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Positive Knots, Closed Braids and the Jones Polynomial

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Abstract. Using the recent Gauß diagram formulas for Vassiliev invariants of Polyak-Viro-Fiedler and combining these formulas with the Bennequin inequality, we prove several inequalities for positive knots relating their Vassiliev invariants, genus and degrees of the Jones polynomial. As a consequence, we prove that for any of the polynomials of Alexander/Conway, Jones, HOMFLY, Brandt-Lickorish-Millett-Ho and Kauffman there are only finitely many positive knots with the same polynomial and no positive knot with trivial polynomial.

We also discuss an extension of the Bennequin inequality, showing that the unknotting number of a positive knot is not less than its genus, which recovers some recent unknotting number results of A'Campo, Kawamura and Tanaka, and give applications to the Jones polynomial of a positive knot.

Mathematics Subject Classification (2000): 57M25 (primary), 05C99 (secondary).

1. – Introduction

Positive knots, the knots having diagrams with all crossings positive, have been for a while of interest for knot theorists, not only because of their intuitive defining property. Such knots have occurred, in the more special case of braid positive knots (in this paper knots which are closed positive braids will be called so) in the theory of dynamical systems [BW], singularity theory [A], [BoW], and in the more general class of quasipositive knots (see [Ru3]) in the theory of algebraic curves [Ru].

Beside the study of some classical invariants of positive knots [Bu], [CG], [Tr], significant progress in the study of such knots was achieved by the discovery of the new polynomial invariants [J], [H], [Ka2], [BLM], [Ho], giving rise to a series of results on properties of these invariants for this knot class [Cr], [Fi], [CM], [Yo], [Zu].

Recently, a conceptually new approach for defining invariants of finite type (Vassiliev invariants) [BL], [BN], [BN2], [BS], [St4], [St], [Vo], [Va] was ini-

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tiated by Fiedler [Fi] and Polyak-Viro [PV] by the theory of small state (or Gauß) sums. Fiedler remarked [Fi4] that the Gauß sum formulas have direct application to the study of positive knots.

This paper aims to work out a detailed account on such applications. Sharpening Fiedler's results, we will prove a number of inequalities for positive knots, relating via the Gauß sum formulas the Vassiliev invariants $v_2(K)$ and $v_3(K)$ of degree 2 and 3 of a positive knot K on the one hand, and classical invariants like its genus $g(K)$, crossing number $c(K)$ and unknotting number $u(K)$ on the other hand. We will use the tables of Rolfsen [Ro, Appendix] and Thistlethwaite [HT] to find examples illustrating and showing the essence of these properties as positivity criteria for knots.

Although all inequalities can be considered in their own right, one of them, which subsequently turned out of central importance, and is thus worth singling out, is the inequality between v_2 and $c(K)$ we will prove in Section 6 (Theorem 6.1). Similar (although harder to prove) inequalities will be first discussed in Section 3 for v_3, improving the one originally given by Fiedler.

THEOREM. *In a positive reduced c crossing knot diagram,*

$$v_2 \geq \frac{c}{4} \qquad \text{and} \qquad v_3 \geq \frac{3}{2}c - 3\,.$$

As a consequence of involving the crossing number into our bounds, we prove:

COROLLARY. *There are only finitely many positive knots with the same Jones polynomial V, and any knot has only finitely many* (*possibly no*) *positive reduced diagrams.*

Thus positivity can always be (at least theoretically) decided for any knot, provided one can identify a knot from a given diagram. A further application of such type of inequality is given in [St7], where it is decisively used to give polynomial bounds of the number of positive knots of fixed genus and given crossing number.

For our results on unknotting numbers it will also turn out useful to apply the machinery of inequalities of Bennequin type [Be, Theorem 3, p. 101] for the (slice) genus. Thus we devote a separate section Section 4 to the discussion of this topic. In particular, there we give an extension of Bennequin's inequality to arbitrary diagrams. An application of this extension is the observation that the unknotting number of a positive knot is not less than its genus (Corollary 4.3). This resolves, *inter alia*, the unknotting numbers of 5 of the undecided knots in Kawauchi's tables [Kw], which have been (partially) obtained by Tanaka [Ta], Kawamura [Km] and A'Campo [A] (Examples 4.1, 4.2 and 4.3). It can also be used to extend some results proved on the genus of positive knots to their unknotting number (see [St7]).

In Section 7 we will use the inequalities derived for v_2 and v_3 together with those given by Morton [Mo2] to give some relations between the values of v_2, v_3 and the HOMFLY polynomial of positive knots.

Braid positive knots, *inter alia*, because of their special importance will be considered in their own right in Section 8, where some further specific inequalities for the Vassiliev invariants will be given. We will also prove, that the minimal degree of the Jones polynomial of a closed positive braid is equal for knots to the genus and is at least a quarter of its crossing number.

Finally, in the Sections 4 and 9 we will review some results and conjectures and summarize some questions, which are interesting within our setting.

Notation

For a knot K denote by $c(K)$ its (minimal) crossing number, by $g(K)$ its genus, by $b(K)$ its braid index, by $u(K)$ its unknotting number, by $\sigma(K)$ its signature. $!K$ denotes the obverse (mirror image) of K. We use the Alexander-Briggs notation and the Rolfsen [Ro] tables to distinguish between a knot and its obverse. "Projection" is the same as "diagram", and this means a knot or link diagram. Diagrams are always assumed oriented.

$\#D$ or $|D|$ denote the cardinality of a (finite) set D.

Let $[P]_{t^a} = [P]_a$ be the coefficient of t^a in a polynomial $P \in \mathbb{Z}[t^{\pm 1}]$. Let

$$\begin{aligned} \min\deg P &= \min\{\, a \in \mathbb{Z} \,:\, [P]_a \neq 0 \,\}\,, \\ \max\deg P &= \max\{\, a \in \mathbb{Z} \,:\, [P]_a \neq 0 \,\}, \\ \min\operatorname{cf} P &= [P]_{\min\deg P} \end{aligned}$$

be the minimal and maximal degree and minimal coefficient of P, respectively. For two sequences (a_n) and (b_n) the notation '$a_n \asymp b_n$' means that $\limsup\limits_{n\to\infty} a_n/b_n < \infty$ and $\liminf\limits_{n\to\infty} a_n/b_n > 0$.

B_n denotes the n-strand braid group. For a braid β denote by $\hat{\beta}$ its closure, by $n(\beta)$ ist strand number and by $[\beta]$ its homology class (or exponent sum), i.e. its image under the homomorphism $[\,.\,] : B_n \to H_1(B_n) = B_n/\langle [B_n, B_n] \rangle \simeq \mathbb{Z}$, given by $[\sigma_i] = 1$, where σ_i are the Artin generators.

The symbol $\square$ denotes the end or the absence of a proof. In latter case it is *assumed* to be evident from the preceding discussion/references; else (and anyway) I'm grateful for any feedback.

"R.h.s." abbreviates "right hand side", "w.r.t." abbreviates "with respect to" and "w.l.o.g." abbreviates "without loss of generality". We will sometimes (although not often) use the logical notation (with functors as $\exists$, $\forall$) for some more complex expressions to avoid verbal misinterpretation.

ACKNOWLEDGEMENTS. I would wish to thank to K. Murasugi for his kindness and help and to T. Fiedler for his remarks and pointing out an extension of Theorem 4.3. I am also especially grateful to S. Tabachnikov, who read carefully parts of this paper and suggested numerous stylistical improvements.

T. Kawamura informed me about the results of T. Tanaka and T. Kanenobu informed me about [K3]. The referee pointed out a few minor mistakes. Finally, I am very grateful to the editors of the Annali for giving me the opportunity to publish this paper, after being turned down by 8 other journals in the course of over 5 years since its initial writing.

2. – Positive knots and Gauß sums

DEFINITION 2.1. The writhe is a number (± 1), assigned to any crossing in a link diagram. A crossing as on Figure 1(a), has writhe 1 and is called positive. A crossing as on Figure 1(b), has writhe -1 and is called negative. A crossing is smoothed out by replacing it by the fragment on Figure 1(c) (which changes the number of components of the link). A crossing as on Figure 1 (a) and 1(b) is smashed to a singularity (double point) by replacing it by the fragment on Figure 1(d). A m-singular diagram is a diagram with m crossings smashed. A m-singular knot is an immersion represented by a m-singular diagram.

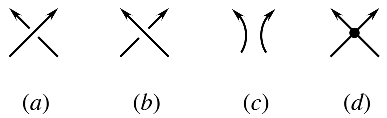


Fig. 1.

DEFINITION 2.2. A knot is called positive, if it has a positive diagram, i.e. a diagram with all crossings positive.

Recall [FS], [PV] the concept of Gauß sum invariants. As they will be the main tool of all the further investigations, we summarize for the benefit of the reader the basic points of this theory.

DEFINITION 2.3 ([Fi3], [PV]). A Gauß diagram (GD) of a knot diagram is an oriented circle with arrows connecting points on it mapped to a crossing and oriented from the preimage of the undercrossing to the preimage of the overcrossing. See Figure 2.

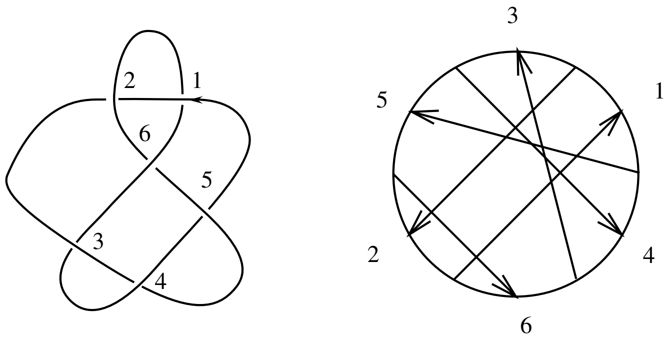
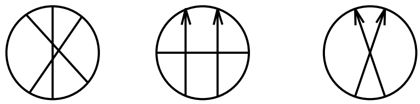


Fig. 2. The knot 6_2 and its Gauß diagram.

Fiedler [Fi3], [FS] found the following formula for (a variation of) the degree-3-Vassiliev invariant using Gauß sums.

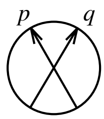
$$v_3 = \sum_{(3,3)} w_p w_q w_r + \sum_{(4,2)0} w_p w_q w_r + \frac{1}{2} \sum_{p,q \text{linked}} (w_p + w_q), \tag{1}$$

where the configurations are



(3, 3) (4, 2) 0 a linked pair

Here chords depict arrows which may point in both directions and w_p denotes the writhe of the crossing p. For a given configuration, the summation in (1) is done over each unordered pair/triple of crossings, whose arrows in the Gauß diagram form that configuration. The terms associated to a pair/triple of crossings occurring in the sums are called *weights*. If no weight is specified, we take by default the product of the writhes of the involved crossings. Thus



means 'sum of $w_p \cdot w_q$ over p, q linked'. In the linked pair of the picture above, call p *distinguished*, that is, the over-crossing of p is followed by the under-crossing of q. For the motivation of this notation, see [FS].

Additionally, one may put a base point on both the knot and Gauß diagram (see [PV]). This is equivalent to distinguishing a cyclic order of the arrow ends, or "cutting" the circle somewhere.

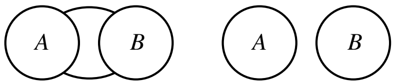
To make precise which variation of the degree-3-Vassiliev invariant we mean, we noted in [FS], that

$$v_3 = -\frac{1}{3} V^{(2)}(1) - \frac{1}{9} V^{(3)}(1),$$

where V is the Jones polynomial [J] and $V^{(n)}$ denotes the n-th derivation of V. The normalization differs by a factor of 4 from the standard one, used in [PV], in which the invariant has values ± 1 on the trefoils. We noted further (and shall use it later), that v_3 is additive under connected knot sum, that is, $v_3(K_1 \# K_2) = v_3(K_1) + v_3(K_2)$.

Definition 2.4. A diagram is composite, if it looks as in Figure 3(a) and both A and B contain at least one crossing. A diagram is split, if it looks as in Figure 3(b) and both A and B are non-empty. A composite link is a link with a composite diagram, in which no one of A and B represent the unknot. A split link is a link with a split diagram.

We will use the synonyms ‘prime’ and ‘connected’ for ‘non-composite’ and ‘disconnected’ for ‘composite’.



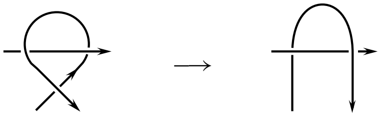
(*a*) (*b*)

Fig. 3.

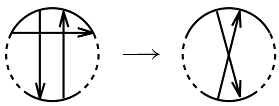
DEFINITION 2.5. A crossing is reducible, if its smoothing out yields a split diagram. A diagram is reduced, if it has no reducible crossings.

DEFINITION 2.6. Call a positive diagram bireduced, if it is reduced and does not admit a move

(2)



To this move we will henceforth refer as a second (reduction) move. On the Gauß diagram this move looks like:



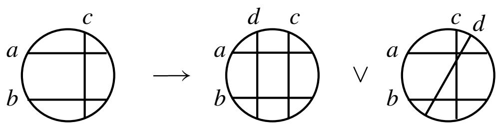
The reason for introducing this move will become clear shortly.

DEFINITION 2.7. The intersection graph of a Gauß diagram is a graph with vertices corresponding to arrows in the Gauß diagram and edges connecting intersecting arrows/vertices.

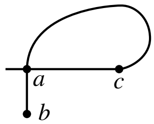
DEFINITION 2.8. For two chords in a Gauß diagram $a \cap b$ means “a intersects b” (or crossings a and b are linked) and $a \not\cap b$ means “a does not intersect b” (or crossings a and b are not linked).

Gauß diagrams have in general two simple properties that will be extensively used in the following, even valence (introduced later in Section 3) and double connectivity.

LEMMA 2.1 (double connectivity). *Whenever in a Gauß diagram $a \cap c$ and $b \cap c$ then either $a \cap b$ or there is an arrow d with $d \cap a$ and $d \cap b$. That is, in the intersection graph of the Gauß diagram any two neighbored edges participate in a cycle of length* 3 *or* 4. *In particular, the Gauß diagram* (*or its intersection graph*) *are doubly connected.*



PROOF. Assume $a \not\cap b$. Consider the plane curve of the projection.



As the curve meets c the second time before doing so with b, it has a segment in the inner part of the above depicted loop between both occurrences of a, and so there must be another crossing between the first and second occurrence of a and the first and second occurrence of b. □

3. – Inequalities for v_3

In this section we shall prove an obstruction to positivity which renders it decidable, whether a given knot has this property. The idea is due to Fiedler, but here we present an improved version of it.

Our goal is now to prove the following two statements.

1) The number of edges in the intersection graph of a non-composite Gauß diagram (= intersections of chords in the Gauß diagram = linked pairs) is at least $3\lfloor\frac{c-1}{2}\rfloor$, where c is the number of vertices in the intersection graph (= chords in the Gauß diagram = crossings in the knot projection).
2) In any positive diagram D of c crossings, $v_3(D) \geq lk(D) \geq c$, where $lk(D)$ is the number of linked pairs in D. If D is bireduced, then $v_3(D) \geq \frac{4}{3}lk(D)$.

We do this in steps and split the arguments into several lemmas. Finally, we summarize the results in a more self-contained form in Theorem 3.1. We start by

LEMMA 3.1. *If K has a positive reduced diagram of c crossings, then $v_3(K) \geq c$.*

PROOF. Consider the Gauß diagram of this projection.

By the additivity of v_3 and c in composite projections (note, that factors in composite positive reduced projections and themselves in positive reduced projections), henceforth assume, the positive diagram is non-composite, i.e. the Gauß diagram of c arrows is connected.

But then this diagram will have at least $c-1$ intersections, i.e. linked pairs, and the above argument (Lemma 2.1) shows that, as the number of arrows is more than 2, the number of chord intersections cannot be equal to $c-1$. As in a positive diagram each linked pair contributes one to the value of the third term in (1) and the first two terms are non-negative, the assertion follows. □

Bounds of this kind render it decidable whether a positive diagram exists.

COROLLARY 3.1. *Any reduced positive diagram of any knot K has at most $v_3(K)$ crossings. In particular, there are only finitely many positive knots with the same v_3 and any knot has only finitely many (possibly no) positive diagrams.* □

Therefore, there are also only finitely many positive knots with the same Jones polynomial, but we will state this fact in greater generality somewhat later.

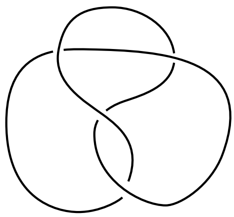
COROLLARY 3.2. *If K is not the unknot, at most one of K and $!K$ can be positive. In particular, no positive non-trivial knot is amphicheral.*

This fact follows in the special case of alternating knots from Thistlethwaite's invariance of the writhe [Ka2] and for Lorenz knots from work of Birman and Williams [BW]. A general proof was first given by Cochran and Gompf [CG, Corollary 3.4, p. 497] and briefly later independently by Traczyk [Tr] using the signature. We will later, in passing by, give an independent argument for the positivity of the signature on positive knots using Gauß diagrams.

PROOF. It follows from $v_3(!K) = -v_3(K)$ which is easy to see from the formula (1): mirroring reverses the orientation of all arrows in the Gauß diagram and all configurations in (1) are invariant under this operation, while the terms in the sum change the sign. □

EXAMPLE 3.1. 4_1 (the figure eight knot) and 6_3 are amphicheral and hence cannot be positive.

REMARK 3.1. The bound of Lemma 3.1 is sharp, as $v_3(!3_1) = 4$ and there is the following reduced positive 4 crossing diagram of the right-hand trefoil:



However, this diagram is not bireduced and here I came to consider this notion.

EXAMPLE 3.2. Beside the standard and the above depicted diagram, there cannot be more positive reduced diagrams of the right-hand trefoil $!3_1$.

EXAMPLE 3.3. Following T. Fiedler, and as indicated in [FS], the knot 6_2 has $v_3 = 4$. So it cannot have any positive diagram (as else it would have a reduced one and this would have to have not more than 4 crossings).

Here is another property of Gauß diagrams we will use in the following to sharpen our bound. Call the *length* of a chord d in a Gauß diagram the smaller number of segments of the circle (between basepoints of two chords) on its two parts separated by d.

LEMMA 3.2 (even valence). *Any chord in a Gauß diagram has odd length* (i.e., *even number of basepoints on both its sides, or equivalently, even number of intersections with other chords, that is, even valence in the intersection graph of the Gauß diagram*).

PROOF. This is, as Lemma 2.1, a consequence of the Jordan curve theorem, and is reflected e.g. also in the definition of the Dowker notation of knot diagrams [DT]. □

Here is the improved bound announced in [FS] under assumption of bireducedness.

LEMMA 3.3. *If K has a positive bireduced diagram D of c crossings, then*

$$v_3(K) \geq \frac{4}{3} lk(D) \geq \frac{4}{3}\,c\,.$$

In particular, $v_3(K) \geq \frac{4}{3}c(K)$ for K positive.

PROOF. We must prove the first inequality (the second one was proved in Lemma 3.1). Assume w.l.o.g. as before the Gauß diagram is connected. We know that the number of intersections in the Gauß diagram (= number of linked pairs) is at least c. So it suffices to prove

$$\#\{\text{matching } (3,3) \text{ and } (4,2)0 \text{ configurations}\} \geq \frac{1}{3} lk(D)\,.$$

To do this, we will construct a map

$$\begin{aligned} m : &\{\text{crossings in the GD (linked pairs)}\} \\ &\longrightarrow \{\text{matching } (3,3) \text{ and } (4,2)0 \text{ configurations}\} \end{aligned}$$

such that each image is realized not more than 3 times.

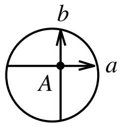
Denote the definition domain of m by $\mathcal{D}$ and its value domain by $\mathcal{E}$. To define m, we distinguish several cases, that is, partition the definition domain $\mathcal{D}$ of m into disjoint sets

$$\mathcal{D} = \biguplus_{i=1}^{n} \mathcal{U}_i\,,$$

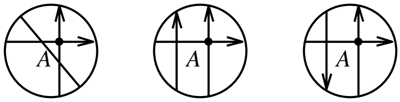
each $\mathcal{U}_i$ corresponding to a particular case. To prove the property of m that $|m^{-1}(e)| \leq 3$ for any $e \in \mathcal{E}$, we prove it successively for $m_i := m|_{\tilde{\mathcal{U}_i}}$ with $\tilde{\mathcal{U}_i} = \bigcup_{j=1}^{i} \mathcal{U}_j$, where $m_i^{-1}(e) = m^{-1}(e) \cap \tilde{\mathcal{U}_i}$. Since $\tilde{\mathcal{U}_n} = \mathcal{D}$, we will be done for $i = n$.

About the definition of m. Set m on a crossing participating in a $(3,3)$ or $(4,2)0$ configuration to one (any arbitrary) of these configurations. So, up to now, all $(4,2)0$ configurations are realized as image under m at most 2 and all $(3,3)$ configurations are realized as image under m at most 3 times.

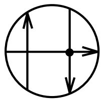
Now look at a crossing A, not participating in *any* $(3, 3)$ and $(4, 2)0$ configuration.



If chord a has length 3 then we have either



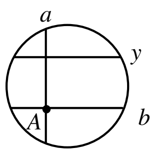
In the first two cases A is in a $(3, 3)$ or $(4, 2)0$ configuration, and in the third case this is exactly the situation of a second move (2). Note: it follows from the positivity of the diagram, that indeed



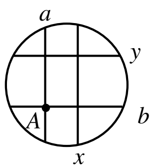
does not exist. Else the diagram part on the right in (2) to be positive, we had to reverse the direction of (exactly) one of the strands, and the crossings would become linked.

So let a have length at least 5.

Case 1. First assume a has only 2 crossings.

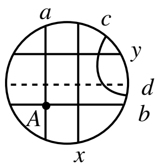


We have $y \not\cap b$ (else $A \in (3, 3)$). By double connectivity $\exists\, x, x \cap y, b$.

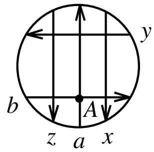


x does not intersect a (else $A \in (3, 3)$).

On the other hand, if for some c, $c \cap y$, then $c \cap b$ and vice versa. (If $c \cap y$ but $c \not\cap b$, by double connectivity on a, c, y we had $\exists d \cap c, d \cap a$. As $b \not\cap c$ we had $d \neq b$, and so d would be a third intersection of a.)

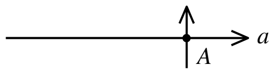


As a is not of length 3, on the side of a not containing x, there must be a chord z which (by assumption of connectedness of the diagram) must intersect one of b or y and therefore (see above) both.



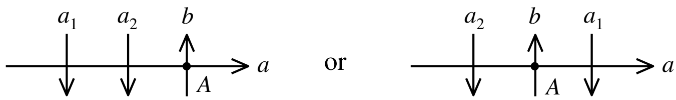
Then z, x must be equally oriented with respect to y and b (else $A \in (4,2)0$), i.e. (z, x, y) and (z, b, x) are of type $(4,2)0$. Define $m(A)$ to be the second one of these configurations. (3)

Case 2. So now let a have at least 4 crossings (remember, each chord has even number of crossings). Look at a:



Beside by b, a is intersected $n \geq 3$ times by (only) downward pointing arrows (else either $A \in (3,3)$ or $A \in (4,2)0$).

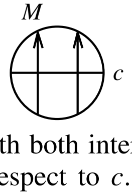
Case 2.1. Two such chords a_1, a_2 do not intersect.



Set

$$m(A) = M := \{a_1, a_2, a\} \in (4,2)0\,. \tag{4}$$

Up to now, $M \in (4,2)0$ has only 2 preimages, unless it was not the object of an assignment of the kind (3) or (4) before. However, there is only at most one such additional preimage A of M, because we can uniquely reconstruct A from M:

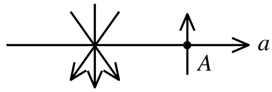


Consider the chord c in M with both intersections on it. Then the other two arrows point in 1 direction with respect to c. A is then the unique intersection point of c with an arrow pointing in the opposite direction than the other two arrows of M do.

Summarizing Cases 1 and 2.1, no $(4,2)0$ configuration received more than 3 preimages as far as m is constructed now.

Case 2.2. All $n \geq 3$ chords intersect. The picture is like this

(5)

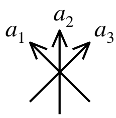


These n chords produce with a $\binom{n}{2}$ configurations of type $(3,3)$ and among themselves $\binom{n}{2}$ intersections. So $n+\binom{n}{2}$ intersection points participate in $\binom{n}{2}$ configurations $(3,3)$ involving a. So there is a relation among these $\binom{n}{2}$ with a preimage under m of at most

$$\frac{n+\binom{n}{2}}{\binom{n}{2}} \leq 2$$

intersections participating in the configuration. Define $m(A)$ to be any of these relations.

Now we count the m-preimages of a configuration of type $(3,3)$. If this configuration was not affected by the so far considered configurations in Case 2.2, it still has at most 3 preimages. If it has been, it had (before Case 2.2) at most 2 preimages among the intersections participating in it. Thus we need to care about how many "A" s could have been assigned to such a configuration K by Case 2.2. If any, K must look like



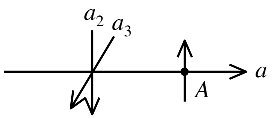
and A must be either on a_3 or a_1 and be the unique intersection point of a chord a' intersecting a_1 (resp. a_3) in the reverse direction of all other chords. Note that among these chords are a_3 (resp. a_1), so that the intersection direction of a' with a_1 (resp. a_3) is uniquely determined. So there are at most 2 such "A"'s and the configuration has at most 4 preimages.

We would like to show now that in fact $(3,3)$ configurations with 4 preimages can always be avoided by a proper choice of $(3,3)$ configurations in Case 2.2.

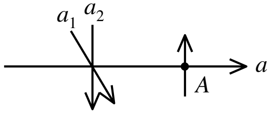
Assume, that at one point in Case 2.2 all configurations $(3,3)$ of a with two downward pointing arrows in (5) already have 3 preimages as a next A has to be added (that is, you are forced to create a fourth preimage to one of the $(3,3)$ configurations). Then there is only one choice. There are exactly 3 chords (which mutually intersect and intersect a), from the resulting 6 crossings and 3 configurations $(3,3)$ involving a, each configuration contains exactly 2 of its points in its preimage (for $n>3$ we have

$$\frac{n+\binom{n}{2}}{\binom{n}{2}} < 2\,,$$

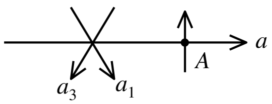
and so there is always a configuration with not more than one of its points in its preimage) and to each of these 3 configurations $(3,3)$ there has already been assigned an "A" by Case 2.2. (Here "A" means an intersection point, which participated as A in some previous application of Case 2.2.) There cannot have been 2 "A"'s added, as A would be the third possible one and we saw that there are no 3 possible ones for the same $(3, 3)$ configuration. Because on each chord of the configuration only one possible "A" can lie, this other "A" (different from our A) must lie on



a_3 for $\{a_3, a_2, a\}$



a_2 for $\{a_1, a_2, a\}$



a_3 for $\{a_3, a_1, a\}$

But this cannot be, because to the "A" on a_3 cannot simultaneously have been assigned both $\{a_3, a_1, a\}$ and $\{a_3, a_2, a\}$ under m. (This "A" on a_3 is unique, because there's always a in the configuration and this "A" must intersect with a_3 in the opposite direction, so that two such "A"'s would $\in (4, 2)0$.) This contradiction shows, that it must be really always possible to define m on an "A" in Case 2.2, not augmenting the number of preimages of a $(3, 3)$ configuration to more than 3.

So now any configuration of type $(3, 3)$ has at most 3 preimages and m is completely defined, and has the desired property. □

But of course, there are in general much more linked pairs than crossings, and so we can go a little further.

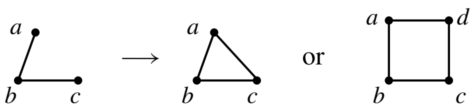
Consider the intersection graph G of a Gauß diagram.

LEMMA 3.4. *In G*

$$\#\,\text{edges} \geq 3\left(\left\lfloor\frac{\#\,\text{vertices} - 1}{2}\right\rfloor\right),$$

if G connected, i.e. the Gauß diagram non-composite.

PROOF. Recall that the intersection graph of a Gauß diagram has the double connectivity property, that each pair of neighbored edges lies in some 3 or 4 cycle.



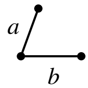
Fix a spanning tree B of the intersection graph G. Denote by c the number of vertices in G. B has $c - 1$ edges, as G is connected. Choose a disjoint cover $\mathcal{U}$ of $2\lfloor\frac{c-1}{2}\rfloor$ edges in B in $\lfloor\frac{c-1}{2}\rfloor$ pairs, so that the two edges in each pair have a common vertex (the reader may easily verify that this is always possible). We call such a pair of edges *neighbored*.

Now we apply (6) to construct a map

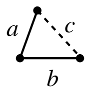
$$m : \mathcal{U}' \longrightarrow \{\text{edges and pairs of edges in } G \text{ outside } B\},$$

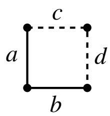
where $\mathcal{U}'$ is an extension (or supercover) of $\mathcal{U}$ (that is, $\forall A \in \mathcal{U}\,\exists\, A' \in \mathcal{U}' : A' \supset A$), yet to be specified.

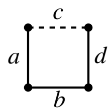
Define m as follows by the same case strategy as in the proof of Lemma 3.3. Fix a pair



in B. There are 3 cases (the full edges belong to B and the dashed edges are outside of B).

1) 

2) 

3) 

Case 1. Set $m(\{a, b\}) := c$.

Case 2. Set $m(\{a, b\}) := \{c, d\}$.

Case 3. Set $m(\{a, b, d\}) := c$.

As all the pairs are disjoint, all triples obtained by extending a pair by one element in Case 3 are distinct, and the map is well-defined. $\mathcal{U}'$ is the cover $\mathcal{U}$, where some of the pairs have been extended to triples by Case 3. We have $\#\mathcal{U}' = \lfloor\frac{c-1}{2}\rfloor$.

As B is without cycle, no edge in $G\backslash B$ has received two preimages by Cases 1 and 3. In the same way, no pair of (neighbored) edges in $G\backslash B$ received two preimages by Case 2. Moreover, if we look at the dual graph G^* of G (where such pairs correspond to edges), the edges in $G\backslash B$ with a preimage by Case 2 from a forest F in G^*. (Convince yourself, using Figure 4, that the existence of a cycle in F implies one in B.) Therefore, for all components C of F the number of involved vertices in C (= edges in $G\backslash B$ involved in one of these pairs) is bigger that the number of edges of C (= pairs of edges in $G\backslash B$ with a preimage by m). Furthermore, for all components C of F at most one of the vertices of C (= edges in $G\backslash B$) has a preimage by 1) or 3) (again as B is a tree). So we see that

$$\#\mathcal{U}' \leq \#\{\text{edges in } G\backslash B\}.$$

So there are at least $\lfloor\frac{c-1}{2}\rfloor$ edges in $G\backslash B$ and at least

$$c - 1 + \left\lfloor\frac{c-1}{2}\right\rfloor \geq 3\left\lfloor\frac{c-1}{2}\right\rfloor$$

edges in G. $\square$

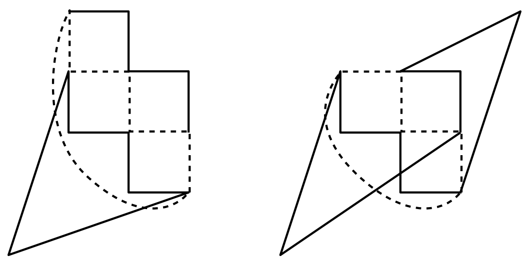


Fig. 4. How a cycle in F implies one in B. Full edges belong to B and dashed outside of B. The vertices in the cycle in F are the edges in the dashed cycle in G. The edges in the cycle in F are pairs of neighbored edges in the dashed cycle in G. The edges in B depicted belong to the pairs of $\mathcal{U}$, which are preimages of the edges in the cycle in F. They are all disjoint and contain together a cycle.

Summarizing, we proved:

THEOREM 3.1. *If K has a positive bireduced non-composite projection of c crossings, then $v_3(K) \geq 4\lfloor\frac{c-1}{2}\rfloor$. If the projection is not bireduced, but reduced and non-composite, we have at least $v_3(K) \geq 3\lfloor\frac{c-1}{2}\rfloor$. If the projection is composite and bireduced, we have $v_3(K) \geq \frac{4}{3}c$, and if it is composite and reduced but not bireduced, $v_3(K) \geq c$.* □

COROLLARY 3.3. *If a prime knot K is positive, it has a positive diagram of not more than $v_3(K)/2 + 2$ crossings.* □

There exist some other generally sharper obstructions to positivity. One is due to Morton and Cromwell [CM]: If P denotes the HOMFLY polynomial [H] (in the convention of [St2]), then for a positive link $P(it, iz)$ must have only non-negative coefficients in z for any $t \in [0, 1]$ (i denotes $\sqrt{-1}$). The special case $t = 1$ is the positivity of the Conway polynomial [Co], proved previously for braid positive links by v. Buskirk [Bu] and later extended to positive links by Cromwell [Cr].

Moreover, in [Cr] it was proved, that for L positive $\min\deg_l(P) = \max\deg_m(P)$.

That these obstructions, although generally sharper, are not always better, shows the following example, coming out of some quest in Thistlethwaite's tables.

EXAMPLE 3.4. The knot 12_{2038} on Figure 5 has the HOMFLY polynomial

$$(-7l^6 - 9l^8 - 3l^{10}) + (13l^6 + 13l^8 + 3l^{10})m^2 + (-7l^6 - 6l^8 - l^{10})m^4 + (l^6 + l^8)m^6.$$

It shows, that the obstructions of [Bu], [Cr] and [CM] are not violated. However, $v_3(12_{2038}) = 8$.

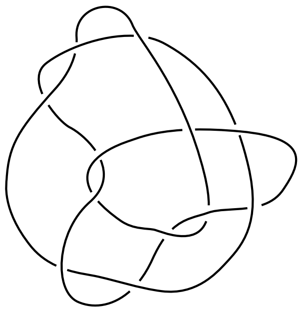


Fig. 5. The knot 12_{2038}.

REMARK 3.2. One may ask, in how far can the given bounds be improved. The answer is, using our arguments, not very much, as the following shows:

EXAMPLE 3.5. Consider the graph G_n, which is the Hasse diagram of the lattice $(\mathcal{P}(\{1, \dots, n\}), \subset)$. That is, its vertices are subsets of $\{1, \dots, n\}$ and A and B are connected by an edge, if $B \subset A$ and $\#(A \setminus B) = 1$. Then G_n satisfies the double connectivity property of Lemma 2.1 and, if n is even, also the even valence property of Lemma 3.2. A Gauß diagram of $c = 2^n$ arrows, with G_n as intersection graph, would yield a value of v_3, asymptotically equivalent modulo constants to $c \cdot (\log_2 c)^2$.

Of course, a simple argument shows, that any graph containing already G_3 as subgraph (i. a., $G_4, G_6, \dots$) cannot be the intersection graph of a Gauß diagram, but evidently closer study of the structure of (intersection graphs of) Gauß diagrams is necessary. Unfortunately, the further conditions will not be that simple and bringing them into the game will make proofs (even more) tedious.

But, in any case, note, that the odd crossing number twist knots ($!3_1$, $!5_2$, $!7_2$, $!9_2, \dots$) show, that we cannot prove more than quadratical growth of v_3 in c. We will see later that even linear growth of v_3 in c is possible (Example 8.5), but will still say something on the twist knots in Section 6.

4. – Unknotting numbers and an extension of Bennequin's inequality

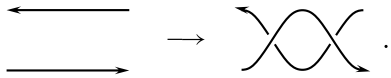
In this section, we introduce the machinery of Bennequin type inequalities and review some results on unknotting numbers, which will subsequently needed to prove further properties of positive knots. Most of these results are well-known. The others are consequences of them that follow straightforwardly, but yet deserve mention in their own right. Alternatively, the reader may consult [N2], [Km2].

Parallelly we will also consider the following question, which naturally arises in the study of knots (and links) via braids.

QUESTION 4.1. Many classical properties of knots are defined by the existence of diagrams with such properties. In how far do these properties carry over, if we restrict ourselves to closed braid diagrams?

We will discuss this question in Section 8 for positivity and, applying our new criteria, give examples that the answer is in general negative. On the other hand, here we will observe for unknotting number the answer to be positive.

To start with, recall the result of Vogel [Vo2] that each diagram is transformable into a closed braid diagram by crossing-augmenting Reidemeister II moves on pairs of reversely oriented strands belonging to *distinct* Seifert circles (henceforth called Vogel moves) only:



As observed together with T. Fiedler, this result has 2 interesting independent consequences. The first one is a "singular" Alexander theorem

THEOREM 4.1. *Each m-singular knot is the closure of an m-singular braid.*

PROOF. Apply the Vogel algorithm to the m-singular diagram, which clearly does not affect the singularities. □

This was, however, also known previously, see e.g., [Bi2].

The other consequence is related to Question 4.1.

THEOREM 4.2. *Each knot realizes its unknotting number in a diagram as a closed braid.*

PROOF. Take a diagram D of K realizing its unknotting number and apply the Vogel algorithm obtaining a diagram D'. As the crossing changes in D commute with the Vogel moves, the same crossing changes unknot K in D'. □

That is, the answer of Question 4.1 for unknotting number is yes!

Combining Vogel's result with the Bennequin inequality [Be], we immediately obtain

THEOREM 4.3. *In each diagram D of a knot K,*

$$|w(D)| + 1 \leq n(D) + 2g(K)\,, \tag{7}$$

where $w(D)$ and $n(D)$ are the writhe and Seifert circle number of D.

PROOF. Bennequin proved the theorem for braid diagrams. From this it follows for all diagrams by the Vogel algorithm, as a Vogel move changes neither the writhe nor the number of Seifert circles. □

This fact for the unknot (which is also a special case of a result of Morton [Mo2], who proved it for all achiral knots) proves (in an independent way than Theorem 3.1) the following

COROLLARY 4.1. *There is no non-trivial positive irreducible diagram of the unknot.*

That is, in positive diagrams the unknot behaves as in alternating ones.

PROOF. For such a diagram D, $n(D) = c(D) + 1$, where $c(D)$ the number of crossings of D. Therefore, each smoothing of a crossing in D augments the number of components. Hence no pair of crossings in the Gauß diagram can be linked, and so all chords are isolated and all crossings are reducible. $\square$

Another more general corollary is originally due to Cromwell [Cr]:

COROLLARY 4.2 (Cromwell). *The Seifert algorithm applied to positive diagrams gives a minimal surface.*

PROOF. This follows from Theorem 4.3 together with the formula for the genus of the Seifert algorithm surface associated to D, which is $(c(D)-n(D)+1)/2$, as $|w(D)| = c(D)$ for D positive. $\square$

Coming finally back to Question 4.1, we see that we have discussed the most interesting cases. For the property of a diagram to be of minimal crossing number, 10_8 is an example that the answer is negative as well. For a diagram to realize the braid index the question does not make much sense, neither it does for Seifert genus. Certainly the Seifert algorithm assigns a surface to each diagram. However, Morton [Mo2] proved that there really exist knots, where in *no* diagram the Seifert algorithm gives a minimal Seifert surface! Posing Question 4.1 on minimality just for canonical Seifert surfaces, that is, Seifert surfaces obtained by the Seifert algorithm, the answer is again negative. The knot 7_4 has a positive diagram, and hence a canonical Seifert surface of (minimal) genus 1, whereas by [BoW] the genus of a canonical Seifert surface in any of its braid diagrams is minorated by its unknotting number 2, calculated by Lickorish [Li] and Kanenobu-Murakami [KM].

The only interesting case to discuss is

QUESTION 4.2. Does each knot realize its bridge number in a closed braid diagram?

Another question coming out of Vogel's result is

QUESTION 4.3. Does each knot realize its unknotting number in a diagram as closed braid of minimal strand number?

Bennequin conjectured (7) also to hold if we replace genus by unknotting number (this is sometimes called the Bennequin unknotting conjecture). This was recently proved by Kronheimer and Mrowka [KMr] (see remarks below) and independently announced by Menasco [Me2], but, to the best of my knowledge, without a published proof. As before, Vogel's algorithm extends this inequality.

THEOREM 4.4. *In each diagram D of a knot K we have the inequality* $|w(D)| + 1 \leq n(D) + 2u(K)$. $\square$

As we observed in Corollary 4.2, (7) is sharp for positive knots and so we obtain

COROLLARY 4.3. *For any positive knot K we have $u(K) \geq g(K)$.* □

This, combined with the inequality of Boileau-Weber-Rudolph [BoW], [Ru] leads to

COROLLARY 4.4. *For any braid positive knot K we have $u(K) = g(K)$.* □

This was conjectured by Milnor [Mi] for algebraic knots, neighborhoods of singularities of complex algebraic curves, which are known to be braid positive. Boileau and Weber [BoW] led it back to the conjecture that the ribbon genus of an algebraic knot is equal to its genus (see Section 4 of [Fi]), which was in turn known by work of Rudolph [Ru, p. 30 bottom] to follow from the Thom conjecture, recently proved by Kronheimer and Mrowka [KMr].

As pointed out by Thomas Fiedler, more generally, Corollary 4.3 also follows from Rudolph's recent result [Ru3] that positive links are (strongly) quasipositive, as he proved [Ru2] that a quasipositive knot bounds a complex algebraic curve in the 4-ball. The genus of such a curve is equal to the lower bound for g in Bennequin's inequality and so not higher than g itself. Hence if a knot, which bounds a complex algebraic curve is positive, then the genus of the knot is equal to the 4-ball genus of the complex algebraic curve that it bounds, which by [KMr] was proved to realize the slice genus of the knot, and this is as well-known always not greater than its unknotting number.

Using Corollary 4.4 we can determine the unknotting number of some knots.

EXAMPLE 4.1. The knots 10_{139} and $!10_{152}$ are braid positive, which is evident from their diagrams in [Ro]. Their Alexander polynomials tell us that they both have genus 4, hence their unknotting number is also 4.

Thus, we recover the result of Kawamura [Km]. However, Corollary 4.3 brings us a step further.

EXAMPLE 4.2. The knots 10_{154} and 10_{162} (the Perko duplication of $!10_{161}$) are positive and have genus 3. Hence their unknotting number is at least 3. Therefore, it is equal to 3, as 3 crossing changes suffice to unknot both knots in their Rolfsen diagrams (the reader is invited to find them). To determine the unknotting numbers in these examples is not possible with the Bennequin unknotting conjecture for itself. Although both knots satisfy (22), a property of braid positive knots we will recall in Section 8, they are both not braid positive. As their genus is 3, a positive n-braid realizing them would have $n+5$ crossings. For $n < 5$ this contradicts their crossing number, and for $n \geq 5$ such a braid would be reducible (getting us back to the case $n < 5$).

EXAMPLE 4.3. Another example is $!10_{145}$. $!10_{145}$ cannot be dealt with directly by Corollary 4.3, as it is not positive (see [Cr]). But it can be dealt with by observing that is differs by one crossing change from $!10_{161}$, or by the original Bennequin unknotting inequality: $!10_{145}$ is a (closed) 11 crossing

4-braid with writhe 7 [J2, appendix]. Thus this knot has unknotting number at least 2. On the other hand, 2 crossing changes suffice to unknot it as evident from its Rolfsen diagram [Ro, appendix].

For all 5 knots in Examples 4.1, 4.2 and 4.3 the inequality $|\sigma(K)/2| \leq u(K)$ is not sharp, hence the signature cannot be used to find out the unknotting number. Therefore, this also disproves a conjecture of Milnor (see [Be]), that $|\sigma(K)/2| = u(K)$ for braid positive knots.

It is, however, striking that all 5 knots are non-alternating. The reason for this is that if the positive diagram of K is also alternating, then indeed $|\sigma(K)/2| = g(K)$, and hence (modulo Question 9.5) Corollary 4.3 (and even the stronger Corollary 1 of [Ru3]) does not give anything more for the unknotting number than the signature. One way to see this is to use the principle of Murasugi and Traczyk (see [Tr] and [Ka2, p. 437]) to compute the signature in alternating diagrams using the checkerboard shading and to observe that if the alternating diagram is simultaneously positive, then the white regions correspond precisely to the Seifert circles.

T. Kawamura informed me that some of the Examples 4.1 and 4.2 have been obtained independently by T. Tanaka [Ta], who also found the unknotting number of $!10_{145}$, inspiring me to give an independent argument in Example 4.3. Very recently, A'Campo informed me that all these examples have also been obtained independently by him in [A].

A further related, and meanwhile very appealing, conjecture was made in [MP]. Using [St5, Remark 3.7], it can be restated as follows.

CONJECTURE 4.1. For any positive fibered knot K, $u(K) = g(K)$.

We have seen this to be true for braid positive knots and also for the two other positive fibered knots in Rolfsen's tables – 10_{154} and $!10_{161}$. The fact that a counterexample must have $u > g$ makes the conjecture hard to disprove. Since for showing $u > g$ any 4-genus estimate is useless, the only still handy way would be to use Wendt's inequality [We] $t(K) \leq u(K)$, where $t(K)$ counts the torsion coefficients of the $\mathbb{Z}$-homology of the double branched cover of S^3 along K. We know from the Seifert matrix that $t(K) \leq 2g(K)$, and for example some (generalized) pretzel knots show, that this inequality is sometimes sharp. Thus knots with $t(K) > g(K)$ exist. However, they are very special and indeed there was no positive fibered prime knot of ≤ 16 crossings with $t(K) > g(K)$ (even $t(K) = g(K)$, where the methods of [St6], [Tr2] may have a chance to work, if all torsion coefficients are divisible by 3 or by 5, was satisfied only by the trefoil).

5. – Further properties of the Fiedler Gauß sum invariant

Here are two properties of v_3 which we will conclude with.

THEOREM 5.1. *If K is a positive knot, then $v_3(K) \geq 4g(K)$.*

PROOF. Take a positive diagram of K. As both the genus of the canonical Seifert surface (which we observed in Section 4 is minimal for positive diagrams) and v_3 are additive under connected sum of diagrams, assume that the diagram is non-composite.

Furthermore assume w.l.o.g., that the diagram cannot be reduced by a Reidemeister I move, which follows a possible sequence of Reidemeister III moves, so it is in particular bireduced. If such a reduction existed, one could reduce the diagram this way, noting that by the above remark this procedure does not change the genus of the canonical Seifert surface.

So we can assume, we have a non-composite bireduced positive diagram of c crossings and n Seifert circles. If $n = 1$ the diagram is an unknot diagram and the result is evident. If $n = 2$ the diagram is of a $(2, m)$-torus knot $K_{2,m}$, m odd and the result follows from a direct calculation of v_3 on $K_{2,m}$ (noting that $g(K_{2,m}) = \frac{|m|-1}{2}$). So now assume $n \geq 3$. Then the genus of the Seifert surface is

$$g(K) = \frac{c-n+1}{2} \leq \frac{c-2}{2}$$

Therefore $4g(K) \leq 2c - 4$. But on the other hand by Theorem 3.1, $v_3(K) \geq 2c - 4$. $\square$

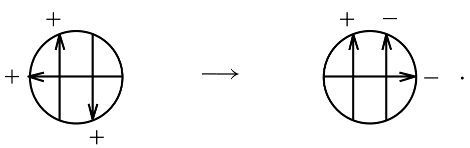
THEOREM 5.2. *Let D be a positive reduced diagram and D' be obtained from D by change of some non-empty set of crossings. Then $v_3(D') < v_3(D)$.*

This fact may not be too surprising, as v_3 in general increases with the number of positive crossings. However, it is not obvious in view of the fact, that v_3 sometimes decreases when a negative crossing is switched to a positive one.

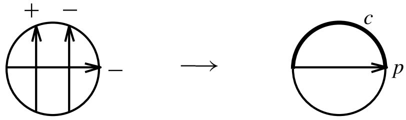
PROOF. To compare $v_3(D')$ and $v_3(D)$, we need to figure out how the configurations in (1) change by switching the positive crossings in D.

The configurations of the first two terms in (1) remain in D' (as orientation of the arrows does not matter) but possibly change their weight. In any case the weight of such a configuration in D' is not higher than (the old weight) 1 and so the contribution of these two configurations to the value of v_3 decreases from D to D'.

Something more interesting happens with the third term. A configuration in D may or may not survive in D'. But even if it does, its weight in D' is not more than one. However, a new configuration of positive weight can be created in D'. It happens if it has exactly two negative arrows and they are linked. This we will call an interesting configuration.



To deal with the interesting configurations, we will find other ones whose negative contributions compensate these of interesting configurations. First note, that any interesting configuration has a canonical pair of a negative arrow p and a half-arc c assigned:



Now consider any such canonical pair in the GD together with all arrows starting outside and ending on the half-arc c. Assume there are l negative and m positive such arrows.

Now beside the interesting configurations there are several ones which which decrease v (see Figure 6; p is always the arrow from left to right and c the upper half-arc).

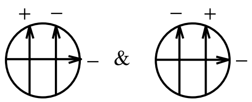
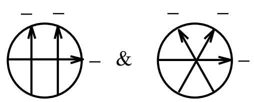
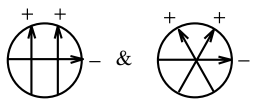
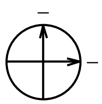
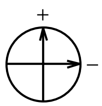
configuration in D'	times appearing	difference of contributions to v_3 from D to D'	times counted
	$\leq l \cdot m$	$+1$	1
	together $\binom{l}{2}$	-2	1, 2
	together $\binom{m}{2}$	-2	1
	l	-2	2
	m	-1	1

Fig. 6.

To compute the total contribution of all these configurations to the change of value of v_3 for one specific canonical pair, we have to multiply their number with the difference of contributions, dividing by the number of counting them with respect to different canonical pairs. The resulting contribution for the configurations in Figure 6 for a given canonical pair does not exceed

$$l \cdot m - \binom{l}{2} - 2\binom{m}{2} - l - m = -\frac{(m-l)^2}{2} - \frac{l}{2} - \frac{m^2}{2},$$

which is negative for $l, m \geq 0$ unless $l = m = 0$. But p is reducible in a canonical pair with $l = m = 0$. This shows the theorem. $\square$

A classical result on the Jones polynomial [Ka3], [Mu], [Th] states that a non-composite alternating and a non-alternating diagram of the same crossing number never belong to the same knot. This is no longer true, if we replace ‘alternating’ by ‘positive’, as we will observe in Section 9. However, it is true if instead of non-compositeness we demand the diagrams to have the same plane curve.

COROLLARY 5.1. *The Jones polynomial always distinguishes a reduced positive and a non-positive diagram with the same plane curve.* □

6. – The Casson invariant on positive knots

Here we shall say a word on the degree-2-Vassiliev invariant v_2, sometimes attributed to Casson because of its relation to the 3-manifold invariant discovered by him, see [AM]. This invariant is the coefficient of z^2 in the Conway polynomial $\nabla(z)$, or alternatively $\Delta''(1)/2$, where Δ is the Alexander polynomial [Al]. Using the Polyak-Viro formula for it, we obtain a similar result for positive knots as for v_3.

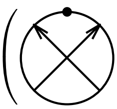
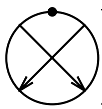
THEOREM 6.1. *In a positive reduced c crossing diagram, $v_2 \geq c/4$.*

This bound is sharp, as again the four crossing diagram of the (positive) trefoil shows. However, under assumption of bireducedness more seems possible.

CONJECTURE 6.1. In a positive bireduced diagram D, $v_2 \geq lk(D)/4$.

As a consequence, if this conjecture is true, for example in a positive bireduced c crossing diagram, $v_2 \geq c/3$. The reason why this bound is suggestive lies in the method we introduce to prove Theorem 6.1 and it will be motivated later.

The proof of Theorem 6.1 uses the Polyak-Viro formula for v_2

(8) $v_2 = \frac{1}{2}\Bigg($  $+$  $\Bigg)$,

obtained by symmetrization (w.r.t. taking the mirror image) from the formula [PV, (4)] (see also the remarks on p. 451 therein).

A similar but somewhat more complicated formula for v_2 was found by Fiedler [Fi3], [Fi4], who uses it to show that for a braid positive knot K one has $v_2(K) \geq g(K)$. This implies Theorem 6.1 for braid positive knots because of the inequality

$$g(K) \geq c/4 \tag{9}$$

in a reduced braid positive c crossing diagram D of a braid positive knot K, which is a consequence of both the Bennequin inequality [Be, Theorem 3, p. 101] and Cromwell's work on homogeneous knots [Cr, Corollaries 5.1 and 5.4], see [St3] or Remark 8.6.

Note, that (9) is not true in general for positive knots, but Fiedler's inequality extends to this case.

THEOREM 6.2. *For positive knots we have* $v_2(K) \geq g(K)$ *and* $v_2(K) \geq u(K)$.

REMARK 6.1. By Corollary 4.3, the first inequality in Theorem 6.2 follows from the second one. But the argument of our proof shows it without involving the slice version of Bennequin's inequality. Therefore, we felt it deserves independent exposition.

Note, that Theorem 6.2 excludes a large class of positive (see [Cr]) polynomials as Conway polynomials of positive knots, $1+z^2+z^4$ is a simple one (belonging, *inter alia*, to the knot 6_3).

In the sequel, we will need the following fact, which we invite the reader to prove.

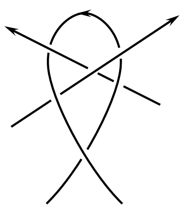
EXERCISE 6.1. Show that in the Gauß diagram of a positive knot diagram D any arrow p is distinguished in exactly half of the pairs in which it is linked.

HINT. Consider the 2-component link diagram obtained from D by smoothing out the crossing of p and apply Jordan's curve theorem.

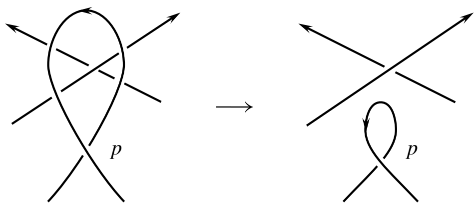
PROOF OF THEOREM 6.1. Call a linked pair in a based Gauß diagram *admissible*, if it is of one of the two kinds appearing in (8).

Fix a reduced positive diagram D. We apply now a series of transformations to D we call *loop moves*, ending at the trivial diagram. What is crucial for our argument, is that (8) is independent of the choice of a base point. That means, as long as we can assure that the Gauß diagram is realizable, that is, corresponds to a knot diagram, we can place for the next loop move on the diagram the base point to some other favorable place.

Now we describe how to perform a loop move. Take a crossing p in D whose smoothing produces a (diagram of a link with a) component K with no self-crossings. In the Gauß diagram this means, that the arrow of p does not have non-linked arrows on both its sides. In D, p looks like this

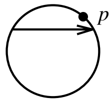


Now switch appropriately at most (but, in fact, exactly) half of the crossings on the loop and remove them from the diagram:



In the Gauß diagram this corresponds to removing the k arrows linked with p. Note, that by even valence $2|k$ and by reducedness $k > 0$. Assume in the resulting diagram D' there are c reducible crossings, p including. Any of them must have been linked in D with some (because of even valence at least 2) crossings linked with p.

Now, we place the basepoint in the Gauß diagram as follows:



.

Henceforth, such a picture means that there is no other end of an arrow between the basepoint and the arrow end to which it is depicted to be close (here the over-crossing of p).

Removing the arrows linked with p in D by Exercise 6.1 removes $k/2$ admissible linked pairs with p and for any other reducible crossing p' in D', at least one admissible linked pair (p' must have been linked in D with some even non-zero number of arrows to be removed and by Exercise 6.1 exactly half of them gives with it an admissible linked pair). Then the procedure of building D' out of D and reducing D' reduces the value of v_2 at least by

$$\frac{1}{2}\left(\frac{k}{2}+c-1\right),$$

an hence, by integrality of v_2, at least by

$$v_2(D)-v_2(D') \geq \left\lfloor \frac{k}{4}+\frac{c}{2} \right\rfloor,$$

whereas it reduces the crossing number of the (reduced) diagram by $k+c$. The ratio

$$\frac{k+c}{\lfloor \frac{k}{4}+\frac{c}{2} \rfloor}$$

for $2|k$, $k, c > 0$ is at most 4, unless $k = 4$ and $c = 1$ (in which case it is 5). We would like to show that in this case v_2 reduces at least by

$$\frac{1}{2}\left(\frac{k}{2}+c\right), \tag{10}$$

and hence, by integrality, at least by $\lfloor \frac{k}{4}+\frac{c+1}{2} \rfloor$.

To do so, now consider some $p'' \neq p$, which is not linked with p and does not become reducible in D'. The loop move reduced the number of arrows linked with p'' by some even number $2l$, possibly 0, such that half of this number (that is, l) of arrows point in either direction w.r.t. p''. Then for each such p'' the loop move reduces v_2 additionally by $l/2$. What we need is that at least for one crossing p'' in D we have $l > 0$. This occurs, unless p belongs to a connected component of D, in which all $p'' \neq p$ are linked with p or linked with all p' linked with p. The connected component would have $c + k$ crossings and would be resolved by the loop move (and the elimination of reducible crossings following it).

But in our case $c+k = 5$ and on the two positive diagrams of 5 crossings v_2 is 2 (for 5_2) resp. 3 (for 5_1). Therefore, (10) follows.

Resolving this case, we have always ensured $v_2(D) - v_2(D') \geq (k+c)/4$, and so the theorem follows inductively over $c(D)$, as it is true for $c(D) = 0$ and any positive diagram can be trivialized by a sequence of the above transformations. □

COROLLARY 6.1. *For any of the polynomials of Alexander/Conway, Jones,* HOMFLY, *the Brandt-Lickorish-Millett-Ho polynomial* Q [BLM], [Ho] *and Kauffman* [Ka2] *only finitely many positive knots have the same polynomial and there is no positive knot with unit polynomial.*

PROOF. Use the inequality for v_2 and the relations

$$-6v_2 := -3\Delta''(1) = V''(1) = Q'(-2)$$

and the well-known specializations for the HOMFLY and Kauffman polynomial. The equality between the Jones and Alexander polynomial is probably due already to Jones [J2, Section 12]. The relation between the Jones and Brandt-Lickorish-Millett-Ho polynomial is proved by Kanenobu in [K3]. □

REMARK 6.2. As a consequence of the result of [Ka3], [Mu], [Th] on the span of the Jones polynomial, only finitely many alternating knots have the same Jones polynomial. On the other hand, collections of such knots (sharing even the same HOMFLY and Kauffman polynomials) of any finite size exist [K]. It would be interesting whether constructions similar to these of Kanenobu are also possible in the positive case for both the Jones and Conway/Alexander polynomial and also to give an infinite series of *alternating* knots having the same Conway/Alexander polynomial, similar to the one (of non-alternating knots) in [K2]. Note, that knots of such a series (except finitely many) can neither be skein equivalent nor (by [Cr]) fibered.

In any case, Kanenobu's examples of [K2] show that the lower bound for the crossing number coming from the span of the Jones polynomial can be arbitrarily bad. Theorem 6.1 gives us a new tool for positive knots.

COROLLARY 6.2. *Let* K *be a knot with a positive reduced diagram of* c *crossings. Then* $c(K) \geq \sqrt{2c}$.

PROOF. Use [PV2, Theorem 1.E]. □

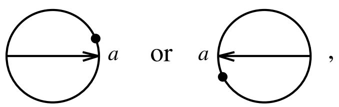
Although we will sharpen it, we already remark the inequality $v_2(K) \geq g(K)/2$ we obtain for the genus of a positive knot K from the inequality $g \leq c/2$.

EXERCISE 6.2. Prove that if D is a positive reduced diagram and D' is obtained from D by change of some but not all of its crossings, then $v_2(D') < v_2(D)$.

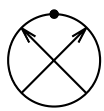
PROOF OF THEOREM 6.2. We use again the inductive step in the proof of Theorem 6.1. Fix a loop in a positive diagram D bounded by a crossing a. Assume the loop has $2c$ crossings on it. Then, switching at most (but, in fact, exactly) c crossings on the loop, it can be pulled above or below all the strands intersecting it.

Now recall the inequality of Bennequin-Vogel (7) of Section 4. The inequality is sharp for D positive, as shows the (therefore minimal) surface coming from the Seifert algorithm. This shows, that the switching of the c crossings in D reduces the absolute writhe at most by $2c$, and so (as it does not affect $n(D)$) g at *most* by c. On the other hand, as we will observe below, it reduces v_2 at *least* by c. The following Reidemeister moves do not change v_2 or g and then the same inductive argument as in the proof of Theorem 6.1 applies to show the first inequality asserted in the theorem. For the second one note, that the procedure describes an unknotting of K (and hence the number of crossing changes is at least its unknotting number).

To see that removing the arrow of a and all its linked arrows in the Gauß diagram to D reduces v_2 at least by c, put the basepoint in the Gauß diagram as follows:



and use the Polyak-Viro formula

(11) $v_2 =$ 

together with Exercise 6.1. □

EXERCISE 6.3. Modify the proof of Theorem 6.2 to show that in a positive diagram D, $lk(D) \geq 3g$, and deduce from this the inequality for any arbitrary diagram.

HINT. Use that beside the pairs linked with a any arrow linked with a must be linked with another arrow in the Gauß diagram to D.

Theorem 6.1 can be sharpened for special types of diagrams. (Note in particular that by the results of [N], [St5] all positive rational knots belong to the class addressed in the theorem.)

THEOREM 6.3. *Let K be a positive knot with a positive irreducible arborescent diagram of c crossings. Then*

$$v_2(K) \geq \frac{c-2}{2}.$$

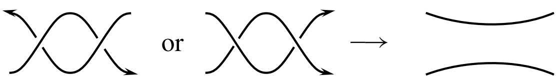
From this we straightforwardly obtain

COROLLARY 6.3. *The positive twist knot diagrams minimize v_2 over all connected irreducible arborescent positive diagrams of odd crossing number.* □

It is likely that a similar statement holds for v_3 (thus coming back to the remark on twist knots from the end of Section 3).

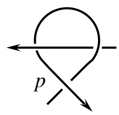
CONJECTURE 6.2. The positive twist knot diagrams minimize v_3 over all connected irreducible arborescent positive diagrams of odd crossing number.

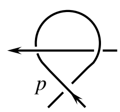
PROOF OF THEOREM 6.3. Every positive arborescent diagram D can be made trivial by resolving positive clasps.

(12) 

.

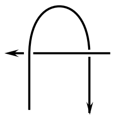
Thus we proceed by induction on c. If after possible flypes D has a clasp whose resolution does not create a reducible crossing, then c decreases by 2 under (12), and v_2 decreases at least by 1, so the claim follows inductively. Otherwise, assume D contains (after possible flypes) a tangle like

(a) 

or (b) 

.

Consider (a). Then switch p and eliminate it, reversing the clasp.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If the diagram has now reducible crossings, then D is the 3 or 4 crossing positive trefoil diagram. Otherwise, again the claim follows by induction, since v_2 and c have decreased both by 1 under switching and eliminating p.

Consider (b). Resolve the clasp, and eliminate p. Using even valence and double connectivity, it is easy to see that, unless D is the positive 4 crossing trefoil diagram, v_2 decreases by at least 2, and the resulting diagram has no reducible crossings. □

REMARK 6.3. Up to additive constants, the same estimate applies to knots with positive diagrams of any given fixed Conway polyhedron [Co].

We finish the discussion of v_2 in its own right by an inequality involving both the crossing and unknotting number of a positive diagram.

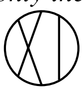
THEOREM 6.4. *Let D be a reduced positive diagram of crossing number $c(D)$ and unknotting number $u(D)$. Then $v_2(D) \geq \frac{c(D)+u(D)}{5}$.*

REMARK 6.4. Replacing the '5' in the denominator by '4', Theorem 6.4 would imply Theorem 6.1, and replacing the '5' by '6', it would follow from it using $u(D) \leq c(D)/2$. Thus '5' is in a sense indeed the interesting denominator. On the other hand, for braid positive knots Theorem 6.1 indeed follows from Theorem 6.4 because of Theorem 8.1, a property of braid positive knots we are going to prove later.

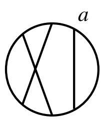
PROOF. We split the proof into two steps recorded as several lemmas. Our strategy will be as follows.

1. Apply loop moves to D, that do not unknot any connected component of D, until one obtains a diagram D'' with the property that any of its connected components gets unknotted by any loop move on it. Show the inequality of Theorem 6.4 for D''.
2. Show that if D' arises from D by a loop move of step 1, then
$$5\big(\, v_2(D) - v_2(D')\,\big) \;\geq\; c(D) - c(D') + k/2\,,$$
where $k/2$ is the number of crossings switched by the loop move (so k is the number of crossings on the loop). The totality of all such crossings switches over all moves of step 1 together with any unknotting sequence for D'' forms an unknotting operation of D (because the removal of a loop after a loop move commutes with all subsequent crossing changes), and the length of this unknotting sequence is $\geq u(D)$.

All diagrams we consider in the sequel will be assumed positive. For the first step we need some preparation.

LEMMA 6.1. *Let D be a connected diagram on which any loop move unknots. We call D loop-minimal. Then and only then D (more exactly its Gauß diagram) has no subdiagram of the kind* (5,1):  .

PROOF. Call an arrow corresponding to a crossing on which a loop move can be applied extreme. If D has a configuration of the kind (5,1)



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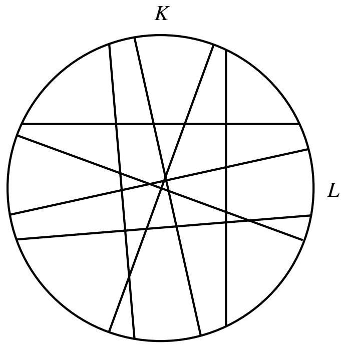
then we can w.l.o.g. find a to be extreme and applying a loop move to a we get a diagram with a linked pair, which is hence knotted. This contradiction shows the direction '$\Longrightarrow$'. To show '$\Longleftarrow$', observe that the absence of a configuration of the kind $(5,1)$ in D means that if for two arrows p and p'' in D, $p \not\cap p''$, then it holds $\forall p' : \, p' \cap p'' \Longrightarrow p' \cap p$ (every p' linked with p'' is also linked with p).

But then all p'' remain reducible after a loop move on p. This shows the other direction, letting p vary over all arrows of D. $\square$

LEMMA 6.2. *Let D be prime and loop-minimal. Then there exists a* (*necessarily disjoint*) *decomposition* $\{$*arrows of* $D\} = K \cup L$, *such that* $K \neq \varnothing \neq L$ *and*

$$\forall p \in K,\ p' \in L : p \cap p' \,. \tag{13}$$

(14)

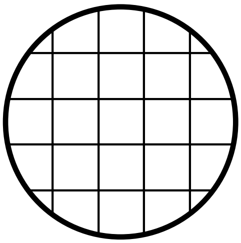


PROOF. Distinguish two cases.

CASE 1. There is no triple of arrows of type $(3, 3)$. Take some chord p. Set

$$K := \{\, p' \,:\, p' \cap p \,\} \quad \text{and} \quad L := \{\, p' \,:\, p' = p \,\vee\, p' \not\cap p \,\}\,.$$

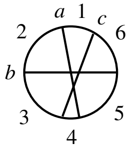
Because of the lack of $(5, 1)$ configurations, any $p \notin K$ is linked with the same set of chords as p, namely K. Thus every chord in K intersects every chord in L. Then, because of the lack of type $(3, 3)$ configurations, no two chords in K or in L can intersect. Thus D has the "grid" form



,

K and L being the set of horizontal and vertical chords, respectively.

CASE 2. There is a triple (a, b, c) of arrows of type $(3, 3)$. Number the segments of the base line, separated by the 6 ends of the arrows, by 1 to 6:



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and let for $i, j \in \{1, \dots, 6\}$

$$[i, j] := \{\text{chords with endpoints on segments } i \text{ and } j\}.$$

By Lemma 6.1, $[i, j] = \varnothing$ if $|i - j| \in \{0, 1, 5\}$, hence there are 9 possibilities left for i, j. Then we can write down some K and L *ad hoc*. Set

$$K := \{b, c, [1,3], [3,5], [3,6], [4,6], [2,6]\} \quad \text{and}$$

$$L := \{a, [1,4], [1,5], [2,4], [2,5]\},$$

and verify the desired property case by case. For example, any element $p \in [1,4]$ must intersect any element p' in $[1,5]$, for if not, a loop move in p' would preserve the linked pair (a, p). □

LEMMA 6.3. *Unless D is the 3 crossing trefoil diagram, K and L in Lemma* 6.2 *can be chosen to have ≥ 2 elements each.*

PROOF. In Case 1 of the proof of Lemma 6.1, the claim is trivial, as the number of chords in K (resp. L) is equal to the number of chords intersecting any arbitrary chord in L (resp. K), and this number is even by even valence. Therefore, $|K|$ and $|L|$ are both even, so in particular ≥ 2.

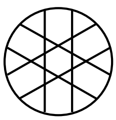
More generally, for Case 2, we can take some chord p and set

$$K' := \{p' : p' \cap p\} \quad \text{and} \quad L' := \{p' : p' \not\cap p \wedge \forall p'' : p'' \cap p \iff p'' \cap p'\} \cup \{p\}.$$

Then by the said in Case 1 of the previous proof, any chord is either in K' or in L', K' and L' still satisfy (13) and by even valence $|K'| \geq 2$. If now $|L'| = 1$, that is, $L' = \{p\}$ for any choice of chord p, then any two arrows are linked in D, and it is a $(2, 2n+1)$ torus knot diagram. But for such a diagram K' and L' with $|K'|, |L'| \geq 2$ are immediately found, unless $n = 1$, which is the 3 crossing trefoil diagram. □

EXERCISE 6.4. Show that in fact all Gauß diagrams without a configuration of type $(5, 1)$ come either from rational knot diagrams of the form $C(p, q)$ with p and q even integers, or from (generalized) pretzel diagrams $P(a_1, \dots, a_n)$, with n and all a_i odd.

HINT. Consider a chord p_1 with maximal valence (number of linked chords) and collect p_1 and all its non-linked chords into a set K_1. Then consider from the rest of the chords again one chord p_2 with maximal valence and build a set K_2 of p_2 and all its non-linked chords, and so on. You obtain a decomposition of the chords into disjoint sets K_i. Using the argument in the last proof then show that two chords intersect if and only if they belong to distinct sets K_i. Use even valence to show that either there is an odd number of sets K_i each one of odd size, which is the pretzel diagram case, or an even number of sets each one of even size. In this case deduce that there are no more than two sets K_i using the non-realizability of

 ,

so that the knot diagram is of the form $C(p, q)$, p and q even.

LEMMA 6.4. *If D is connected, reduced and loop-minimal, then $v_2(D) \geq \frac{c(D)-2}{2}$.*

PROOF. Assume $c(D) \geq 4$, as the trefoil diagrams are easily checked. By Lemma 6.2 and Lemma 6.3 we have the decomposition of the arrows into K and L with $k := |K|$, $l := |L| \geq 2$. Then in the picture (14) we can w.l.o.g., modulo rotating the diagram by 90° (swopping K and L) and mirroring, assume that $\geq k/2$ arrows in K point upward, and $\geq l/2$ arrows in L point from left to right. Then placing the basepoint above the arrows in L and to the right of the arrows in K and using the formula for v_2 in (11), we see

$$v_2 \geq \frac{kl}{4} \geq \frac{k(c-k)}{4} \geq \frac{2(c-2)}{4} = \frac{c-2}{2},$$

as $k \geq 2$, $c-k \geq 2$ and $c \geq 4$. □

LEMMA 6.5 (First step). *If D is a loop-minimal connected prime or composite diagram, then $v_2(D) \geq \frac{c(D)+u(D)}{5}$.*

PROOF. Assume first that D is prime. As $u(D) \leq c(D)/2$ it suffices to show $v_2 \geq \frac{3c(D)}{10}$. But Lemma 6.4 gives $v_2 \geq c(D)/2 - 1$, which is a better estimate unless $c(D) < 5$, in which case the claim can be directly checked.

If D is composite, use that $c(D)$, $u(D)$ and $v_2(D)$ are additive under connected sum of *diagrams* (not knots[(1)]!). □

LEMMA 6.6 (Second step). *If a loop move $D \to D'$ does not unknot a connected component of D, then*

$$5\big(v_2(D) - v_2(D')\big) \geq c(D) - c(D') + k/2$$

where $k/2$ is the number of crossings switched by the loop move (so k is the number of crossings on the loop).

PROOF. By the proof of Theorem 6.2 we have

$$v_2(D) - v_2(D') \geq \frac{k}{2},$$

and by the proof of Theorem 6.1 we have

$$v_2(D) - v_2(D') \geq \frac{k}{4} + \frac{c}{2},$$

[(1)]Beware that the question whether unknotting and crossing number of *knots* are additive under connected sum is a 100 year old problem, that no one knows how to solve, except in special cases, and even these partial results have been remarkable achievements.

again denoting by c the number of reducible crossings after the move. Hence by arithmetic mean

$$\begin{aligned} v_2(D) - v_2(D') &\geq \frac{1}{2}\left(\frac{k}{4} + \frac{c}{2} + \frac{k}{2}\right) \\ &= \frac{3k}{8} + \frac{c}{4} \\ &= \left(\frac{c+k}{4}\right) + \frac{k/2}{4}, \end{aligned}$$

and as $c + k = c(D) - c(D')$, already

$$4\big(\,v_2(D) - v_2(D')\big) \;\geq\; c(D) - c(D') + k/2\,,$$

that certainly remains true when replacing the factor '4' by '5'. □

Using the strategy outlined in the beginning, Lemmas 6.5 and 6.6 prove Theorem 6.4. □

REMARK 6.5. It is striking that the whole proof goes through with denominator '4' instead of '5', except at one point: the case $c(D) = 4$ in Lemma 6.5 (the positive 4 crossing trefoil diagram). This is, however, fatal for our argument, because we would need to control how many such factors occur in the diagram D'' after Step 1. One hope to get out of the dilemma would be to find loop moves, such that connectedness is always preserved, but one can find examples, where this is not possible.

Moreover, along similar lines one shows that $3v_2(D)$ decreases not slower than $c(D)$ under the moves of Step 2. So the motivation for Conjecture 6.1 is again the problem how to handle Step 1. Similarly to connectedness, it is difficult to make the loop move behave well w.r.t. bireducedness.

COROLLARY 6.4. If K is positive, then $5v_2(K) \geq \max\deg V(K)$.

PROOF. Use that by [Ka3], [Mu], [Th], $\operatorname{span} V(K) \leq c(D)$ on a positive (or any other) diagram D of K, and that $u(D) \geq u(K) \geq g(K) = \min\deg V(K)$ by Corollary 4.3 and [St5, Theorem 3.1]. □

It is interesting to remark that this is an entirely combinatorial statement that heavily relies on this deep topological fact – the truth of the (local) Thom conjecture. It would be nice to know whether it cannot be derived also completely combinatorially.

REMARK 6.6. The work done in this paragraph also recovers in an easy manner for positive knots the mentioned result of Cochran–Gompf and Traczyk on the positivity of the signature. For this it suffices to remark that a loop move, consisting of switching positive crossings to negative, never reduces the signature, and that it is positive on the knots of Exercise 6.4 by direct calculation.

7. – Relations between v_2, v_3 and the HOMFLY polynomial

The Polyak-Viro-Fiedler formulas also allow to relate both the degree-2 and degree-3 Vassiliev invariants to each other in positive diagrams, giving a lower bound for their crossing number.

In the following we give inequalities resulting from such combined applications of the various formulas, which, while not completely sharp, hardly seem provable using other arguments.

Let

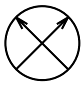
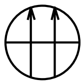
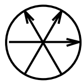
$$l_i := \#\{\text{crossings linked with crossing } i\}$$

in some fixed positive diagram D of c crossings.

LEMMA 7.1. *In a positive diagram D of c crossings,*

$$v_3 \geq \sum_{i=1}^{c} \binom{l_i/2}{2} + \frac{l_i}{2} = \sum_{i=1}^{c} \frac{(l_i+1)^2-1}{8}\,. \tag{15}$$

PROOF. We have in a positive diagram

$v_3(D) \geq$  $+$  $_i\ +$  $_i$.

The first term on the right gives the second summand in (15) (note, that a linked pair is counted twice for both arrows in it). Numbering the horizontal chord in terms 2 and 3 by i, we see that the sum of terms 2 and 3 is the count of pairs of equally oriented arrows with respect to arrow i. Now by exercise 6.1 for each i and for each orientation there are two collections of $l_i/2$ equally oriented arrows with respect to arrow i, giving $2\binom{l_i/2}{2}$ possible choices of pairs of equally oriented arrows. As linked equally oriented arrows are counted twice, we factor out the '2' and obtain the formula (15). □

LEMMA 7.2. *In a positive diagram D,*

$$4v_2 \leq \sum_{i=1}^{c} l_i\,. \tag{16}$$

PROOF. This is obviously a consequence of (8). □

THEOREM 7.1. *In a positive reduced c crossing diagram ($c > 0$) we have $v_3 > v_2$ and*

$$c \geq \frac{2v_2^2}{v_3 - v_2}\,.$$

PROOF. In view of (16), the right hand side of (15) is minimized by $l_i := 4v_2/c$, in which case it becomes

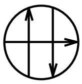
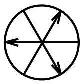
$$\frac{c\left(\frac{4v_2}{c}+1\right)^2}{8}-\frac{c}{8}\,.$$

So

$$v_3 \;\geq\; \frac{2v_2^2}{c}+v_2\,, \tag{17}$$

from which the assertion follows, as by (8) and (1) one always has $v_3 > v_2$ (even $v_3 \geq 2v_2$). □

REMARK 7.1. The Polyak-Viro formula for $v_3/4$

$$\frac{v_3}{4} = \frac{1}{2}$$  $+$ 

proves similarly as (15) the inequality

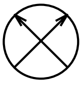
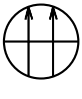
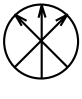
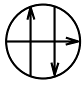
$$\frac{v_3}{4} \;\leq\; \frac{1}{2}\sum_{i=1}^{c}\left(\frac{l_i}{2}\right)^2\,.$$

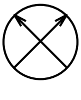
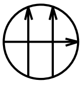
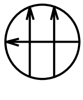
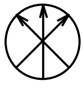
Adding $\frac{v_3}{4}-\sum\frac{l_i^2}{8}$ to the right hand side of (15), we obtain $\frac{3}{4}v_3 \;\geq\; \sum_{i=1}^{c}\frac{l_i}{4}$, so by Lemma 7.2, $v_3 \;\geq\; \frac{4v_2}{3}$. This inequality is weaker than (17), if $v_2 \;\geq\; \frac{c}{6}$, which we showed always holds in reduced positive diagrams.

THEOREM 7.2. *In a positive diagram of c crossings,*

$$\frac{3}{4}v_3 \;\leq\; v_2\,\cdot\,c\,.$$

PROOF. As before, combining the Fiedler and Polyak-Viro formulas, we have

$$\frac{3}{4}v_3 =$$  $+$  $+$  $-\frac{1}{2}$ 

$\leq$  $+$  $+$  $+$  .

But for positive diagrams the four terms on the right are equal to the first four terms in the numerator of the v_2 formula [PV, p. 450 bottom]. □

All previous calculations suggest that, from the point of view of Gauß sums, the following invariant plays some key role.

DEFINITION 7.1. Define the linked pair number $lk(K)$ of a knot K as the minimal number of linked pairs in all its diagrams.

What can we say about $lk(K)$? As a consequence of (8) or (16), $lk(K) \geq 2v_2(K)$ for any knot K. However, $v_2(K)$ may be sometimes negative. A non-negative lower bound is $3g(K)$, following from Exercise 6.3. In fact, Exercise 6.3 shows $lk(K) \geq 3\tilde{g}(K)$, where $\tilde{g}(K)$ is the weak Seifert genus of K, that is, the minimal genus of a surface, obtained by applying the Seifert algorithm to any diagram of K. As we noted, sometimes $\tilde{g}(K) > g(K)$. Morton showed [Mo2], that

$$\tilde{g}(K) \geq \max\deg_m\ P(K)/2\,,$$

so

$$lk(K) \geq \frac{3}{2}\max\deg_m\ P(K)\,.$$

On the other hand, for K positive we proved $v_3(K) \geq \frac{4}{3}lk(K)$, so we obtain a self-contained inequality

PROPOSITION 7.1. *For a positive knot K we have*

$$v_3(K) \geq 2\max\deg_m\ P(K)\,. \qquad \square$$

This condition is also violated by our previous example 12_{2038}. We also obtain

PROPOSITION 7.2. $v_3(K) \geq \frac{8}{3}v_2(K)$ *for K positive.* □

As simple examples show, except for the low crossing number cases and connected sums thereof these inequalities are far from being sharp, so significant improvement seems possible. The problem with pushing further our inductive arguments in Section 6 is that it appears hard to control how often these low crossing number cases occur as connected components in intermediate steps of trivializing a positive diagram with our move.

A final nice relation between v_2 and v_3 is unrelated to Gauß sums and bases on an observation of X.-S. Lin [L]. Let $w_\pm$ denote the untwisted double operation of knots with positive (resp. negative) clasp.

PROPOSITION 7.3. $v_3(w_\pm(K)) = \pm 8v_2(K)$.

PROOF. The dualization $w_\pm^*$ of $w_\pm$ is a nilpotent endomorphism of $\mathcal{V}^n$, the space of Vassiliev invariants of degree at most n. But $\mathcal{V}^3/\mathcal{V}^2$ and $\mathcal{V}^2/\mathcal{V}^1$ are one-dimensional and hence are killed by $w_\pm^*$. Therefore, $w_\pm^*$ maps v_2 to a constant and checking it on the unknot we find that it is zero (this also follows from $\Delta = 1$ for an untwisted Whitehead double of any knot). v_3 is taken to something in degree at most 2, so $v_3(w_\pm(K)) = c_1^\pm\, v_2(K) + c_0^\pm$. That $c_0^\pm = 0$ follows from taking the unknot, and to see $c_1^\pm = \pm 8$ check that v_3 is 8 on the positive-clasped untwisted Whitehead double of one of the trefoils. □

Combining this with our Gauß sum inequalities we immediately obtain

COROLLARY 7.1. *An untwisted Whitehead double of a positive knot has non-self-conjugate Jones polynomial. In particular, the knot is chiral and has non-trivial Jones polynomial. Moreover, there are only finitely many positive knots, whose untwisted Whitehead doubles* (*or similarly, twisted Whitehead doubles with any fixed framing*) *have the same Jones polynomial.* □

8. – Braid positive knots

The following section deals with the more specific subclass of positive knots, namely those with positive braid representations. First, as a digression from the Gauß sum approach, we improve some inequalities of Fiedler [Fi] on the degree of the Jones polynomial of such knots, and later we write down certain inequalities for the Casson invariant of knots with positive braid representations, giving some applications.

DEFINITION 8.1. A knot is called braid positive, if it has a positive diagram as a closed braid.

NOTE. The term "braid positive" is self-invented and provided to give a naturally seeming name for such knots and links, distinguishing them from the ones we call 'positive'. However, braid positive knots are called sometimes "positive knots" elsewhere in the literature, so beware of confusion!

First we will recall and sharpen an obstruction of Fiedler [Fi] to braid positivity.

LEMMA 8.1 ([Fi]). *For any braid positive k component link L without trivial split components, we have* $\min\deg V(L) > 0$ *and* $\min\mathrm{cf}\, V(L) = (-1)^{k-1}$.

Here is our improved version of Fiedler's result.

THEOREM 8.1. *If L is a non-split k component link, $L = \hat{\beta}$, with β a positive reduced braid of c crossings, then*

$$\min\deg V(L) \geq c/4 - \frac{k-1}{2} \geq c(L)/4 - \frac{k-1}{2} \tag{18}$$

and $\min\mathrm{cf}\, V(L) = (-1)^{k-1}$.

To prove the theorem, let's start with the

LEMMA 8.2. *If a positive braid diagram of a prime knot is reducible, then it admits a reducing Markov* II [Bi] *move, see Figure* 7. *So, if a prime knot has a positive* (*closed*) *braid diagram, it also has a reduced one.*

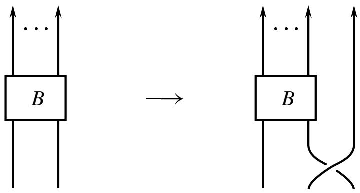


Fig. 7. The Markov II move.

PROOF. Take a reducible crossing in the closed braid diagram and smooth it out. As the knot is prime, assume w.l.o.g. that the right one of the two resulting closed braid diagrams belongs to the unknot. If we know, that each positive braid diagram of the unknot is either trivial or reducible, repeat this procedure, ending up with a trivial (braid) diagram of the unknot on the right. Then the last smoothed crossing is the one corresponding to a reducing Markov II move.

For positive braids it follows from work of Birman and Menasco [BM] and also from the Bennequin inequality [Be, Theorem 3, p. 101], that if $\hat{\beta}$ is the unknot, then $|[\beta]| < n(\beta)$. Therefore, if β is positive, it must contain each generator exactly once, so all its crossings are reducible. □

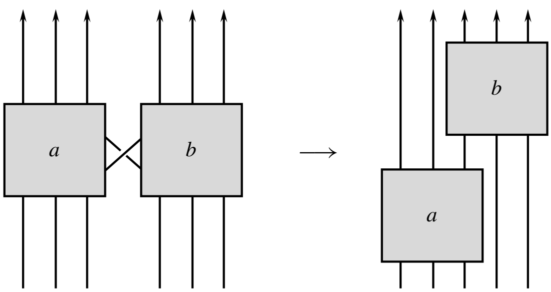
REMARK 8.1. Note, that our capability to control positive braid diagrams of the unknot so well by these (deeper) results, is rather surprising, as in general there exist extremely complicated braid representations of the unknot [Mo], [Fi2].

REMARK 8.2. A similar statement is also true for alternating diagrams. To see the fact, that each alternating braid diagram of the unknot is either trivial or reducible, recall the result of Kauffman [Ka3], Murasugi [Mu] and Thistlethwaite [Th], that all alternating diagrams of the unknot are either trivial or reducible.

The assertion in Lemma 8.2 in the positive case is also true for composite knots and links.

LEMMA 8.3. *Any braid positive link has a reduced braid positive diagram.*

PROOF. In the braid positive diagram use the iteration of the procedure



which gives a reduced diagram.

REMARK 8.3. Note, that, however, for alternating diagrams the above procedure does not work. The granny knot $!3_1\#!3_1$ has a reducible alternating diagram as closed 4-braid, but no alternating diagram as closed 3-braid.

PROOF OF THEOREM 8.1. Take equation (10) of [Fi] for positive β.

$$\min\deg V(L) = \frac{1}{2}([\beta]+1-n(\beta)) \tag{19}$$

As β is w.l.o.g. by Lemma 8.3 reduced, and generators appearing only once in β correspond to reducible crossings in the closed braid diagram, we have

$$[\beta] \geq 2(n(\beta)-k), \tag{20}$$

so

$$\min\deg V(L) \geq \frac{n(\beta)-k}{2}. \tag{21}$$

On the other hand, as β is positive, $[\beta]=c$, so

$$\min\deg V(L) \geq \frac{c+1-n(\beta)}{2}.$$

Therefore

$$\min\deg V(L) \geq \min_n \max\left(\frac{n(\beta)-1}{2}, \frac{c+1-n(\beta)}{2}\right) - \frac{k-1}{2} = \frac{c}{4} - \frac{k-1}{2}.$$

The second assertion follows directly from [Fi, Theorem 2]. □

REMARK 8.4. Applying $n \geq b(\hat\beta)$ in (21), or taking the inequality $c(L) \geq 2(b(L)-k)$ of Ohyama [Oh] in (18), we also obtain the weaker inequality

$$\min\deg V(\hat\beta) \geq \frac{b(\hat\beta)-k}{2}.$$

REMARK 8.5. Considering $L = K$ to be a knot, the first inequality in (18) is evidently sharp, as a braid with each generator appearing twice shows. Concerning the second inequality and demanding the braid to be irreducible (i.e. not conjugate to a braid with an isolated generator), the inequality (20) can be further improved a little by observing, that a positive braid with exactly $2(n(\beta)-1)$ crossings is still transformable modulo Yang-Baxter relation (that is, a transformation of the kind $\sigma_{i-1}\sigma_i\sigma_{i-1} = \sigma_i\sigma_{i-1}\sigma_i$) into one with isolated generators. So we can add a certain constant on the r.h.s. of (20), and to our bound, maybe excluding some low crossing cases ($!3_1$ and $!5_1$ show that $\lceil c(K)/4\rceil$ at least is sharp.)

However, for an improvement of the last estimate in (18) beyond $c(K)/4+2$ there will be substantial subtleties to deal with, as for $[\beta] = 2n(\beta)+6$ there is a series of examples of braids $\{\beta_n \mid n \text{ odd}\}$ with

$$\beta_n = \left((\sigma_1\sigma_3 \ldots \sigma_{n-4}\sigma_{n-2}^3)\,(\sigma_2^3\sigma_4 \ldots \sigma_{n-3}\sigma_{n-1})\right)^2$$

or schematically

$$\beta_n = \begin{array}{ccccccccc} & 3 & & 1 & \ldots & & 1 & & 1 \\ 1 & & 1 & & \ldots & 1 & & 3 & \\ & 3 & & 1 & \ldots & & 1 & & 1 \\ 1 & & 1 & & \ldots & 1 & & 3 & \\ \sigma_1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \sigma_{n-1} \end{array}$$

which do not admit a Yang-Baxter relation modulo cyclic permutation and close to a knot. Of course, this is far away from saying that β_n are irreducible or that even $\hat{\beta}_n$ is a minimal diagram (which would mean, that the second bound is also sharp) but I don't know how to decide this.

REMARK 8.6. The expression appearing on the r.h.s. of (19) is equal to

$$\frac{1-\chi(L)}{2}$$

where $\chi(L)$ is the is the maximal Euler characteristic of a (n orientable) spanning Seifert surface for L. This follows from the (classical) formula for the genus of the canonical Seifert surface, together with the fact that this Seifert surface is (of) minimal (genus) in positive diagrams, see Corollary 4.2. (This observation is treated in more detail and generalized in [St5].) For a braid positive knot K we have $1-\chi = 2g$, so that

$$\max\deg\Delta(K) \;=\; g(K) \;=\; \min\deg V(K) \;\geq\; c(K)/4\,, \tag{22}$$

where Δ is the Alexander polynomial and the first equality comes from the fiberedness of the knot. The condition (22) is not sufficient, though. We have seen this in Example 4.2.

As a braid positive knot K by Lemma 8.2 always has a reduced braid positive diagram, and a reduced braid positive diagram by Theorem 8.1 does have not more than $4\min\deg V(K)$ crossings, we see that braid positivity can always be decided. This, of course, works with the results of the previous section as well, but the present bound is *considerably* sharper.

Here we shall observe that braid positive is really stronger than positive, so our Definition 8.1 is justified.

EXAMPLE 8.1. The knot $!5_2$ is positive (see, e.g., [Ka2]). We have (see, e.g., [Ad, Appendix] or [St2]) $\min\deg V(!5_2) = 1 < 5/4$. So, although positive, $!5_2$ can never be represented as a closed positive braid. The same is true for

$!7_2$ and 7_4. Note, that in all 3 examples the conclusion of non-braid positivity would not have been possible with Fiedler's weaker criterion.

REMARK 8.7. It is known, that closed positive braids are fibered, and that fibered knots have monic Alexander polynomial [Ro, p. 259], (i.e., with minimal/maximal coefficients ± 1), so the monicity of the Alexander polynomial is also an obstruction to braid positivity, and applies in the above 3 examples $!5_2, !7_2$ and 7_4 as well. Another way to deal with these cases is to use the observation, that they all have genus 1 (which can be seen by applying the Seifert algorithm to their alternating diagrams [Ga]), and the fact (following from the Bennequin inequality [Be, Theorem 3, p. 101]), that the only braid positive genus 1 knot is the positive trefoil. A special way to exclude 7_4 is to use the fact that it has unknotting number 2 [Ad], contradicting the inequality $u(K) \leq g(K)$ for braid positive knots K due to Boileau and Weber [BoW] and Rudolph [Ru, prop. on p. 30], see also [Be].

EXAMPLE 8.2. The 10 crossing knot $!10_2$ is fibered and its minimal degree of the Jones polynomial is positive, but it is 1, so $!10_2$ is not a closed positive braid. $!10_2$, however, can also be dealt with by the non-positivity of its Conway polynomial [Bu].

EXAMPLE 8.3. On the other hand, the knots 7_3 and $!7_5$ are positive, but the minimal degree of their Jones polynomial, being equal to 2, does not tell us, that they are not braid positive. But they have non-monic Alexander polynomial, and so they cannot even be fibered.

The variety of existing obstructions to (braid) positivity makes it hard to find a case, where our condition is universally better. Here is a somewhat stronger example, coming out of some quest in Thistlethwaite's tables.

EXAMPLE 8.4. The knot $!12_{1930}$ on Figure 8 has the HOMFLY polynomial

$$(4l^8 + 2l^{10} - l^{12}) + (-4l^4 + 2l^6 - 4l^8 + l^{10})m^2 + l^4m^4\,.$$

It shows, that no one of the above mentioned (braid) positivity obstructions of [Ro], [Bu], [Cr], [CM], [Fi] is violated, but ours is. Note that, although monic, the Alexander polynomial can also be indirectly used to show non-braid positivity because of (22).

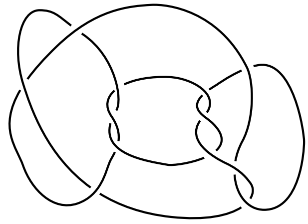


Fig. 8. The knot $!12_{1930}$.

We now give some improvements of the inequalities for positive knots for closed positive braids of given strand number. It is obvious that without this restriction not more than a linear lower bound for v_2 and v_3 in c can be expected, as the iterated connected sum of trefoils shows. (We will shortly construct more such examples.)

THEOREM 8.2. *If β is a positive braid of exponent sum (or crossing number) $[\beta]$, and n strands, closing to a knot, then*

$$v_2(\hat{\beta}) \geq \frac{[\beta]^2}{4n(n-1)} - \frac{(2n-3)(n-1)}{8}. \tag{23}$$

PROOF. Consider $l_{ij} = lk(i, j)$ for $1 \leq i < j \leq n$, the linking number of strands i and j in β. Then $[\beta] = \sum_{i<j} l_{ij}$ and each pair of strands i and j contributes to the Gauß diagram of $\hat{\beta}$ a collection of l_{ij} mutually linked arrows. If l_{ij} is odd, the contribution of these arrows to the Gauß sum is (independently of the choice of basepoint) the one of the $(2, l_{ij})$ torus knot, namely $(l_{ij}^2 - 1)/8$, while for l_{ij} even, the contribution depends on the choice of basepoint (changes by ∓ 1), but is in any case at least $(l_{ij}^2 - 4)/8$. The bound on the right of (23) is obtained by taking all l_{ij} equal, namely $[\beta]/\binom{n}{2}$, and using that at least $n-1$ of the l_{ij} are odd, as any one- cycle permutation of n elements has length at least $n-1$. □

COROLLARY 8.1. *If β is a braid of n strands, closing to a knot, $[\beta]_-$ the number of negative crossings in β, and $[\beta]_0 = [\beta] + 2[\beta]_-$ the total number of crossings of β, then*

$$v_2(\hat{\beta}) \geq \frac{[\beta]_0^2}{4n(n-1)} - \frac{(2n-3)(n-1)}{8} - \frac{[\beta]_-\big([\beta]_0 - [\beta]_-\big)}{2}.$$

PROOF. Use the expression of v_2 in (8), showing that switching $[\beta]_-$ positive crossings in any diagram of $[\beta]_0$ crossings, decreases v_2 at most by the third term on the right. □

This means, that for $[\beta]_-$ sufficiently small, we have $v_2(\hat{\beta}) > 0$, implying as before that in particular $\hat{\beta}$ has non-trivial Δ, V and Q polynomial, and untwisted Whitehead doubles with non-trivial V polynomial.

In a similar way one proves

THEOREM 8.3. *If β is a positive braid of n strands, closing to a knot, then*

$$v_3(\hat{\beta}) \geq C_1 \frac{[\beta]^3}{n^4} - C_2 n^2, \tag{24}$$

for some (effectively computable and independent on β and n) constants $C_{1,2} > 0$. □

Theorems 8.3 and 8.2 imply a positive solution to Willerton's problem 5 in [Wi, Section 4] for positive braids of given strand number.

COROLLARY 8.2. *If* (β_i) *are distinct positive braids of n strands, then* $v_3(\hat\beta_i)\asymp v_2(\hat\beta_i)^{3/2}$, *in particular,* $\lim\limits_{i\to\infty}\log_{v_2(\hat\beta_i)} v_3(\hat\beta_i) = 3/2$. □

Such a property can be used to show that certain special positive braids are not Markov equivalent to positive braids of given strand number, where the calculation of the Homfly polynomial (and the bound of the Morton-Williams-Franks inequality [Mo2], [FW]) can be tedious.

EXAMPLE 8.5. Let β_i and β_i' be positive braids of length $O(i^{1/6-\epsilon})$ in some B_n with $\phi(\beta_i) = (1\,2)$, $\phi : B_n \to S_n$ being the permutation homomorphism, n independent on i. Let $\{\,.\,\}_j$ be the shift map $\sigma_i \mapsto \sigma_{i+j}$. Set

$$\beta_{[j]} = \prod_{i=1}^{j}\{\beta_i\}_{i-1} \cdot \{\beta_j'\}_j \in B_{n+j}\,,$$

so that $\phi(\beta_{[j]}) = (1\,2\,\dots\,n+j)^{-1}$. Then all but finitely many of the knots $\hat\beta_{[j]}$ have no positive braid representations of some fixed strand number. To see this, use that any crossing in $\{\beta_i\}_{i-1}$ in the diagram $\hat\beta_{[j]}$ is linked only with crossings in $\{\beta_{i'}\}_{i'-1}$ for $|i'-i|\le n$. Thus any such crossing has $O(i^{1/6-\epsilon})$ linked crossings, from which the v_3 formula shows

$$v_3(\hat\beta_{[j]}) = \sum_{i=1}^{j}\left(O(i^{1/6-\epsilon})^3\right) = O(j^{3/2-3\epsilon})\,,$$

so if $\log_{v_2(\hat\beta_{[j]})} v_3(\hat\beta_{[j]}) \to x$, we must have $x \le 3/2-3\epsilon$, contradicting Corollary 8.2.

This example also shows that Willerton's problem cannot be solved positively in general for braid positive knots. We can already take the iterated connected sum of trefoils, but we also see how to construct prime examples using [Cr2]. For example, taking all $\beta_i = \sigma_1^2\sigma_2^2\sigma_1 \in B_3$ and $\beta_i' = \sigma_1$, one obtains $v_2(\hat\beta_{[j]}) = O(j) = v_3(\hat\beta_{[j]})$.

9. – Questions on positive knots

After alternating knots have been well understood, it's interesting to look for another class of knots. The positive knots provide many interesting questions in analogy to alternating knots.

Here are some appealing questions which come in analogy when thinking of alternating knots.

By [Ka3], [Mu], [Th] any alternating reduced diagram is minimal. We saw that, ignoring the second reduction move, this is not true for positive knots. Is it true with the second reduction move (and all its cablings)? It seems, however, that things are not that easy with positive knots (or the other way around – it makes them the more challenging!).

EXAMPLE 9.1. Consider the knot, which is the closed rational tangle with the Conway notation $(-1, -2, -1, -2, -5)$. Its diagram as closed $(-1, -2, -1, -2, -5)$ tangle is reduced and alternating, and hence minimal. The knot, however, has also a positive diagram as closed $(1, 2, 1, 2, 1, 1, -1, -3)$ tangle, which is bireduced, but non-minimal. (This is one of a series of such examples I found by a small computer program.)

Conversely, for alternating prime knots, any minimal diagram is alternating. As the example of the Perko pair [Ka2, fig. 10], shows, this is not true for positive knots. So we can ask:

QUESTION 9.1. Does every positive knot have at least *one* positive minimal diagram? If so, is there a set of local moves reducing a positive diagram to a positive minimal diagram?

REMARK 9.1. I tried to find counterexamples to Question 9.1 using the following (common) idea: Consider the Conway notation $a = (a_1, \dots, a_n)$ of a (diagram of a) rational tangle A, closing to a positive (diagram of some) knot K. Then take some expression $c = (c_1, \dots, c_m)$ of its iterated fraction

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{a_{n-3} + \dots}}} = c_m + \cfrac{1}{c_{m-1} + \cfrac{1}{c_{m-2} + \cfrac{1}{c_{m-3} + \dots}}}$$

with all c_i of the same sign. The (diagram of the) tangle C with Conway notation c is equivalent to A [Ad], closes to an alternating diagram $\bar{C}$ of K. K is also prime (e.g. by [Me], as $\bar{C}$ is non-composite and alternating). Therefore any minimal diagram of this knot is alternating and if $\bar{C}$ is not a positive diagram, by Thistlethwaite's invariance of the writhe [Ka2] it would follow that, as K has one non-positive minimal diagram, no minimal diagram can be positive (and also it didn't matter which c you chose). My computer program revealed, however, that there is no such a with $|a| \le 26$ (where $|a| := \sum_{i=1}^{n} |a_i|$; note, that by minimality of alternating diagrams always $|c| < |a|$). Is there such an a at all?

QUESTION 9.2. Is (something like) the Tait flyping conjecture [MT] true for positive knots, i.e. are minimal positive diagrams transformable by flypes?

QUESTION 9.3. Menasco [Me], [Ad]/Aumann [Ad, p. 150] proved that composite/split alternating links appear composite/split in *any* alternating diagram. Using the linking number, it's easy to see that for split links latter is also true in the positive case. But what is the case with composite knots?

In view of Corollary 4.2, this is a special case of a conjecture of Cromwell [Cr2, Conjecture 1.6]. Note, that answering Questions 9.1 and 9.3 in the affirmative we would prove the additivity of the crossing number for positive knots under connected sum.

QUESTION 9.4. If Question 9.3 has a negative answer, is still the weaker statement true that positive composite knots have (only or at least one) positive prime factor(s)?

QUESTION 9.5. Is it possible to classify alternating positive knots? Does an alternating positive knot always have a (simultaneously) alternating (and) positive diagram? (Note, that this question for prime knots and Question 9.1 for prime alternating knots are the same.)

A question on unknotting numbers is

QUESTION 9.6. Does every positive knot realize its unknotting number in a positive diagram?

If the answer were yes, by arguments analogous to those in the proof of Theorem 6.2, the inequality of Bennequin-Vogel (7) would imply that $u(K) \geq g(K)$ independently from Menasco's announced result, so the question is consistent with it.

A final question is suggested by the comparison between the growth rates of v_2 and v_3 on positive knots.

QUESTION 9.7. What can be said about the sets

$$S := \{ \log_{v_2(K)} v_3(K) \;:\; K \neq ! 3_1 \text{ positive} \}$$

and

$$SB := \{ \log_{v_2(K)} v_3(K) \;:\; K \neq ! 3_1 \text{ braid positive} \}\,?$$

We have shown that $1 \in \tilde{S} = \bar{S} \backslash \operatorname{disc} S \subset [1, 3]$, and similarly for $\widetilde{SB}$, where $\bar{S}$ denotes closure and $\operatorname{disc} S$ the subset of discrete points of S and '$\subset$' denotes a not necessarily proper inclusion. Is $\tilde{S} \subset [1, 2]$ or even $\tilde{S} = [1, 2]$? Is $\widetilde{SB}$ equal to or at least contained in $[1, 3/2]$?

REFERENCES

[A] N. A'CAMPO, *Generic immersions of curves, knots, monodromy and gordian number*, Inst. Hautes Études Sci. Publ. Math. **88** (1998), 151–169.

[Ad] C. C. ADAMS, "The knot book", W. H. Freeman & Co., New York, 1994.

[AM] S. AKBULUT – J. D. MCCARTHY, "Casson's invariant for oriented 3-spheres", Mathematical notes **36**, Princeton, 1990.

[Al] J. W. ALEXANDER, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), 275–306.

[BN] D. BAR-NATAN, *On the Vassiliev knot invariants*, Topology **34** (1995), 423-472.

[BN2] D. BAR-NATAN, *Bibliography of Vassiliev invariants*, available from the web site http://www.math.toronto.edu/drorbn/VasBib/VasBib.html.

[BS] D. BAR-NATAN – A. STOIMENOW, *The Fundamental Theorem of Vassiliev invariants*, In “Geometry and Physics”, Lecture Notes in Pure & Appl.Math. **184**, M.Dekker, New York, 1996, 101-134.

[Be] D. BENNEQUIN, *Entrelacements et équations de Pfaff*, Soc. Math. de France, Astérisque **107-108** (1983), 87-161.

[Bi] J. S. BIRMAN, “Braids, links and mapping class groups”, Ann. of Math. Studies **82**, Princeton, 1976.

[Bi2] J. S. BIRMAN, *New Points of View in Knot Theory*, Bull. Amer. Math. Soc. **28** (1993), 253-287.

[BL] J. S. BIRMAN – X. S. LIN, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), 225-270.

[BM] J. S. BIRMAN – W. W. MENASCO, *Studying knots via braids V: The unlink*, Trans. Amer. Math. Soc. **329** (1992), 585-606.

[BW] J. S. BIRMAN – R. F. WILLIAMS, *Knotted periodic orbits in dynamical systems - I, Lorenz's equations*, Topology **22** (1983), 47-82.

[BoW] M. BOILEAU – C. WEBER, *Le problème de J. Milnor sur le nombre gordien des nœuds algébriques*, Enseign. Math. **30** (1984), 173-222.

[BLM] R. D. BRANDT – W. B. R. LICKORISH – K. MILLETT, *A polynomial invariant for unoriented knots and links*, Invent. Math. **84** (1986), 563-573.

[Bu] J. V. BUSKIRK, *Positive links have positive Conway polynomial*, Springer Lecture Notes in Math. **1144** (1983), 146-159.

[CG] T. D. COCHRAN – R. E. GOMPF, *Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and Property P*, Topology **27** (1988), 495-512.

[Co] J. H. CONWAY, *On enumeration of knots and links*, In “Computational Problems in abstract algebra”, J. Leech (ed.), 329-358. Pergamon Press, 1969.

[Cr] P. R. CROMWELL, *Homogeneous links*, J. London Math. Soc. (series 2) **39** (1989), 535-552.

[Cr2] P. R. CROMWELL, *Positive braids are visually prime*, Proc. London Math. Soc. **67** (1993), 384-424.

[CM] P. R. CROMWELL – H. R. MORTON, *Positivity of knot polynomials on positive links*, J. Knot Theory Ramif. **1** (1992), 203-206.

[DT] C. H. DOWKER – M. B. THISTLETHWAITE, *Classification of knot projections*, Topol. Appl. **16** (1983), 19-31.

[Fi] T. FIEDLER, *On the degree of the Jones polynomial*, Topology **30** (1991), 1-8.

[Fi2] T. FIEDLER, *A small state sum for knots*, Topology **32** (1993), 281-294.

[Fi3] T. FIEDLER, “Gauss sum invariants for knots and links”, Kluwer Academic Publishers, Mathematics and Its Applications Vol. **532**, 2001.

[Fi4] T. FIEDLER, *Die Casson-Invariante eines positiven Knotens ist nicht kleiner als sein Geschlecht*, talk given at the knot theory workshop in Siegen, Germany, 1993.

[FS] T. FIEDLER – A. STOIMENOW, *New knot and link invariants*, Proceedings of the International Conference on Knot Theory “Knots in Hellas, 98”, Series on Knots and Everything **24**, World Scientific, 2000.

[FW] J. FRANKS – R. F. WILLIAMS, *Braids and the Jones-Conway polynomial*, Trans. Amer. Math. Soc. **303** (1987), 97-108.

[H] P. FREYD – J. HOSTE – W. B. R. LICKORISH – K. MILLETT – A. OCNEANU – D. YETTER, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239-246.

[Ga] D. GABAI, *Genera of the alternating links*, Duke Math. J. **53** (1986), 677-681.

[Ho] C. F. HO, *A polynomial invariant for knots and links – preliminary report*, Abstracts Amer. Math. Soc. **6** (1985), 300.

[HT] J. HOSTE – M. THISTLETHWAITE, *KnotScape*, a knot polynomial calculation and table access program, available at http://www.math.utk.edu/~morwen.

[J] V. F. R. JONES, *A polynomial invariant of knots and links via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103-111.

[J2] V. F. R. JONES, *Hecke algebra representations of of braid groups and link polynomials*, Ann. of Math. **126** (1987), 335-388.

[K] T. KANENOBU, *Kauffman polynomials for 2-bridge knots and links*, Yokohama Math. J. **38** (1991), 145-154.

[K2] T. KANENOBU, *Examples of polynomial invariants for knots and links*, Math. Ann. **275** (1986), 555-572.

[K3] T. KANENOBU, *An evaluation of the first derivative of the Q polynomial of a link*, Kobe J. Math. **5** (1988), 179-184.

[KM] T. KANENOBU – H. MURAKAMI, *2-bridge knots of unknotting number one*, Proc. Amer. Math. Soc. **98(3)** (1986), 499-502.

[Ka] L. H. KAUFFMAN, “Knots and physics” (second edition), World Scientific, Singapore, 1993.

[Ka2] L. H. KAUFFMAN, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417-471.

[Ka3] L. H. KAUFFMAN, *New invariants in the theory of knots*, Amer. Math. Mon. **3** (1988), 195-242.

[Km] T. KAWAMURA, *The unknotting numbers of* 10_{139} *and* 10_{152} *are 4*, Osaka J. Math. **35**, (3) (1998), 539-546.

[Km2] T. KAWAMURA, *Relations among the lowest degree of the Jones polynomial and geometric invariants for a closed positive braid*, Comment. Math. Helv. **77** (1), (2002), 125-132.

[Kw] A. KAWAUCHI, “A survey of Knot Theory”, Birkhäuser, Basel-Boston-Berlin, 1991.

[KMr] P. B. KRONHEIMER – T. MROWKA, *On the genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994), 797-808.

[Li] W. B. R. LICKORISH, *The unknotting number of a classical knot*, In “Contemporary Mathematics” **44** (1985), 117-119.

[L] X.-S. LIN, *Finite type link invariants of 3-manifolds*, Topology **33**, (1) (1994), 45-71.

[Me] W. W. MENASCO, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1986), 37-44.

[Me2] W. W. MENASCO, *The Bennequin-Milnor Unknotting Conjectures*, C. R. Acad. Sci. Paris Sér. I Math. **318** (1994), 831-836.

[MT] W. W. MENASCO – M. B. THISTLETHWAITE, *The Tait flyping conjecture*, Bull. Amer. Math. Soc. **25** (1991), 403-412.

[Mi] J. MILNOR, "Singular points of complex hypersurfaces", Annals of Math. Studies **61** (1968).

[Mo] H. R. MORTON, *An irreducible 4-string braid with unknotted closure*, Math. Proc. Camb. Phil. Soc. **93** (1983), 259-261.

[Mo2] H. R. MORTON, *Seifert circles and knot polynomials*, Proc. Cambridge Philos. Soc. **99** (1986), 107-109.

[Mu] K. MURASUGI, *Jones polynomial and classical conjectures in knot theory*, Topology **26** (1987), 187-194.

[MP] K. MURASUGI – J. PRZYTYCKI, *The skein polynomial of a planar star product of two links*, Math. Proc. Cambridge Philos. Soc. **106** (1989), 273-276.

[N] T. NAKAMURA, *Positive alternating links are positively alternating*, J. Knot Theory Ramif. **9**, (1) (2000), 107-112.

[N2] T. NAKAMURA, *Four-genus and unknotting number of positive knots and links*, Osaka J. Math. **37**, (2) (2000), 441-451.

[Oh] Y. OHYAMA, *On the minimal crossing number and the braid index of links*, Canad. J. Math. **45**, (1) (1993), 117-131.

[PV] M. POLYAK – O. VIRO, *Gauss diagram formulas for Vassiliev invariants*, Int. Math. Res. Notes **11** (1994), 445-454.

[PV2] M. POLYAK – O. VIRO, *On the Casson knot invariant*, J. Of Knot Theory and Its Ram. **10** (2001) (Special volume of the International Conference on Knot Theory "Knots in Hellas, 98"), 711-738.

[Ro] D. ROLFSEN, "Knots and links", Publish or Perish, 1976.

[Ru] L. RUDOLPH, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helv. **58** (1983), 1-37.

[Ru2] L. RUDOLPH, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. **29** (1993), 51-59.

[Ru3] L. RUDOLPH, *Positive links are strongly quasipositive*, "Geometry and Topology Monographs" **2** (1999), Proceedings of the Kirbyfest, 555-562. See also http://www.maths.warwick.ac.uk/gt/GTMon2/ paper25.abs.html.

[St] A. STOIMENOW, *Gauss sum invariants, Vassiliev invariants and braiding sequences*, J. Knot Theory Ramif. **9** (2000), 221-269.

[St2] A. STOIMENOW, *Polynomials of knots with up to 10 crossings*, available on http://www.math.toronto. edu/stoimeno/.

[St3] A. STOIMENOW, *Genera of knots and Vassiliev invariants*, J. Of Knot Theory and Its Ram. **8** (2) (1999), 253-259.

[St4] A. STOIMENOW, *A Survey on Vassiliev Invariants for knots*, "Mathematics and Education in Mathematics", Proceedings of the XXVII. Spring Conference of the Union of Bulgarian Mathematicians, 1998, 37-47.

[St5] A. STOIMENOW, *On some restrictions to the values of the Jones polynomial*, preprint.

[St6] A. STOIMENOW, *Polynomial values, the linking form and unknotting numbers*, preprint.

[St7] A. STOIMENOW, *Knots of genus one*, Proc. Amer. Math. Soc. **129**, (7) (2001), 2141-2156.

[Ta] T. TANAKA, *Unknotting numbers of quasipositive knots*, Topology and its Applications **88**, (3) (1998), 239-246.

[Th] M. B. THISTLETHWAITE, *A spanning tree expansion for the Jones polynomial*, Topology **26** (1987), 297-309.

[Tr] P. TRACZYK, *Non-trivial negative links have positive signature*, Manuscripta Math. **61** (1988), 279-284.

[Tr2] P. TRACZYK, *A criterion for signed unknotting number*, Contemporary Mathematics **233** (1999), 215-220.

[Va] V. A. VASSILIEV, *Cohomology of knot spaces*, "Theory of Singularities and its Applications" (Providence) V. I. Arnold (ed.), Amer. Math. Soc., Providence, 1990.

[Vo] P. VOGEL, *Algebraic structures on modules of diagrams*, to appear in Invent. Math.

[Vo2] P. VOGEL, *Representation of links by braids: A new algorithm*, Comment. Math. Helv. **65** (1990), 104-113.

[We] H. WENDT, *Die Gordische Auflösung von Knoten*, Math. Z. **42** (1937), 680-696.

[Wi] S. WILLERTON, *On the first two Vassiliev invariants*, preprint math.GT/0104061.

[Yo] Y. YOKOTA, *Polynomial invariants of positive links*, Topology **31** (1992), 805-811.

[Zu] L. ZULLI, *The rank of the trip matrix of a positive knot diagram*, J. Knot Theory Ramif. **6** (1997), 299-301.

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