{aligned}$$

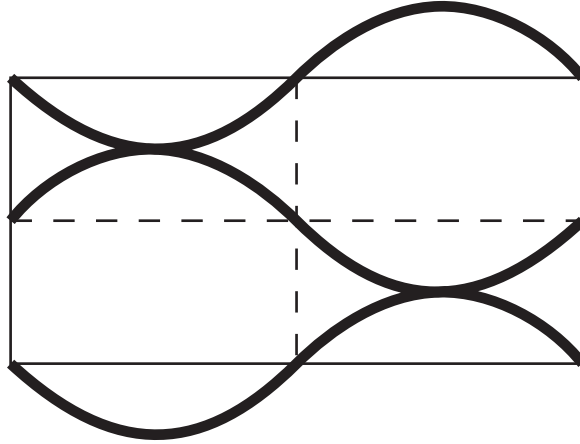


Figure 4.3. Nodal lines of φ_{t_0} .

Since u is axis-symmetric and φ_{t_0} is odd we also have

$$\begin{aligned} \int_{D_{t_0}} e^u &= \int_{R_4} e^u + \int_{-R_1} e^u + \int_{R_2-2a} e^u + \int_{-R_3} e^u + \int_{(-a,0)\times(-b,0)} e^u \\ &= \int_{R_4} e^u + \int_{R_1} e^u + \int_{R_2} e^u + \int_{R_3} e^u + \int_{(-a,0)\times(-b,0)} e^u \\ &= 2 \int_R e^u = \frac{1}{2} \int_{\Omega} e^u. \end{aligned}$$

Therefore the set D_{t_0} fulfills the conditions (4.4). □

We are now able to prove that any Steiner symmetric solution is one-dimensional whenever $\rho \leq 8\pi$.

Proof of Theorem 1.2. The proof of part (a) is the content of Proposition 2.3. To prove the statement (b) of the theorem, let u be a Steiner symmetric solution and assume that $\frac{\partial u}{\partial x_i} \neq 0$ for $i = 1, 2$. Choose $t_0 \in (0, 1)$, φ_{t_0} and the simply connected domain D_{t_0} as given by Proposition 4.1. By setting $v := u + \log(\frac{\rho}{\int_{\Omega} e^u})$, note first

$$-\Delta v = e^v - \frac{\rho}{|T|} < e^v \text{ in } \overline{D}_{t_0} \quad \text{and} \quad \int_{D_{t_0}} e^v \leq 4\pi. \tag{4.8}$$

Furthermore in D_{t_0} , φ_{t_0} satisfies

$$-\Delta \varphi_{t_0} = e^v \varphi_{t_0}, \quad \varphi_{t_0} \in C^1(\overline{D}_{t_0}), \quad \varphi_{t_0} = 0 \text{ in } \partial D_{t_0}. \tag{4.9}$$

Hence (4.8), (4.9) together with Proposition 3.3 show that $\varphi_{t_0} \equiv 0$. Therefore $\nabla u(x) = \frac{\partial u}{\partial x_2}(x) \begin{pmatrix} 1 - t_0^{-1} \\ 1 \end{pmatrix}$, *i.e.* the level sets of u are parallel straight lines. Since u is axis symmetric in the periodic cell Ω , the level sets must actually be parallel to either the x_1 or x_2 -axis. This would contradict $\frac{\partial u}{\partial x_i} \neq 0$ for $i = 1, 2$. So we deduce that u is one-dimensional.

To prove the last statement of the theorem, we note that for $\rho \leq \lambda_1(T)|T|$ a result of [23] shows that $u \equiv 0$ is the unique one-dimensional solution of Problem (1.2). □

Clearly Theorem 1.1 stated in the introduction follows immediately from Theorem 1.2. As a consequence we derive the inequality

$$\frac{1}{2} \int_T |\nabla u|^2 - \rho_0 \log \left(\frac{1}{|T|} \int_T e^u \right) \geq 0, \quad \forall u \in \mathring{H}(T), \tag{4.10}$$

with equality holding if and only if $u \equiv 0$. The above inequality can also be rewritten as (1.4) stated in the introduction, and holds for ρ_0 replaced by any $\rho \leq \rho_0$. To see the sharpness of inequality (4.10), we notice the following:

- (a) For $\rho > 8\pi$, test functions of the type $\delta_\mu(x) = \log \frac{8\mu^2}{(1+\mu^2|x|^2)^2}$ can be used to show that $\inf\{J^\rho(u) : u \in \mathring{H}(T)\} = -\infty$ (see [24]);
- (b) When $\lambda_1(T)|T| < 8\pi$ and $\rho \in (\lambda_1(T)|T|, 8\pi)$, by taking the function $\epsilon\Psi_1$ with Ψ_1 an eigenfunction associated to the first eigenvalue of the Laplacian $\lambda_1(T)$, we see easily that $J^\rho(\epsilon\Psi_1) < 0$ when ϵ is small enough.

This shows that inequality (1.4) is sharp.

A. Appendix: Two connectedness properties

Consider a C^1 -manifold M of dimension m compact without boundary and a family of functions $F : [0, 1] \times M \rightarrow \mathbb{R}$. Under suitable assumptions, we prove in this appendix that the property for two fixed points of being connected in the sets:

$$\mathcal{Z}_t := \{x \in M : F(t, x) = 0\}, \tag{A.1}$$

is preserved as t varies from $t = 0$ to $t = 1$. Such a result is used in Section 4 when the manifold is a two dimensional torus.

Let us first recall some known facts. We say that two points $A, B \in \mathcal{Z}_t$ are “connected in \mathcal{Z}_t ” if there is a connected set $\Gamma \subseteq \mathcal{Z}_t$ containing both A, B . Given a sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$ of subsets of a topological space X , we define the sets $\limsup\{\Gamma_n\}$ and $\liminf\{\Gamma_n\}$ as follows:

- (a) $x \in \limsup\{\Gamma_n\}$ if and only if each neighborhood of x intersects Γ_n for infinitely many indices $n \in \mathbb{N}$,
- (b) $x \in \liminf\{\Gamma_n\}$ if and only if each neighborhood of x intersects Γ_n but for finitely many indices $n \in \mathbb{N}$.

If $\{\Gamma_n\}_{n \in \mathbb{N}}$ is a sequence of *connected* sets in a metric space satisfying

$$\liminf\{\Gamma_n\} \neq \emptyset \quad \text{and} \quad \bigcup_{n \in \mathbb{N}} \Gamma_n \text{ is relatively compact,} \tag{A.2}$$

then $\limsup\{\Gamma_n\}$ is a nonempty connected set (see [26, Theorem 9.1], or [27, p. 39]).

Lemma A.1. *Let $F : [0, 1] \times M \rightarrow \mathbb{R}$ be a family of continuous maps. Assume two points $A, B \in M$ are connected in \mathcal{Z}_{t_n} with $\lim_{n \rightarrow \infty} t_n = \hat{t}$. Then A, B are connected in $\mathcal{Z}_{\hat{t}}$.*

Proof. For each $n \in \mathbb{N}$, consider a connected set $\Gamma_n \subseteq \mathcal{Z}_{t_n}$ containing A, B . Clearly $A, B \in \liminf\{\Gamma_n\}$, Γ_n is connected and since M is compact the set $\bigcup_{n \in \mathbb{N}} \Gamma_n$ is relatively compact in M . Thus (A.2) holds in the metric space M . Therefore $\limsup\{\Gamma_n\}$ is *connected* (by [26, 27]) and contains the points A, B . Furthermore by continuity of F , we check easily that $\limsup\{\Gamma_n\} \subseteq \mathcal{Z}_{\hat{t}}$. Therefore A, B are connected in $\mathcal{Z}_{\hat{t}}$. □

The above result may fail if M is not compact. For example in \mathbb{R}^2 consider the family of functions $F(t, x) = x_1(1 - x_1) - tx_2^2$. For each $t > 0$ the set of zeros of $F(t, \cdot)$ connects the points $(0, 0)$ and $(1, 0)$, but this is not anymore the case at $t = 0$.

The following lemma is a consequence of the implicit function theorem.

Lemma A.2. *Let $A, B \in M$ and $F : [0, 1] \times M \rightarrow \mathbb{R}$ be a C^1 -mapping such that*

$$A, B \in \mathcal{Z}_t, \quad \forall t \in [0, 1], \tag{A.3}$$

Assume that for some $\hat{t} \in [0, 1]$ the points A, B are connected in $\mathcal{Z}_{\hat{t}}$ and

$$\partial_2 F(\hat{t}, z) \neq 0, \quad \forall z \in \mathcal{Z}_{\hat{t}}. \tag{A.4}$$

Then there exists an interval $\hat{I} = (\hat{t} - \varepsilon, \hat{t} + \varepsilon) \cap [0, 1]$ such that A, B are connected in \mathcal{Z}_t for any $t \in \hat{I}$.

Proof. By using assumption (A.4), at each point $p \in \mathcal{Z}_{\hat{t}}$ there is a neighborhood U_p and a homeomorphism $\varphi_p : (-1, 1)^m \rightarrow U_p$ such that

$$\frac{\partial(F_t \circ \varphi_p)}{\partial x_m}(x) \neq 0, \quad \forall x \in \varphi_p^{-1}(\mathcal{Z}_{\hat{t}} \cap U_p).$$

By applying the implicit function theorem to the mapping

$$\tilde{F}_p : [0, 1] \times (-1, 1)^{m-1} \times (-1, 1) \rightarrow \mathbb{R}, \quad (t, x', x_m) \mapsto F(t, \varphi_p(x', x_m)),$$

we can find an interval $I_p := (\hat{t} - \varepsilon_p, \hat{t} + \varepsilon_p) \cap [0, 1]$ and a unique C^1 -mapping

$$\Phi_p : I_p \times (-\delta_p, \delta_p)^{m-1} \rightarrow \mathbb{R}$$

such that:

$$\tilde{F}_p(t, x', \Phi_p(t, x')) = 0, \quad \forall (t, x') \in I_p \times (-\delta_p, \delta_p)^{m-1}.$$

Hence by setting $\tilde{U}_p := \varphi_p((-\delta_p, \delta_p)^{m-1} \times (-1, 1))$, we deduce that $\mathcal{Z}_t \cap \tilde{U}_p$ is connected whenever $t \in I_p$.

Consider the family $\mathcal{F} = \{(\tilde{U}_p, I_p, \Phi_p) : p \in \mathcal{Z}_{\hat{t}}\}$ and let $\Gamma_{\hat{t}} \subseteq \mathcal{Z}_{\hat{t}}$ be a compact connected set containing the points A, B . By compactness and connectedness we can find a finite number of points $p_1, \dots, p_k \in \Gamma_{\hat{t}}$ such that

$$\Gamma_{\hat{t}} \subseteq \bigcup_{i=1}^k \tilde{U}_{p_i}, \quad \tilde{U}_{p_i} \cap \tilde{U}_{p_{i+1}} \neq \emptyset, \quad (A, B) \in \tilde{U}_{p_1} \times \tilde{U}_{p_k}, \quad (\text{A.5})$$

$$\mathcal{Z}_t \cap \tilde{U}_{p_i} \text{ connected} \quad \forall t \in I_{p_i}, \quad i = 1, \dots, k. \quad (\text{A.6})$$

Furthermore by using the continuity of each function Φ_{p_i} , we may choose a smaller interval $\hat{I} \subseteq \bigcap_{i=1}^k I_{p_i}$ in order to also have

$$\mathcal{Z}_t \cap (\tilde{U}_{p_i} \cap \tilde{U}_{p_{i+1}}) \neq \emptyset, \quad \forall t \in \hat{I}, \quad i = 1, \dots, k - 1.$$

Hence with this choice of open sets $\tilde{U}_{p_i} \subset M$ and of interval $\hat{I} \subseteq [a, b]$, the following conditions are satisfied for each $t \in \hat{I}$,

$$\begin{cases} \mathcal{Z}_t \cap \tilde{U}_{p_i} \text{ is connected, } \mathcal{Z}_t \cap (\tilde{U}_{p_i} \cap \tilde{U}_{p_{i+1}}) \neq \emptyset \text{ (for } i = 1, \dots, k - 1), \\ A \in \mathcal{Z}_t \cap \tilde{U}_{p_1}, \quad B \in \mathcal{Z}_t \cap \tilde{U}_{p_k}. \end{cases}$$

We infer that $\bigcup_{i=1}^k (\mathcal{Z}_t \cap \tilde{U}_{p_i})$ is a connected set containing A, B whenever $t \in \hat{I}$. □

References

- [1] C. BANDLE, “Isoperimetric Inequalities and Applications”, Pitman, London, 1980.
- [2] G. BOL, *Isoperimetriscche Ungleichungen fur Bereiche auf Flächen*, Jahresber. Deutschen Math. Vereinigung **51** (1941), 219–257.
- [3] X. CABRÉ, M. LUCIA and M. SANCHÓN, *A mean field equation on a torus: one-dimensional symmetry of solutions*, Comm. Partial Differential Equations **30** (2005), 1315–1330.
- [4] S.-Y. A. CHANG, C.-C. CHEN and C.-S. LIN, *Extremal functions for a mean field equation in two dimension*, In: “Lecture on Partial Differential Equations”, New Stud. Adv. Math., 2, Int. Press, Somerville, MA, 2003, 61–93.
- [5] S. CHANILLO and M. KIESSLING, *Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry*, Comm. Math. Phys. **160** (1994), 217–238.
- [6] C.-C. CHEN and C.-S. LIN, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math. **55** (2002), 728–771.
- [7] C.-C. CHEN and C.-S. LIN, *Topological degree for a mean field equation on Riemann surfaces*, Comm. Pure Appl. Math. **56** (2003), 1667–1727.
- [8] C.-C. CHEN, C.-S. LIN and G. WANG, *Concentration phenomena of two-vortex solutions in a Chern-Simons model*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (2004), 367–397.
- [9] S.-Y. CHENG, *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51** (1976), 43–55.
- [10] W. DING, J. JOST, J. LI and G. WANG, *The differential equation $\Delta u = 8\pi - 8\pi e^u$ on a compact Riemann surface*, Asian J. Math. **1** (1997), 230–248.
- [11] L. FONTANA, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helv. **68** (1993), 415–454.
- [12] M. E. GURTIN and H. MATANO, *On the structure of equilibrium phase transitions within the gradient theory of fluids*, Quart. Appl. Math. **46** (1988), 301–317.
- [13] C.-W. HONG, *A best constant and the Gaussian curvature*, Proc. Amer. Math. Soc. **97** (1986), 737–747.
- [14] B. KAWOHL, “Rearrangements and Convexity of Level Sets in PDE”, Lecture Notes in Mathematics, Vol. 1150, Springer-Verlag, 1985.
- [15] C.-S. LIN, *Uniqueness of solutions to the mean field equations for the spherical Onsager vortex*, Arch. Ration. Mech. Anal. **153** (2000), 153–176.
- [16] C.-S. LIN, *Topological degree for mean field equations on S^2* , Duke Math. J. **104** (2000), 501–536.
- [17] C.-S. LIN and M. LUCIA, *Uniqueness of solutions for a mean field equation on the torus*, J. Differential Equations **229** (2006), 172–185.
- [18] M. H. A. NEWMAN, “Elements of the Topology of Plane Sets of Points”, 2nd ed., University Press, Cambridge, 1951.
- [19] M. NOLASCO and G. TARANTELLO, *On a sharp Sobolev-type inequality on two-dimensional compact manifolds*, Arch. Ration. Mech. Anal. **145** (1998), 161–195.
- [20] E. ONOFRI, *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Phys. **86** (1982), 321–326.
- [21] L. E. PAYNE, *On two conjectures in the fixed membrane eigenvalue problem*, Z. Angew. Math. Phys. **24** (1973), 721–729.
- [22] G. PÓLYA and G. SZEGÖ, “Isoperimetric Inequalities in Mathematical Physics. Annals of Mathematics Studies”, Vol. 27, Princeton University Press, Princeton, N. J., 1951.
- [23] T. RICCIARDI and G. TARANTELLO, *On a periodic boundary value problem with exponential nonlinearities*, Differential Integral Equations **11** (1998), 745–753.
- [24] M. STRUWE and G. TARANTELLO, *On multivortex solutions in Chern-Simons Gauge theory*, Boll. Unione Mat. Ital. Sez. B, Artic. Mat. **B (8) 1** (1998), 109–121.
- [25] T. SUZUKI, *Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), 367–397.

- [26] G. T. WHYBURN, “Topological Analysis”, Princeton Mathematical Series, Vol. 23, Princeton University Press, Princeton, N. J., 1958.
- [27] G. WHYBURN and E. DUDA, “Dynamic Topology”, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1979.

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