Transport Equations with Partially *BV* Velocities

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Abstract. We prove the uniqueness of weak solutions for the Cauchy problem for a class of transport equations whose velocities are partially with bounded variation. Our result deals with the initial value problem $\partial_t u + Xu = f, u_{|t=0} = g$, where X is the vector field

$$a_1(x_1) \cdot \partial_{x_1} + a_2(x_1, x_2) \cdot \partial_{x_2}, \quad a_1 \in BV(\mathbb{R}^{N_1}_{x_1}), \quad a_2 \in L^1_{x_1}\left(BV(\mathbb{R}^{N_2}_{x_2})\right),$$

with a boundedness condition on the divergence of each vector field a_1, a_2. This model was studied in the paper [LL] with a $W^{1,1}$ regularity assumption replacing our BV hypothesis. This settles partly a question raised in the paper [Am]. We examine the details of the argument of [Am] and we combine some consequences of the Alberti rank-one structure theorem for BV vector fields with a regularization procedure. Our regularization kernel is not restricted to be a convolution and is introduced as an unknown function. Our method amounts to commute a pseudo-differential operator with a BV function.

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1. – Introduction

In this article, we want to study some transport equations whose velocities are partially with bounded variation. More precisely, we intend to prove the uniqueness of weak solutions u of

$$\partial_t u + Xu = 0, \quad u_{|t=0} = 0,$$

for vector fields X of the following type,

$$(1.1) \quad \begin{cases} X = a_1(x_1) \cdot \partial_{x_1} + a_2(x_1, x_2) \cdot \partial_{x_2}, \quad a_1 \in BV(\mathbb{R}^{N_1}_{x_1}), \quad a_2 \in L^1_{x_1}\left(BV(\mathbb{R}^{N_2}_{x_2})\right), \\ \mathrm{div}_1\, a_1 \in L^\infty(\mathbb{R}^{N_1}), \quad \mathrm{div}_2\, a_2 \in L^\infty(\mathbb{R}^{N_1+N_2}). \end{cases}$$

Note that the BV vector field a_1 depends only on the x_1-variables, but that the vector field a_2 is only L^1 with respect to the x_1-variables (and BV with respect to x_2). Note also that our condition on the divergence is stronger that $\operatorname{div} X \in L^\infty$ since we want to control both divergences of the vector fields a_1, a_2. This type of question is tackled in a recent paper by C. Le Bris and P. L. Lions [LL], in which they examine vector fields of type (1.1), where the BV regularity is replaced by a $W^{1,1}$ assumption.

A short historical account of the problem

Let us recall briefly a part of the recent history of this problem. In 1989, R. DiPerna and P. L. Lions proved in [DL] that the $W^{1,1}$ regularity of a vector field, (along with a condition of boundedness on the divergence and a global condition) is enough to ensure the uniqueness of weak solutions. In 1998, P. L. Lions introduced in [Li] the so-called piecewise $W^{1,1}$ class and extended the results of [DL] for this type of regularity. In 2001, F. Bouchut studied in [Bo] some cases of BV regularity corresponding essentially to $W^{1,1}$ singularities occurring on hyperplanes. The paper [CL2] introduced the invariantly defined class *conormal BV*, for which the authors prove the uniqueness of weak solutions; moreover, their definition is simplified by the remark that closed sets whose $(N-1)$-Hausdorff measure is zero are unimportant for locally bounded vector fields. Finally in 2003, L. Ambrosio fully proved in [Am] the conjecture formulated in [DL] that BV vector fields (with bounded divergence) do have a flow. The main new ingredient brought forward by the article [Am] is a deep structure result on BV-vector-valued functions due to G. Alberti [Al]. Although the full strength of the Alberti theorem is not needed as noted in the Remark 3.7 of [Am] and also below in our Remark 3.4, it is nevertheless a very helpful tool for our investigation. It should also be noted that the classical counterexample of M. Aizenman [Ai], the recent counterexamples of N. Depauw [De] and of F. Colombini, T. Luo, J. Rauch [CLR] indicate that the BV regularity is close to optimality for the uniqueness property.

The renormalization property

Following the method introduced in [DL], the main tool for the proof of all these uniqueness results for vector fields X is a commutation lemma devised to ensure that a (bounded) solution u of the equation $Xu = 0$ should be also such that $X(u^2) = 0$ and more generally should satisfy the renormalization property

$$(1.2) \qquad\qquad X\big(\beta(u)\big) = \beta'(u)Xu$$

for any C^1 function β (to get uniqueness, it is enough to prove the Leibniz formula $X(u^2) = 2uX(u)$). The property (1.2) could fail even if both sides of the equality make sense, as shown by the counterexample constructed in [De]. In that paper, N. Depauw shows that there exists a bounded measurable vector

field $a \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x^2; \mathbb{R}^2)$ with null divergence and a bounded measurable function $u \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x^2)$, supported in $\{t \geq 0\}$ such that

$$\partial_t u + \partial_x \cdot (au) = 0, \quad u^2 = \mathbf{1}_{\mathbb{R}_+}(t).$$

For that particular vector field $X_D = \partial_t + a(t, x) \cdot \partial_x$ and that function u, we have $X_D(u^2) = \delta(t)$ and $2uX_D(u) = 0$, violating (1.2), in spite of the fact that $X_D(u^2)$ and $2uX_D(u)$ are both meaningful. A somewhat equivalent approach to checking the property (1.2) is the fact that to get uniqueness for a vector field X, one should be able to prove that it behaves like an ordinary vector field with respect to the Leibniz formula, namely, assuming for instance that X is in L_{loc}^1 with L_{loc}^1 divergence, and u, v are L_{loc}^∞ functions such that $X(u), X(v)$ are in L_{loc}^1, we have to check

$$X(uv) = uX(v) + vX(u).$$

Checking (1.2) can be reduced to a commutation problem. In fact, assuming that X is a L_{loc}^1 vector field with null divergence and u is a L_{loc}^∞ function such that $Xu = 0$, checking $X(u^2) = 0$ amounts to examine the bracket of duality (φ is a test function in C_c^1)

$$\langle Xu^2, \varphi \rangle = -\int u^2(x)(X\varphi)(x)dx$$

since the divergence of X is zero. Now assuming that $R_\epsilon u$ is C^1, bounded and converging pointwise a.e. to u when ϵ goes to zero, we get

$$\langle Xu^2, \varphi \rangle = -\lim_{\epsilon \to 0} \int u(x) \ (R_\epsilon u)(x) \ (X\varphi)(x)dx = \langle X(uR_\epsilon u), \varphi \rangle.$$

Now since $R_\epsilon u$ is C^1, one can use Leibniz formula $X(uR_\epsilon u) = (Xu)(R_\epsilon u) + uX(R_\epsilon u)$ and since $Xu = 0$, we get

$$\langle Xu^2, \varphi \rangle = \lim_{\epsilon \to 0} \langle uX(R_\epsilon u), \varphi \rangle = \lim_{\epsilon \to 0} \int \varphi(x) \ u(x) \ (XR_\epsilon u)(x) \ dx.$$

Using again that $Xu = 0$, we obtain

$$\langle Xu^2, \varphi \rangle = \lim_{\epsilon \to 0} \int (\varphi u)(x) \ \big([X, R_\epsilon]u\big)(x) \ dx.$$

Since the function φu is bounded with compact support, to obtain $Xu^2 = 0$ is thus reduced to proving that the commutator $[X, R_\epsilon]u$ goes to zero in L_{loc}^1.

- If the vector field X is $W_{\text{loc}}^{1,1}$, one can take the regularization operator R_ϵ as any convolution by a C_c^1 function $\rho(x/\epsilon)\epsilon^{-N}$ (with integral 1).
- If the vector field has some singularities on affine submanifolds, for instance on $\{x_1 = 0\}$, this translation invariance property leaves open the choice of

a convolution operator, but with a structure respecting the geometry such as

$$\rho\left(\frac{x_1}{\epsilon_1}, \frac{x_2}{\epsilon_2}\right)\epsilon_1^{-N_1}\epsilon_2^{-N_2}, \quad 0 < \epsilon_1 \ll \epsilon_2.$$

- If a simple jump for X occurs on a curved hypersurface, no convolution operator will do the job, which is quite natural after all, since no translation invariance is preserved. In fact one has to look for a regularizing kernel of a more general form

(1.3) $$(R_\epsilon u)(x) = \int \rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N}u(y)dy.$$

More pedantically, one could say that we intend to commute a pseudo-differential operator of order 1 with a BV function. Our approach in the present paper will be to take the kernel ρ in (1.3) as an unknown function, and we shall see that the commutation property of $[X, R_\epsilon]$ can essentially be expressed as some first-order PDE on that kernel ρ.

Note also that going from the property (1.2) to the uniqueness is now rather standard a fact, since, taking for instance $\beta(u) = u^2$, we produce non-negative solutions, whose uniqueness is easy to establish (see e.g. Lemma 3.1 in [CL2], Lemma 2.2 in [LL]).

A sketch of our paper

Our goal here is in fact twofold. First of all, we wish to revisit the Ambrosio's argument of [Am] by following our approach of commuting our vector field with a regularizing operator of type (1.3), checking which constraints occur on the unknown kernel ρ (this is done in our Section 3). However, our method will follow closely the arguments of [Am] and we shall try to be as explicit as possible in our construction. In particular, if X is our vector field, the canonical decomposition of its derivative can be written as

$$DX = DX^{ac} + DX^s, \quad |DX^{ac}| \ll m, \quad DX^s \perp m$$
$$(m \text{ is the Lebesgue measure on } \mathbb{R}^N).$$

Using the polar decomposition of the singular part, we get $DX^s = M|DX^s|$, where $M(x)$ is a $N \times N$ matrix. An "ideal" kernel $\rho(x, z)$ to be used in (1.3) should satisfy

(1.4) $$\frac{\partial \rho}{\partial z}(x, z)M(x)z = 0$$

which is a PDE in the variable z, which should be satisfied $|DX^s|$-a.e in the variable x. Also the support in the z-variable should be compact. These notions have to be clarified, at least for questions of regularity, and it is done in details

in Section 3. However it is interesting to note that if M is an antisymmetric matrix, one can choose ρ as a convolution kernel (i.e. independent of the variable x) depending only on $|z|^2$, and the equation (1.4) becomes ${}^t z M z = 0$ which is satisfied since M is antisymmetric: so we recover also the remark made in [CP] that we could also generalize. Anyhow, to get compactly supported solutions (in z) for the equation (1.4) requires some spectral condition on the matrix M, and at least spectrum $M \subset i\mathbb{R}$ (naturally that condition is satisfied by an antisymmetric matrix). At any rate, the spectral structure of the matrix $M(x)$ is playing a key role and we shall use some consequences of Alberti's rank-one theorem [Al]. More details are given in our Remark 3.4 below.

Our second aim is to use that constructive approach to tackle vector fields of type (1.1) and to obtain a generalization to BV regularity of the results of [LL]. This gives also a partial answer to the Remark 3.8(3) of [Am]. In our Section 4, we concentrate our attention on the proof of the renormalization property for vector fields of type (1.1). We shall use a regularizing kernel in (1.3) of type $\rho(x_1, x_2, z_2)$, which means in particular that we regularize only in the x_2-variable but in a way depending on the point (x_1, x_2). We have to deal with another commutation problem between the vector field $a_1(x_1)\partial_{x_1}$ in (1.1) and our regularization operator. Moreover, we follow the construction in Section 3 with parameters x_1 and we use the disintegration of the measure $\partial a_2 / \partial x_2$. Also in our Remark 5.5, we give some invariance properties of the matrix M under $C^{1,1}$ diffeomorphism.

2. – Statement of the results

We concentrate our attention on the so-called *renormalization property* for the vector field

$$(2.1) \qquad X = \underbrace{a_1(x_1)\partial_{x_1}}_{X_1} + \underbrace{a_2(x_1, x_2)\partial_{x_2}}_{X_2},$$

$$a_1 \in BV_{\text{loc}}(\mathbb{R}^{N_1}), \quad a_2 \in L^1_{\text{loc}}\big(\mathbb{R}^{N_1}; BV_{\text{loc}}(\mathbb{R}^{N_2})\big)$$

satisfying also

$$(2.2) \qquad \operatorname{div} X_1 \in L^1_{\text{loc}}(\mathbb{R}^{N_1}), \quad \operatorname{div} X_2 \in L^1_{\text{loc}}(\mathbb{R}^{N_1 + N_2}).$$

THEOREM 2.1. *Let N, N_1, N_2 be non-negative integers such that $N = N_1 + N_2$. Let X be a vector field on $\mathbb{R}^N$ satisfying (2.1)-(2.2) and let w be a $L^\infty_{\text{loc}}(\mathbb{R}^N)$ function such that $Xw \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then, with $\alpha \in C^1(\mathbb{R}; \mathbb{R})$,*

$$(2.3) \qquad X\big(\alpha(w)\big) = \alpha'(w) X w.$$

The proof of this theorem is given in Section 4. Theorem 2.1 above along with Lemma 3.1 in [CL1] imply readily the following local uniqueness result.

THEOREM 2.2. *Let X be a vector field satisfying the assumptions of Theorem 2.1 such that $X \in L^\infty_{\text{loc}}$, div $X \in L^\infty_{\text{loc}}$. Let Ω be an open subset of $\mathbb{R}^N$ and S be a C^1 hypersurface of Ω such that X is transverse to S. Let c and w be $L^\infty_{\text{loc}}(\Omega)$ functions such that $Xw = cw$ on Ω, and $\operatorname{supp} w \subset S_+$ (here S_+ is the "half-space" above S). Then $w = 0$ on a neighborhood of S.*

Theorem 2.1 is also the key step to get the uniqueness of bounded solutions for transport equations of type $\partial_t + X$. Let $T > 0$ be given and X be a vector field as above. Let c be a $L^1_{\text{loc}}(\mathbb{R}^{N+1})$ function and w be a $L^\infty_{\text{loc}}(\mathbb{R}^{N+1})$ function. Let us recall that the following equation

$$(2.4) \qquad \begin{cases} \partial_t w + Xw = cw & \text{on } (0, T) \times \mathbb{R}^N, \\ w(0, x) = 0 & \text{on } \mathbb{R}^N, \end{cases}$$

holds weakly means that

$$(2.4)' \;\; \forall \varphi \in C^\infty_c\big([0, T) \times \mathbb{R}^N\big), \quad \int_0^T \int_{\mathbb{R}^N} w\big(\partial_t\varphi + X\varphi + \varphi \operatorname{div} X + c\varphi\big) dx dt = 0.$$

The following theorem is a consequence of Theorem 2.1 above and of Lemma 3.3 in [CL1].

THEOREM 2.3. *Let X be a vector field satisfying the assumptions of Theorem 2.1 such that*

$$(2.5) \qquad \frac{a_1(x_1)}{1 + |x_1|} \in L^1(\mathbb{R}^{N_1}), \qquad \frac{a_2(x_1, x_2)}{1 + |x_2|} \in L^1\big(\mathbb{R}^N\big),$$

$$(2.6) \qquad \operatorname{div} X_1 \in L^\infty(\mathbb{R}^{N_1}), \quad \operatorname{div} X_2 \in L^\infty(\mathbb{R}^N).$$

Let T be a positive number. Let $c(t, x)$ and $w(t, x)$ be $L^\infty\big((0, T) \times \mathbb{R}^N\big)$ functions such that (2.4) holds weakly (i.e. (2.4)'). Then $w = 0$ on $(0, T) \times \mathbb{R}^N$.

We refer the reader to the paper [LL] for the statements of similar uniqueness theorems that we are able to generalize by replacing in *(H1)* and *(H4)* of [LL] the $W^{1,1}$ regularity by the BV regularity.

In our Remark 5.6, we point out that an invariant formulation can be found to express an assumption such as (2.1).

3. – Following Ambrosio's argument with some modifications

In this section, we follow closely Ambrosio's argument in [Am], based on the Alberti rank-one theorem (Theorem 2.3 in [Am]). We give a few modifications and we take the regularizing kernel as an unknown function.

THEOREM 3.1. *Let Ω be an open subset of $\mathbb{R}^N$, X be a real BV_{loc} vector field on Ω such that* $\mathrm{div}\, X \in L^1_{\mathrm{loc}}$ *and* $w \in L^\infty_{\mathrm{loc}}$ *such that* $Xw \in L^1_{\mathrm{loc}}$. *Then, with* $\alpha \in C^1(\mathbb{R}; \mathbb{R})$,

$$(3.1) \qquad\qquad X\big(\alpha(w)\big) = \alpha'(w)Xw.$$

REMARKS 3.2. Note that proving the uniqueness of weak solutions requires only to get $X(w^2) = 0$ whenever $Xw = 0$ for $w \in L^\infty_{\mathrm{loc}}$ (see e.g. Lemma 3.1 in [CL1]). Also to obtain (3.1) on the open set Ω, it is enough to prove it on an open subset Ω_0 such that

$$\mathcal{H}^{N-1}(\Omega\backslash\Omega_0) = 0,$$

provided that the vector field X is also locally bounded as well as its divergence (here $\mathcal{H}^{N-1}$ stands for the $N-1$ Hausdorff measure). For a L^1_{loc} vector field $X = \sum a_j\partial_j$ with an L^1_{loc} divergence and a L^∞_{loc} function w, the distribution Xw is defined as

$$(3.2) \qquad\qquad Xw = \sum_{1 \le j \le n} \partial_j(a_j w) - w\,\mathrm{div}\, X.$$

PROOF OF THEOREM 3.1.
Step 1: Preliminaries.
Proving (3.1) amounts to checking that for any test function $\varphi \in C^1_c(\Omega)$,

$$\int \alpha(w)\big(X\varphi + \varphi\,\mathrm{div}\, X\big)dm + \int \alpha'(w)(Xw)\varphi dm = 0,$$

where dm is the Lebesgue measure on $\mathbb{R}^N$. Let χ be a $C^1_c(\Omega)$ function identically equal to 1 on the support of φ. Then for $x \in \mathrm{supp}\,\varphi$, $\alpha\big(w(x)\big) = \alpha\big(\chi(x)w(x)\big)$, so that we need only to check

$$(3.3) \qquad -\int \alpha(\chi w)\big(X\varphi + \varphi\,\mathrm{div}\, X\big)dm = \int \alpha'(\chi w)\big(X(\chi w)\big)\varphi dm.$$

Note that, from the assumptions of Theorem 3.1,

$$v = \chi w \in L^\infty_{\mathrm{comp}}, \quad Xv = wX\chi + \chi Xw \in L^1_{\mathrm{comp}},$$

and we can use a mollifier with the properties of Lemma 5.2 of our appendix to write

$$-\int \alpha(v)\big(X\varphi + \varphi\,\mathrm{div}\, X\big)dm = -\lim_{\epsilon\to 0}\int \alpha\big(R_\epsilon v\big)\big(X\varphi + \varphi\,\mathrm{div}\, X\big)dm$$

$$= \lim_{\epsilon\to 0}\left\{\int \alpha'\big(R_\epsilon v\big)\big(X(R_\epsilon v) - R_\epsilon(Xv)\big)\varphi dm + \int \alpha'\big(R_\epsilon v\big)R_\epsilon(Xv)\varphi dm\right\}.$$

Lemma 5.2 shows that $R_\epsilon(Xv)$ converges to Xv in L^1 and since $R_\epsilon v$ is bounded independently of ϵ, and converges almost everywhere toward v, proving (3.3) amounts to proving

$$\lim_{\epsilon \to 0} \int \varphi \alpha'(R_\epsilon v)[X, R_\epsilon]v dm = 0.$$

Using Lemma 5.3 in the appendix, we get

$$\int \varphi \alpha'(R_\epsilon v)[X, R_\epsilon]v dm = \int \varphi \alpha'(R_\epsilon v) T_{\epsilon,\rho} v dm + \int \overbrace{\alpha'(R_\epsilon v)}^{\text{bounded}} \overbrace{\varphi N_\epsilon v}^{\substack{\to 0 \\ \text{in } L^1}} dm$$

so that we need only to prove

(3.4) $$\lim_{\epsilon \to 0} \int \varphi \alpha'(R_\epsilon v) T_{\epsilon,\rho} v dm = 0,$$

that is, using (5.3)

$$\lim_{\epsilon \to 0} \iint \partial_2 \rho(x,z) \epsilon^{-1} (X(x) - X(x - \epsilon z))(v(x - \epsilon z) - v(x)) \varphi(x) \alpha'((R_\epsilon v)(x)) dx dz = 0.$$

Now if $(v_k)_{k \in \mathbb{N}}$ is a sequence of C_c^1 functions converging almost everywhere to the L_{comp}^∞ function v so that $\|v_k\|_{L^\infty} \leq \|v\|_{L^\infty}$, we set

$$\omega(\epsilon, k) = \iint \partial_2 \rho(x,z) \epsilon^{-1} (X(x) - X(x - \epsilon z))(v_k(x - \epsilon z) - v_k(x))$$

(3.5)
$$\times \varphi(x) \alpha'((R_\epsilon v)(x)) dx dz = \iiint_0^1 \partial_2 \rho(x,z) DX(x - \epsilon \theta z) z$$
$$\times (v_k(x - \epsilon z) - v_k(x)) \varphi(x) \alpha'((R_\epsilon v)(x)) dx dz d\theta,$$

which makes sense as a bracket of duality since the distribution derivative DX is of order ≤ 1. We have to prove

(3.6) $$\lim_{\epsilon \to 0} \left(\lim_{k \to \infty} \omega(\epsilon, k) \right) = 0.$$

Step 2: Getting rid of the absolutely continuous part.

So far our discussion required only that X and div X should belong to $L_{\text{loc}}^1(\Omega)$. In fact our assumption on X in Theorem 3.1 makes DX a Radon measure. We consider now the canonical decomposition of that measure DX

$$DX = DX^a + DX^s, \quad |DX^a| \ll m, \quad |DX^s| \perp m,$$

where m is the Lebesgue measure on $\mathbb{R}^N$. We note that defining

$$
\omega_0(\epsilon, k) = \iiint_0^1 \partial_2 \rho(x, z) DX^a(x - \epsilon\theta z) z \big(v_k(x - \epsilon z) - v_k(x) \big)
$$
$$
\times \varphi(x) \alpha' \big((R_\epsilon v)(x) \big) dx dz d\theta,
$$

we get with

$$
(3.7) \qquad C_0 = \sup_{|s| \le \|v\|_{L^\infty}} |\alpha'(s)|,
$$

using that $DX^a \in L^1$,

$$
\limsup_{k \to \infty} |\omega_0(\epsilon, k)| \le C_0 \iiint_0^1 |DX^a(x - \epsilon\theta z)| |(\tau_{\epsilon z} v - v)(x)| |\varphi(x)| dx |z| \rho_0(z) dz d\theta
$$

and using Lemma 5.1, we get (3.6) for ω_0.

Step 3: Handling the singular part.

We are left with the bracket of duality (that we write as an integral),

$$
\omega_1(\epsilon, k) = \iiint_0^1 \partial_2 \rho(x + \epsilon\theta z, z) DX^s(x) z \big(\tau_{\epsilon z - \epsilon\theta z} v_k - \tau_{-\epsilon\theta z} v_k \big)(x)
$$
$$
\times \tau_{-\epsilon\theta z} \varphi(x) \alpha' \big((\tau_{-\epsilon\theta z} R_\epsilon v)(x) \big) dx dz d\theta,
$$

so that, using the polar decomposition of the measure $DX^s = M|DX^s|$ with the notation $\mu = |DX^s|$, we get

$$
\omega_1(\epsilon, k) = \iiint_0^1 \partial_2 \rho(x + \epsilon\theta z, z) M(x) z \big(\tau_{\epsilon z - \epsilon\theta z} v_k - \tau_{-\epsilon\theta z} v_k \big)(x)
$$
$$
\times (\tau_{-\epsilon\theta z} \varphi)(x) \alpha' \big((\tau_{-\epsilon\theta z} R_\epsilon v)(x) \big) d\mu(x) dz d\theta.
$$

Using now that $\sup_{k,x} |v_k(x)| \le \|v\|_{L^\infty}$ and that the measure μ is positive, we obtain

$$
|\omega_1(\epsilon, k)| \le C_0 2 \|v\|_{L^\infty} \iiint_0^1 |\partial_2 \rho(x + \epsilon z, z) M(x) z|
$$
$$
\times |\varphi(x + \epsilon\theta z)| d\mu(x) \mathbf{1}_{\mathrm{supp}\,\rho_0}(z) \, dz d\theta.
$$

Since $\partial_2 \rho$ and φ are continuous functions, $|M(x)| \le 1$, μ-a.e., the dominated convergence theorem for the measure $d\mu dz d\theta$ gives

$$
(3.8) \quad \limsup_{\epsilon \to 0} \big(\sup_k |\omega_1(\epsilon, k)| \big) \le C_0 2 \|v\|_{L^\infty} \iint |\partial_2 \rho(x, z) M(x) z| |\varphi(x)| d\mu(x) dz.
$$

We reach now the main point of the proof which amounts to choosing properly the mollifier ρ so that the z-integral of the dot product $Mz \cdot \partial_2\rho$ in (3.8) is small. Let us consider the function

$$(3.9) \qquad \rho(x, z) = F_0\big(U(x)z\big)|\det U(x)|$$

where, using the notation $\mathbb{M}_N(\mathbb{R})$ for the $N \times N$ real matrices, and C_b^1 for the C^1 functions bounded as well as their first derivatives,

$$(3.10) \qquad U \in C_b^1(\mathbb{R}^N; \mathbb{M}_N(\mathbb{R})), \quad {}^tU(x)U(x) \geq \mathrm{Id},$$

and $F_0 \in C_c^1(\mathbb{R}^N; \mathbb{R}_+)$, $\int F_0(\zeta)d\zeta = 1$. Note that the function ρ given by (3.9) satisfies the assumptions of Lemma 5.2; to check (5.1) we assume $\mathrm{supp}\, F_0 \subset B(0, r_0)$ and we get, since $|z| \leq |U(x)z|$,

$$\sup_x \rho(x, z) \leq \sup |F_0|\mathbf{1}_{B(0, r_0)}(z)\, \|\det U\|_{L^\infty}\, .$$

Similar estimates are true for $\sup_x |d_{x,z}\rho(x, z)|$. We get

$$\iint |\partial_2\rho(x, z)M(x)z||\varphi(x)|d\mu(x)dz$$

$$= \iint |F_0'(U(x)z)U(x)M(x)z||\det U(x)||\varphi(x)|d\mu(x)dz$$

$$\leq \int \|U(x)M(x)U(x)^{-1}\||\varphi(x)|d\mu(x) \int |F_0'(\zeta)||\zeta|d\zeta,$$

so that we obtain from (3.8) that we need only to prove

$$(3.11) \qquad 0 = \inf_{\substack{U \\ \text{satisfying (3.10)}}} \int \|U(x)M(x)U(x)^{-1}\||\varphi(x)|d\mu(x).$$

It is possible to simplify further that condition by getting rid of the continuity properties of U, U' required in (3.10). First of all, (3.11) is a consequence of

$$(3.12) \quad 0 = \inf_{V \in C_c^1(\mathbb{R}^N; \mathbb{M}_N(\mathbb{R}))} \int \|\big(\mathrm{Id} + {}^tV(x)V(x)\big)M(x)\big(\mathrm{Id} + {}^tV(x)V(x)\big)^{-1}\||\varphi(x)|d\mu(x),$$

since the matrix $U(x) = \mathrm{Id} + {}^tV(x)V(x)$ satisfies (3.10) for $V \in C_c^1(\mathbb{R}^N; \mathbb{M}_N(\mathbb{R}))$. We claim now that it is enough to obtain

$$(3.13) \quad 0 = \inf_{V \in L^\infty(|\varphi|d\mu)} \int \|\big(\mathrm{Id} + {}^tV(x)V(x)\big)M(x)\big(\mathrm{Id} + {}^tV(x)V(x)\big)^{-1}\||\varphi(x)|d\mu(x).$$

To prove that claim, we consider a matrix $V \in L^\infty(|\varphi|d\mu)$; since $|\varphi|d\mu$ is a finite Radon measure on $\mathbb{R}^N$, can find a sequence $(V_l) \in C_c^0(\mathbb{R}^N; \mathbb{M}_N(\mathbb{R}))$ converging to V in $L^1(|\varphi|d\mu)$ with

$$\sup_x \|V_l(x)\| \leq \|V\|_{L^\infty(|\varphi|d\mu)}\, .$$

Regularizing by a standard mollifier the matrices V_l, we may suppose that they are in $C_c^1(\mathbb{R}^N; \mathbb{M}_N(\mathbb{R}))$ and, extracting a subsequence, we may also assume that they converge pointwise $|\varphi| d\mu$-a.e. to V. We note that $|\varphi| d\mu$-a.e.,

$$\|(\mathrm{Id} +{}^t V_l(x) V_l(x)) M(x) (\mathrm{Id} +{}^t V_l(x) V_l(x))^{-1}\| \leq \|\mathrm{Id} +{}^t V_l(x) V_l(x)\|$$
$$\leq 1 + \|V\|^2_{L^\infty_{(|\varphi| d\mu)}},$$

so that the Lebesgue dominated convergence theorem for the measure with finite mass $|\varphi| d\mu$ gives

$$(3.14) \quad \lim_{l \to +\infty} \int \|(\mathrm{Id} +{}^t V_l(x) V_l(x)) M(x) (\mathrm{Id} +{}^t V_l(x) V_l(x))^{-1}\| |\varphi(x)| d\mu(x)$$
$$= \int \|(\mathrm{Id} +{}^t V(x) V(x)) M(x) (\mathrm{Id} +{}^t V(x) V(x))^{-1}\| |\varphi(x)| d\mu(x).$$

Thus the infimum in the rhs of (3.13), a priori smaller than the rhs of (3.12) is actually the same, proving our claim. We are then reduced to proving (3.13). The key argument relies on

THEOREM 3.3 (Alberti's rank one theorem [Al]). *Let Ω be an open subset of $\mathbb{R}^N$, $a \in BV(\Omega, \mathbb{R}^{N'})$ and let $Da = M|Da|$ be the polar decomposition of its distribution derivative. Then $M(x)$ has rank one, i.e.*

$$M(x) = \xi(x) \otimes \eta(x), \quad |D^s a| \text{ almost everywhere.}$$

The product $\xi \otimes \eta$ is the linear map defined by $\langle \xi, T \rangle \eta$ and if a is a vector field on Ω ($\Omega \subset \mathbb{R}^N$, $N' = N$), the divergence of a is $\langle \xi, \eta \rangle |Da|$ so that the absolute continuity of the divergence with respect to the Lebesgue measure amounts to the orthogonality of the unit vectors ξ, η, $|D^s a|$ almost everywhere. We apply this theorem to the matrix-valued measure $DX^s = M|DX^s|$, $\mu = |DX^s|$, and we use the notation $M = \xi \otimes \eta$, μ-almost everywhere. We choose now the $L^\infty(\mu)$ matrix $V(x) = \gamma^{1/2} M(x)$, where $\gamma \geq 0$, and we note that, from Lemma 5.4,

$$(3.15) \qquad \|(\mathrm{Id} +{}^t V(x) V(x)) M(x) (\mathrm{Id} +{}^t V(x) V(x))^{-1}\| \leq (1 + \gamma)^{-1}.$$

Since γ is an arbitrary positive number and $|\varphi| d\mu$ is finite, we obtain (3.13), completing the proof of Theorem 3.1. □

REMARK 3.4. Looking at our proof, it seems quite obvious that the full strength of Alberti's theorem is not needed. For instance, in the paper [CL2], the key argument could be modified to rely on the fact that the matrix M is triangular with zeros on the diagonal. It is also pointed out in Remark 3.7 of [Am] that a recent still unpublished proof of Alberti is using only the absolute continuity of the divergence with respect to the Lebesgue measure, that is $\mathrm{Tr}\, M = 0$. As far as our proof is concerned, from the analysis in (3.8), for

fixed x, we need to have a compactly supported (in z) solution $\rho(x, z)$ of the equation

(3.16)
$$\frac{\partial \rho}{\partial z}(x, z) M(x) z = 0.$$

The previous equation is simply given by a vector field in the z-variables, with coefficients depending linearly on z (and with parameters x),

$$\sum_{1 \leq j \leq N} \left(\sum_{1 \leq k \leq N} M_{kj}(x) z_j \right) \frac{\partial \rho}{\partial z_j}(x, z) = 0.$$

For this vector field, to have a compactly supported solution require the spectral condition,

(3.17)
$$\text{spectrum } (M(x)) \subset i\mathbb{R}.$$

However, the equation (3.16) need not to be satisfied exactly, and (3.17) may certainly be relaxed.

REMARK 3.5. If we follow our Remark 5.5 below, we see that, although the matrix $\mathcal{M}(x) = (\frac{\partial a_j}{\partial x_k})_{1 \leq j,k \leq N}$ does not carry any geometric meaning, its class modulo L^1_{loc} has actually some invariance properties, so that it is not hopeless to expect that a spectral condition of type (3.17) could be meaningful, even for a vector field $\sum_j a_j \partial_j$ more singular than BV.

4. – Sum of Leibnizian vector fields

In this section, we prove that the renormalization property holds for the vector field

(4.1)
$$X = \underbrace{a_1(x_1) \partial_{x_1}}_{X_1} + \underbrace{a_2(x_1, x_2) \partial_{x_2}}_{X_2},$$
$$a_1 \in BV_{\text{loc}}(\mathbb{R}^{N_1}), \quad a_2 \in L^1_{\text{loc}}\left(\mathbb{R}^{N_1}; BV_{\text{loc}}(\mathbb{R}^{N_2})\right)$$

provided that

(4.2)
$$\text{div } X_1 \in L^1_{\text{loc}}(\mathbb{R}^{N_1}), \quad \text{div } X_2 \in L^1_{\text{loc}}(\mathbb{R}^{N_1+N_2}).$$

In Remark 5.6 below, we point out that an invariant formulation of our statement can be given, using a codimension N_1-foliation of the reference open set.

THEOREM 4.1. *Let N, N_1, N_2 be non-negative integers such that $N = N_1 + N_2$. Let X be a vector field on $\mathbb{R}^N$ satisfying (4.1)-(4.2) and $w \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ such that $Xw \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then, with $\alpha \in C^1(\mathbb{R}; \mathbb{R})$,*

$$(4.3) \qquad X\big(\alpha(w)\big) = \alpha'(w) X w.$$

PROOF. To simplify our argument, we shall only prove that, if $w \in L^\infty(\mathbb{R}^N)$ satisfies $Xw = 0$, then $X(w^2) = 0$. Also, we shall assume that both divergences in (4.2) are vanishing identically. Let us consider $\rho \in C^1(\mathbb{R}^{N_1}_{x_1} \times \mathbb{R}^{N_2}_{x_2} \times \mathbb{R}^{N_2}_{z_2}; \mathbb{R}_+)$ such that

$$(4.4) \qquad \int_{\mathbb{R}^{N_2}} \rho(x_1, x_2, z_2) dz_2 = 1,$$

$$(4.5) \qquad \sup_{x_1, x_2} \big(|\rho(x_1, x_2, z_2)| + |d_{x_1,x_2,z_2} \rho(x_1, x_2, z_2)|\big) = \rho_0(z_2) \in L^\infty_{\text{comp}}.$$

We define also for $\epsilon > 0$ the operator R_ϵ by

$$(4.6) \qquad (R_\epsilon u)(x_1, x_2) = \int_{\mathbb{R}^{N_2}} \rho\big(x_1, x_2, \epsilon^{-1}(x_2 - y_2)\big) \epsilon^{-N_2} u(x_1, y_2) dy_2.$$

We need now to commute R_ϵ with the vector field X_1.

LEMMA 4.2. *Let X_1, R_ϵ, w be as above. Then, $X_1 R_\epsilon w$ and $R_\epsilon X_1 w$ belong to L^1_{loc} and $\lim_{\epsilon \to 0_+}[X_1, R_\epsilon]w = 0$ in L^1_{loc}.*

PROOF OF THE LEMMA. We have, since ρ is C^1,

$$\begin{aligned}
X_1 R_\epsilon w &= \frac{\partial}{\partial x_1} \cdot \int a_1(x_1) \rho\big(x_1, x_2, \epsilon^{-1}(x_2 - y_2)\big) \epsilon^{-N_2} w(x_1, y_2) dy_2 \\
&= \int a_1(x_1) \cdot \frac{\partial \rho}{\partial x_1}\big(x_1, x_2, \epsilon^{-1}(x_2 - y_2)\big) \epsilon^{-N_2} w(x_1, y_2) dy_2 \\
&\quad + \int \rho\big(x_1, x_2, \epsilon^{-1}(x_2 - y_2)\big) \epsilon^{-N_2} \frac{\partial}{\partial x_1} \cdot \big(a_1(x_1) w(x_1, y_2)\big) dy_2 \\
&= \int a_1(x_1) \cdot \frac{\partial \rho}{\partial x_1}(x_1, x_2, z_2)\big(w(x_1, x_2 - \epsilon z_2) - w(x_1, x_2)\big) dz_2 \\
&\quad + a_1(x_1) \cdot \underbrace{\int \frac{\partial \rho}{\partial x_1}(x_1, x_2, z_2) dz_2}_{=0 \text{ from } (4.4)} w(x_1, x_2) + R_\epsilon X_1 w,
\end{aligned}$$

which entails $[X_1, R_\epsilon]w \in L^1_{\text{loc}}$ and from Lemma 5.1 that $\lim_{\epsilon \to 0}[X_1, R_\epsilon]w = 0$ in L^1_{loc}. Moreover, we have, using that ρ is C^1 and the equation $Xw = 0$,

$$\begin{aligned}
R_\epsilon X_1 w &= -R_\epsilon X_2 w = -\int \rho\big(x_1, x_2, \epsilon^{-1}(x_2 - y_2)\big) \epsilon^{-N_2} (\partial_2 \cdot a_2 w)(x_1, y_2) dy_2 \\
&= -\int \frac{\partial \rho}{\partial z_2}\big(x_1, x_2, \epsilon^{-1}(x_2 - y_2)\big) \epsilon^{-N_2 - 1} \cdot (a_2 w)(x_1, y_2) dy_2
\end{aligned}$$

which belongs to L^1_{loc}, completing the proof of the lemma. □

We have, using Theorem 3.1 and Lemma 4.2 (that is the fact that X_1 is Leibnizian, $R_\epsilon w \in L^\infty_{\text{loc}}$, $X_1 R_\epsilon w \in L^1_{\text{loc}}$)

$$X\big((R_\epsilon w)^2\big) = X_1\big((R_\epsilon w)^2\big) + X_2\big((R_\epsilon w)^2\big) = 2(R_\epsilon w)(X_1 R_\epsilon w) + X_2\big((R_\epsilon w)^2\big).$$

We note also that the function R_ϵ is C^1 with respect to x_2, so that the ordinary Leibniz formula can be used to get $X_2\big((R_\epsilon w)^2\big) = 2(R_\epsilon w)(X_2 R_\epsilon w)$. We obtain,

$$\begin{aligned}
X\big((R_\epsilon w)^2\big) &= 2(R_\epsilon w)([X_1, R_\epsilon]w) \\
&\quad + 2(R_\epsilon w)(R_\epsilon X_1 w) + 2(R_\epsilon w)([X_2, R_\epsilon]w) + 2(R_\epsilon w)R_\epsilon X_2 w \\
&= 2(R_\epsilon w)([X_1, R_\epsilon]w) + 2(R_\epsilon w)(R_\epsilon X w) + 2(R_\epsilon w)([X_2, R_\epsilon]w)
\end{aligned}$$

which gives, since $Xw = 0$,

$$(4.7) \qquad X\big((R_\epsilon w)^2\big) = 2(R_\epsilon w)([X_1, R_\epsilon]w) + 2(R_\epsilon w)([X_2, R_\epsilon]w).$$

Since the term $[X_1, R_\epsilon]w$ goes to zero in L^1_{loc} from the Lemma 4.2 and the function $R_\epsilon w$ is locally bounded from Lemma 5.2 we are left with the bracket $[X_2, R_\epsilon]w$. We recall that, with $\varphi \in C^1_c(\mathbb{R}^N)$,

$$\begin{aligned}
\langle (X_1 + X_2)(w^2), \varphi \rangle_{\mathcal{D}'^{(1)}, C^1_c} &= -\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} (R_\epsilon w)^2 (X\varphi)dm \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} X\big((R_\epsilon w)^2\big)\varphi dm \\
&= 2\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} (R_\epsilon w)([X_2, R_\epsilon]w)\varphi dm.
\end{aligned}$$

We need only to prove that the last term above is 0. We define

$$\begin{aligned}
\widetilde{\omega}(\epsilon) &= \int_{\mathbb{R}^N} (R_\epsilon w)([X_2, R_\epsilon]w)\varphi dm \\
&= \iiint (R_\epsilon w)(x_1, x_2)\varphi(x_1, x_2)\frac{\partial \rho}{\partial z_2}(x_1, x_2, z_2)\epsilon^{-1}\big(X_2(x_1, x_2) - X_2(x_1, x_2 - \epsilon z_2)\big) \\
&\qquad \big(w(x_1, x_2 - \epsilon z_2) - w(x_1, x_2)\big)dx_2 dz_2 dx_1.
\end{aligned}$$

Now, we consider a sequence of continuous functions w_k bounded by $\|w\|_{L^\infty}$ converging a.e. to w and we define

$$\begin{aligned}
\widetilde{\omega}(\epsilon, k) &= \iiint (R_\epsilon w)(x_1, x_2)\varphi(x_1, x_2)\frac{\partial \rho}{\partial z_2}(x_1, x_2, z_2)\epsilon^{-1}\big(X_2(x_1, x_2) \\
&\quad - X_2(x_1, x_2 - \epsilon z_2)\big)\big(w_k(x_1, x_2 - \epsilon z_2) - w_k(x_1, x_2)\big)dx_1 dx_2 dz_2.
\end{aligned}$$

We need only to prove that

$$(4.8) \qquad \lim_{\epsilon \to 0}\Big(\lim_{k \to \infty} \widetilde{\omega}(\epsilon, k)\Big) = 0.$$

We have, using an integral notation for the bracket of duality,

$$\widetilde{\omega}(\epsilon,k) = \int_0^1 \iiint (R_\epsilon w)(x_1, x_2)\varphi(x_1, x_2)\frac{\partial \rho}{\partial z_2}(x_1, x_2, z_2)\frac{\partial a_2}{\partial x_2}(x_1, x_2 - \epsilon\theta z_2)z_2$$
$$\big(w_k(x_1, x_2 - \epsilon z_2) - w_k(x_1, x_2)\big)d\theta dx_1 dx_2 dz_2$$
$$= \int_0^1 \iiint (R_\epsilon w)(x_1, x_2 + \epsilon\theta z_2)\varphi(x_1, x_2 + \epsilon\theta z_2)\frac{\partial \rho}{\partial z_2}(x_1, x_2 + \epsilon\theta z_2, z_2)$$
$$\times \frac{\partial a_2}{\partial x_2}(x_1, x_2)z_2\big(w_k(x_1, x_2 + \epsilon\theta z_2 - \epsilon z_2) - w_k(x_1, x_2 + \epsilon\theta z_2)\big)d\theta dx_1 dx_2 dz_2.$$

From our assumption on the $L^1(\mathbb{R}^N)$ function a_2, that we shall make globally for simplicity, we know that for m_{N_1}-almost all x_1 in $\mathbb{R}^{N_1}$, the function $\mathbb{R}^{N_2} \ni x_2 \mapsto a_2(x_1, x_2) \in \mathbb{R}^{N_2}$ is in $BV(\mathbb{R}^{N_2})$ with an L^1 divergence and

$$(4.9) \qquad \int \Big[\|a_2(x_1, \cdot)\|_{BV(\mathbb{R}^{N_2})} + \|\text{div } a_2(x_1, \cdot)\|_{L^1(\mathbb{R}^{N_2})}\Big]dx_1 < \infty.$$

As a consequence, from the canonical decomposition of $D_2 a_2(x_1, \cdot)$, the polar decomposition of $D_2 a_2(x_1, \cdot)^s$, and the Theorem 3.3 along with Lemma 5.4, we get that for m_{N_1}-almost all x_1 in $\mathbb{R}^{N_1}$,

$$(4.10) \qquad D_2 a_2(x_1, \cdot) = (D_2 a_2(x_1, \cdot))^{ac} + (D_2 a_2(x_1, \cdot))^s,$$

$$(4.11) \qquad (D_2 a_2(x_1, \cdot))^s = M_{x_1}(x_2)\mu_{x_1}(x_2), \quad \mu_{x_1} = |(D_2 a_2(x_1, \cdot))^s|$$

and
$$(4.12)$$
$$\big\|\big(\text{Id} + \gamma^t M_{x_1}(x_2)M_{x_1}(x_2)\big)M_{x_1}(x_2)\big(\text{Id} + \gamma^t M_{x_1}(x_2)M_{x_1}(x_2)\big)^{-1}\big\| \leq (1+\gamma)^{-1},$$

with

$$(4.13) \qquad \int_{\mathbb{R}^{N_1}} \Big[\big\|D_2 a_2(x_1, \cdot)^{ac}\big\|_{L^1(\mathbb{R}^{N_2})} + \big\|D_2 a_2(x_1, \cdot)^s\big\|_{\mathcal{M}(\mathbb{R}^{N_2})}\Big]dx_1 < \infty.$$

In particular, setting $k_{x_1}(x_2) = (D_2 a_2(x_1, \cdot))^{ac}$ we get that $\int_{\mathbb{R}^{N_1}} \|k_{x_1}\|_{L^1(\mathbb{R}^{N_2})} dx_1 < \infty$ and thus the function $(x_1, x_2) \mapsto k_{x_1}(x_2)$ belongs to $L^1(\mathbb{R}^N)$. Let us recall the standard (see e.g. Theorem 2.28 in [AFP])

LEMMA 4.3 (disintegration of the measure $\partial a_2/\partial x_2$). *Let N, N_1, N_2 as above and $a_2 \in L^1\big(\mathbb{R}^{N_1}_{x_1}; BV(\mathbb{R}^{N_2}_{x_2})\big)$. We denote by π_1 the projection $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}^{N_1}$, by v the measure $\partial a_2/\partial x_2$ and we set $\lambda = \pi_{1*}(|v|)$. Our assumption implies that $|v|(\mathbb{R}^N) < \infty$. The disintegration theorem gives*

$$v = \lambda \otimes v_{x_1}$$

where for λ-almost all $x_1 \in \mathbb{R}^{N_1}$, the $\mathbb{M}_{N_2}$-valued measure v_{x_1} is such that

$$|v_{x_1}|(\mathbb{R}^{N_2}) = 1.$$

It means that for $F(x_1, x_2) \in L^1(\mathbb{R}^N, dv)$, we have, for λ-almost all $x_1 \in \mathbb{R}^{N_1}$, $F(x_1, \cdot) \in L^1(\mathbb{R}^{N_2}, d|v_{x_1}|)$, $x_1 \mapsto \int_{\mathbb{R}^{N_2}} F(x_1, x_2) dv_{x_1}(x_2)$ belongs to $L^1(\mathbb{R}^{N_1}, d\lambda)$ and

$$\int_{\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}} F(x_1, x_2) dv(x_1, x_2) = \int_{\mathbb{R}^{N_1}} \left(\int_{\mathbb{R}^{N_2}} F(x_1, x_2) dv_{x_1}(x_2) \right) d\lambda(x_1).$$

We note also that the measure λ is absolutely continuous with respect to the Lebesgue measure: let A be a Borelian subset of $\mathbb{R}^{N_1}$ of Lebesgue measure 0. We have from (4.9)

$$\lambda(A) = |v|(A \times \mathbb{R}^{N_2}) \leq \int_A \overbrace{\||D_2 a_2|(x_1, \cdot)\|_{\mathcal{M}(\mathbb{R}^{N_2})}}^{\in L^1(\mathbb{R}^{N_1})} dx_1 = 0.$$

We thus obtain that, with $h \in L^1(\mathbb{R}^{N_1})$, $v = \lambda \otimes v_{x_1} = h m_{N_1} \otimes v_{x_1}$, and thus we have the disintegration formulas

$$(4.14) \qquad \frac{\partial a_2}{\partial x_2} = v = m_{N_1} \otimes h(x_1) v_{x_1} = m_{N_1} \otimes M_{x_1} \mu_{x_1} + L^1(\mathbb{R}^N).$$

In fact for $F \in C_c^0(\mathbb{R}^N)$, we have with the notations of (4.10-13)

$$\int_{\mathbb{R}^{N_1}} \left(\int_{\mathbb{R}^{N_2}} F(x_1, x_2) M_{x_1}(x_2) d\mu_{x_1}(x_2) \right) dx_1 + \int_{\mathbb{R}^{N_1}} \left(\int_{\mathbb{R}^{N_2}} F(x_1, x_2) k_{x_1}(x_2) dx_2 \right) dx_1$$

$$= \int_{\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}} F(x_1, x_2) dv(x_1, x_2)$$

$$= \int_{\mathbb{R}^{N_1}} \left(\int_{\mathbb{R}^{N_2}} F(x_1, x_2) dv_{x_1}(x_2) \right) h(x_1) dx_1.$$

The term belonging to $L^1(\mathbb{R}^N)$ in (4.14) can be given the same treatment as DX^a in Section 3 and the same method along with (4.14) gives

$$(4.15) \quad \limsup_{\epsilon \to 0} \widetilde{\omega}(\epsilon)$$

$$\leq 2 \|w\|_{L^\infty}^2 \iint \left(\int |\varphi(x_1, x_2)| \left| \frac{\partial \rho}{\partial z_2}(x_1, x_2, z_2) M_{x_1}(x_2) z_2 \right| d\mu_{x_1}(x_2) \right) dz_2 dx_1.$$

We inspect then the arguments of Section 3, between (3.9) and (3.15). We consider a function

$$(4.16) \qquad \rho(x_1, x_2, z_2) = F_0(U(x_1, x_2) z_2) |\det U(x_1, x_2)|$$

where $U \in C_b^1(\mathbb{R}^N; \mathbb{M}_{N_2}(\mathbb{R}))$ is such that ${}^tU(x)U(x) \geq \mathrm{Id}$, and $F_0 \in C_c^1(\mathbb{R}^{N_2}; \mathbb{R}_+)$ satisfies $\int_{\mathbb{R}^{N_2}} F_0(\zeta)d\zeta = 1$. We would like to choose

$$U(x) = \mathrm{Id} + \gamma\, {}^tM_{x_1}(x_2)M_{x_1}(x_2),$$

but, as in Section 3, it is not directly possible because of the lack of regularity of that function. The matrices $M_{x_1}(x_2)$ have norm ≤ 1. For all x_1, we can find a sequence $(V_{x_1,l}(x_2))_{l\in\mathbb{N}}$ of functions in $C_b^1(\mathbb{R}^{N_2};$ unit ball of $\mathbb{M}_{N_2}(\mathbb{R}))$ converging pointwise to the bounded matrix $M_{x_1}(x_2)$. We can also regularize these matrices with respect to x_1 by a standard mollifier, which will be enough since the integral in the variables x_1, z_2 takes place on a fixed compact set. We can conclude as in Section 3 by using (4.12). The proof of Theorem 4.1 is complete. $\qquad\square$

5. – Appendix

LEMMA 5.1. *Let $a \in L^1(\mathbb{R}^N)$ and $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, we have*

$$\lim_{t\to 0} \int |a(x)||v(x+t) - v(x)|dx = 0.$$

PROOF. We have for $\lambda > 0$,

$$\int |a(x)||v(x-t) - v(x)|dx \leq \lambda \int_{|a|\leq\lambda} |v(x-t) - v(x)|dx + 2\,\|v\|_{L^\infty} \int_{|a|\geq\lambda} |a(x)|dx$$

$$\leq \lambda\,\|\tau_t v - v\|_{L^1} + 2\,\|v\|_{L^\infty} \int \mathbf{1}_{|a|\geq\lambda}|a(x)|dx,$$

so that, since $v \in L^1$, $\limsup_{t\to 0} \int |a(x)||v(x-t) - v(x)|dx \leq 2\,\|v\|_{L^\infty} \int \mathbf{1}_{|a|\geq\lambda} |a(x)|dx$, which gives the result by taking the limit of the rhs when λ goes to infinity. $\qquad\square$

REMARK. Assuming $v \in L^\infty$ is enough, as proven in the Lemma 5.1 of [CL2].

LEMMA 5.2. *Let $\rho \in C^1(\mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}_+)$ such that $\int_{\mathbb{R}^N} \rho(x,z)dz = 1$ and*

$$(5.1) \qquad \rho_0(z) = \sup_x |\rho(x,z)| + \sup_x |d_{x,z}\rho(x,z)| \in L^\infty_{\mathrm{comp}}.$$

For $\epsilon > 0$, we consider the operator R_ϵ with kernel $\rho\big(x, \epsilon^{-1}(x-y)\big)\epsilon^{-N}$, defined for $u \in L^1_{\mathrm{loc}}$ by

$$(5.2) \qquad (R_\epsilon u)(x) = \int \rho\big(x, \epsilon^{-1}(x-y)\big)\epsilon^{-N}u(y)dy.$$

Let $1 \leq p < +\infty$ and $u \in L^p$; then $\lim_{\epsilon\to 0} R_\epsilon u = u$ in L^p. If $u \in L^\infty$, $\|R_\epsilon u\|_{L^\infty} \leq \|u\|_{L^\infty}$. If u belongs to L^1_{loc}, the function $R_\epsilon u$ belongs to $C^1(\mathbb{R}^N)$ and for almost all $x \in \mathbb{R}^N$, $\lim_{\epsilon\to 0}(R_\epsilon u)(x) = u(x)$.

PROOF. This lemma is classical for a convolution. We check here that this more general regularizing kernel does not introduce any new difficulty. Let us first assume that $u \in L^p$ with $1 \leq p < +\infty$. We have

$$(R_\epsilon u)(x) - u(x) = \int \rho(x, z)\big(u(x - \epsilon z) - u(x)\big)dz$$

so that, defining $\alpha = \|\rho_0\|_{L^1}$, we get from Jensen's inequality

$$\|R_\epsilon u - u\|_{L^p}^p \leq \int \left(\int \rho_0(z)|(\tau_{\epsilon z}u - u)(x)|dz\right)^p dx \leq \alpha^{p-1}\int \rho_0(z)\,\|\tau_{\epsilon z}u - u\|_{L^p}^p\,dz.$$

Since $\lim_{\epsilon \to 0}\|\tau_{\epsilon z}u - u\|_{L^p} = 0$ and $\|\tau_{\epsilon z}u - u\|_{L^p} \leq 2\|u\|_{L^p}$, Lebesgue's dominated convergence theorem gives the result. Let us assume now that $u \in L^\infty$; the estimate on $\|R_\epsilon u\|_{L^\infty}$ is trivial. Moreover, we have for $u \in L^1_{\text{loc}}$

$$|(R_\epsilon u)(x) - u(x)| \leq \int_{z \in \text{supp}\,\rho_0} |u(x - \epsilon z) - u(x)|dz\,\|\rho_0\|_{L^\infty}$$

and since $\text{supp}\,\rho_0$ is compact, the Lebesgue differentiation theorem gives that this quantity is going to 0 with ϵ for almost all x. Assuming $u \in L^1_{\text{loc}}$, the function

$$y \mapsto \rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N}u(y)$$

belongs to L^1 for all x since

$$\int \epsilon^{-N}\rho_0\big(\epsilon^{-1}(x - y)\big)|u(y)|dy \leq \epsilon^{-N}\,\|\rho_0\|_{L^\infty}\int_{x - \epsilon\,\text{supp}\,\rho_0} |u(y)|dy < \infty.$$

Moreover the function $x \mapsto \rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N}u(y)$ is continuously differentiable and for K compact

$$\int \sup_{x \in K}\left|\partial_1\rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N} + \epsilon^{-1}\partial_2\rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N}\right| |u(y)|dy$$

$$\leq 2\,\|\rho_0\|_{L^\infty}\,\epsilon^{-N-1}\int_{K - \epsilon\,\text{supp}\,\rho_0} |u(y)|dy < \infty.$$

We obtain that the function $R_\epsilon u$ belongs to $C^1(\mathbb{R}^N)$, completing the proof of the lemma. $\qquad\square$

LEMMA 5.3. *Let Ω be an open subset of $\mathbb{R}^N$ and let X be a L^1_{loc} vector field on Ω such that* $\text{div}\,X \in L^1_{\text{loc}}(\Omega)$. *Let v be in $L^\infty_{\text{comp}}(\Omega)$ and R_ϵ be given by Lemma 5.1. Then we have*

$$(XR_\epsilon v - R_\epsilon Xv)(x) = (T_{\epsilon,\rho}v)(x) + (N_\epsilon v)(x)$$

with $\lim_{\epsilon \to 0} N_\epsilon v = 0$ *in* L^1_{loc} *and*

(5.3) $(T_{\epsilon,\rho} v)(x) = \int \partial_2 \rho(x, z) \epsilon^{-1} \big(X(x) - X(x - \epsilon z)\big) \big(v(x - \epsilon z) - v(x)\big) dz.$

PROOF. To avoid confusion between the vector field X (for each $x \in \Omega$, $X(x)$ is a vector tangent to Ω at x) and the operator X acting on functions, we shall denote by $\mathbb{X}$ the operator: we use the notation $(\mathbb{X}w)(x) = dw(x)X(x)$ (a scalar quantity as the product of the $1 \times N$ covector dw with the $N \times 1$ vector X) and we write

$$\mathbb{X}R_\epsilon v = \int \partial_1 \rho\big(x, \epsilon^{-1}(x - y)\big) \epsilon^{-N} v(y) dy\, X(x)$$

$$+ \int \partial_2 \rho\big(x, \epsilon^{-1}(x - y)\big) \epsilon^{-1-N} v(y) dy\, X(x).$$

We note that $\partial_2 \rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-1} = -\frac{\partial}{\partial y}\Big(\rho\big(x, \epsilon^{-1}(x - y)\big)\Big)$ which gives, using the identity $\int \rho(x, z) dz \equiv 1$,

$$\mathbb{X}R_\epsilon v = \overbrace{\int \partial_1 \rho(x, z) X(x)\big(v(x - \epsilon z) - v(x)\big) dz}^{:=N_{\epsilon,1}v(x)} + \overbrace{\int \partial_1 \rho(x, z) dz\, X(x) v(x)}^{=0}$$

$$+ \int \frac{\partial}{\partial y}\Big(\rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N}\Big)\big(X(y) - X(x)\big) v(y) dy$$

$$- \int \frac{\partial}{\partial y}\Big(\rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N}\Big) X(y) v(y) dy,$$

so that

$$\mathbb{X}R_\epsilon v = N_{\epsilon,1}v(x) + \int \partial_2 \rho(x, z)\epsilon^{-1}\big(X(x) - X(x - \epsilon z)\big) v(x - \epsilon z) dz$$

$$+ \int \rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N} \frac{\partial}{\partial y} \cdot \big(X(y) v(y)\big) dy,$$

where the last term is in fact a bracket of duality. Since we have from (3.2)

$$\frac{\partial}{\partial y} \cdot \big(X(y) v(y)\big) = (\mathbb{X}v)(y) + (\operatorname{div} X)(y) v(y),$$

we obtain

$$\mathbb{X}R_\epsilon v = N_{\epsilon,1}v + \overbrace{\int \partial_2 \rho(x, z)\epsilon^{-1}\big(X(x) - X(x - \epsilon z)\big)\big(v(x - \epsilon z) - v(x)\big) dz}^{=(T_{\epsilon,\rho}v)(x)}$$

$$+ \int \partial_2 \rho(x, z)\epsilon^{-1}\big(X(x) - X(x - \epsilon z)\big) dz\, v(x)$$

$$+ \int \rho\big(x, \epsilon^{-1}(x - y)\big)\epsilon^{-N}\big((\mathbb{X}v)(y) + (\operatorname{div} X)(y) v(y)\big) dy,$$

which implies

$$\mathbb{X}R_\epsilon v = N_{\epsilon,1}v + T_{\epsilon,\rho}v + \int \partial_2\rho(x,z)\epsilon^{-1}\big(X(x) - X(x - \epsilon z)\big)dz v(x)$$
$$+ R_\epsilon\mathbb{X}v + \mathbb{R}_\epsilon v \operatorname{div} X.$$

Since $z \mapsto \rho(x,z)$ is compactly supported and C^1, the covector $\int \partial_2\rho(x,z)dz = 0$. Moreover, we have

$$-\int \partial_2\rho(x,z)\epsilon^{-1}X(x - \epsilon z)dz v(x) + (\mathbb{R}_\epsilon v \operatorname{div} X)(x)$$

$$= -v(x)\int \rho(x,z)(\operatorname{div} X)(x - \epsilon z) + \int \rho(x,z)v(x - \epsilon z)(\operatorname{div} X)(x - \epsilon z)dz$$

$$= \int \rho(x,z)\big(v(x - \epsilon z) - v(x)\big)(\operatorname{div} X)(x - \epsilon z)dz := N_{\epsilon,2}v(x)$$

and consequently

$$\mathbb{X}R_\epsilon v = N_{\epsilon,1}v + T_{\epsilon,\rho}v + N_{\epsilon,2}v + R_\epsilon\mathbb{X}v.$$

We have for K compact subset of $\mathbb{R}^N$ and $\rho_0 \in L^1(dz)$ given in (5.1)

$$\int_K |N_{\epsilon,1}(x)|dx \leq \overbrace{\int\int |\mathbf{1}_K(x)X(x)||(\tau_{\epsilon z}v - v)(x)|dx}^{\substack{\text{goes to 0 with } \epsilon \text{ from Lemma 5.1,} \\ \text{bounded above by } 2\|\mathbf{1}_K X\|_{L^1}\|v\|_{L^\infty}}} \rho_0(z)dz,$$

so that Lebesgue's dominated convergence theorem gives $\lim_{\epsilon \to 0}\|N_{\epsilon,1}v\|_{L^1(K)} = 0$. We have for K compact subset of $\mathbb{R}^N$

$$\int_K |N_{\epsilon,2}v(x)|dx \leq \iint \mathbf{1}_{K - \epsilon \operatorname{supp} \rho_0}(x)|(\operatorname{div} X)(x)||v(x) - v(x + \epsilon z)|\rho_0(z)dzdx$$

which goes to zero with ϵ from Lemma 5.1. The proof of Lemma 5.3 is complete. $\qquad\square$

LEMMA 5.4. *Let E be a real Euclidean finite dimensional vector space. Let M be an endomorphism of E such that $M = \xi \otimes \eta$ with ξ, η orthogonal unit vectors. Then for all $\gamma \geq 0$,*

$$(5.4) \qquad \left\|(\operatorname{Id} + \gamma\,^t M M)M(\operatorname{Id} + \gamma\,^t M M)^{-1}\right\| \leq (1 + \gamma)^{-1}.$$

PROOF. Since for $T \in E$, we have $MT = \langle \xi, T\rangle\eta$ we get $M^2T = \langle \xi, T\rangle\langle \xi, \eta\rangle\eta = 0$ and

$$\langle M\,^t M M T_1, T_2\rangle = \langle \xi,\,^t M M T_1\rangle\langle \eta, T_2\rangle = \langle \eta, \eta\rangle\langle \xi, T_1\rangle\langle \eta, T_2\rangle = \langle MT_1, T_2\rangle,$$

which means $M = M\,^t M M$ and implies $(1 + \gamma)M = M + \gamma M\,^t M M = M(\operatorname{Id} + \gamma\,^t M M)$ and

$$(5.5) \qquad (\operatorname{Id} + \gamma\,^t M M)M(\operatorname{Id} + \gamma\,^t M M)^{-1} = M(\operatorname{Id} + \gamma\,^t M M)^{-1} = (1 + \gamma)^{-1}M$$

implying (5.4). $\qquad\square$

REMARK 5.5. It worth noticing that for a BV_{loc} vector field $X = \sum_{1 \leq j \leq N} a_j(x) \partial_{x_j}$, the matrix

$$(5.6) \qquad \mathcal{M}(x) = \left(\frac{\partial a_j}{\partial x_k} \right)_{1 \leq j,k \leq N}$$

has some invariance properties, under C^1 diffeomorphism, at least modulo L^1_{loc} matrices. In fact if $x = \kappa(y)$, $y = \nu(x)$ is such a diffeomorphism, the vector field X in the y-chart is

$$X = \sum_{1 \leq j,k \leq N} a_j(\kappa(y)) \frac{\partial y_k}{\partial x_j} \partial_{y_k} = \sum_{1 \leq k \leq N} b_k(y) \partial_{y_k}, \quad b_k = \sum_{1 \leq j \leq N} a_j \frac{\partial y_k}{\partial x_j},$$

$$\text{so that} \frac{\partial b_k}{\partial y_l} = \sum_{1 \leq j \leq N} \frac{\partial a_j}{\partial x_m} \frac{\partial x_m}{\partial y_l} \frac{\partial y_k}{\partial x_j} + L^1_{\text{loc}},$$

which means

$$(5.7) \qquad \mathcal{N}(y) = \left(\frac{\partial b_k}{\partial y_l} \right)_{1 \leq k,l \leq N} = \nu'(x) \mathcal{M}(x) \kappa'(y) + L^1_{\text{loc}}$$

and since $\nu'(\kappa(y)) \kappa'(y) = \text{Id}$, we find that matrices $\mathcal{M}(x)$ and $\mathcal{N}(y)$ are equivalent, modulo a L^1_{loc} matrix.

REMARK 5.6. Let us point out here that an invariant formulation of our statement of Theorem 2.1 can be given, using a codimension N_1-foliation of the reference open set. Let Ω be an open subset of $\mathbb{R}^N$ equipped with a codimension N_1 foliation (in our coordinates the leaves are $x_1 = cte$). A vector field T is tangent to the foliation means, in our coordinates, that $T = \beta(x_1, x_2) \partial_{x_2}$ since $T(x_1)$ should be identically 0. Let us call $\mathcal{T}$ the vector fields tangent to the foliation. We introduce a vector field X such that,

$$(5.8) \qquad \forall T \in \mathcal{T}, \quad [X, T] \in \mathcal{T}.$$

In our coordinates, it means if $X = \alpha_1(x_1, x_2) \partial_{x_1} + \alpha_2(x_1, x_2) \partial_{x_2}$ and $T = \beta(x_1, x_2) \partial_{x_2}$ is any tangent vector field

$$[X, \beta(x_1, x_2) \partial_{x_2}] = \alpha_1 \partial_{x_1}(\beta) \partial_{x_2} - \beta \partial_{x_2}(\alpha_1) \partial_{x_1} + \alpha_2 \partial_{x_2}(\beta) \partial_{x_2} - \beta \partial_{x_2}(\alpha_2) \partial_{x_2}$$

is tangent to the foliation. So to ask for this commutator to be tangent is the requirement $\beta \partial_{x_2}(\alpha_1) = 0$ for all β, which means $\partial_{x_2}(\alpha_1) = 0$, so in our coordinates

$$X = a_1(x_1) \partial_{x_1} + a_2(x_1, x_2) \partial_{x_2}.$$

If the open set Ω is equipped with a Riemannian structure, X can be decomposed in the sum of a tangential part to the foliation $(a_2(x_1, x_2) \partial_{x_2})$ and a normal part $(a_1(x_1) \partial_{x_1})$; to get the divergence property, we shall assume that the divergence of both parts is zero. The geometric hypothesis (5.8) allows us to produce an invariant result.

Index of notations

- Let $X = \sum_{1 \le j \le N} a_j \frac{\partial}{\partial x_j}$ be a vector field on an open set of $\mathbb{R}^N$. DX stands for the matrix $(\frac{\partial a_j}{\partial x_k})_{1 \le j,k \le N}$. When X is a BV_{loc} vector field, DX is a matrix of Radon measures and we can write the canonical decomposition

$$DX = DX^a + DX^s, \quad |DX^a| \ll m, \quad \mu = |DX^s| \perp m$$

  where m is the Lebesgue measure on $\mathbb{R}^N$. The polar decomposition of the matrix DX^s is

$$DX^s = M\mu.$$

- For $y \in \mathbb{R}^N$ and $u \in \mathcal{D}'(\mathbb{R}^N)$,

$$\langle \tau_y u, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \langle u, \tau_{-y}\varphi \rangle_{\mathcal{D}',\mathcal{D}}, \qquad (\tau_{-y}\varphi)(x) = \varphi(x+y).$$

- C_b^1 stands for the C^1 functions bounded as well as their first derivatives.

- For $\xi, \eta \in \mathbb{R}^N$, the product $\xi \otimes \eta$ is the linear map $\mathbb{R}^N \ni T \mapsto \langle \xi, T \rangle \eta \in \mathbb{R}^N$.

REFERENCES

[Ai]   M. AIZENMAN, *On vector fields as generators of flows: a counterexample to Nelson's conjecture*, Ann. of Math. **107** (1978), 287-296.

[Al]   G. ALBERTI, *Rank one properties for derivatives of functions with bounded variations*, Proc. Roy. Soc. Edinburgh sect. A **123** (1993), 239-274.

[Am]   L. AMBROSIO, *Transport equations and Cauchy problem for BV vector fields*, Invent. Math. **158** (2003), 227-260.

[AFP]   L. AMBROSIO – N. FUSCO – D. PALLARA, "Functions of bounded variations and free discontinuity problems", Oxford Mathematical Monographs, 2000.

[Bo]   F. BOUCHUT, *Renormalized solutions to the Vlasov equation with coefficients of bounded variation*, Arch. Ration. Mech. Anal. **157** (2001), 75-90.

[BD]   F. BOUCHUT – L. DESVILLETTES, *On two-dimensional Hamiltonian transport equations with continuous coefficients*, Differential Integral Equations **14** (2001), 1015-1024.

[BJ]   F. BOUCHUT – F. JAMES, *One dimensional transport equations with discontinuous co-efficients*, Nonlinear Anal. **32** (1998), 891-933.

[CP]   I. CAPUZZO DOLCETTA – B. PERTHAME, *On some analogy between different approaches to first-order PDE's with non smooth coefficients*, Adv. Math. Sci. Appl. **6** (1996), 689-703.

[ChL]   J.-Y. CHEMIN – N. LERNER, *Flot de champ de vecteurs non lipschitziens et équations de Navier-Stokes*, J. Differential Equations **121** (1995), 314-328.

[CL1]   F. COLOMBINI – N. LERNER, *Uniqueness of continuous solutions for BV vector fields*, Duke Math. J. **111** (2002), 357-384.

[CL2] F. COLOMBINI – N. LERNER, *Uniqueness of L^∞ solutions for a class of conormal BV vector fields*, to appear in: "Geometric Analysis of PDE and Several Complex Variables", 2003, S. Chanillo – P. Cordaro – N. Hanges – J. Hounie – A. Meziani (eds.).

[CLR] F. COLOMBINI – T. LUO – J. RAUCH, *Uniqueness and nonuniqueness for nonsmooth divergence free transport*, Séminaire XEDP, Ecole Polytechnique (2003-04).

[De] N. DEPAUW, *Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d'un hyperplan.*, C.R. Math. Acad. Sci. Paris **337** (2003), 249-252.

[DL] R. J. DIPERNA – P.-L.LIONS, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), 511-547.

[Fe] H. FEDERER, "Geometric measure theory", Grund. der Math. Wiss. **153**, Springer-Verlag, 1969.

[LL] C. LE BRIS – P.-L. LIONS, *Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications*, Ann. Mat. Pura Appl. **183** (2004), 97-130.

[Li] P.-L.LIONS, *Sur les équations différentielles ordinaires et les équations de transport*, C.R. Acad. Sc. Paris, Série I, **326** (1998), 833-838.

[Tr] F. TREVES, "Topological vector spaces, distributions and kernels", Pure & Appl. Math. Ser., Academic Press, 1967.

[Vo] A. I. VOL'PERT, *The space BV and quasi-linear equations*, Math. USSR Sbornik **2** (1967), 225-267.

[Zi] W. P. ZIEMER, "Weakly differentiable functions", Graduate texts in mathematics, Springer-Verlag, Vol. 120, 1989.

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